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Inflation

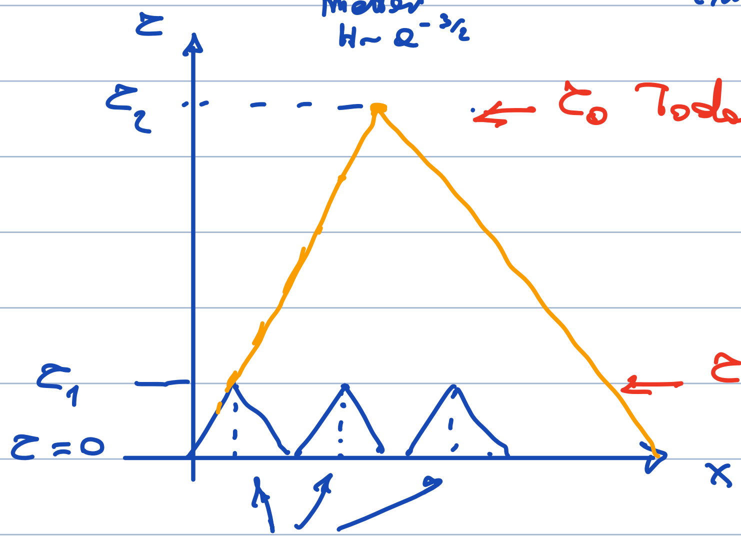
Problems of the Big Bang

1) Horizon

Particle horizon $(K=0)$: $dr = d\tau = \frac{dt}{a} = \frac{1}{aH}\, d\ln a$

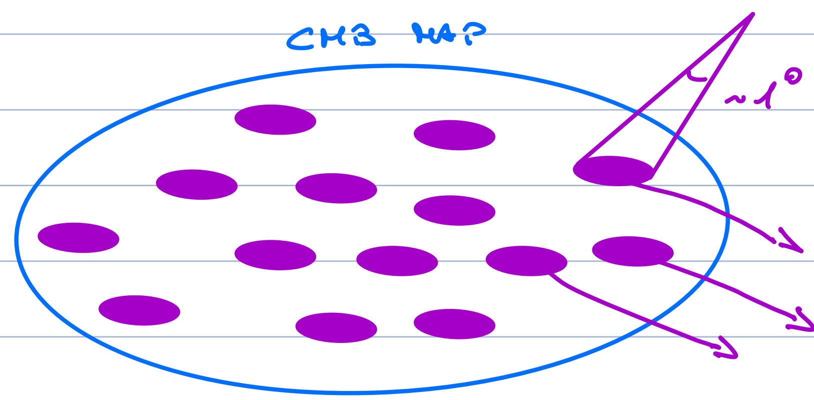
$$r_H \sim \frac{1}{aH} \quad \begin{cases} \text{rad } H\sim a^{-2}: & a \\ \text{matter } H\sim a^{-3/2}: & a^{1/2} \end{cases}$$

grows both in Rad Dom and in matter dom.



Disconnected universes

$$\frac{r_H(z_0)}{r_H(z_{CMB})} = \frac{a\, H_{CMB}}{a_0 H_0} = \left(\frac{1+z_{Dec}}{1}\right)^{1/2} \simeq 33$$



causally disconnected at CMB decoupling

* why T is the same in all the domains?

(later, we found an answer also to "why $\Delta T/T \sim 10^{-5}$?")

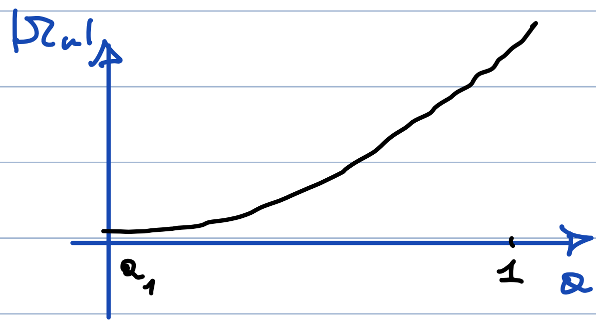
2) Flatness (or kinetic energy problem)

$$|\Omega_K| = \left|1 - \frac{\rho}{\rho_c}\right| = \frac{|K|}{\underbrace{a^2 H^2}}$$ Friedmann eq

$$\rho_c \equiv \frac{3H^2}{8\pi G}$$

same combination as in $r_H \sim \frac{1}{aH}$

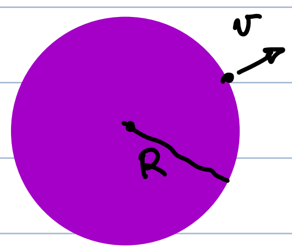
if $K \neq 0$ $\quad |\Omega_K| \begin{matrix} \xrightarrow{R.D.} a^2 \\ \xrightarrow[MD]{} a \end{matrix}$



Flat universe ($K=0$) is an "unstable fixed point"!

Notice that $-K$ can be interpreted as $-K = E_{TOT} = T + V$

kinetic energy $\quad$ potential energy



$$T = \frac{1}{2} v^2 \qquad V = -\frac{GM(R)}{R}$$

$K = 0 \Rightarrow E_{TOT} = E_{escape}$

$K > 0 \Rightarrow E_{TOT} < E_{escape} \rightarrow$ future collapse

$K < 0 \Rightarrow E_{TOT} > E_{escape} \rightarrow$ expand forever

To get $|\Omega| < O(1)$ Today we need

$$|\Omega_{GUT}| = \left(\frac{a_{GUT}}{a_{EQ}}\right)^2 a_{EQ} = \left(\frac{T_{EQ}}{T_{GUT}}\right)^2 \frac{T_0}{T_{eq}} \approx 10^{-54} \left(\frac{10^{16}\,GeV}{T_{GUT}}\right)^2$$

$|\Omega_{ew}| \simeq 10^{-26}$ $|\Omega_{BBN}| \simeq 10^{-16}$ ...

$\begin{cases} \text{Why was the Universe so flat?} \\ \text{" was T so close to } -V\text{ ?} \end{cases}$

3) Unwanted relics $\nearrow$ particles $\searrow$ topological defects

EXAMPLE: GRAVITINOS $T_D \sim \frac{m_G^3}{M_P^2} \to T_D \sim 100\,\text{eV} \left(\frac{m_G}{100\,\text{GeV}}\right)^{3/2}$

$m_G \sim m_p$ Decay into photons $\to$ destroy Deuterium from BBN

EXAMPLE: MONOPOLES from GUT phase transitions

$\rho_{MON} \sim a^{-3}$ dominate over ρ_{RAD} too early!

we need to get rid of all these, very common, unwanted relics.

Inflation: the idea

Look at horizon and flatness problems. They both come from the fact that $\frac{1}{aH}$ ($\propto r_H, |\Omega_k|^{1/2}$) grows in time.

Inflation is an epoch defined by:

$$\boxed{\frac{d}{dt}\left(\frac{1}{aH}\right) < 0} \iff \ddot{a} > 0 \iff -\frac{\dot{H}}{H} < H$$

$H = \frac{\dot{a}}{a}$ $\hookrightarrow$ universe accelerates

Also, since $a \sim t^{\frac{2}{3(w+1)}}$ $\quad \ddot{a} \sim -(1+3w) > 0 \Rightarrow \boxed{w < -1/3}$

Finally, define $\frac{\ddot{a}}{a} = H^2(1-\varepsilon)$ $\quad \boxed{\text{INFLATION} \Leftrightarrow \varepsilon < 1}$

$\hookrightarrow -\frac{\dot{H}}{H^2}$

How inflation solves problems: consider $w = -1$ for definiteness

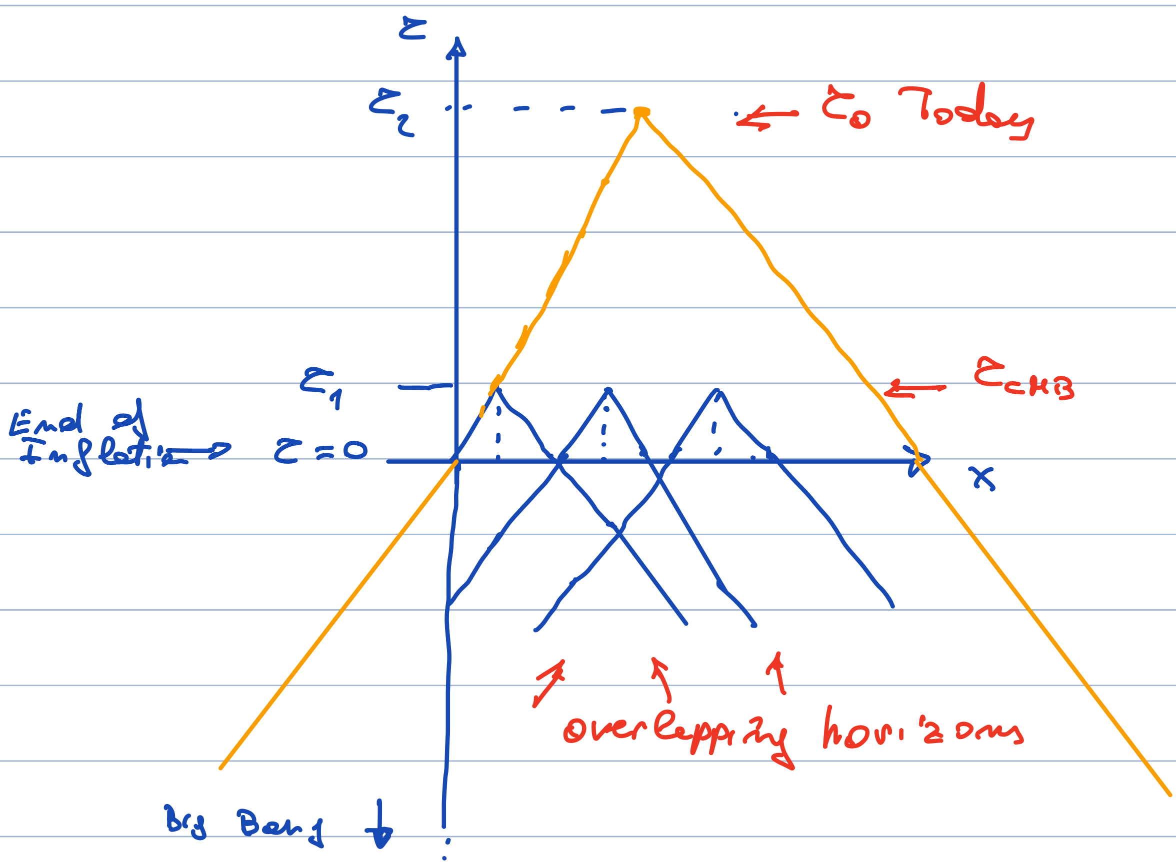
$\hookrightarrow$ de Sitter

$w = -1 \to H^2 \propto \rho \sim a^{-3(w+1)}$ constant $\to \frac{1}{aH} \sim 1/a$

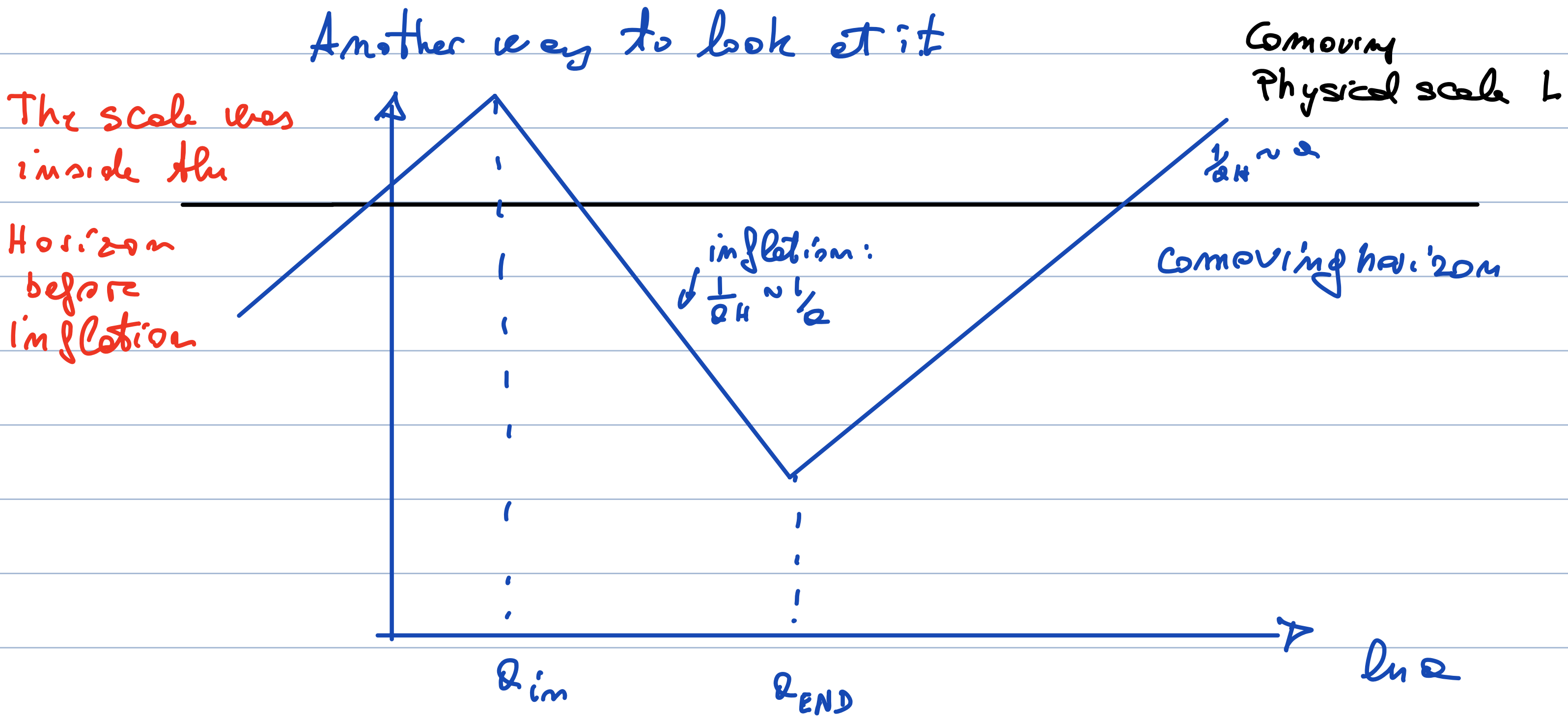
$$\tau_f - \tau_i = \int_{t_i}^{t_f} \frac{dt}{a(t)} = \int_{a_i}^{a_f} \frac{1}{aH}\frac{da}{a} \underset{w=-1}{=} \frac{1}{H}\left(\frac{1}{a_f} - \frac{1}{a_{in}}\right)$$

$$\boxed{\tau = -\frac{1}{Ha} + \text{const}}$$

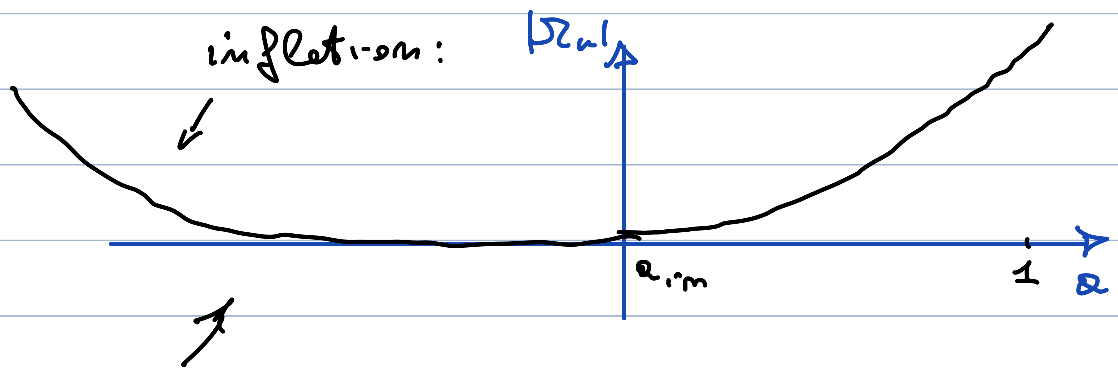
Big Bang $a \to 0 \Rightarrow \tau \to -\infty$!



Another way to look at it



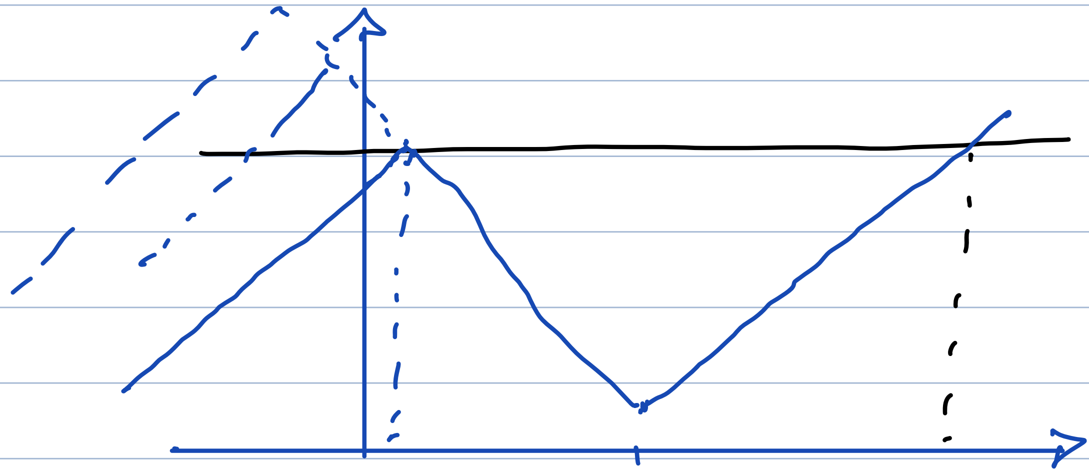
For flatness problem: $|\Omega| \sim \frac{1}{a^2}$ during inflation



$|\Omega|_k \to 0$ attractive fixed point!

How much inflation?

Requirement: the horizon scale today was inside the horizon at the beginning of inflation



Maximum acceptable a_{IN}

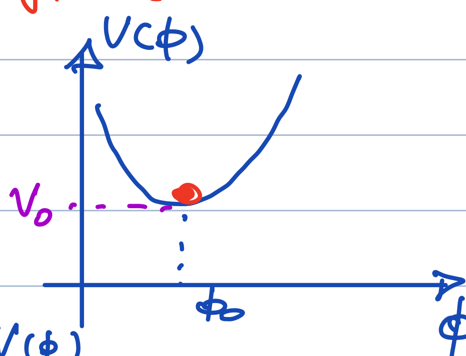
$N = \ln a$ e-folding number

$$\Delta N_{TOT} = N_{END} - N_{IN} = \ln \frac{a_{END}}{a_{IN}} \gtrsim 60-70$$

acceptable inflation

How to make the Universe inflate

Simplest possible way:



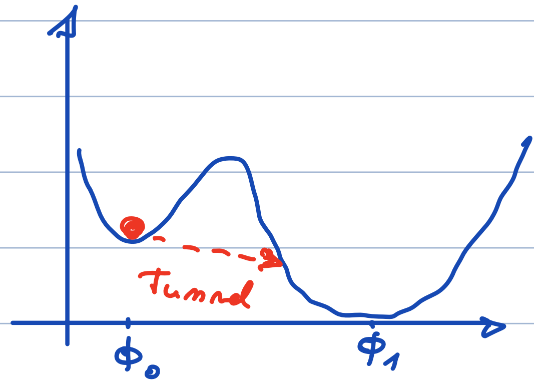
For a scalar field $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$

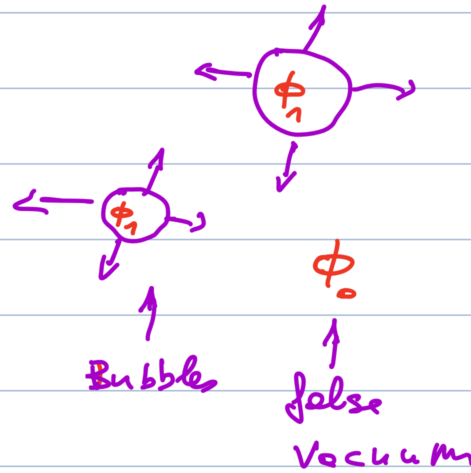
$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$\Rightarrow$ in ϕ_0 $\quad w = \dfrac{P}{\rho} = \dfrac{-V_0}{V_0} = -1$ $\rightarrow$ De Sitter!

The problem is that this inflation never ends!

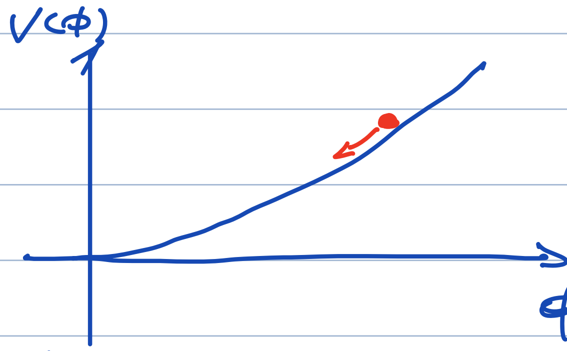
Try with:





does not work either: bubbles do not "percolate" because the false vacuum is expanding

Slow-roll inflation



$$w = \frac{\frac{\dot\phi^2}{2} - V}{\frac{\dot\phi^2}{2} + V} \qquad \frac{\dot\phi^2}{2} \ll V \longrightarrow w \simeq -1 \qquad \text{SLOW ROLL}$$

Eq of motion:

$$\begin{cases} H^2 = \dfrac{8\pi G}{3}\left(\dfrac{\dot\phi^2}{2} + V(\phi)\right) \\ \ddot\phi + 3H\dot\phi + \dfrac{dV}{d\phi} = 0 \end{cases}$$

$\phi = \phi(t)$

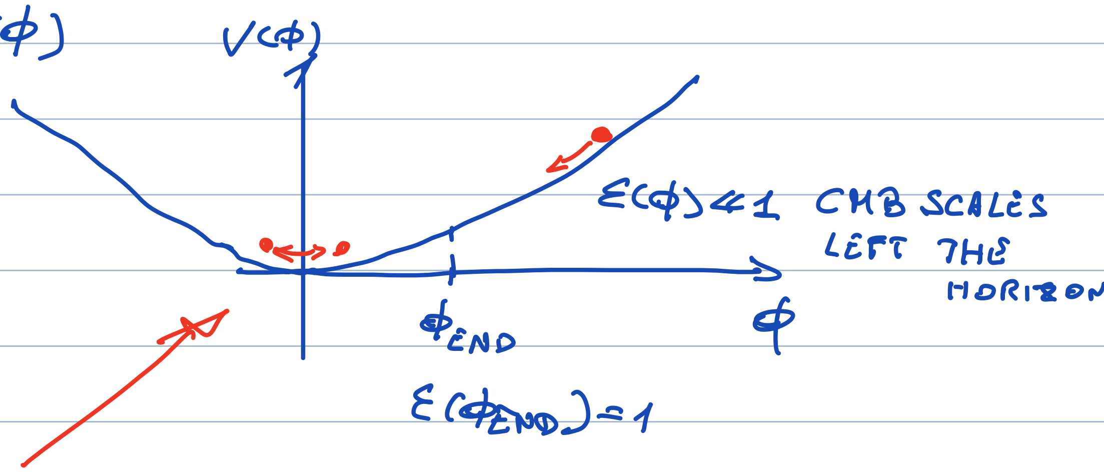
uniform field

Slow-roll approx:

$$\begin{cases} H^2 \simeq \dfrac{8\pi G}{3} V \\ \dot\phi \simeq -\dfrac{1}{3H}\dfrac{dV}{d\phi} \end{cases}$$

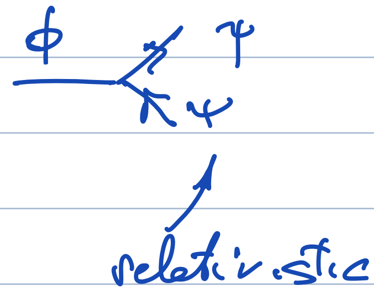
$$\Rightarrow \quad \varepsilon = -\frac{\dot H}{H^2} \simeq \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \qquad 8\pi G = \frac{1}{M_P^2}$$

$$\Rightarrow \quad \varepsilon = \varepsilon(\phi)$$



End of inflation → reheating: coherent oscillations with damping caused by coupling with other fields

Ex:



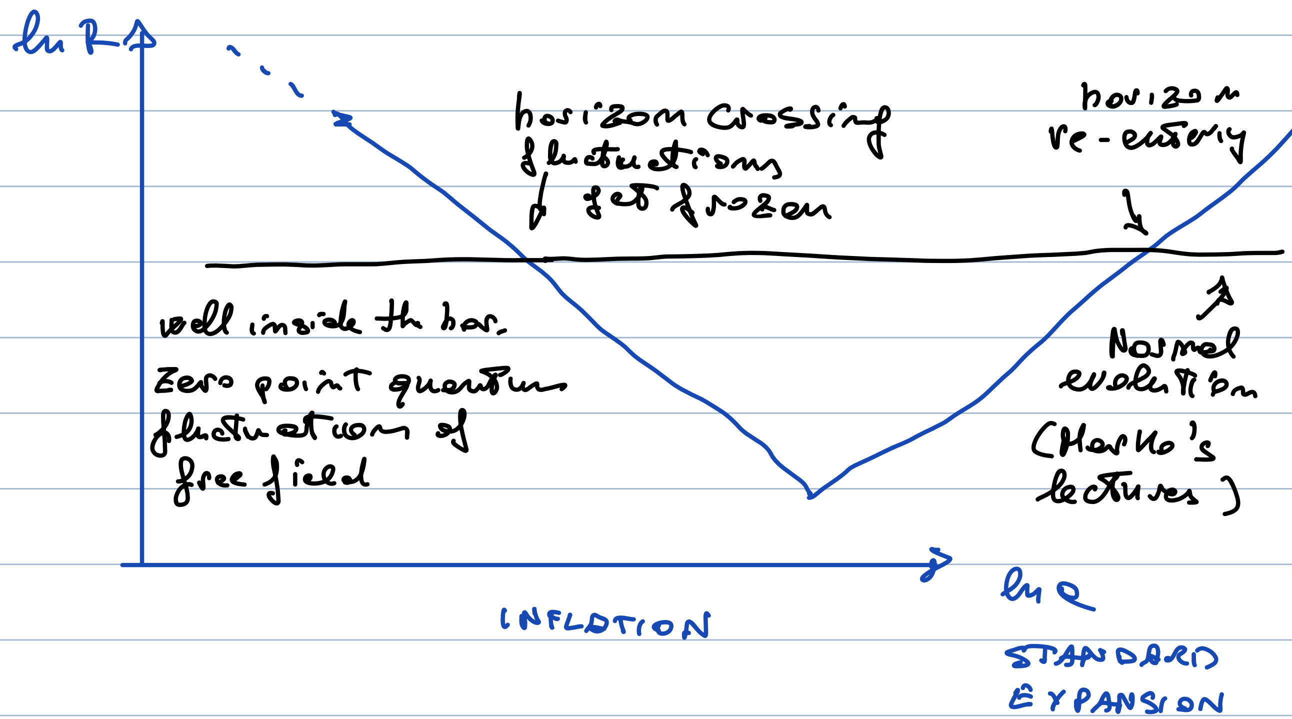
Γ_ϕ $\quad \rightarrow \quad \ddot\phi + (3H + \Gamma_\phi)\dot\phi + V_{,\phi} = 0$

Instant reheating $\quad \rho_\psi \sim T_{RH}^4 \sim V_\phi \quad \left(\text{if } \Gamma_{RH} > H(T_{RH})\right)$

Cosmological perturbations at inflation

The inflaton is a quantum field. Its quantum fluctuations perturbe the homogeneity of FLRW!

Intuitive picture:



Scalar field in de Sitter background:

$$\ddot{\varphi} + 3H\dot{\varphi} - a^{-2}\nabla^2\varphi + \frac{\partial V}{\partial \varphi} = 0 \qquad \varphi(\vec{x},t) = \bar{\varphi}(t) + \delta\varphi(\vec{x},t)$$

- Fourier $\delta\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{x}}\, \delta\varphi_k(t)$
- conformal time $d\tau = \frac{1}{aH}\, d\ln a \xrightarrow[H=\text{const}]{} \tau = -\frac{1}{aH} \in [-\infty, 0]$
- $\chi_k(\tau) \equiv a\, \delta\varphi_k(\tau)$

$\Rightarrow$ Mukhanov-Sasaki eq: $\chi_k'' + \omega_k^2(\tau)\chi_k = 0$

$( )' \equiv \frac{d}{d\tau}$

$$\omega_k^2(\tau) \equiv k^2 - \frac{2}{\tau^2} = k^2 - 2a^2H^2$$

$aH \ll k$ $(\tau \to -\infty)$ — scale $1/k$ is subhorizon $\Rightarrow$ $\chi_k'' + k^2\chi_k = 0$

harmonic oscillator

$\chi_k \sim \frac{e^{-ik\tau}}{\sqrt{2k}}$

$aH \gg k$ $(\tau \to 0)$ — scale $1/k$ is superhorizon $\Rightarrow$ $\chi_k'' = \frac{2}{\tau^2}\chi_k$

$\Rightarrow \chi_k \sim \frac{-i}{\sqrt{2k}} \frac{1}{k\tau}$

$$\Rightarrow |\delta\varphi_k|^2 \sim \left|\frac{\chi_k}{a}\right|^2 \sim \frac{1}{a^2\tau^2} \sim H^2$$

~ constant in time (frozen)

~ scale independent (almost, as H evolves slowly)

$$\Rightarrow \quad P_{\delta\varphi}(k) = \frac{k^3}{2\pi^2}|\delta\varphi_k|^2 \xrightarrow[aH \gg k]{} \left(\frac{H_k}{2\pi}\right)^2$$

power spectrum

value of H when $aH = k$

$$\bar{\varphi}(t) = \bar{\varphi}(t + \delta t(\vec{x}, t)) \simeq \bar{\varphi}(t) + \underbrace{\dot{\bar{\varphi}}\,\delta t}_{\delta\varphi(\vec{x}, t)}$$

FLRW $\quad a(t) \to a(t+\delta t) = a(t) + \frac{\dot{a}}{a} a\,\delta t$

$$= a(t)\left(1+\zeta(\vec{x},t)\right)$$

$\zeta(\vec{x},t) = H\,\delta t(\vec{x},t)$ curvature perturbation

$$= H\frac{\delta\varphi}{\dot{\varphi}}$$

$$= H\frac{\delta\rho}{\dot{\rho}}$$

$$\boxed{P_\zeta = \frac{H^2}{\dot{\varphi}^2} P_{\delta\varphi} = \frac{1}{8\pi^2\varepsilon}\frac{H^2}{M_{Pl}^2}\Bigg|_{k=aH}}$$

depends on k through H and ε

$$\boxed{\frac{d\ln P_\zeta}{d\ln k} \equiv n_s - 1 = -2\varepsilon - \eta}$$

$\eta = \varepsilon - \frac{1}{2\varepsilon}\frac{d\varepsilon}{d\ln a}$

$$P_\zeta = A_s\left(\frac{k}{k_*}\right)^{n_s-1}$$

Amplitude $= P_\zeta(k_*)$ $\quad$ ↳ Pivot scale

Gravitational wave background

Tensor perturbations of the metric obey the Mukhanov Sasaki equation too. (each polarization separately)

$$\Rightarrow P_h(k) = \frac{8}{M_{Pl}^2}\left(\frac{H_k}{2\pi}\right)^2$$

Note that it is ε-independent

→ directly related to ε

Observational constraints :

From CMB :

Amplitude $\underbrace{\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle}_{\sim P_s} \sim 10^{-10} \Rightarrow \left(\frac{V}{\varepsilon}\right)^{1/4} \simeq 6.6 \cdot 10^{16}\ \mathrm{GeV}$

spectral tilt:

$$n_s - 1 \simeq 2\eta - 6\varepsilon \simeq -0.0335 \pm 0.0038$$

running $\frac{d n_s}{d \ln k}$ = compatible with zero

Tensor-to scalar ratio $r = \frac{P_h}{P_s} < 0.065$