IBP Reduction using Gröbner bases

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P H
Particle Physics Phenomenology after the Higgs Discovery

based on	M. Barakat, R. Brüser, J. Piclum, TH	2207.09275 (PoS),
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Outline

- Introduction / motivation
- Mathematical stuff
- Gröbner basis of the ideal of IBP relations in the double-shift algebra
- Special IBP relations
- Linear algebra ansatz
- Conclusion and outlook

Motivation

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- IBP-based integral reduction has vastly served the community over the past $\sim$ two decades
  - Many tools exist [Anastasiou,Lazopoulos'04]

[Manteuffel,Studerus'12; Smirnov et al.'08+; Lee'12+; Maierhöfer,Usovitsch,Uwer'17]
[Klappert,Lange,Maierhöfer,Usovitsch'20; Marquard,Seidel; Liu,Ma,Guan; . . .]

- Mostly based on Laporta's algorithm [Laporta'01]
  - Solves IBP equations for numerical values of indices with Gaussian elimination.
- Several refinements exist, e.g.
  - Parallelization
  - Methods from finite fields

[v. Manteuffel,Schabinger'14; Smirnov,Chukharev'19]
[Peraro'16'19; Klappert,Klein,Lange'19'20]

- Drawbacks
  - Compute many more integrals than required
  - Large storage required for results of $10^{\sim 4-6}$ integrals

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  [Peraro'16'19; Klappert,Klein,Lange'19'20]
- Drawbacks
  - Compute many more integrals than required
  - Large storage required for results of $10^{\sim 4-6}$ integrals
- New ideas from
  - syzygy equations

    [Kosower et al.'10'18; Schabinger et al.'11'20; Ita'15; Böhm et al.'17]
  - algebraic geometry and module intersection

    [Larsen,Zhang'14; Böhm et al.'18'19; Wu,Böhm,Ma,Xu,Zhang'23]
  - intersection numbers

    [Mastrolia,Mizera'18; Frellesvig et al.'19'20; Weinzierl'20]
- Our approach
  - Leave propagator powers symbolic
  - Find a Gröbner basis of the left ideal of IBP relations in the rational double-shift algebra
  - Derive normal form IBPs from Gröbner basis or linear algebra
- Previous work on Gröbner bases in integral reduction [Tarasov'98'04; Gerdt,Robertz'05'06; Smirnov,Smirnov'05-'08; Lee'08]

Reminder about mathematical quantities

Let R be a ring over a field $\mathbb{K}$.

- A subset $I \subseteq R$ is an **ideal** if it forms an additive group and fulfils

$$x \in R \wedge y \in I \implies xy \in I \ \wedge \ yx \in I.$$

  - Example: Set of even integers is an ideal in the ring of integers.
  - Left and right ideal analogous

- Multi-index notation

$$x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

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$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

- A **monomial order** on R is a total order $>$ s.t.

$$x^\alpha > x^\beta \implies x^\gamma x^\alpha > x^\gamma x^\beta \quad \forall\, \alpha, \beta, \gamma \in \mathbb{N}^n$$

  - **Lexicographic order**

$$x^\alpha >_{\text{lex}} x^\beta \quad \iff \quad \text{first nonzero entry of } \alpha - \beta > 0.$$

  - **Degree lexicographic order**

$$x^\alpha >_{\text{dlex}} x^\beta \iff \deg x^\alpha > \deg x^\beta \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and first nonzero entry of } \alpha - \beta > 0).$$

  - **Degree reverse lexicographic order**

$$x^\alpha >_{\text{drlex}} x^\beta \iff \deg x^\alpha > \deg x^\beta \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and last nonzero entry of } \alpha - \beta < 0).$$

  - Example: $x_1^2 x_2 x_3^3 >_{\text{lex}} x_1 x_2^3 x_3^2 >_{\text{lex}} x_1 x_2 x_3^5$

$$x_1 x_2 x_3^5 >_{\text{dlex}} x_1^2 x_2 x_3^3 >_{\text{dlex}} x_1 x_2^3 x_3^2$$

$$x_1 x_2 x_3^5 >_{\text{drlex}} x_1 x_2^3 x_3^2 >_{\text{drlex}} x_1^2 x_2 x_3^3$$

Reminder about mathematical quantities

- For $f \in R$, the **leading term** $\boldsymbol{L}_>(f)$ w.r.t. $>$ is the largest term in f w.r.t. $>$.
- A finite subset $G = \{g_1, \ldots, g_r\} \subset I$ is a **Gröbner basis for** $\boldsymbol{I}$ if

$$\boldsymbol{L}_>(I) = \boldsymbol{L}_>(G)\,,$$

  - i.e. the leading submodule of I is generated by the leading terms of the elements of G.
  - Hence G generates I.
- One way of computing Gröbner bases is via **Buchberger's algorithm**.
  - Also applicable to the non-commutative case (left ideal).
- The **remainder** h of

$$g = \sum_{i=1}^{r} f_i\, g_i + h$$

  is uniquely determined by g, I, and $>$.
- Call $\mathrm{NF}_{I,>}(g) = \mathrm{NF}_G(g) = h$ the **normal form** of g mod I w.r.t. $>$.

Operators and algebra

- Introduce operators with **partial right** action. For each $i = 1, \ldots, n$ have

$$I(\ldots, z_i, \ldots) \bullet D_i = I(\ldots, z_i - 1, \ldots), \qquad \underbrace{I(\ldots, z_i, \ldots)}_{\text{not scaleless}} \bullet D_i^- = I(\ldots, z_i + 1, \ldots),$$

[note that $D_i \sim \boldsymbol{i}^-$ and $D_i^- \sim \boldsymbol{i}^+$]

$$I(\ldots, z_i, \ldots) \bullet a_i = z_i I(\ldots, z_i, \ldots), \qquad I(\ldots, \underbrace{z_i}_{\neq 0}, \ldots) \bullet a_i^{-1} = \frac{1}{z_i} I(\ldots, z_i, \ldots).$$

- The following computations will take place in the **non-commutative rational** double-shift algebra

$$Y := \mathbb{Q}(d, s_{ij}, m_i^2)(a_1, \ldots, a_n)\langle D_j, D_j^- \mid j = 1, \ldots, n\rangle / (D_i D_i^- = 1 = D_i^- D_i \mid i = 1, \ldots, n)$$

in the indeterminates $a_1, \ldots, a_n, D_1, \ldots, D_n, D_1^-, \ldots, D_n^-$ with relations

$$[a_i, D_j] = \delta_{ij} D_i\,, \qquad [a_i, D_j^-] = -\delta_{ij} D_i^-\,, \qquad D_i D_i^- = 1,$$

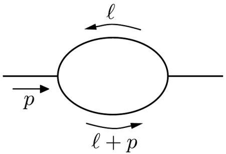
[no summation over repeated indices]

$$[a_i, a_j] = [D_i, D_j] = [D_i^-, D_j^-] = [D_i, D_j^-] = 0 \quad .$$

Example: one-loop bubble

- Start with one-loop massless bubble

[figures courtesy by Robin Brüser]



$(p^2 \neq 0)$

$$F(a_1, a_2) = \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}} \frac{1}{D_1^{a_1}\, D_2^{a_2}}\,, \qquad a_i \in \mathbb{Z}$$

$$\begin{aligned} D_1 &= -\ell^2 \\ D_2 &= -(\ell+p)^2 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \ell^2 &= -D_1 \\ \ell\cdot p &= \frac{1}{2}(D_1 - D_2 - p^2) \end{aligned}$$

- IBP equations

[Tkachov'81;Chetyrkin,Tkachov'81]

$$\int \frac{d^d \ell_1}{i\pi^{d/2}} \cdots \int \frac{d^d \ell_L}{i\pi^{d/2}} \frac{\partial}{\partial \ell_i^\mu} \frac{v_j^\mu}{D_1^{a_1} \cdots D_n^{a_n}} = 0$$

- Standard IBPs for one-loop bubble

$\underline{v = \ell :}$

$$\begin{aligned} 0 = {} & (d - a_2 - 2a_1)F(a_1, a_2) - a_2 p^2 F(a_1, a_2 + 1) \\ & - a_2 F(a_1 - 1, a_2 + 1) \end{aligned}$$

$\underline{v = p :}$

$$\begin{aligned} 0 = {} & (a_1 - a_2)F(a_1, a_2) + a_2 p^2 F(a_1, a_2 + 1) - a_1 p^2 F(a_1 + 1, a_2) \\ & + a_2 F(a_1 - 1, a_2 + 1) - a_1 F(a_1 + 1, a_2 - 1) \end{aligned}$$

IBPs as left ideal in the double-shift algebra

- Write standard IBP relations in terms of operators, e.g. for one-loop bubble

$$
\begin{aligned}
0 &= (d - z_2 - 2z_1)\,F(z_1,z_2) - z_2\,p^2\,F(z_1,z_2+1) - z_2\,F(z_1-1,z_2+1) \\
&= F(z_1,z_2) \bullet \underbrace{\left[(d - a_2 - 2a_1) - p^2\,a_2 D_2^- - a_2\,D_1\,D_2^-\right]}_{=\,r_1}
\end{aligned}
$$

- Similarly $\quad r_2 = -a_1\,D_1^-\,D_2 + a_2\,D_1^-\,D_2 - p^2\,a_1\,D_2^- + a_1 - a_2$

IBPs as left ideal in the double-shift algebra

- Write standard IBP relations in terms of operators, e.g. for one-loop bubble

$$0 = (d - z_2 - 2z_1)\, F(z_1, z_2) - z_2\, p^2\, F(z_1, z_2 + 1) - z_2\, F(z_1 - 1, z_2 + 1)$$

$$= F(z_1, z_2) \bullet \underbrace{\left[(d - a_2 - 2a_1) - p^2\, a_2 D_2^- - a_2\, D_1\, D_2^-\right]}_{= r_1}$$

- Similarly $\quad r_2 = -a_1\, D_1^-\, D_2 + a_2\, D_1^-\, D_2 - p^2\, a_1\, D_2^- + a_1 - a_2$

The IBP relations generate a **left** ideal in the non-commutative rational double-shift algebra Y!

$$I_{\text{IBP}} := \langle r_i \mid i = 1, \ldots, L(L+E) \rangle \lhd Y$$

Formulation of the algebra and ideal was crucial for successful computation

- For the one-loop bubble $\quad I_{\text{IBP}} = \langle r_1, r_2 \rangle_Y = \{u_1\, r_1 + u_2\, r_2 | u_{1,2} \in Y\}$
- By construction, have $\quad F(z_1, z_2) \bullet r = 0 \quad$ for $\quad r \in I_{\text{IBP}}$

Goal: Compute a Gröbner basis for the left ideal I_{IBP} in Y

- Remainder (normal form) corresponds to result of reduction

Computing a Gröbner basis for $I_{\mathrm{IBP}} \lhd Y$

- The computations are done in the GAP package `LoopIntegrals`.

  https://homalg-project.github.io/pkg/LoopIntegrals

  It computes IBP relations among loop integrals.

- Dependencies
  - the computer algebra system SINGULAR for **commutative** Gröbner bases in **polynomial** rings, [Decker,Greuel,Pfister,Schönemann'19]
  - its subsystem PLURAL for **non-commutative** Gröbner bases in the double-shift algebra with **polynomial** coefficients, [Levandovskyy,Schönemann'03]
  - Chyzak's Maple package `Ore_algebra` for **noncommutative** Gröbner bases in the double-shift algebra with **rational** coefficients, [Chyzak'98]
  - the Julia package HECKE for simulating the reduction w.r.t. Gröbner bases in the rational double-shift algebra using linear algebra over the field of rational functions.
    HECKE uses finite-field methods to compute GCDs.

Gröbner basis of left ideal of IBPs in the double-shift algebra

- Rational Gröbner basis of one-loop massless bubble has 4 elements and was computed in <1s.

$$G = \Big\{ (d-a_1-a_2)(d-2a_1-2a_2+2)D_2 - (a_2-1)(d-2a_2)\,p^2,$$

$$(d-a_1-a_2)(d-2a_1-2a_2+2)D_1 - (a_1-1)(d-2a_1)\,p^2,$$

$$a_2\,p^2\,(d-2a_2-2)D_2^- - (d-a_1-a_2-1)(d-2a_1-2a_2),$$

$$a_1\,p^2\,(d-2a_1-2)D_1^- - (d-a_1-a_2-1)(d-2a_1-2a_2)\Big\}$$

  - Leaves propagator powers parametric

- Particular interesting are IBP operators of the form ("normal form IBPs")

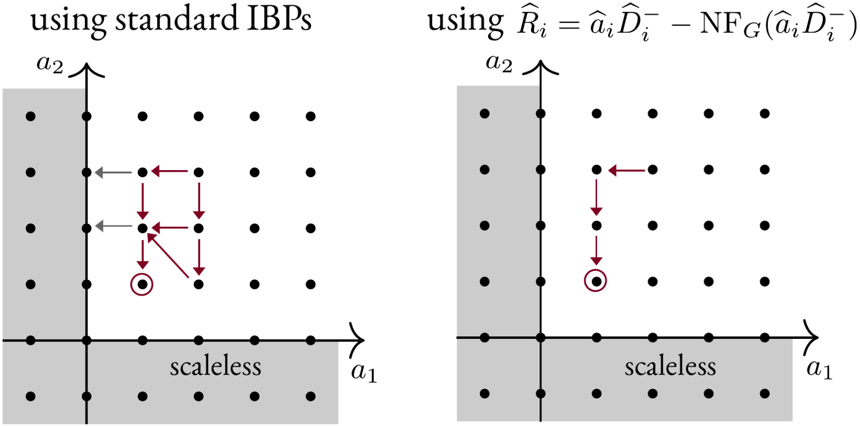
$$R_i = a_i\,D_i^- - \mathrm{NF}_G\left(a_i D_i^-\right) \in I_{\mathrm{IBP}} \qquad \text{[no summation over } i\text{]}$$

  - e.g. $$\mathrm{NF}_G\left(a_1 D_1^-\right) = \frac{(d-a_1-a_2-1)(d-2a_1-2a_2)}{p^2(d-2a_1-2)}$$
  - Allow for a straightforward reduction

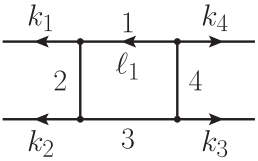
Standard IBPs vs. Gröbner basis

- Compare reduction with standard IBPs vs. Gröbner basis for the one-loop massless bubble

Reduction of $F(3,2)$:



One-loop massless box



$$\begin{aligned} P_1 &= -\ell_1^2, \\ P_2 &= -(\ell_1 - k_1)^2, \\ P_3 &= -(\ell_1 - k_1 - k_2)^2, \\ P_4 &= -(\ell_1 + k_4)^2. \end{aligned}$$

- Massless one-loop box, $k_i^2 = 0$
- Independent Mandelstam variables s_{12}, s_{14}
- Four standard IBPs

$$\begin{aligned} r_1 &= -a_2 D_1 D_2^- - a_3 D_1 D_3^- - a_4 D_1 D_4^- - s_{12} a_3 D_3^- + (D - 2a_1 - a_2 - a_3 - a_4), \\ r_2 &= a_1 D_1^- D_2 - a_2 D_1 D_2^- - a_3 D_1 D_3^- + a_3 D_2 D_3^- - a_4 D_1 D_4^- + a_4 D_2 D_4^- - s_{12} a_3 D_3^- + s_{14} a_4 D_4^- - a_1 + a_2, \\ r_3 &= -a_1 D_1^- D_2 + a_1 D_1^- D_3 + a_2 D_2^- D_3 - a_3 D_2 D_3^- - a_4 D_2 D_4^- + a_4 D_3 D_4^- + s_{12} a_1 D_1^- - s_{14} a_4 D_4^- - a_2 + a_3, \\ r_4 &= a_2 D_1 D_2^- + a_3 D_1 D_3^- - a_1 D_1^- D_4 - a_2 D_2^- D_4 - a_3 D_3^- D_4 + a_4 D_1 D_4^- - s_{14} a_2 D_2^- + s_{12} a_3 D_3^- + a_1 - a_4 \end{aligned}$$

One-loop massless box

- Reduced Gröbner basis over *rational* double-shift algebra has 9 elements

$$
\begin{aligned}
G = \Big\{ & D_4 - D_2 + \frac{(a_2 - a_4)s_{14}}{D - a_{1234}} 1, \\
& D_3 - D_1 + \frac{(a_1 - a_3)s_{12}}{D - a_{1234}} 1, \\
& \ldots, \\
& 4(a_2 - 1)(D - a_{1234})D_3 - 2(D - 2a_{134})(D - a_{1234})D_4 + (D - 2a_{14} - 2)(D - 2a_{234})s_{12}1 \\
& - 2(D - 2a_{134})(a_2 - a_4)s_{14}1 - \frac{(D - 2a_{14} - 2)(D - 2a_{34} - 2)a_4 s_{12} s_{14}}{D - a_{1234} - 1} D_4^-, \\
& - 2(D - 2a_{1234} + 4)(D - a_{1234} + 1)D_3^2 + (D - 2a_{123} + 2)(D - 2a_{134} + 2)s_{14}D_3 \\
& - 2(a_1 - a_3 + 1)(D - 2a_{1234} + 4)s_{12}D_3 + 4(a_1 - 1)(a_3 - 1)s_{12}D_4 \\
& - \frac{(D - 2a_{123} + 2)(a_3 - 1)(D - 2a_{34})s_{12}s_{14}}{D - a_{1234}} 1 \Big\},
\end{aligned}
$$

- G is rational in D, a_i, s_{ij} and polynomial in D_i, D_i^-.

One-loop massless box

- A **standard monomial** w.r.t. the Gröbner basis G of I_{IBP} is a monomial m in the D_i, D_j^- such that $\text{NF}_G(m) = m$.
- The set of standard monomials is a basis for the finite dimensional $\mathbb{K}$-vector space Y/I_{IBP}
- It corresponds to a set of master integrals with respect to e.g. the corner integral

$$
\begin{aligned}
\text{NF}_G(D_1) =& D_3 + \frac{(a_1 - a_3)s_{12}}{D - a_{1234}} && \rightsquigarrow D_1 \text{ is a nonstandard monomial} \\
\text{NF}_G(D_2) =& D_4 + \frac{(a_2 - a_4)s_{14}}{D - a_{1234}} && \rightsquigarrow D_2 \text{ is a nonstandard monomial} \\
\text{NF}_G(D_3) =& D_3 && \rightsquigarrow D_3 \text{ is a standard monomial} \\
\text{NF}_G(D_4) =& D_4 && \rightsquigarrow D_4 \text{ is a standard monomial}
\end{aligned}
$$

- The set of standard monomials with respect to G is $\{1, D_3, D_4\}$
- Corresponds to three master integrals

$$\{I(1,1,1,1), I(1,1,0,1), I(1,1,1,0)\}.$$

- Can also verify that $D_1D_2, D_1D_4, D_2D_3, D_3D_4$ are the minimal scaleless monomials w.r.t. $I(1,1,1,1)$.

One-loop massless box

- Again compute IBP relations of the form $R_i = a_i D_i^- - \mathrm{NF}_G(a_i D_i^-) \in I_{\mathrm{IBP}}$
- Normal form of operators $a_i D_i^-$ w.r.t. the Gröbner basis G of the left ideal I_{IBP}

$$\begin{aligned}
\mathrm{NF}_G(a_1 D_1^-) = &-\frac{2\,(D-2a_{124})\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{12}-2)\,(D-2a_{14}-2)\,s_{12}s_{14}}\,D_3\\
&+\frac{4\,(a_3-1)\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{12}-2)\,(D-2a_{14}-2)\,s_{12}s_{14}}\,D_4\\
&+\frac{(D-2a_{134})\,(D-a_{1234}-1)}{(D-2a_{14}-2)\,s_{12}}\,1,
\end{aligned}$$

$$\begin{aligned}
\mathrm{NF}_G(a_2 D_2^-) = &\frac{4\,(a_4-1)\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{12}-2)\,(D-2a_{23}-2)\,s_{12}s_{14}}\,D_3\\
&-\frac{2\,(D-2a_{123})\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{12}-2)\,(D-2a_{23}-2)\,s_{12}s_{14}}\,D_4\\
&+\frac{(D-2a_{234})\,(D-a_{1234}-1)}{(D-2a_{23}-2)\,s_{14}}\,1,
\end{aligned}$$

$$\begin{aligned}
\mathrm{NF}_G(a_3 D_3^-) = &-\frac{2\,(D-2a_{234})\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{23}-2)\,(D-2a_{34}-2)\,s_{12}s_{14}}\,D_3\\
&+\frac{4\,(a_1-1)\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{23}-2)\,(D-2a_{34}-2)\,s_{12}s_{14}}\,D_4\\
&+\frac{(D-2a_{134})\,(D-a_{1234}-1)}{(D-2a_{34}-2)\,s_{12}}\,1\\
&-\frac{2\,(a_1-a_3)\,(D-2a_{234})\,(D-a_{1234}-1)}{(D-2a_{23}-2)\,(D-2a_{34}-2)\,s_{14}}\,1,
\end{aligned}$$

$$\begin{aligned}
\mathrm{NF}_G(a_4 D_4^-) = &\frac{4\,(a_2-1)\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{14}-2)\,(D-2a_{34}-2)\,s_{12}s_{14}}\,D_3\\
&-\frac{2\,(D-2a_{134})\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{14}-2)\,(D-2a_{34}-2)\,s_{12}s_{14}}\,D_4\\
&+\frac{(D-2a_{234})\,(D-a_{1234}-1)}{(D-2a_{34}-2)\,s_{14}}\,1\\
&-\frac{2\,(a_2-a_4)\,(D-2a_{134})\,(D-a_{1234}-1)}{(D-2a_{14}-2)\,(D-2a_{34}-2)\,s_{12}}\,1
\end{aligned}$$

- In denominators of $\mathrm{NF}_G(a_i D_i^-)$, all a_j appear together with D.
- $\mathrm{NF}_G(a_i D_i^-)$ are $\mathbb{K}$-linear combinations of the standard monomials

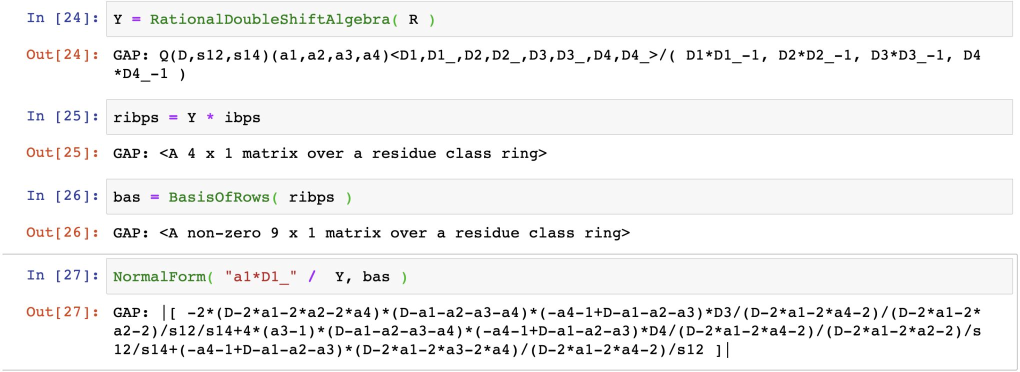
One-loop massless box

- The Jupyter notebook can be found at https://homalg-project.github.io/nb/1LoopBox/

```
In [17]: LoadPackage( "LoopIntegrals" )

In [18]: R = RingOfLoopDiagram( LD )

Out[18]: GAP: Q[D,s12,s14][D1,D2,D3,D4]

In [19]: Ypol = DoubleShiftAlgebra( R )

Out[19]: GAP: Q[D,s12,s14][a1,a2,a3,a4]<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4
         *D4_-1 )

In [20]: ibps = MatrixOfIBPRelations( LD )

Out[20]: GAP: <A 4 x 1 matrix over a residue class ring>

In [21]: r1 = ibps[1,1]

Out[21]: GAP: |[ -a2*D1*D2_-s12*a3*D3_-a3*D1*D3_-a4*D1*D4_+D-2*a1-a2-a3-a4 ]|

In [22]: bas_pol = BasisOfRows( ibps )

Out[22]: GAP: <A non-zero 28 x 1 matrix over a residue class ring>

In [23]: NormalForm( "a1*D1_" /  Ypol, bas_pol )

Out[23]: GAP: |[ a1*D1_ ]|
```

One-loop massless box

The following command needs Chyzak's Maple package `Ore_algebra` for the noncommutative Gröbner bases of the rational double-shift algebra:



```
In [24]: Y = RationalDoubleShiftAlgebra( R )

Out[24]: GAP: Q(D,s12,s14)(a1,a2,a3,a4)<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4
         *D4_-1 )

In [25]: ribps = Y * ibps

Out[25]: GAP: <A 4 x 1 matrix over a residue class ring>

In [26]: bas = BasisOfRows( ribps )

Out[26]: GAP: <A non-zero 9 x 1 matrix over a residue class ring>

In [27]: NormalForm( "a1*D1_" /  Y, bas )

Out[27]: GAP: |[ -2*(D-2*a1-2*a2-2*a4)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D3/(D-2*a1-2*a4-2)/(D-2*a1-2*
         a2-2)/s12/s14+4*(a3-1)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D4/(D-2*a1-2*a4-2)/(D-2*a1-2*a2-2)/s
         12/s14+(-a4-1+D-a1-a2-a3)*(D-2*a1-2*a3-2*a4)/(D-2*a1-2*a4-2)/s12 ]|
```

One-loop massless box

```
In [28]: NormalFormWrtInitialIntegral( "D1_" / Y, bas )

Out[28]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [29]: NormalFormWrtInitialIntegral( "D3_" / Y, bas )

Out[29]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [30]: NormalFormWrtInitialIntegral( "D1*D2" / Y, bas )

Out[30]: GAP: |[ 0 ]|
```

One-loop massless box

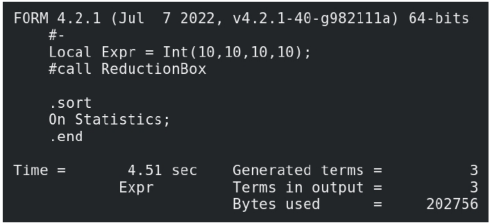
```
In [28]: NormalFormWrtInitialIntegral( "D1_" / Y, bas )

Out[28]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [29]: NormalFormWrtInitialIntegral( "D3_" / Y, bas )

Out[29]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [30]: NormalFormWrtInitialIntegral( "D1*D2" / Y, bas )

Out[30]: GAP: |[ 0 ]|
```

- Virtues of the Gröbner basis / normal form
  - Contains the entire information required for reduction
  - Recognizes scaleless sectors (and in certain cases symmetries)
  - No new bottom-up reduction required for new/additional integrals
  - Ideally suited for storage, e.g. in a database

One-loop massless box

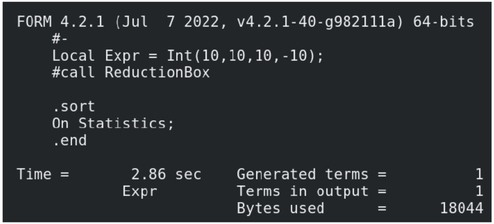
- Also easy to implement in Mathematica or FORM
    - well-suited for parallelization
    - Allows for fast reduction, also of high propagator powers

$$F(10,10,10,10)\ , \quad \sim 5\ \text{sec.}$$



```
FORM 4.2.1 (Jul  7 2022, v4.2.1-40-g982111a) 64-bits
    #-
    Local Expr = Int(10,10,10,10);
    #call ReductionBox

    .sort
    On Statistics;
    .end

Time =       4.51 sec    Generated terms =           3
            Expr         Terms in output =           3
                         Bytes used      =      202756
```

$$F(10,10,10,-10)\ , \quad \sim 3\ \text{sec.}$$

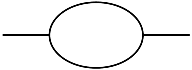


```
FORM 4.2.1 (Jul  7 2022, v4.2.1-40-g982111a) 64-bits
    #-
    Local Expr = Int(10,10,10,-10);
    #call ReductionBox

    .sort
    On Statistics;
    .end

Time =       2.86 sec    Generated terms =           1
            Expr         Terms in output =           1
                         Bytes used      =       18044
```

Special IBPs

- Look for IBP relations that do not increase the propagator powers, e.g. for the one-loop bubble

$$\int \frac{d^d\ell}{i\pi^{d/2}} \frac{\partial}{\partial \ell^\mu} \frac{v_i^\mu}{D_1^{a_1}\, D_2^{a_2}} = 0$$



- Vector in numerator can be a linear combination of loop and external momenta

$$v_i^\mu = \underset{\uparrow}{C_1^{(i)}}(D_1, D_2)\,\ell^\mu + \underset{\uparrow}{C_2^{(i)}}(D_1, D_2)\,p^\mu \quad = \sum_j C_j^{(i)}\, q_j^\mu$$

polynomial dependence

- Derivative acting on $D_k^{-a_k}$ increases propagator power

$$v_i^\mu \frac{\partial}{\partial \ell^\mu} \frac{1}{D_1^{a_1}\, D_2^{a_2}} = \sum_k v_i^\mu \frac{\partial D_k}{\partial \ell^\mu} \boxed{\frac{\partial}{\partial D_k} \frac{1}{D_1^{a_1}\, D_2^{a_2}}} = \sum_{k,j} C_j^{(i)} \underbrace{q_j^\mu \frac{\partial D_k}{\partial \ell^\mu}}_{:=\mathcal{E}_{kj}^{(i)}} \frac{\partial}{\partial D_k} \frac{1}{D_1^{a_1}\, D_2^{a_2}}$$

⟵ IBP-generating matrix

Special IBPs

$$v_i^\mu \frac{\partial}{\partial \ell^\mu} \frac{1}{D_1^{a_1}\, D_2^{a_2}} = \sum_k \boxed{v_i^\mu \frac{\partial D_k}{\partial \ell^\mu}}\ \boxed{\frac{\partial}{\partial D_k} \frac{1}{D_1^{a_1}\, D_2^{a_2}}} = \sum_{k,j} C_j^{(i)} \underbrace{q_j^\mu \frac{\partial D_k}{\partial \ell^\mu}}_{:=\mathcal{E}_{kj}^{(i)}} \frac{\partial}{\partial D_k} \frac{1}{D_1^{a_1}\, D_2^{a_2}}$$

$\longleftarrow$ IBP-generating matrix

- Increase of propagator power survives unless

[Kosower et al.'10'18; Schabinger et al.'11'20; Ita'15; Böhm et al.'17]

$$v_i^\mu \frac{\partial D_k}{\partial \ell^\mu} \propto D_k \qquad \forall\, k$$

Amounts to computing column syzygy matrix S of $\mathcal{E}$ modulo $\mathrm{diag}(D_1, D_2)$

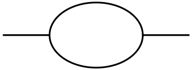
$$\sum_j \mathcal{E}_{kj}^{(i)}\, S_{ji} \propto D_k \qquad \forall k, i$$

Baikov represenation

- Basic idea: use propagators as integration variables

[Baikov'96]

$$I_{a_1 a_2} = \text{prefactor} \times \int dD_1\, dD_2\, \frac{\mathcal{B}^\gamma}{D_1^{a1}\, D_2^{a2}}$$



- Exponent: $\gamma = \dfrac{d - L - E - 1}{2}$

- Baikov polynomial $\mathcal{B}$ is the Gram determinant of loop and external momenta

$$\mathcal{B} = \begin{vmatrix} \ell \cdot \ell & \ell \cdot p \\ p \cdot \ell & p \cdot p \end{vmatrix} = \frac{1}{4}\left(-D_1^2 + 2 D_1\, D_2 - D_2^2 - 2\, D_1\, p^2 - 2\, D_2\, p^2 - p^4\right)$$

- IBP relations from

[Zhang, Larsen; Lee; Frellesvig,Papadopoulos]

$$\int dD_1\, dD_2 \sum_k \frac{\partial}{\partial D_k}\left(\underset{\substack{\uparrow \\ \text{polynomial}}}{\boxed{u_k(D_1, D_2)}}\, \frac{\mathcal{B}^\gamma}{D_1^{a1}\, D_2^{a2}}\right) = \int dD_1\, dD_2 \left[\gamma\, \underset{\substack{\uparrow \\ \text{dimension shift}}}{\boxed{\mathcal{B}^{\gamma-1}}} \left(\sum_k \frac{\partial \mathcal{B}}{\partial D_k}\, u_k\right) \frac{1}{D_1^{a1}\, D_2^{a2}} + \mathcal{B}^\gamma \sum_k \underset{\substack{\uparrow \\ \text{additional dots}}}{\boxed{\frac{\partial}{\partial D_k}\, \frac{u_k}{D_1^{a1}\, D_2^{a2}}}}\right] = 0$$

Special IBPs

$$\int dD_1\, dD_2 \left[\gamma \boxed{\mathcal{B}^{\gamma-1}} \boxed{\left(\sum_k \frac{\partial \mathcal{B}}{\partial D_k}\, u_k\right)} \frac{1}{D_1^{a_1}\, D_2^{a_2}} + \mathcal{B}^{\gamma} \sum_k \boxed{\frac{\partial}{\partial D_k} \frac{\boxed{u_k}}{D_1^{a_1}\, D_2^{a_2}}}\right] = 0$$

↑ dimension shift

↑ additional dots

- Avoid additional dots if $u_k = \tilde{u}_k(D_1, D_2) \times D_k \xrightarrow{\text{solutions}}$ module $M_1 = \langle D_1, D_2 \rangle$
- Avoid dimension shift if $\sum_k \frac{\partial \mathcal{B}}{\partial D_k} u_k = \text{poly}(D_1, D_2) \times \mathcal{B} \xrightarrow{\text{solutions}}$ module M_2 [Böhm et al.'17]
- Obtain special IBPs via module intersection $M_1 \cap M_2$ [Bendle et al.'20]

Special IBPs

$$\int dD_1\, dD_2 \left[\gamma\, \boxed{\mathcal{B}^{\gamma-1}}\, \boxed{\left(\sum_k \frac{\partial \mathcal{B}}{\partial D_k}\, u_k\right)} \frac{1}{D_1^{a_1}\, D_2^{a_2}} + \mathcal{B}^{\gamma} \sum_k \boxed{\frac{\partial}{\partial D_k} \frac{\boxed{u_k}}{D_1^{a_1}\, D_2^{a_2}}}\right] = 0$$

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- Avoid additional dots if $u_k = \tilde{u}_k(D_1, D_2) \times D_k$ $\underrightarrow{\text{solutions}}$ module $M_1 = \langle D_1, D_2 \rangle$
- Avoid dimension shift if $\displaystyle\sum_k \frac{\partial \mathcal{B}}{\partial D_k}\, u_k = \mathrm{poly}(D_1, D_2) \times \mathcal{B}$ $\underrightarrow{\text{solutions}}$ module M_2 [Böhm et al.'17]
- Obtain special IBPs via module intersection $M_1 \cap M_2$ [Bendle et al.'20]

Important observation

Computation is identical in Baikov and in momentum space representation!

$$M_1 \cap M_2 \quad \overset{\longleftrightarrow}{\text{equivalent}} \quad \sum_j \mathcal{E}_{kj}^{(i)}\, S_{ji} \propto D_k$$

Special IBPs

- In particular, have $M_2 = \langle \mathcal{E} \rangle$, i.e.

$$\sum_k \frac{\partial \mathcal{B}}{\partial D_k} \mathcal{E}_{kj}^{(i)} \propto \mathcal{B}$$

$$\uparrow$$

no dimension-shift

$$\sum_{k,j} \frac{\partial \mathcal{B}}{\partial D_k} \mathcal{E}_{kj}^{(i)} S_{ji} \propto \mathcal{B}$$

$$\uparrow$$

no dim.-shift and no dots

- I.e. we don't have to compute M_2
- M_1 is generated by our special IBPs $\sum_j \mathcal{E}_{kj}^{(i)} S_{ji}$

Special IBPs

- In particular, have $M_2 = \langle \mathcal{E} \rangle$, i.e.

$$\sum_k \frac{\partial \mathcal{B}}{\partial D_k} \mathcal{E}^{(i)}_{kj} \propto \mathcal{B}$$

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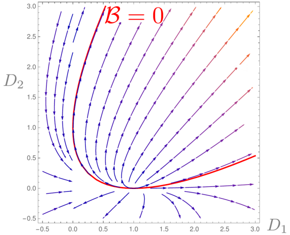
$$\uparrow$$

no dim.-shift and no dots

- I.e. we don't have to compute M_2
- M_1 is generated by our special IBPs $\sum_j \mathcal{E}^{(i)}_{kj} S_{ji}$

- Solutions form a generating system of the logarithmic vector fields of the Baikov kernel, e.g. [see also Schulze et al.'17]

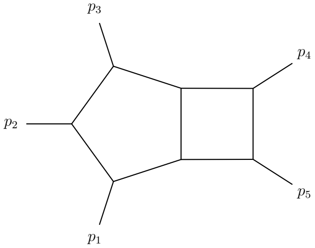
$$\sum_j \mathcal{E}^{(2)}_{kj} S_{j2} = \begin{pmatrix} p^2 D_1 + D_1^2 + D_1 D_2 \\ 2 D_1 D_2 \end{pmatrix}_k$$



- Public programs
  - NeatIBP [Wu,Böhm,Ma,Xu,Zhang'23]
  - LoopIntegrals [Barakat,Brüser,Fieker,Piclum,TH'22]

Examples

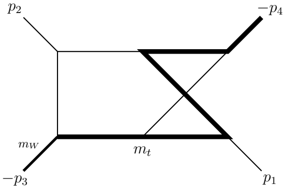
- Two-loop massless planar penta-box



[from 2305.08783]

- Compute also Gröbner basis of syzygies w.r.t. deg.rev.lex.
- 4h runtime
- $<$ 10 GB of memory
- no degree-bound

- Two-loop massive non-planar box



[from 2305.08783]

- under construction

Linear algebra ansatz

- We observed that $R_i = a_i D_i^- - \mathrm{NF}_G(a_i D_i^-)$ are well-suited for reduction
  - in all considered problems normal-form IBPs generate the left ideal I_{IBP} (most likely not true for arbitrary problems)

Idea: Use linear algebra to compute R_i when Gröbner basis is not available.

- Consider again one-loop bubble

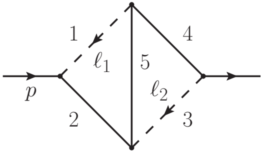
$$
\begin{aligned}
\widehat{r}_1 &= (d - \widehat{a}_2 - 2\widehat{a}_1) - p^2 \widehat{a}_2 \widehat{D}_2^- - \widehat{a}_2 \widehat{D}_1 \widehat{D}_2^- \\
\widehat{r}_2 &= -\widehat{a}_1 \widehat{D}_1^- \widehat{D}_2 + \widehat{a}_2 \widehat{D}_1 \widehat{D}_2^- - p^2 \widehat{a}_1 \widehat{D}_2^- + (\widehat{a}_1 - \widehat{a}_2) \\
\widehat{D}_1^- \widehat{r}_1 &= (d - \widehat{a}_2 - 2\widehat{a}_1 - 2) \widehat{D}_1^- - p^2 \widehat{a}_2 \widehat{D}_1^- \widehat{D}_2^- - \widehat{a}_2 \widehat{D}_2^- \\
&\vdots \longleftarrow (\widehat{D}_1^-)^{j_1} (\widehat{D}_2^-)^{j_2} \widehat{r}_i
\end{aligned}
$$

treat as unknowns

Solve for $\widehat{D}_1^-$ and $\widehat{D}_2^-$

- Question: Which values for j_1, j_2 ??
- Observation: Special IBPs require lower max. values of $j_{1,2}$ than standard IBPs

Two-loop on-shell kite



$$
\begin{aligned}
P_1 &= -\ell_1^2, \\
P_2 &= m^2 - (\ell_1 + p)^2, \\
P_3 &= m^2 - (\ell_2 + p)^2, \\
P_4 &= -\ell_2^2, \\
P_5 &= m^2 - (\ell_1 + \ell_2 + p)^2
\end{aligned}
$$

- On-shell kite, $p^2 = m^2 \equiv s$

- Gröbner basis not yet available
- Simulating, using linear algebra, the computation of the normal form of $a_i D_i^-$ w.r.t. a Gröbner basis

$$\mathrm{NF}(a_1 D_1^-) = \frac{p_{10}}{4 d_1 d_2 d_3 d_4 d_7 d_8 s} + \frac{p_{12} D_2 + p_{13} D_3 + p_{14} D_4 + p_{15} D_5}{16 d_1 d_2 d_3 d_4 d_7 d_8 d_9 s^2}, \ \ldots$$

- p_i and d_i are expressions in the field $\mathbb{Q}(D, s_{ij}, m_i^2)(a_1, \ldots, a_n)$.
- The p_i are too large for printing, the d_i are small, e.g. $d_1 = 2a_1 + a_2 + a_3 + 2a_4 + a_5 - 2D + 1$.
- Allows for reduction of top-level sector

Conclusion

Conclusion

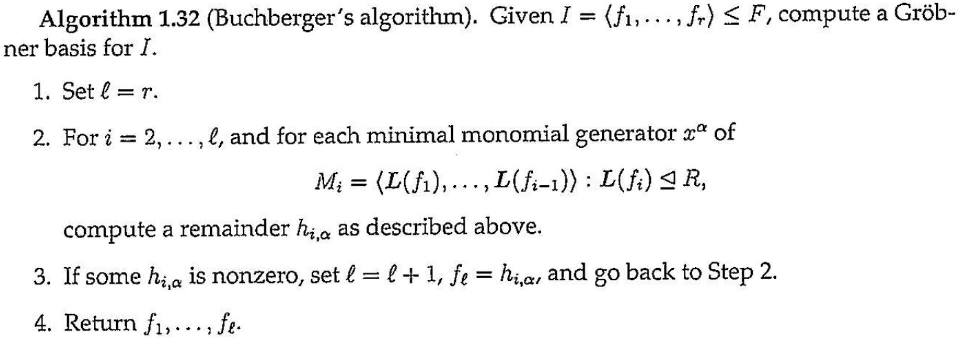
- We established IBP relations as a left ideal in the rational double-shift algebra
- For simple problems, the reduction is completely solved using the Gröbner basis
    - Fast reduction, easy parallelization
    - However: Derivation of GB computationally expensive for more complicated problems.
- Module intersection corresponds to column syzygies of IBP-generating matrix $\mathcal{E}$
- Derivation of normal-form IBPs possbible with linear algebra if GB is not available

Outlook / wishlist

- In general: want more loops, more legs, more scales
    - However: New ideas required to deal with enormous expression swell with increasing complexity.
- Combine normal-form IBPs with existing reduction algorithms
- Establish a database to store results of Gröbner bases or normal forms.

Backup slides

Buchberger's algorithm



Algorithm 1.32 (Buchberger's algorithm). Given $I = \langle f_1, \dots, f_r \rangle \le F$, compute a Gröbner basis for I.

1. Set $\ell = r$.
2. For $i = 2, \dots, \ell$, and for each minimal monomial generator x^α of

$$M_i = \langle L(f_1), \dots, L(f_{i-1}) \rangle : L(f_i) \trianglelefteq R,$$

   compute a remainder $h_{i,\alpha}$ as described above.

3. If some $h_{i,\alpha}$ is nonzero, set $\ell = \ell + 1$, $f_\ell = h_{i,\alpha}$, and go back to Step 2.
4. Return $f_1, \dots, f_\ell$.