Resultants and difference equations

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$P(x)$ polynomial

$$I(N) = \int_{x_1}^{x_2} dx \; P^N(x)$$

x_1, x_2 zeroes of $P(x)$ or $|x_i| = \infty$, where we apply i.b.p. without surface terms

Consider the polynomial of degree 2:

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

We use integration by parts (i.b.p.) identities to find the recurrence relation

$$\left(4a_2 \left(N + \tfrac{1}{2}\right)\right) I(N) + (a_1^2 - 4a_0 a_2) N I(N-1) = 0$$

which is a *difference equation* in the variable N.

degree 3

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \qquad I(N) = \int_{-\infty}^{+\infty} dx \; P_3^N(x)$$

$$-27a_3^2 \left(N^2 - \tfrac{1}{9}\right) I(N) + 2\left(27a_0 a_3^2 - 9a_1 a_2 a_3 + 2a_2^3\right) N \left(N - \tfrac{1}{2}\right) I(N-1)$$
$$+ \left(-27a_0^2 a_3^2 + 18a_0 a_1 a_2 a_3 - 4a_0 a_2^3 - 4a_1^3 a_3 + a_1^2 a_2^2\right) N(N-1) I(N-2) = 0,$$

Resultant of 2 polynomials

$$P(x) = a_m \prod_{i=1}^{m} (x - \mu_i) \qquad \text{Polynomial 1 with zero } \mu_i$$

$$Q(x) = b_n \prod_{i=1}^{n} (x - \nu_i) \qquad \text{Polynomial 2 with zero } \nu_i$$

$$\text{Resultant}(P, Q) = a_m^n b_n^m \prod_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} (\mu_i - \nu_j) = a_m^n \prod_{i=1}^{m} Q(\mu_i) = b_n^m \prod_{i=1}^{n} P(\nu_i)$$

- =Product of all the differences between the zeroes of both polynomials
- =Product of the value of a polynomial in the zeroes of the other polynomial
- Resultant=0 only if two zeroes of the polynomials are coincident
- Method of calculation: determinant of the Sylvester matrix of the two polynomials (simple but not efficient for large degree)
- Method of calculation: from the remainders of polynomial division (GCD of two polynomials, Euclid's algorithm)

Example: Sylvester matrix of two polynomial of degree 4 and 3:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$Q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$\begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

$\leftarrow$ Sylvester's matrix

The polynomials of the previous slide are Resultant of $P(x)$ and its derivative $\frac{dP}{dx}$

$$\begin{vmatrix} a_2 & a_1 & a_0 \\ 2a_2 & a_1 & 0 \\ 0 & 2a_2 & a_1 \end{vmatrix} = (-a_2)\left(a_1^2 - 4a_0 a_2\right)$$

$$\begin{vmatrix} a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 \\ 3a_3 & 2a_2 & a_1 & 0 & 0 \\ 0 & 3a_3 & 2a_2 & a_1 & 0 \\ 0 & 0 & 3a_3 & 2a_2 & a_1 \end{vmatrix} = (-a_3)\left(-27a_0^2 a_3^2 + 18a_0 a_1 a_2 a_3 - 4a_0 a_2^3 - 4a_1^3 a_3 + a_1^2 a_2^2\right)$$

(Discriminant: $\mathrm{Disc}(P) = \frac{(-1)^{n(n-1)/2}}{a_n}\mathrm{Resu}(P, P')$ avoids unwanted prefactors) degree 2

$$P_2(x) = a_0 + a_1 x + a_2 x^2 \qquad I(N) = \int_{-\infty}^{+\infty} dx\ P_2^N(x)$$

i.b.p.

$$\left((4a_2)\left(N+\frac{1}{2}\right)\right) I(N) + (a_1^2 - 4a_0 a_2) N I(N-1) = 0$$

$$\left[\frac{\partial}{\partial a_0}\text{Resu}\left(P_2(x), P_2'(x)\right)\right]\left(N+\frac{1}{2}\right) I(N) - \left[\text{Resu}\left(P_2(x), P_2'(x)\right) N I(N-1)\right] = 0$$

$$\text{Resu}\left(P_2(x), P_2''(x)\right) = \tfrac{\partial}{\partial a_0}\text{Resu}\left(P_2(x), P_2'(x)\right)$$

degree 3

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \qquad I(N) = \int_{-\infty}^{+\infty} dx\ P_3^N(x)$$

$$\left[\frac{1}{2!}\frac{\partial^2}{\partial a_0^2}\text{Resu}\left(P_3(x), P_3'(x)\right)\right]\left(N^2 - \tfrac{1}{9}\right) I(N) - \left[\frac{\partial}{\partial a_0}\text{Resu}\left(P_3(x), P_3'(x)\right)\right] N\left(N - \tfrac{1}{2}\right) I(N-1)$$
$$+ \left[\text{Resu}\left(P_3(x), P_3'(x)\right)\right] N(N-1) I(N-2) = 0$$

$$\text{Resu}\left(P_3(x), P_3'''(x)\right) = 4\tfrac{\partial^2}{\partial a_0^2}\text{Resu}\left(P_3(x), P_3'(x)\right) \qquad \text{Resu}\left(P_3(x), P_3''(x)\right) = -\tfrac{1}{2}\tfrac{\partial}{\partial a_0}\text{Resu}\left(P_3(x), P_3'(x)\right)$$

$$c_0 I(N)+c_1 I(N-1)+c_2 I(N-2)+c_3 I(N-3)=0 \quad \text{third order} \qquad I(N)=\int_{-\infty}^{\infty} \frac{dx}{P_4(x)^N} \quad P_4(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$$

$$c_0 = -\delta\frac{1}{3}\left(\alpha(3N-8)-a_4\beta\right)(N-1)(N-2)(N-3)$$

$$c_1 = \left[\frac{\delta'}{6}(2N-3)\left(\alpha(3N-8)-a_4\beta\right)-3\delta a_4\gamma\right](N-2)(N-3)$$

$$c_2 = \left[\delta''\left(-16\alpha(3N-8)(3N-7)(3N-5)+a_4\beta(593+12N(-49+12N))\right)-6a_4\gamma\left(84\delta'(7-3N)+\beta\gamma(-89+36N)\right)\right]\frac{N-3}{864}$$

$$c_3 = \frac{8}{3}\frac{\delta'''}{1536}\left(\alpha(3N-5)-a_4\beta\right)(-5+2N)(-13+4N)(-11+4N)$$

$$\delta = 256a_0^3a_4^3-27a_0^2a_3^4-192a_0^2a_3a_4^2a_1-128a_0^2a_2^2a_4^2+144a_0^2a_2a_3^2a_4-80a_0a_2^2a_3a_4a_1-4a_0a_2^3a_3^2+16a_0a_2^4a_4$$
$$+144a_0a_2a_4^2a_1^2-6a_0a_3^2a_4a_1^2+18a_0a_2a_3^3a_1+18a_2a_3a_4a_1^3+a_2^2a_3^2a_1^2-4a_2^3a_4a_1^2-4a_3^3a_1^3-27a_4^2a_1^4$$

$$\delta=\frac{1}{a_4}\mathrm{Resu}(P_4,P_4') \qquad \delta'=\frac{1}{a_4}\frac{d}{da_0}\mathrm{Resu}(P_4,P_4') \qquad \delta''=\frac{1}{a_4}\frac{d^2}{da_0^2}\mathrm{Resu}(P_4,P_4') \qquad \delta'''=\frac{1}{a_4}\frac{d^3}{da_0^3}\mathrm{Resu}(P_4,P_4')=1536a_4^3$$

$$\delta=\frac{4}{27}I_2^3-\frac{1}{27}I_3^2 \qquad \alpha=\frac{1}{16a_4}\mathrm{Resu}(P_4',P_4'') \qquad \beta=-I_3 \qquad \frac{\alpha-4a_4\beta}{3\gamma}=I_2; \qquad \gamma=\frac{1}{144a_4}\mathrm{Resu}(P_4'',P_4''')=-\frac{1}{9}\frac{d}{da_0}\beta$$

$$\frac{1}{16a_4}\mathrm{Resu}(P_4,P_4'')=\frac{3}{2}\delta'+a_4I_2^2 \qquad \frac{1}{16\cdot 81a_4}\mathrm{Resu}(P_4,P_4''')=\frac{1}{6}\delta''+\frac{2}{3}\gamma^2$$

I_2 I_3 invariants (see arXiv:0903.2595, 0911.5278 math-ph)

$$I_2=a_2^2-3a_1a_3+12a_0a_4$$

$$I_3=-2a_2^3+9a_1a_3a_2+72a_0a_4a_2-27a_0a_3^2-27a_1^2a_4$$

$$\Phi(n) = c_0 I(N) + c_1 I(N-1) + c_2 I(N-2) + c_3 I(N-3) = 0 \quad \text{third order} \qquad I(N) = \int_{-\infty}^{\infty} \frac{dx}{P_4(x)^N}$$

$$c_0 = -\delta \frac{1}{3} (\alpha(3N-8) - a_4\beta)(N-1)(N-2)(N-3)$$

$$c_1 = \left[\frac{\delta'}{6}(2N-3)\left(\alpha(3N-8) - a_4\beta\right) - 3\delta a_4 \gamma\right](N-2)(N-3)$$

$$c_2 = [\delta'' \left(-16\alpha(3N-8)(3N-7)(3N-5) + a_4\beta(593 + 12N(-49+12N))\right) - 6a_4\gamma \left(84\delta'(7-3N) + \beta\gamma(-89+36N)\right)] \tfrac{N-3}{864}$$

$$c_3 = \frac{8}{3}\frac{\delta'''}{1536}(\alpha(3N-5) - a_4\beta)(-5+2N)(-13+4N)(-11+4N)$$

Polynomials in N (in red) with zeros depending on coefficients: *apparent* singularity.

Desingularization (ex.: Maple): take suitable combination with rational coefficients r_1, r_2

$$r_1(N,a_i) * \Phi(N-1) + r_2(N,a_i)\Phi(N) = c'_0 I(N) + c'_1 I(N-1) + c'_2 I(N-2) + c'_3 I(N-3) + c'_4 I(N-4) = 0 \quad \text{3rd} \to \text{4th order}$$

such that the undesired coefficients (in red) are factored out

$$c'_0 = +\delta(4\alpha - a_4\beta)N(N-1)(N-2)$$

$$c'_1 = \left[\left(-\delta'(4\alpha - a_4\beta) - 9\delta a_4\gamma\right)\left(N - \tfrac{1}{2}\right) + \tfrac{63}{2}a_4\delta\gamma\right](N-1)(N-2)$$

$$c'_2 = \left[\tfrac{1}{2}\left(N-\tfrac{2}{3}\right)\left(N-\tfrac{4}{3}\right)\left((4\alpha - a_4\beta)\delta'' + 18\delta' a_4\gamma\right) - \tfrac{63}{2}\delta' a_4\gamma\left(N - \tfrac{29}{18}\right) + \tfrac{5}{96}a_4\beta(\delta'' + 22\gamma^2)\right](N-2)$$

$$c'_3 = -\left(\tfrac{1}{6}\delta'''(4\alpha - a_4\beta) + \tfrac{9}{2}\delta'' a_4\gamma\right)\left(N-\tfrac{8}{3}\right)\left(N-\tfrac{7}{3}\right)(N-3) + \tfrac{7}{12}\delta'''(-4\alpha + a_4\beta)N^2 + \delta'''\left(7\alpha(-\tfrac{5379}{3456} + \tfrac{4979}{3456}N) + 2a_4\beta(\tfrac{5601}{3456} - \tfrac{4756}{3456}\right.$$

$$c'_4 = \frac{-\delta'''}{6}\left(-\frac{3}{2}a_4\gamma\right)\left(N-\tfrac{2}{5}\right)\left(N-\tfrac{13}{4}\right)\left(N-\tfrac{11}{4}\right)$$

No more factors depending of N and a_i

leading behaviour for $N \to \infty$: extract leading powers in N

$$c_0 = -\delta \frac{1}{3} \left(\alpha(3N)\right)(N)(N)(N) = -\delta \alpha N^4$$

$$c_1 = \left[\frac{\delta'}{6} (2N) \left(\alpha(3N)\right) \right] (N)(N) = +\delta' \alpha N^4$$

$$c_2 = \left[\delta'' \left(-16\alpha(3N)(3N)(3N) + \right)\right] \frac{N}{864} = -\frac{\delta''}{2!} \alpha N^4$$

$$c_3 = \frac{8}{3} \frac{\delta'''}{1536} \left(\alpha(3N)\right)(2N)(4N)(4N) = +\frac{\delta'''}{3!} \alpha N^4$$

$$\to \left\{ -\delta, +\delta,' - \frac{\delta''}{2!}, + \frac{\delta'''}{3!} \right\} \alpha N^4$$

- Degree 5: 3 invariants I_4, I_8, I_{12}
- Degree n: $n-2$ invariants
- Main discriminant → rational prefactor Δ^n
- Other invariants: argument of (multivariate) hypergeometric functions (conjectured)

coefficients of the highest power give the characteristic (algebraic) equation of the difference equation → scales μ_i^N

General case degree n

coefficients of the highest power of N:

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \left[\frac{\partial^k}{\partial a_{00}^k} \mathrm{Resu}(P_n, P'_n) \right] I(N+1-n+k)$$

$$P_{22}(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} \qquad I(N) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\; P_{22}^N(x,y)$$

$$\left(a_{11}^2 - 4a_{02}a_{20}\right) I(N)(N+1) + \left(a_{00}a_{11}^2 - a_{01}^2 a_{20} - a_{10}^2 a_{02} + a_{01}a_{10}a_{11} + 4a_{00}a_{02}a_{20}\right) N I(N-1) = 0$$

$$\mathrm{Disc}_x(P) = \frac{(-1)^{n(n-1)/2}}{a_n}\mathrm{Resu}_x(P,P')$$

$$\mathrm{Disc}_y \mathrm{Disc}_x P_{22}(x,y) = -16a_{20}\left(a_{00}a_{11}^2 - a_{01}^2 a_{20} - a_{10}^2 a_{02} + a_{01}a_{10}a_{11} + 4a_{00}a_{02}a_{20}\right)$$

$$\mathrm{Disc}_x \mathrm{Disc}_y P_{22}(x,y) = -16a_{02}\left(a_{00}a_{11}^2 - a_{01}^2 a_{20} - a_{10}^2 a_{02} + a_{01}a_{10}a_{11} + 4a_{00}a_{02}a_{20}\right)$$

$$\Delta_{22}(a_{00}) = a_{00}a_{11}^2 - a_{01}^2 a_{20} - a_{10}^2 a_{02} + a_{01}a_{10}a_{11} + 4a_{00}a_{02}a_{20}$$

$$\frac{\partial}{\partial a_{00}}\Delta(a_{00}) I(N)(N+1) + \Delta_{22}(a_{00}) N I(N-1) = 0$$

$$P_{33}(x,y) = a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3+a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+a_{00} \qquad I(N) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\; P_{33}^N(x,y)$$

4-th order difference equation

$$I(N)p_1(N,a) + I(N-1)p_2(N,a) + I(N-2)p_3(N,a) + I(N-3)p_4(N,a) + I(N-4)p_5(N,a) = 0$$

$$\text{Disc}_y\text{Disc}_x P_{33}(x,y) = -a_{30}q_1(a)\Delta_{33}$$

$$\text{Disc}_x\text{Disc}_y P_{33}(x,y) = -a_{03}q_2(a)\Delta_{33}$$

The discriminant Δ_{33} has 2040 terms. (degree 4 in a_{00} $\to$ 4th order difference equation)

$$\Delta_{33} = 80621568a_{00}^4a_{03}^4a_{30}^4 + \ldots - 4096a_{03}^2a_{10}^3a_{11}^4a_{12}a_{20}^2 = \frac{64}{27}\left(I_6^2 - I_4^3\right)$$

I_4, I_6 invariants 25, 103 terms

$$I_4 = 144a_{00}a_{02}a_{12}a_{30} - 48a_{00}a_{02}a_{21}^2 - 216a_{00}a_{03}a_{11}a_{30} + 144a_{00}a_{03}a_{20}a_{21} + 24a_{00}a_{11}a_{12}a_{21} - 48a_{00}a_{12}^2a_{20} - 48a_{01}^2a_{12}a_{30}$$
$$+16a_{01}^2a_{21}^2 + 24a_{01}a_{02}a_{11}a_{30} - 16a_{01}a_{02}a_{20}a_{21} + 144a_{01}a_{03}a_{10}a_{30} - 48a_{01}a_{03}a_{20}^2 - 16a_{01}a_{10}a_{12}a_{21} - 8a_{01}a_{11}^2a_{21} + 24a_{01}a_{11}a_{12}a_{20}$$
$$-48a_{02}^2a_{10}a_{30} + 16a_{02}^2a_{20}^2 + 24a_{02}a_{10}a_{11}a_{21} - 16a_{02}a_{10}a_{12}a_{20} - 8a_{02}a_{11}^2a_{20} - 48a_{03}a_{10}^2a_{21} + 24a_{03}a_{10}a_{11}a_{20} + 16a_{10}^2a_{12}^2$$
$$-8a_{10}a_{11}^2a_{12} + a_{11}^4$$

$$
\begin{aligned}
I_6 = {}& a_{11}^6 - 12a_{10}a_{12}a_{11}^4 - 12a_{02}a_{20}a_{11}^4 - 12a_{01}a_{21}a_{11}^4 + 36a_{03}a_{10}a_{20}a_{11}^3 + 36a_{01}a_{12}a_{20}a_{11}^3 + 36a_{02}a_{10}a_{21}a_{11}^3 + 36a_{00}a_{12}a_{21}a_{11}^3 \\
&+ 36a_{01}a_{02}a_{30}a_{11}^3 + 540a_{00}a_{03}a_{30}a_{11}^3 + 48a_{10}^2a_{12}^2a_{11}^2 + 48a_{02}^2a_{20}^2a_{11}^2 - 72a_{01}a_{03}a_{20}^2a_{11}^2 + 48a_{01}^2a_{21}^2a_{11}^2 - 72a_{00}a_{02}a_{21}^2a_{11}^2 \\
&- 72a_{00}a_{12}^2a_{20}a_{11}^2 + 24a_{02}a_{10}a_{12}a_{20}a_{11}^2 - 72a_{03}a_{10}^2a_{21}a_{11}^2 + 24a_{01}a_{10}a_{12}a_{21}a_{11}^2 + 24a_{01}a_{02}a_{20}a_{21}a_{11}^2 - 648a_{00}a_{03}a_{20}a_{21}a_{11}^2 \\
&- 72a_{02}^2a_{10}a_{30}a_{11}^2 - 648a_{01}a_{03}a_{10}a_{30}a_{11}^2 - 72a_{01}^2a_{12}a_{30}a_{11}^2 - 648a_{00}a_{02}a_{12}a_{30}a_{11}^2 - 144a_{02}a_{03}a_{10}a_{20}^2a_{11} - 144a_{01}a_{02}a_{12}a_{20}^2a_{11} \\
&+ 864a_{00}a_{03}a_{12}a_{20}^2a_{11} - 144a_{01}a_{02}a_{10}a_{21}^2a_{11} + 864a_{00}a_{03}a_{10}a_{21}^2a_{11} - 144a_{00}a_{01}a_{12}a_{21}^2a_{11} - 144a_{01}a_{10}a_{12}^2a_{20}a_{11} \\
&- 144a_{03}a_{10}^2a_{12}a_{20}a_{11} - 144a_{00}a_{10}a_{12}^2a_{21}a_{11} - 144a_{02}a_{10}^2a_{12}a_{21}a_{11} - 144a_{02}^2a_{10}a_{20}a_{21}a_{11} + 720a_{01}a_{03}a_{10}a_{20}a_{21}a_{11} \\
&- 144a_{01}^2a_{12}a_{20}a_{21}a_{11} + 720a_{00}a_{02}a_{12}a_{20}a_{21}a_{11} + 864a_{02}a_{03}a_{10}^2a_{30}a_{11} + 864a_{00}a_{01}a_{12}^2a_{30}a_{11} + 720a_{01}a_{02}a_{10}a_{12}a_{30}a_{11} \\
&- 1296a_{00}a_{03}a_{10}a_{12}a_{30}a_{11} - 144a_{01}a_{02}^2a_{20}a_{30}a_{11} + 864a_{01}^2a_{03}a_{20}a_{30}a_{11} - 1296a_{00}a_{02}a_{03}a_{20}a_{30}a_{11} + 864a_{00}a_{02}^2a_{21}a_{30}a_{11} \\
&- 144a_{01}^2a_{02}a_{21}a_{30}a_{11} - 1296a_{00}a_{01}a_{03}a_{21}a_{30}a_{11} - 64a_{10}^3a_{12}^3 - 64a_{02}^3a_{20}^3 - 864a_{00}a_{03}^2a_{20}^3 + 288a_{01}a_{02}a_{03}a_{20}^3 - 64a_{01}^3a_{21}^3 \\
&+ 288a_{00}a_{01}a_{02}a_{21}^3 - 864a_{00}^2a_{03}a_{21}^3 + 216a_{03}^2a_{10}^2a_{20}^2 + 216a_{01}^2a_{12}^2a_{20}^2 - 576a_{00}a_{02}a_{12}^2a_{20}^2 + 96a_{02}^2a_{10}a_{12}a_{20}^2 - 144a_{01}a_{03}a_{10}a_{12}a_{20}^2 \\
&+ 216a_{02}^2a_{10}^2a_{21}^2 - 576a_{01}a_{03}a_{10}^2a_{21}^2 + 216a_{00}^2a_{12}^2a_{21}^2 + 96a_{01}^2a_{10}a_{12}a_{21}^2 - 144a_{00}a_{02}a_{10}a_{12}a_{21}^2 - 576a_{00}a_{02}^2a_{20}a_{21}^2 + 96a_{01}^2a_{02}a_{20}a_{21}^2 \\
&+ 864a_{00}a_{01}a_{03}a_{20}a_{21}^2 - 864a_{00}a_{02}^3a_{30}^2 + 216a_{01}^2a_{02}^2a_{30}^2 - 5832a_{00}^2a_{03}^2a_{30}^2 - 864a_{01}^3a_{03}a_{30}^2 + 3888a_{00}a_{01}a_{02}a_{03}a_{30}^2 + 288a_{00}a_{10}a_{12}^3a_{20} \\
&+ 96a_{02}a_{10}^2a_{12}^2a_{20} + 96a_{01}a_{10}^2a_{12}^2a_{21} + 96a_{01}a_{02}^2a_{20}^2a_{21} - 576a_{01}^2a_{03}a_{20}^2a_{21} + 864a_{00}a_{02}a_{03}a_{20}^2a_{21} + 288a_{03}a_{10}^3a_{12}a_{21} \\
&- 144a_{02}a_{03}a_{10}^2a_{20}a_{21} - 144a_{00}a_{01}a_{12}^2a_{20}a_{21} + 48a_{01}a_{02}a_{10}a_{12}a_{20}a_{21} - 1296a_{00}a_{03}a_{10}a_{12}a_{20}a_{21} - 864a_{03}^2a_{10}^3a_{30} - 864a_{00}^2a_{12}^3a_{30} \\
&- 576a_{01}^2a_{10}a_{12}^2a_{30} + 864a_{00}a_{02}a_{10}a_{12}^2a_{30} - 576a_{02}^2a_{10}^2a_{12}a_{30} + 864a_{01}a_{03}a_{10}^2a_{12}a_{30} + 288a_{02}^3a_{10}a_{20}a_{30} + 3888a_{00}a_{03}^2a_{10}a_{20}a_{30} \\
&- 1296a_{01}a_{02}a_{03}a_{10}a_{20}a_{30} + 864a_{00}a_{02}^2a_{12}a_{20}a_{30} - 144a_{01}^2a_{02}a_{12}a_{20}a_{30} - 1296a_{00}a_{01}a_{03}a_{12}a_{20}a_{30} - 144a_{01}a_{02}^2a_{10}a_{21}a_{30} \\
&+ 864a_{01}^2a_{03}a_{10}a_{21}a_{30} - 1296a_{00}a_{02}a_{03}a_{10}a_{21}a_{30} + 288a_{01}^3a_{12}a_{21}a_{30} - 1296a_{00}a_{01}a_{02}a_{12}a_{21}a_{30} + 3888a_{00}^2a_{03}a_{12}a_{21}a_{30}
\end{aligned}
$$

Polynomial degree n, in v variables

From dimension counting the expected number of invariants $\mathcal{N}$ is

$$\mathcal{N}(v,n) = -v(2+v) + \frac{(v+n)!}{v!n!}$$
$$\mathcal{N}(1,n) = n - 3$$
$$\mathcal{N}(2,n) = \frac{n}{2}(n+3) + 7$$

$$\ldots$$

$\mathcal{N}$ grows fast for large n, v;

Example: $\mathcal{N}(5,5) = 217$

Summary

- Resultants, discriminants and invariants appear in difference equations for integrals of powers of polynomials
- Similar results appear for non-gaussian integrals (exponential of multivariate symmetric polynomials)
- It can be useful study existing algorithms for the calculations of resultants and discriminants in order to get useful ideas to implement in other approaches (e.g. intersection theory)