Collaborative Research Center TRR 257

Particle Physics Phenomenology after the Higgs Discovery

New facets of pySecDec

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MathemAmplitudes

Padova, Sep 27, 2023

Karlsruhe Institute of Technology

based on work in collaboration with

Stephen Jones, Matthias Kerner, Vitaly Magerya,
Anton Olsson, Johannes Schlenk, et al.

https://arxiv.org/abs/2305.19768

https://arxiv.org/abs/2108.10807

https://secdec.readthedocs.io

also (not my work): https://arxiv.org/abs/2211.14845

pySecDec Collaboration 2023

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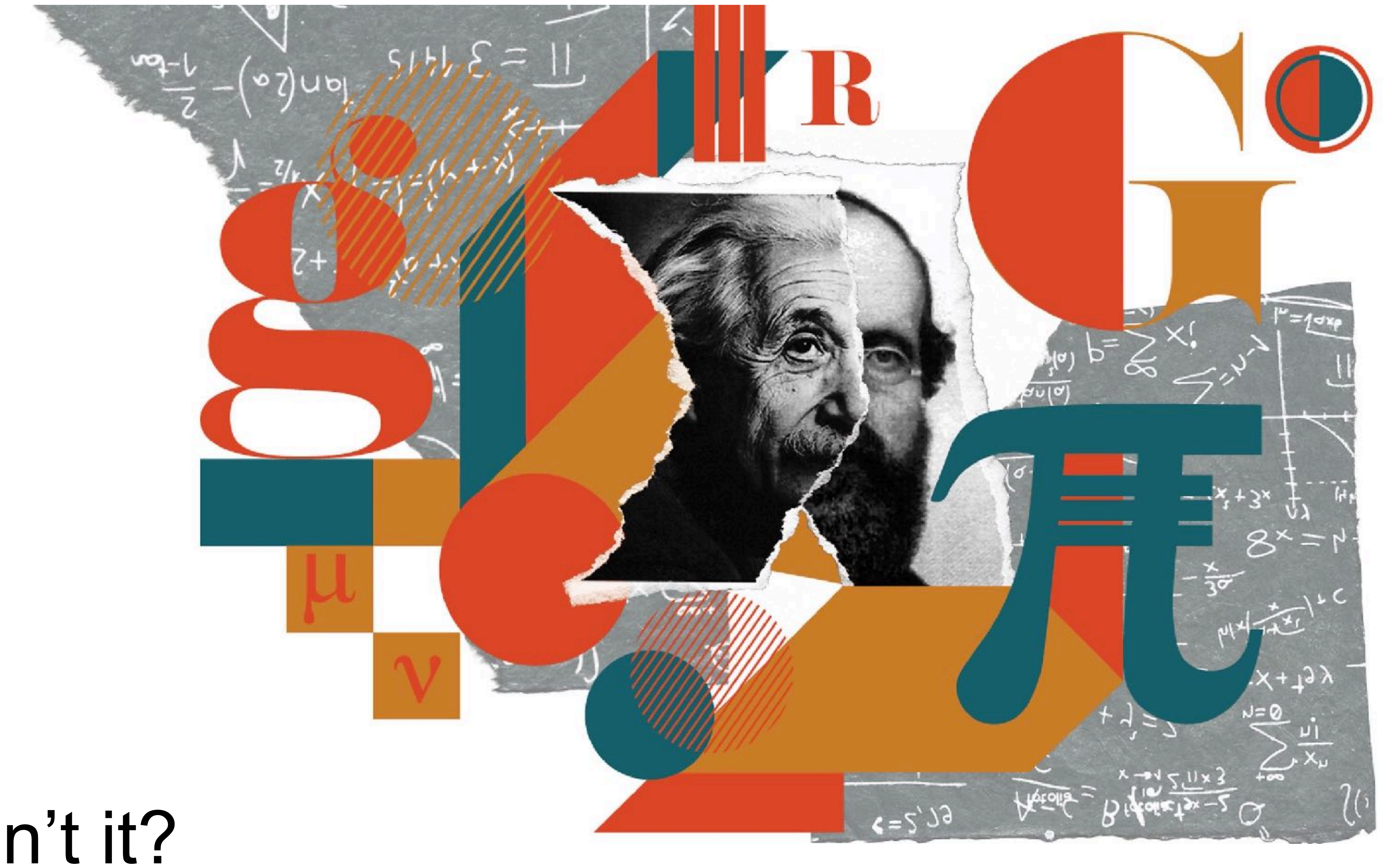
Motivation

- The interplay between mathematics and physics was often fruitful in the history of science

- The story is ongoing, insights gained with scattering amplitudes are a prime example

- However, pySecDec is just number crunching … isn't it?



Artwork by Sandbox Studio, Chicago for The Symmetry Magazine

Geometric formulation of sector decomposition

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Feynman integral in the Lee-Pomeransky representation in D space-time dimensions:

$$I(\nu_1 \ldots \nu_N) = \frac{(-1)^{N_\nu}\Gamma(D/2)}{\Gamma((L+1)D/2 - N_\nu)\prod_j \Gamma(\nu_j)} \int_0^\infty \left(\prod_{j=1}^{N} dz_j\, z_j^{\nu_j - 1}\right) (\mathcal{U}+\mathcal{F})^{-D/2}$$

$$\mathcal{U}(\vec{x}) = \sum_{T\in\mathcal{T}_1}\Big[\prod_{j\in\mathcal{C}(T)} x_j\Big] \;,\quad \mathcal{F}_0(\vec{x}) = \sum_{\hat{T}\in\mathcal{T}_2}\Big[\prod_{j\in\mathcal{C}(\hat{T})} x_j\Big](-s_{\hat{T}}) \;,\quad \mathcal{F}(\vec{x}) = \mathcal{F}_0(\vec{x}) + \mathcal{U}(\vec{x})\sum_{j=1}^{N} x_j m_j^2 \;,\quad N_\nu = \sum_{i=1}^{N}\nu_i$$

structure: $$I \sim \int_{\mathbb{R}^N_{>0}} \frac{\mathrm{d}\mathbf{x}}{\mathbf{x}}\, \mathbf{x}^{\nu} \left[\sum_{i=1}^{m} c_i \mathbf{x}^{\mathbf{p}_i}\right]^{-\frac{D}{2}}$$

important object:

Newton polytope

defined by exponent vectors $\mathbf{p}_i$

integral over a polynomial to some power, $\mathbf{x}^{\mathbf{a}} = \prod_{j=1}^{N} x_j^{a_j}$

Geometric formulation of sector decomposition

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structure:

$$I \sim \int_{\mathbb{R}^N_{>0}} \frac{\mathrm{d}\mathbf{x}}{\mathbf{x}}\, \mathbf{x}^{\nu} \left[\sum_{i=1}^m c_i \mathbf{x}^{\mathbf{p}_i}\right]^{-\frac{D}{2}}$$

integral over a polynomial to some power, $\mathbf{x}^{\mathbf{a}} = \prod_{j=1}^N x_j^{a_j}$

important object:

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see talks by Felix Tellander, Claudia Fevola, Simon Telen

Geometric formulation of sector decomposition

Karlsruhe Institute of Technology

Newton polytope:

$$\mathcal{N}(I) = \mathrm{convHull}\,(\mathbf{p}_1, \mathbf{p}_2, \dots) = \left\{ \sum_j \alpha_j \mathbf{p}_j \;\middle|\; \alpha_j \geq 0 \wedge \sum_j \alpha_j = 1 \right\}$$

can be written as intersection of hyperplanes

$$\mathcal{N}(I) = \bigcap_{f \in F} \{\mathbf{m} \in \mathbb{R}^{N+1} \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0\} \quad , \quad a_f \in \mathbb{Z}$$

F : set of polytope facets with inward-pointing normal vectors, $\mathbf{n}_f$: normal vectors

Geometric formulation of sector decomposition

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- a cone σ is defined as $$\sigma = \bigcap_{f \in F} \{\mathbf{m} \in \mathbb{R}^{N+1} \mid \langle \mathbf{m}, \mathbf{n}_f \rangle \geq 0\}$$

- cones are simplicial if their extreme rays are linearly independent, otherwise a triangulation should be performed

- the set of simplicial cones forms the basis for the sector functions

- the normal vectors define local coordinates on each facet of the simplicial cones $$x_i = \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{e}_i \rangle}$$

- this transformation leads to the decomposed form

$$I \sim \sum_{\sigma \in \Delta_N^T(f),\, \mathrm{dim}\sigma = N} \left(\prod_{f \in \sigma} \int_0^1 \frac{\mathrm{d}y_f}{y_f}\, y_f^{\langle \mathbf{n}_f, \nu \rangle + a_f \frac{D}{2}} \right) \left[\sum_i c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{p}_i \rangle + a_f} \right]^{-\frac{D}{2}}$$

Bogner, Weinzierl 2007
Kaneko, Ueda 2009
Schlenk 2016

Geometric formulation of sector decomposition

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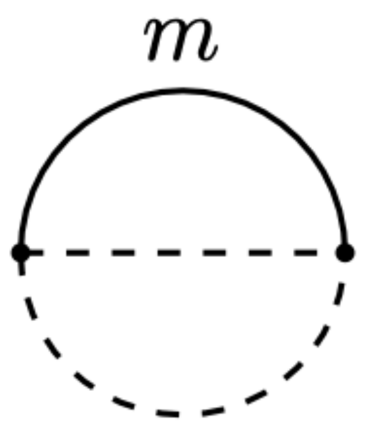
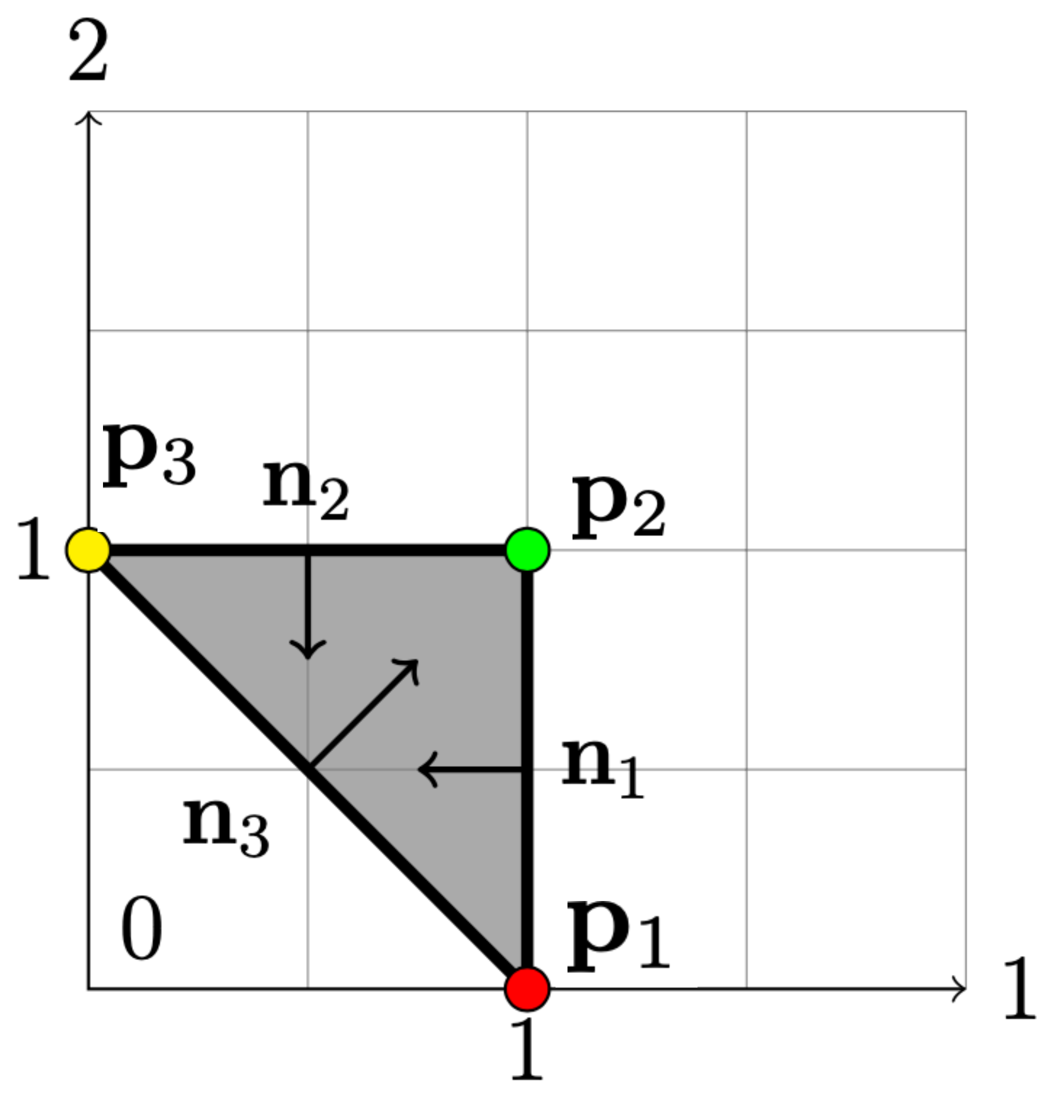
Example:



$$I = \text{[diagram]} = \frac{(-1)^{\nu}\Gamma(\nu - LD/2)}{(m^2)^{\nu - LD/2}\prod_i \Gamma(\nu_i)} \int_0^\infty \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{x_1 x_2} x_1^{\nu_1} x_2^{\nu_2} \left(x_1^1 x_2^0 + x_1^1 x_2^1 + x_1^0 x_2^1\right)^{-\frac{D}{2}}$$

Johannes Schenk '16



$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \mathbf{p}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \mathbf{p}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; ; \; \mathbf{n}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \; \mathbf{n}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \; \mathbf{n}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$a_1 = 1\,, \; a_2 = 1\,, \; a_3 = -1$$

maximal cones are defined by $\{\mathbf{n}_3, \mathbf{n}_1\}, \{\mathbf{n}_1, \mathbf{n}_2\}, \{\mathbf{n}_2, \mathbf{n}_3\}$

incident to vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$

define variable transformations, e.g. $\mathbf{p}_1 : x_1 = y_1^{-1} y_3^1,\; x_2 = y_1^0 y_3^1$

leading to

$$I = \frac{(-1)^{\nu}\Gamma(\nu - LD/2)}{(m^2)^{\nu - LD/2}\prod_i \Gamma(\nu_i)} \int_0^1 \frac{\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3}{y_1 y_2 y_3} y_1^{-\nu_1 + \frac{D}{2}}\, y_2^{-\nu_2 + \frac{D}{2}}\, y_3^{\nu_1 + \nu_2 - \frac{D}{2}}$$
$$(y_1 + y_2 + y_3)^{-\frac{D}{2}} \left[\delta(1 - y_2) + \delta(1 - y_3) + \delta(1 - y_1)\right]$$

Expansion by regions

pioneered by Beneke, Smirnov '97; see also Pak, Smirnov '10; Jantzen '11

idea:

- exploit hierarchies between kinematic scales
- expand integrand in small parameter, e.g. m^2/p^2

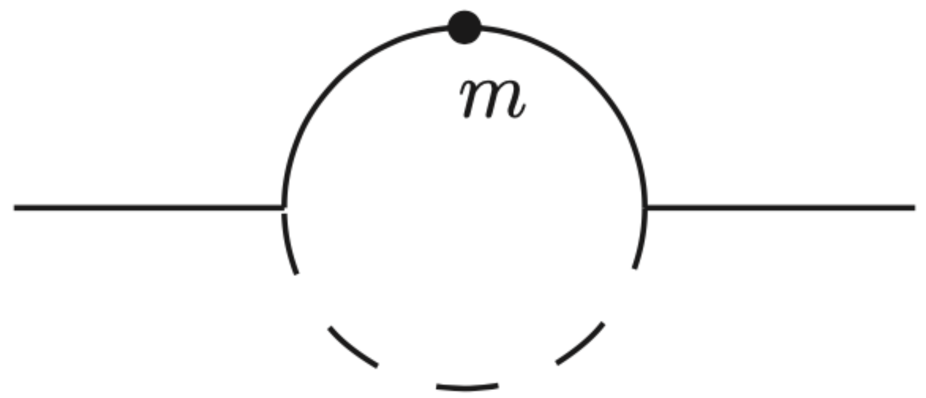
→ integrals easier to evaluate

under certain conditions:

integrating expanded integrands over full integration range and summing over all regions gives **full result**

Expansion by regions in momentum space

Example $$I_2 = \mu^{2\epsilon} \int d\kappa \frac{1}{(k+p)^2(k^2-m^2)^2}$$



$d\kappa = d^D k / i\pi^{\frac{D}{2}}$

two regions: hard: $|k^2| \gg m^2$ soft: $|k^2|, |k \cdot p| \ll p^2$

$$(h): \quad \frac{1}{(k+p)^2(k^2-m^2)^2} \to \frac{1}{(k+p)^2\,(k^2)^2}\left(1 + 2\frac{m^2}{k^2} + \ldots\right)$$

$$(s): \quad \frac{1}{(k+p)^2(k^2-m^2)^2} \to \frac{1}{p^2\,(k^2-m^2)^2}\left(1 - \frac{k^2+2p\cdot k}{p^2} + \ldots\right)$$

Geometric formulation of expansion by regions

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polynomials contain additional "smallness parameter" *t, e.g.* m^2/s in small mass expansion

$$P(\mathbf{x},t) = \sum_{i=1}^{m} c_i t^{p_{i,0}} x_1^{p_{i,1}} \ldots\, x_N^{p_{i,N}} \qquad\qquad c_i \geq 0$$

$$I = \int_0^\infty \frac{\mathrm{d}\mathbf{x}}{\mathbf{x}} t^{\nu_0} \mathbf{x}^{\nu} \left[\sum_{i=1}^{m} c_i t^{p_{i,0}} \mathbf{x}^{\mathbf{p}_i}\right]^{-\frac{D}{2}}$$

Newton polytope Δ' of the polynomial:

convex hull of exponent vectors $\mathbf{p}_i' \equiv (p_{i,0}, \mathbf{p}_i)$

Expansion by regions in parameter space

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procedure:

- find regions
- expand in smallness parameter *t*
- sum over regions and integrate

automated in FIESTA
A.V. Smirnov et al.
and
ASPIRE
Ananthanarayan et al. '18
and
pySecDec
2108.10807

two ways to do the expansion: (a) $t \to zt\;,\;x_j \to z^{v_j} x_j$

Taylor expand in z, then set z=1

(b) $t \to t\;,\;x_j \to t^{v_j} x_j$

$\mathbf{v} = (1, v_1, \ldots, v_N)$ region vector

Expansion by regions geometrically

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write Newton polytope Δ' as convex hull of exponent vectors $\mathbf{p}'_i \equiv (p_{i,0}, \mathbf{p}_i)$

$F^+ = \{f \in F \mid (\mathbf{n}_f)_0 > 0\}$ facets with normal vectors pointing into positive t-direction

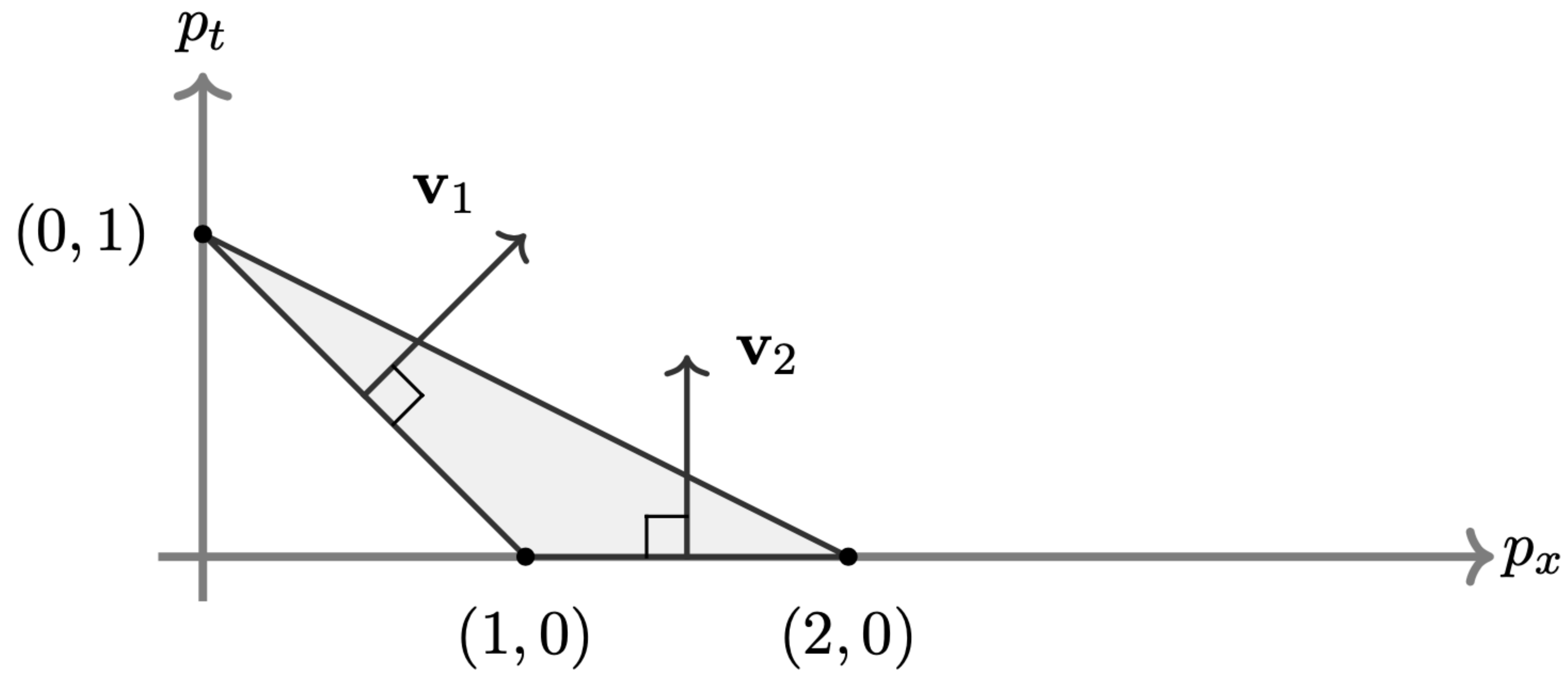
region vectors are given by vectors in F^+ (method of regions projects onto facets of Δ')

change variables $t \to z_f^{(\mathbf{n}_f)_0} t \;,\; x_i \to z_f^{(\mathbf{n}_f)_i} x_i \;,\; f \in F^+$

example:

$$P(x,t) = t + x + x^2$$

$$\mathbf{v}_1 = (1,1), \mathbf{v}_2 = (1,0)$$



Method of regions and pySecDec

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after rescaling with "smallness parameter":

$$I=\left(\prod_{f\in F^+} z_f^{\langle \mathbf{n}_f,\nu'\rangle+\frac{D}{2}a_f}\right)\int_0^\infty \frac{\mathrm{d}\mathbf{x}}{\mathbf{x}}\mathbf{x}^\nu t^{\nu_0}\left[\sum_i c_i \mathbf{x}^{\mathbf{p}_i} t^{p_{i,0}} \prod_{f\in F^+} z_f^{\langle \mathbf{n}_f,\mathbf{p}'_i\rangle+a_f}\right]^{-\frac{D}{2}}$$

$$I=\sum_{f\in F^+} I_f\,,\quad I_f:$$ results of expansion in z_f

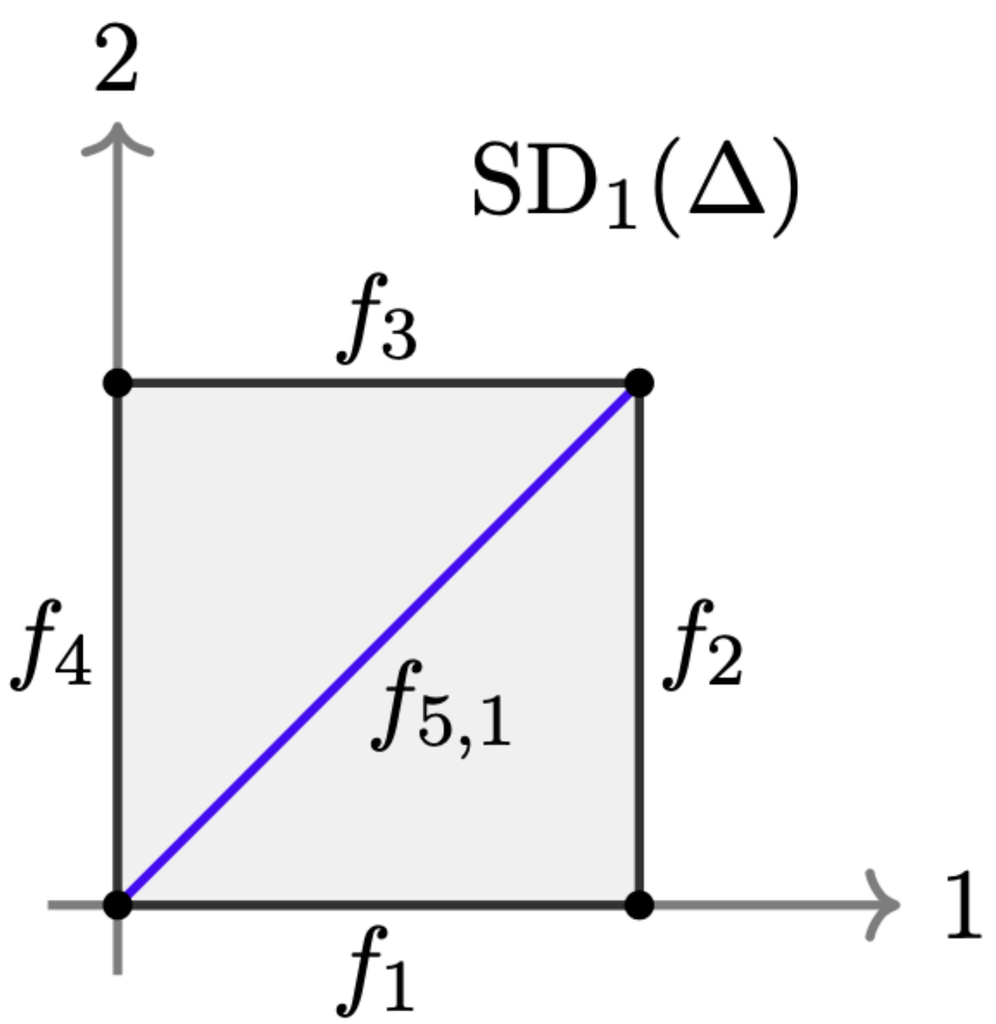
local coordinates on each facet lead to form

$$I_f\sim\left(\prod_{j\in f}\int_0^1 \frac{\mathrm{d}y_j}{y_j}\, y_j^{\langle \mathbf{n}_j,\nu\rangle+a_j\frac{D}{2}}\right)\left[\sum_i c_i \prod_{j\in f} y_j^{\langle \mathbf{n}_j,\mathbf{p}_i\rangle+a_j}\right]^{-\frac{D}{2}}$$

for individual integrals occurring in the expansion by regions,

a_j can be zero

internal facet with $a_j=0$



Method of regions and pySecDec

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NEW: pySecDec version 1.6, 2305.19768

- method of regions can lead to integrals which are **not regulated by dim. reg.**
- these integrals need an additional regulator that cancels when summing over regions
- since pySecDec version 1.6:

> ✳ detects automatically if extra regulators are needed
>
> ✳ tells the user which of the Feynman parameters need an extra regulator

example 1-loop box in high energy expansion $m_H, m_t \ll s, |t|$

```
extra_regulator_constraints():
```
$$\nu_2 - \nu_4 \neq 0, \quad \nu_1 - \nu_3 \neq 0$$

```
suggested_extra_regulator_exponent():
```
$$\{\delta\nu_1, \delta\nu_2, \delta\nu_3, \delta\nu_4\} = \{0, 0, \eta, -\eta\}$$

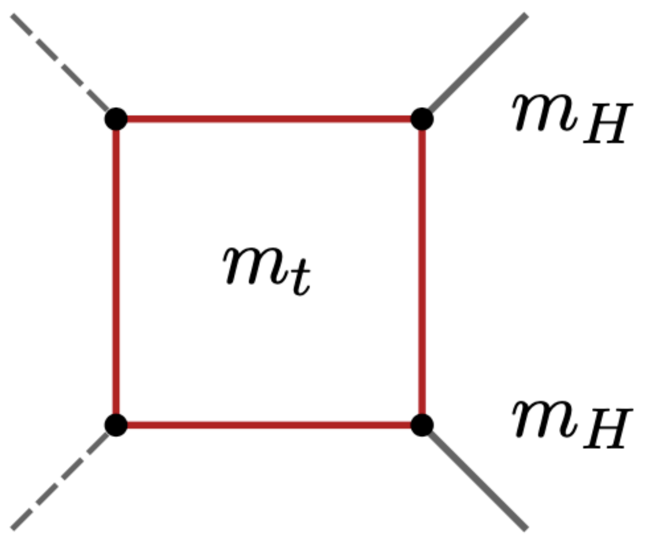


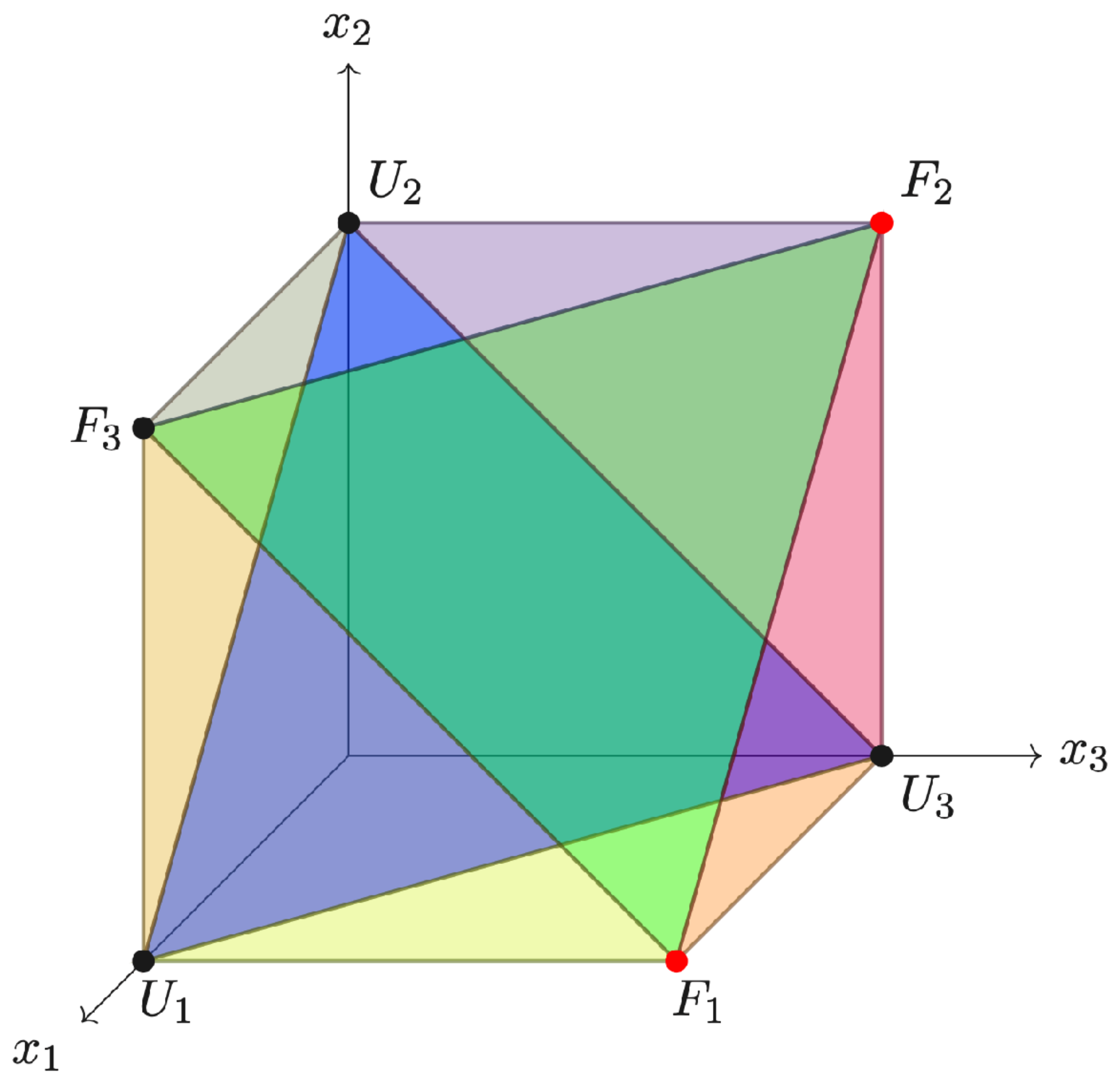
figure: Stephen Jones

Landau equations and on-shell expansion

Gardi, Herzog, Jones, Ma, Schlenk '22

see also
Arkani-Hamed, Hillmann, Mizera '22,
Mizera, Telen '21,
Dlapa, Helmer, Papathanasiou, Tellander '23

- consider **regions** of Feynman integrals with massless propagators and on-shell expansion of external momenta
- identify each region with a solution of Landau equations, or as a **facet of the Newton polytope**
- leads to necessary and sufficient conditions to classify infrared regions
- allows to identify infrared regions at the Feynman graph level
- valid to all orders in the power expansion



New developments in pySecDec: disteval

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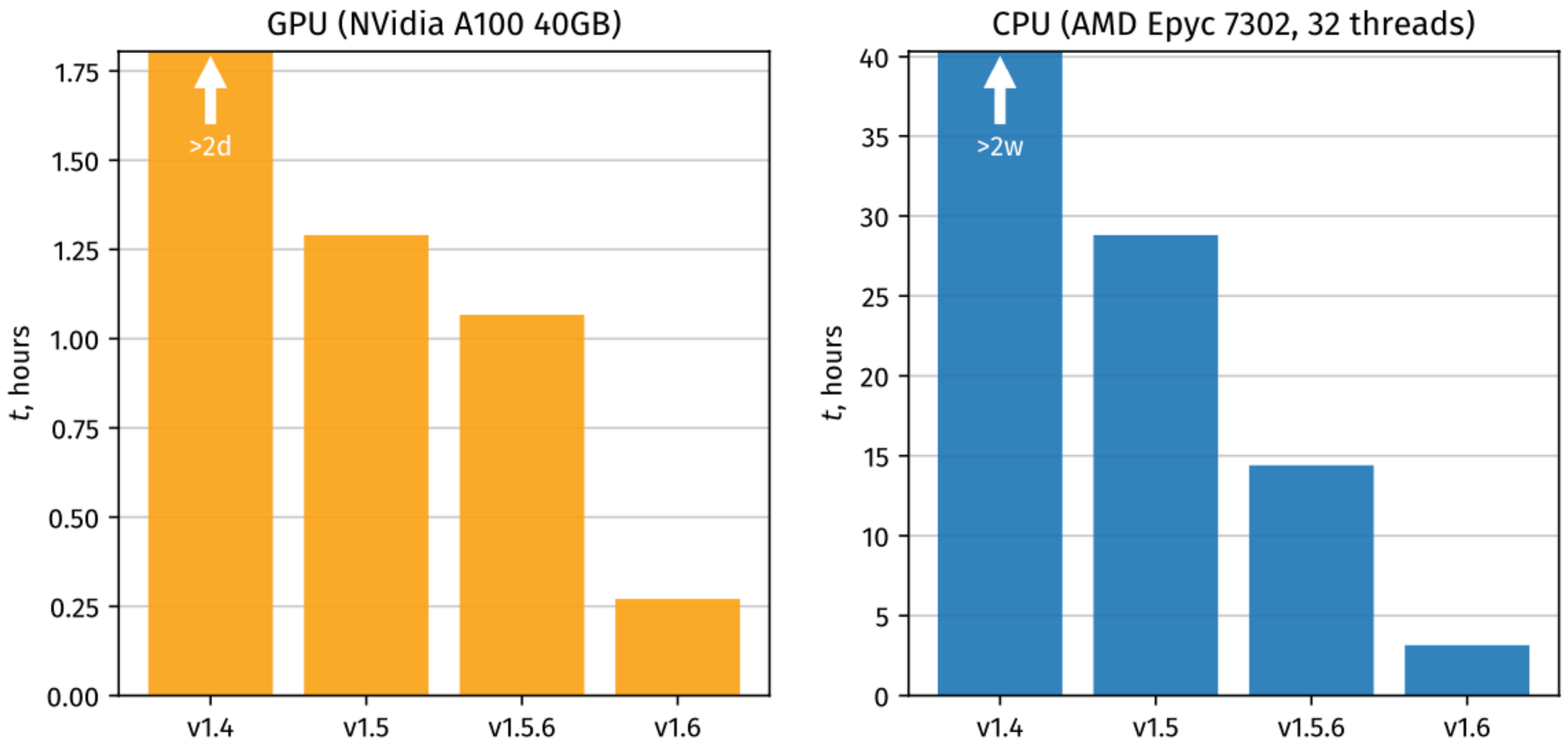
Time to integrate  to 7 digits of precision with pySecDec:



Vitaly Margerya, RADCOR 2023

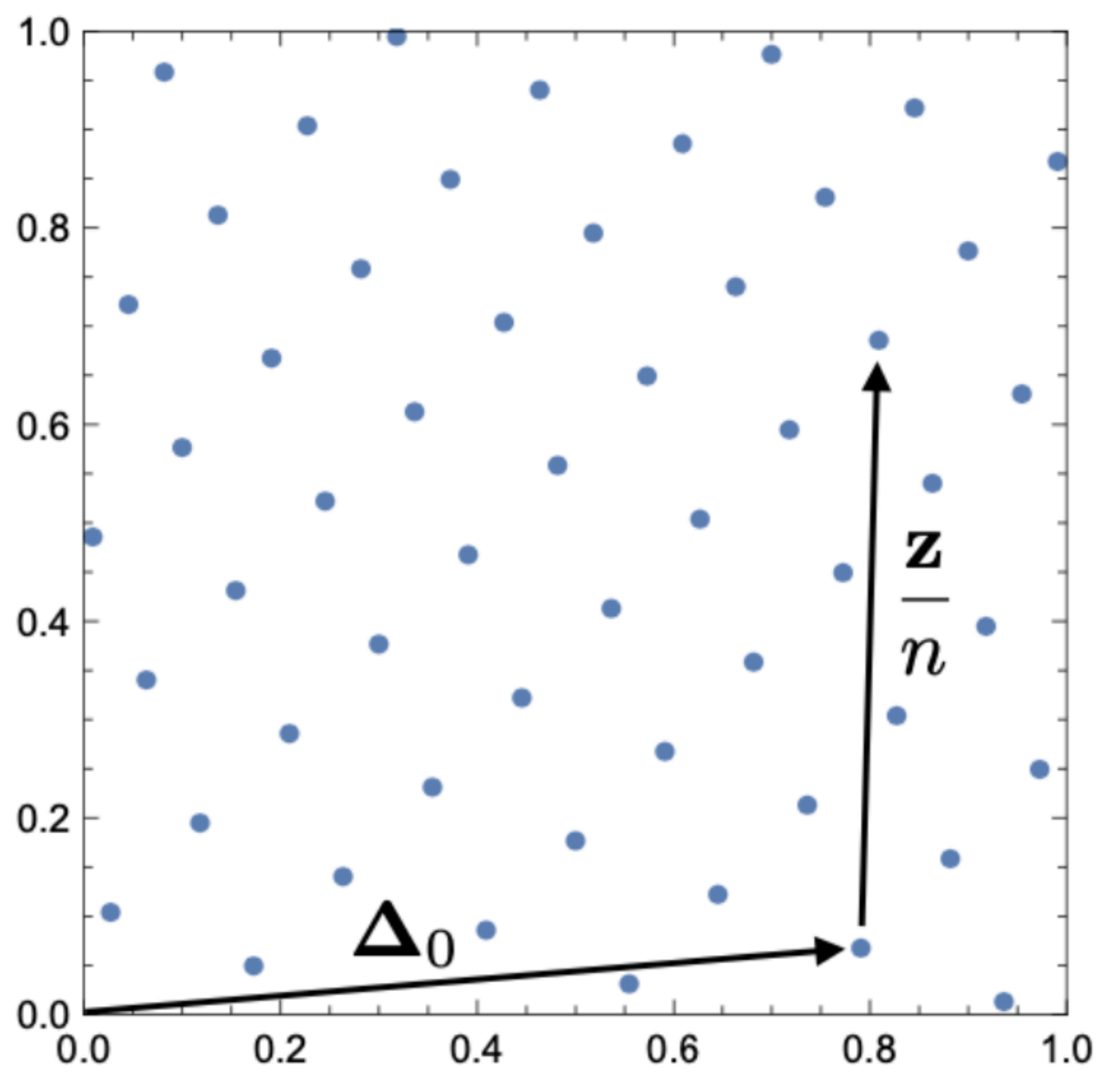
- v1.5: weighted sampling of sums
- v1.6: new quasi-Monte-Carlo integrator **disteval** (more distributed evaluation, performance improvements)

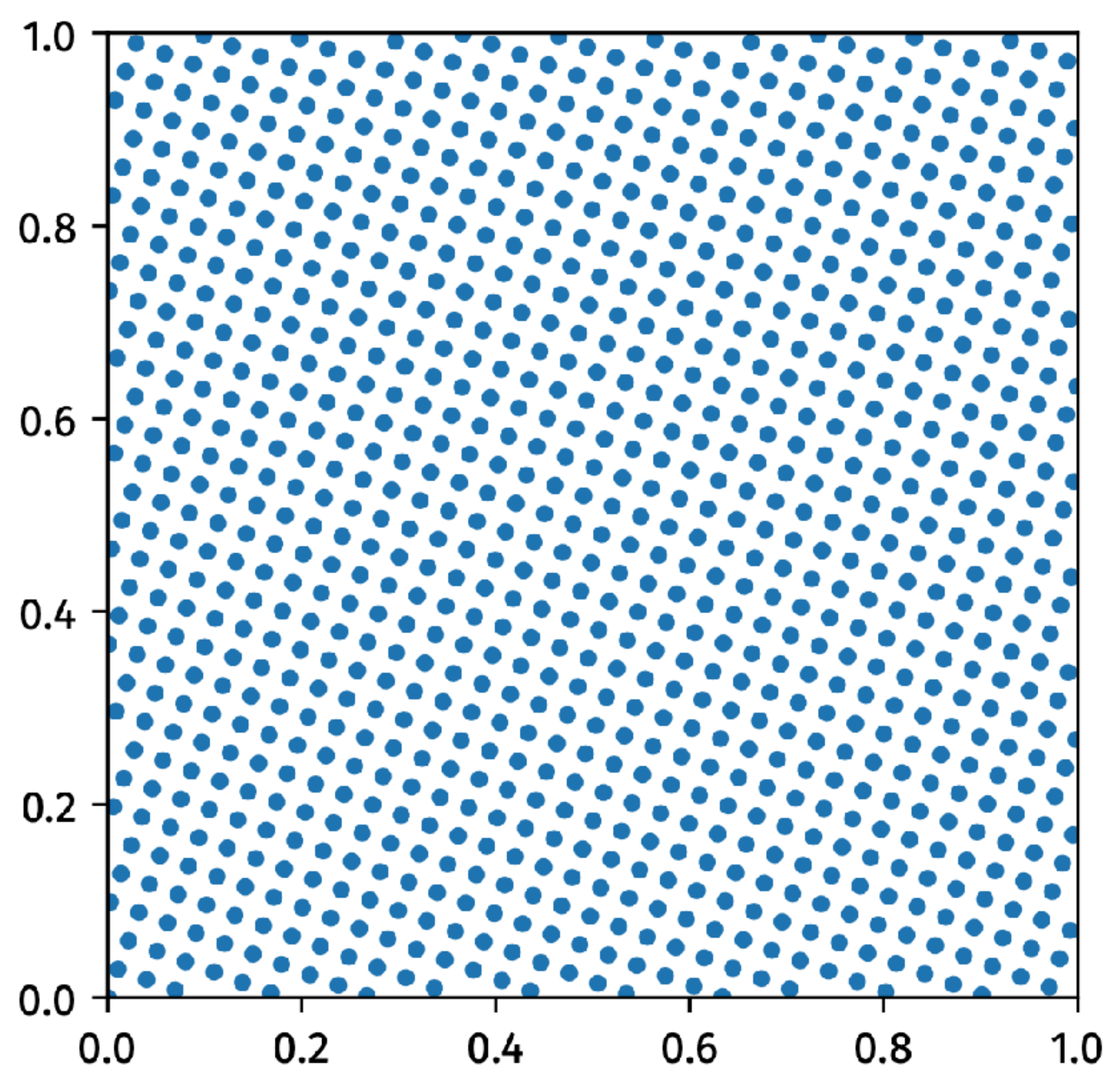
Quasi-Monte-Carlo method

$$I[f] = \int_0^1 d^d\vec{x}\, f(\vec{x})$$

$$I[f] \approx \bar{Q}_{n,m}[f] \equiv \frac{1}{m}\sum_{k=0}^{m-1} Q_n^{(k)}[f]$$

$$Q_n^{(k)}[f] \equiv \frac{1}{n}\sum_{i=0}^{n-1} f\left(\left\{\frac{i\mathbf{z}}{n} + \mathbf{\Delta}_k\right\}\right)$$

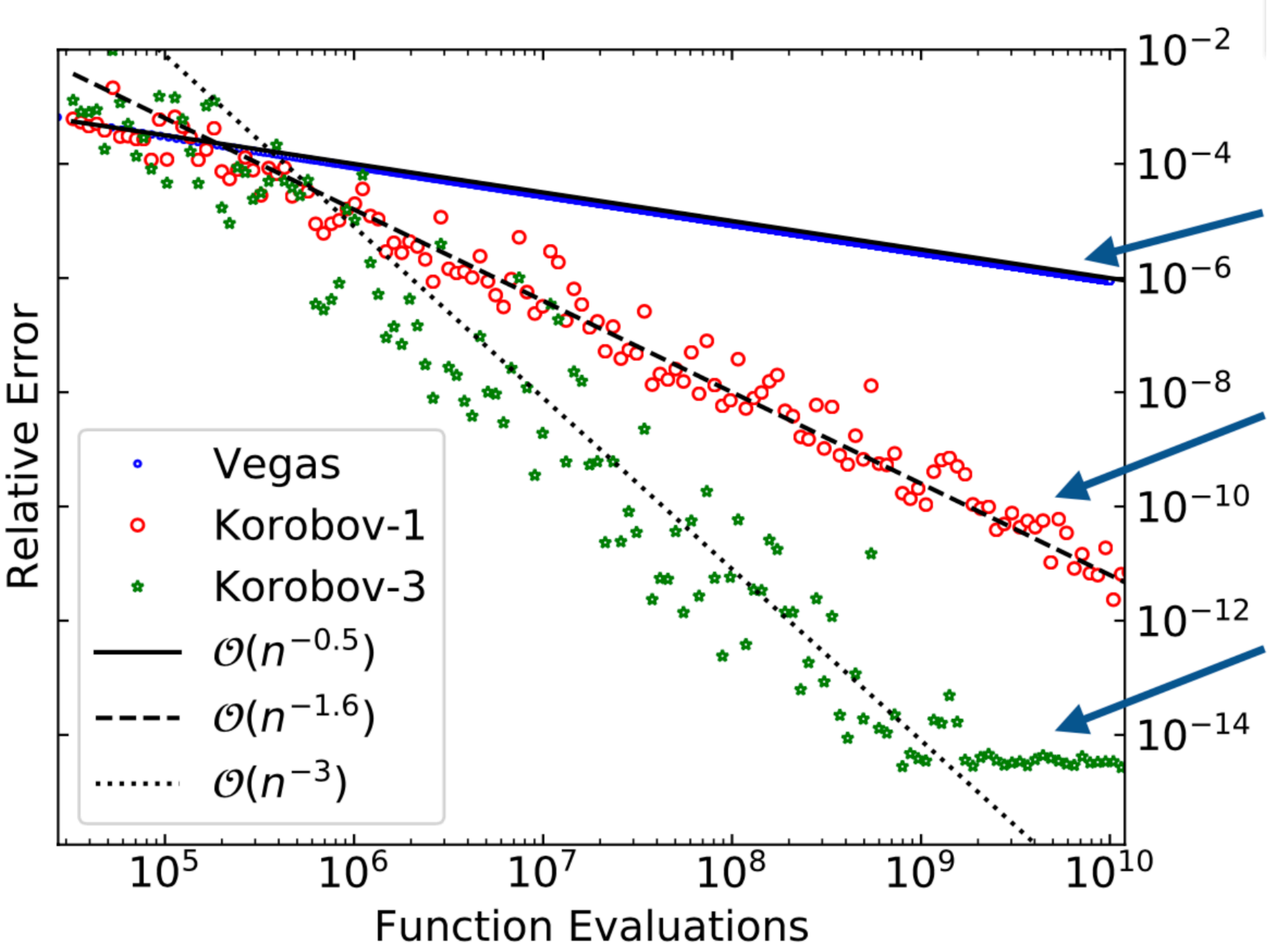




n lattice points, m random shifts, $\mathbf{z} \in \mathbb{N}^n$ generating vector, Δ_k random shift vector

error scaling $\sim 1/n^\alpha$ if $\partial_x^{(\alpha)} f(\vec{x})$ is square-integrable and periodic

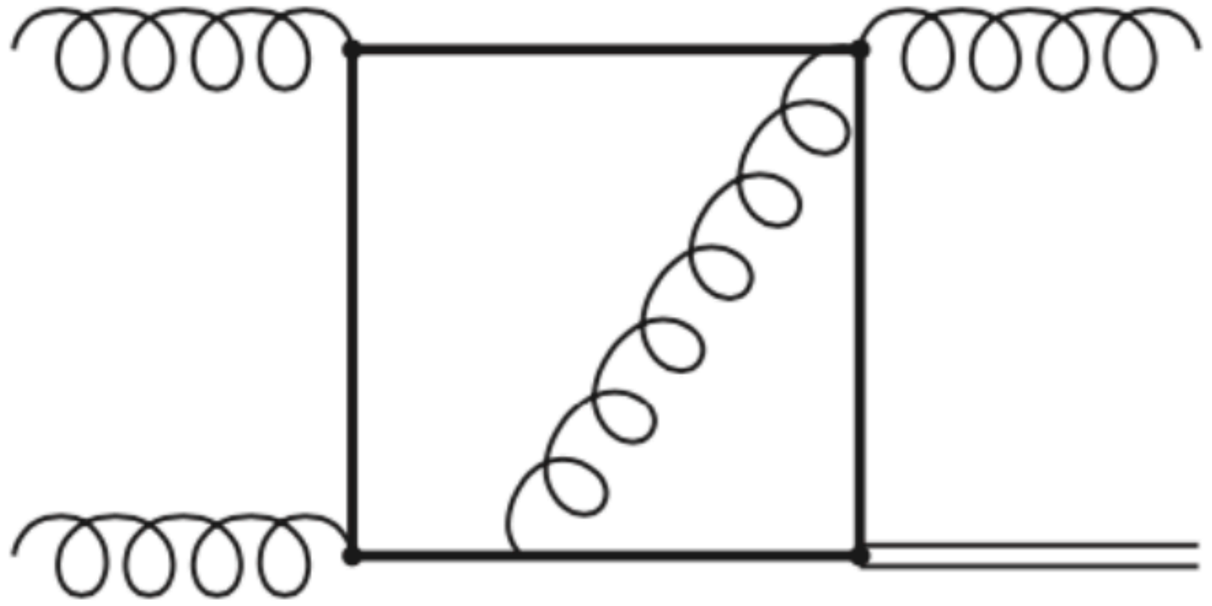
Error scaling



Monte Carlo scaling

Better than ``guaranteed' n^{-1} scaling

Limited by machine precision (double)

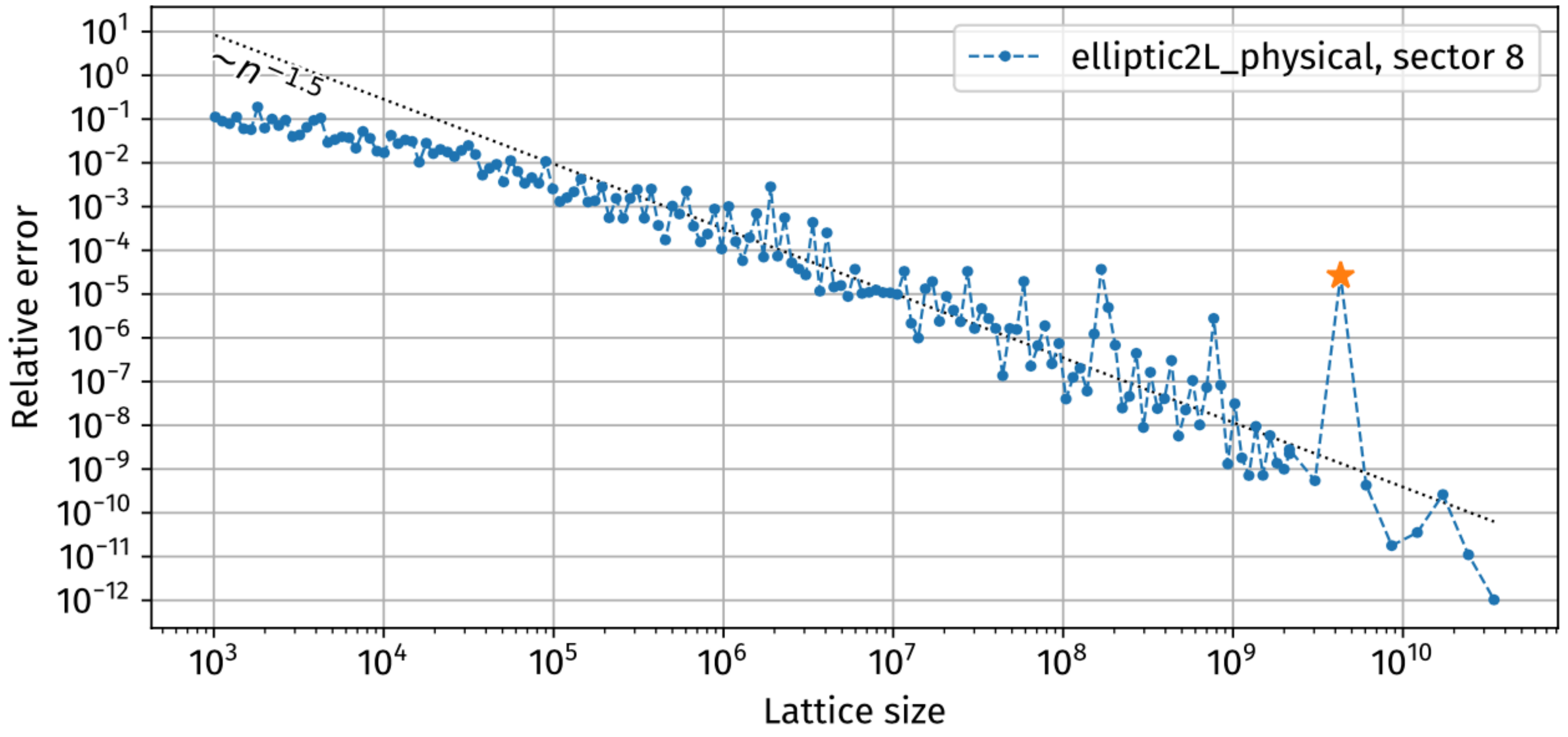


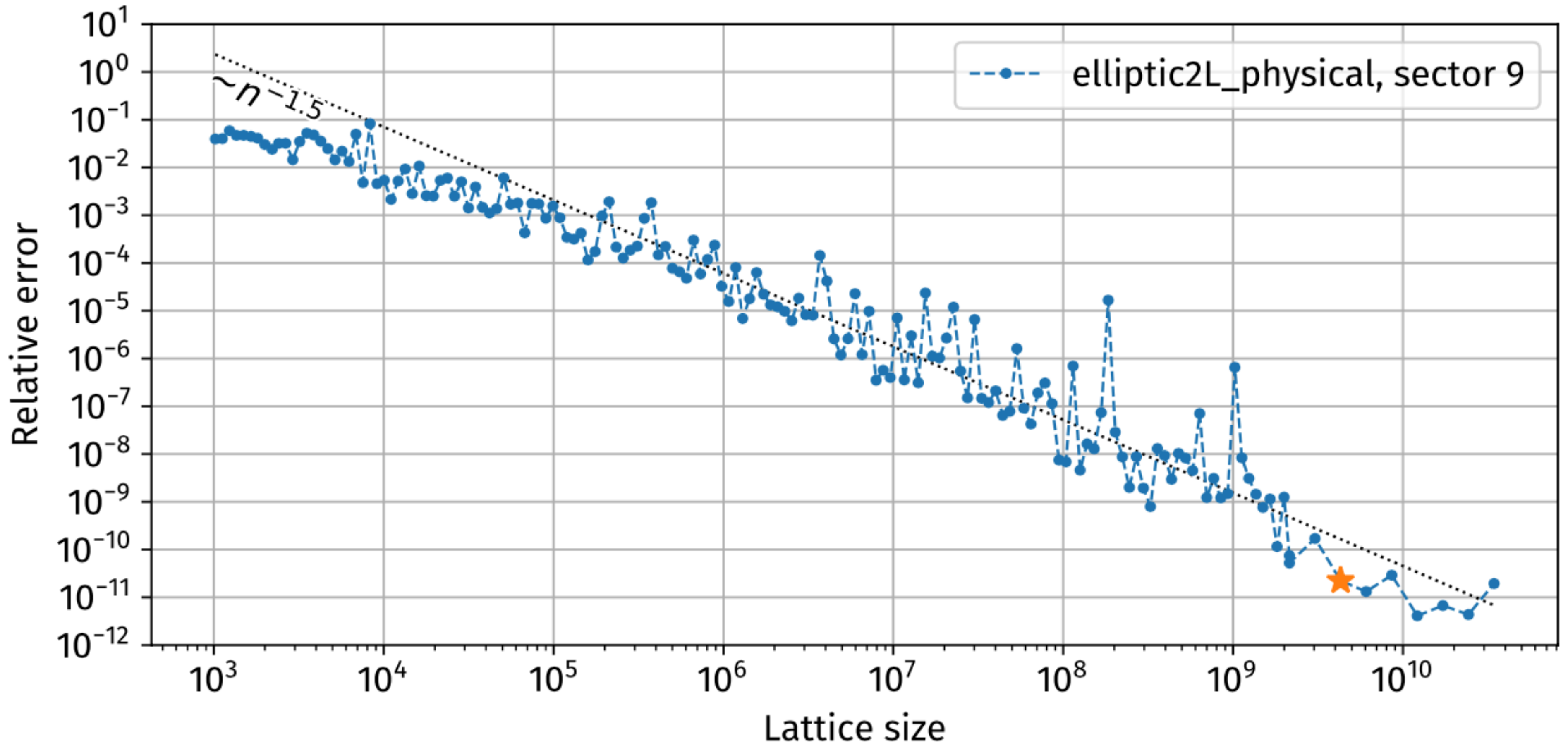
contains elliptic functions

figure: S. Jones, M. Kerner

New developments in pySecDec: median QMC

however for some lattices and functions sudden precision drop





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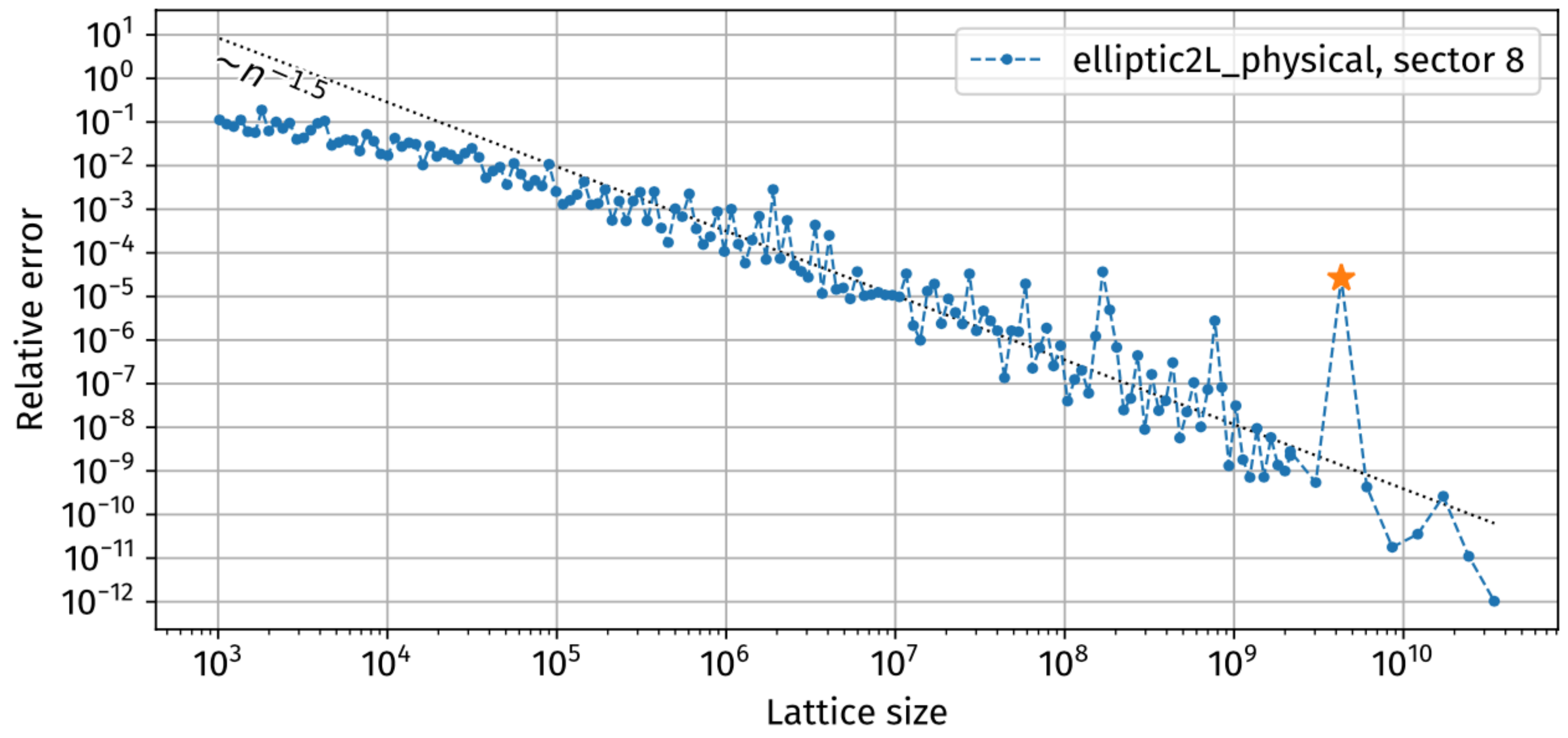
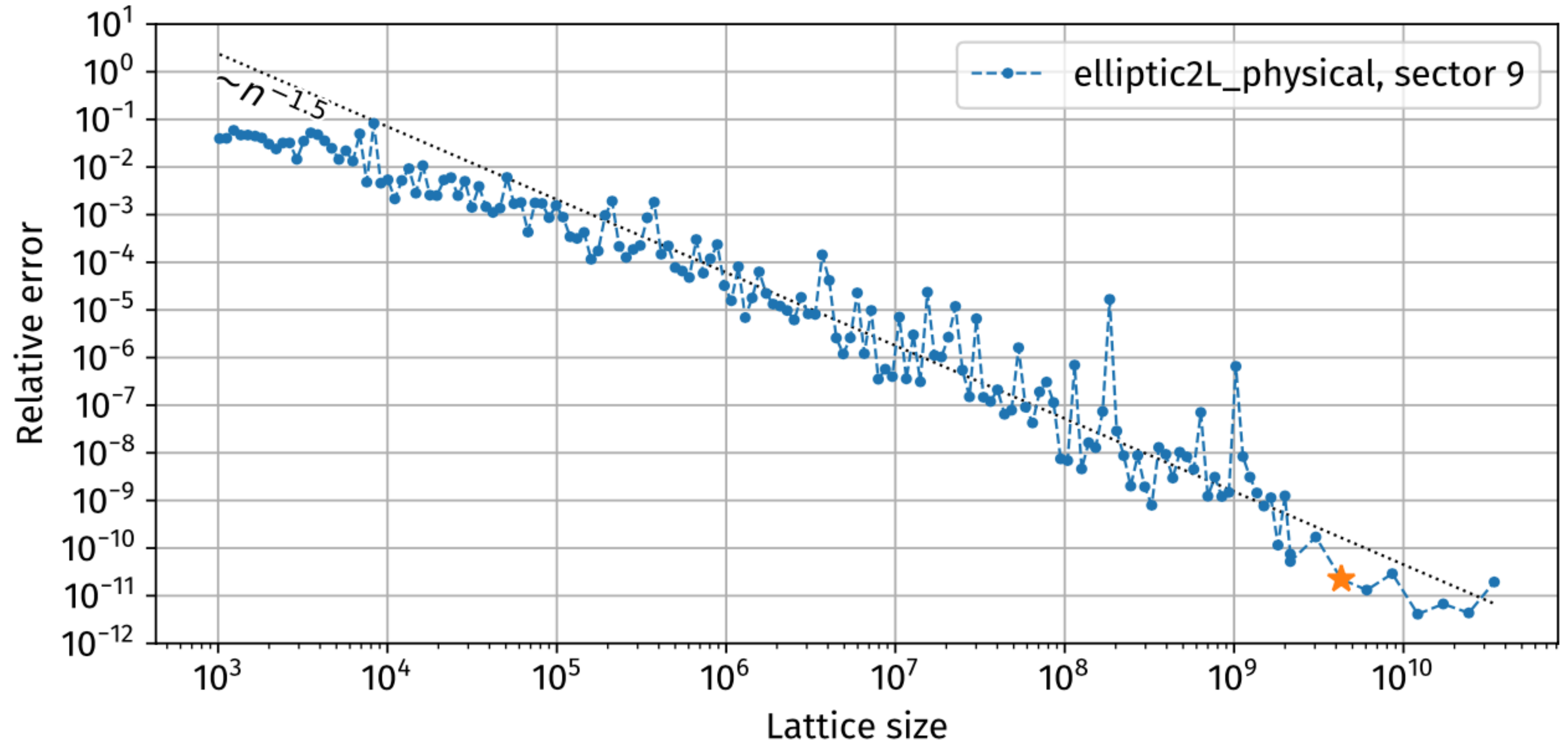
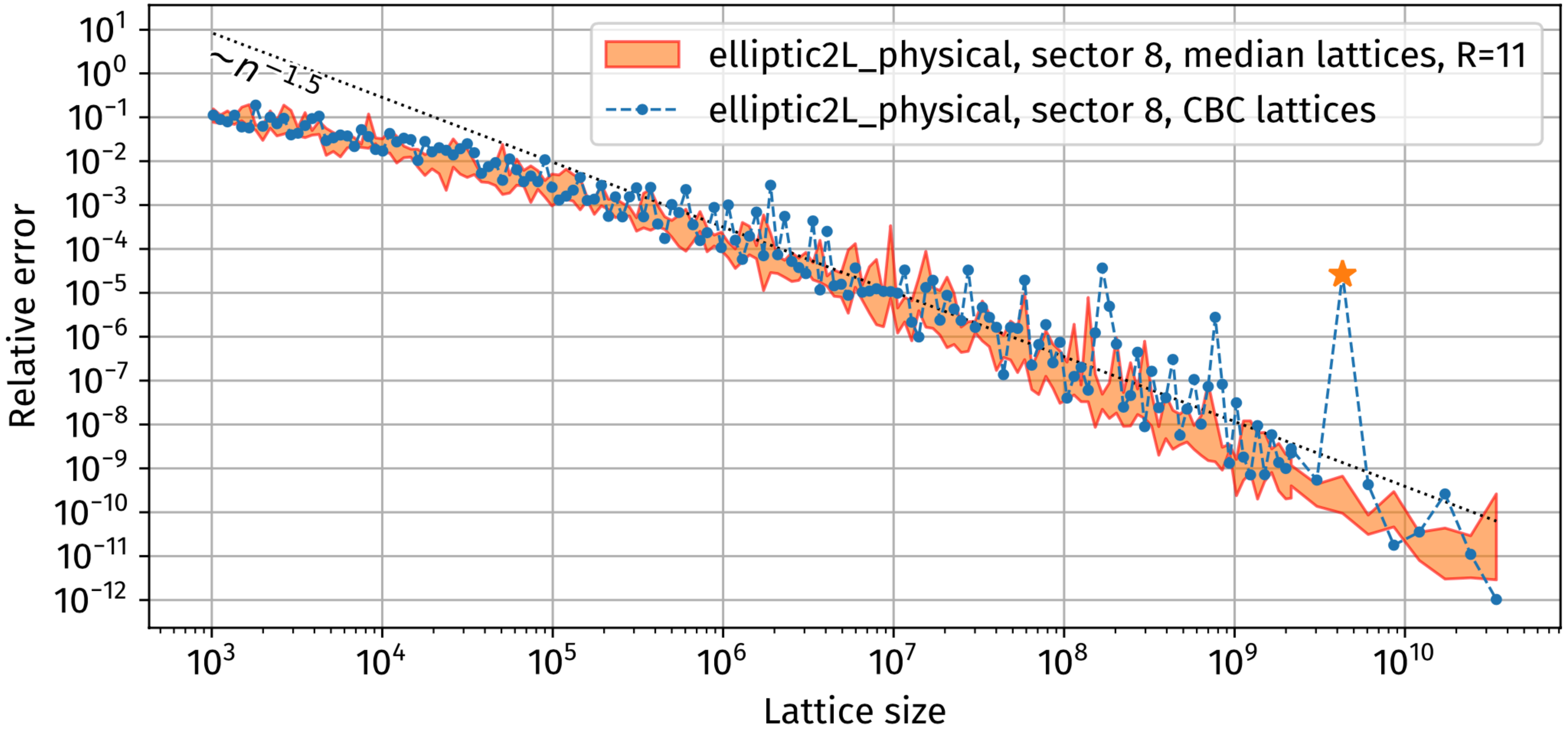




remedy: **median lattices**

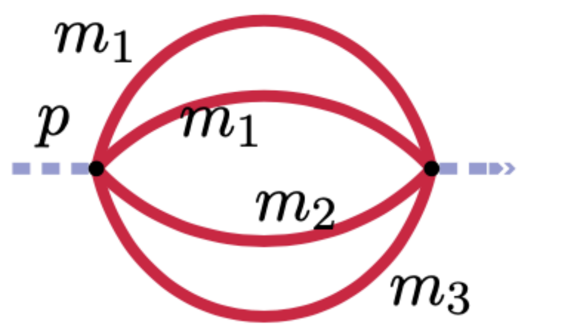
(based on R generating vectors)

Goda, L'Ecuyer '22
M. Kerner et al. '23 (pySecDec)

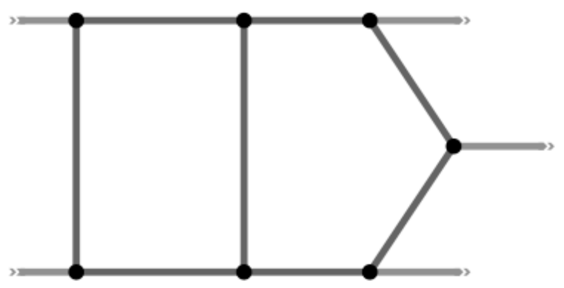


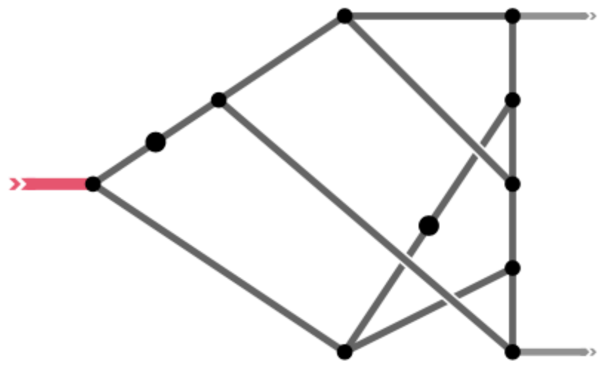
Examples for speed improvements

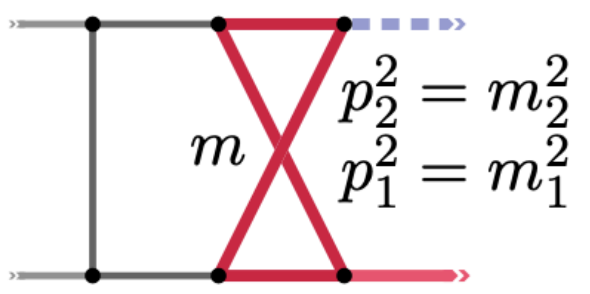
Integrator \ Accuracy		10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
banana_3mass	DISTEVAL	2.1 s	2.1 s	2.4 s	2.6 s	2.6 s	2.9 s	3.6 s
	INTLIB	5.0 s	4.9 s	6.4 s	7.2 s	8.5 s	8.5 s	13.8 s
	Ratio	2.3	2.3	2.7	2.7	3.2	3.0	3.9
bubble6L	DISTEVAL	1.8 m	1.8 m	1.8 m	2.1 m	3.8 m	10.2 m	1.2 h
	INTLIB	39.5 m	38.8 m	39.6 m	43.8 m	85.1 m	170.7 m	11.6 h
	Ratio	22	22	22	21	22	17	10
formfactor4L	DISTEVAL	4.1 m	4.1 m	4.1 m	4.4 m	7.7 m	14.6 m	0.96 h
	INTLIB	74 m	73 m	73 m	74 m	136 m	246 m	10.9 h
	Ratio	18	18	18	17	18	17	11
elliptic2L_physical	DISTEVAL	1.6 s	1.5 s	1.7 s	1.9 s	4.0 s	19 s	7.6 m
	INTLIB	3.1 s	4.8 s	4.9 s	7.3 s	13.8 s	53 s	4.3 m
	Ratio	1.9	3.1	2.8	3.9	3.4	2.9	0.6
hz2L_nonplanar	DISTEVAL	2.1 s	2.6 s	4.6 s	30.4 s	2.2 m	5.1 m	27.1 m
	INTLIB	9 s	17 s	41 s	163 s	9.6 m	16.0 m	27.3 m
	Ratio	1.8	3.4	4.6	4.4	4.2	3.0	1.0
Nbox2L_split_b	DISTEVAL	2.7 s	9.8 s	16.8 s	0.58 m	2.4 m	9.1 m	20 m
	INTLIB	24 s	73 s	223 s	6.6 m	26 m	43 m	93 m
	Ratio	3.0	4.6	9.7	9.9	10.5	4.8	4.7
pentabox_fin	DISTEVAL	5 s	8 s	11 s	0.71 m	3.7 m	18.5 m	1.1 h
	INTLIB	45 s	65 s	88 s	3.2 m	11.3 m	74.8 m	4.6 h
	Ratio	8.6	7.9	7.7	4.5	3.1	4.0	4.2



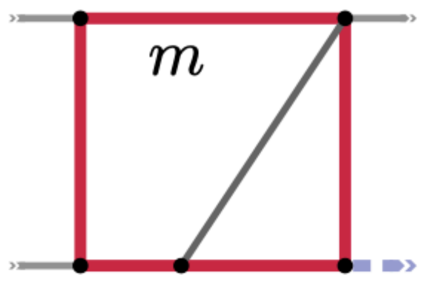
(a) banana_3mass


(b) pentabox_fin

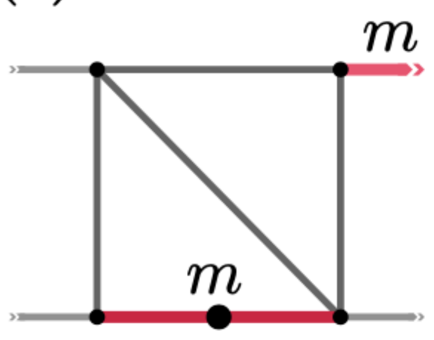

(c) formfactor4L



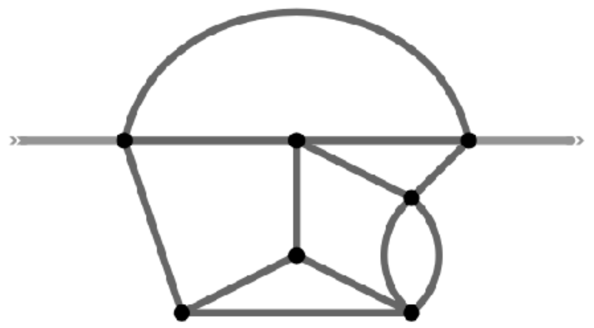
(d) hz2L_nonplanar



(e) elliptic2L_physical



(f) Nbox2L_split_b


bubble6L

integration timings on a GPU, Nvidia A100 80G

Summary

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- Formulation of Feynman integrals in terms of algebraic geometry leads to very useful insights, e.g. (blue: work in progress)
  - how to avoid infinite recursion in sector decomposition
  - when an extra regulator is needed in expansion by regions
  - finding minimal number of sectors
  - relation to Landau equations
- Numerics:
  - new integrator in pySecDec: **disteval**
  - **median** quasi-Monte-Carlo rules
- Fruitful interplay between physics and mathematics!

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