A semi-analytic method for one-scale Feynman integrals

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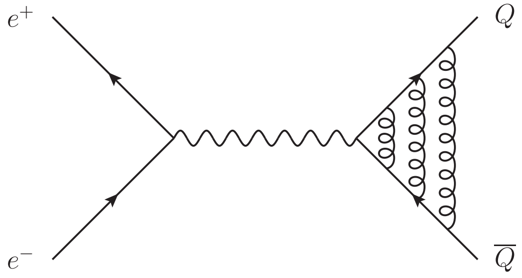
in collaboration with M. Egner, F. Lange, K. Schönwald, M. Steinhauser

MARIE CURIE ACTIONS

Funded by the European Union

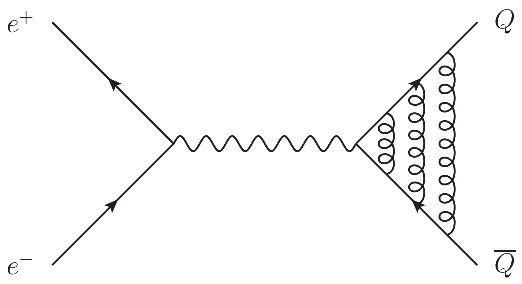
- **The method**
- **Application I: QCD massive form factors at 3 loops**
- **Application II: B-meson decays**

Solving Feynman integrals via differential equations



qgraf

Solving Feynman integrals via differential equations



qgraf

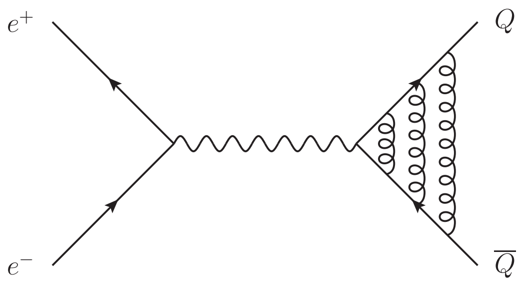
```
Symbol x;
Local E = f(1,2,x,3,4);

id f(?a,x,?b) = f(?b,?a);

Print;
.end
```

FORM

Solving Feynman integrals via differential equations



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```

FORM

AIR, Kira, FIRE, Reduze
Blade, NeatIBP, FiniteFlow

- $d = 4 - 2\epsilon$
- $\vec{x} = \{x_1, \ldots, x_m\}$ with e.g. $x_i = m/M, x = s/M^2$
- $\vec{j}(\vec{x}, \epsilon) = \{j_1(\vec{x}, \epsilon), \ldots, j_N(\vec{x}, \epsilon)\}$
- $M_i(\vec{x}, \epsilon)$ from IBP and dimensional shift relations

$$\frac{\partial \vec{j}}{\partial x_i} = M_i(\vec{x}, \epsilon)\, \vec{j}$$

Kotikov, Phys.Lett.B 254 (1991) 158
Gehrmann, Remiddi, Nucl.Phys.B 580 (2000) 485

- **Analytic solution**
  - Solve in terms of known constants/functions
  - Function properties well understood
  - Known analytic structures and series expansions
  - Fast and generic numerical evaluation tools

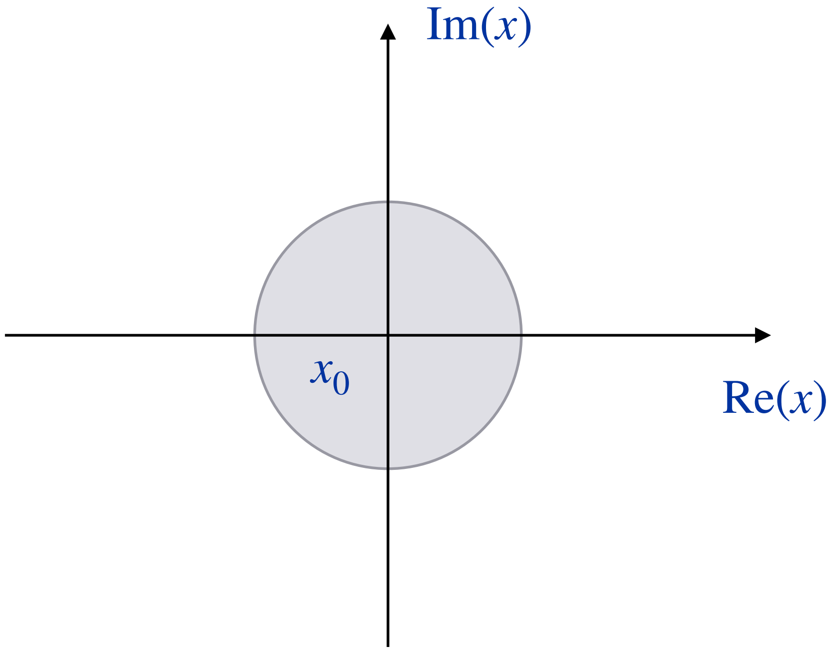
- **Numerical solution**
  - Oriented to phenomenological studies
  - Applicable to larger class of problems
  - Finite numerical accuracy

A semi-analytic method for one-scale problem

$$\frac{\partial \vec{j}}{\partial x} = M(x,\epsilon)\,\vec{j}$$

- We consider one dimensionless variable x
- We can compute **boundary conditions for** $x = x_0$
- **GOAL**: construct in the complex plane a series expansion around some point x_0 (and ϵ)

$$j_a(x,\epsilon) = \sum_{m=m_{\min}}^{m_{\max}} \sum_{n=0}^{n_{\max}} c_{a,mn}\epsilon^m (x-x_0)^n$$

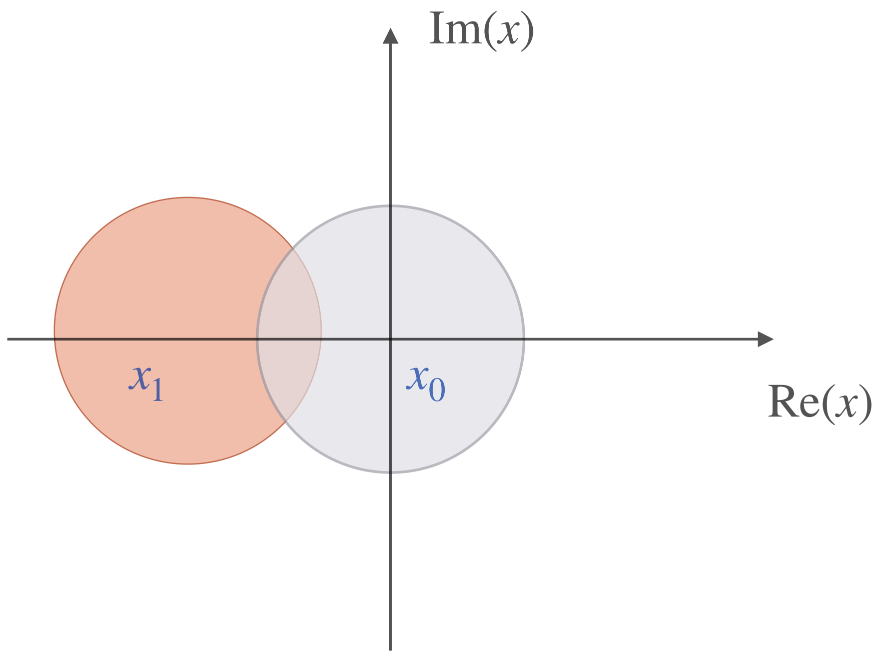


A semi-analytic method for one-scale problem

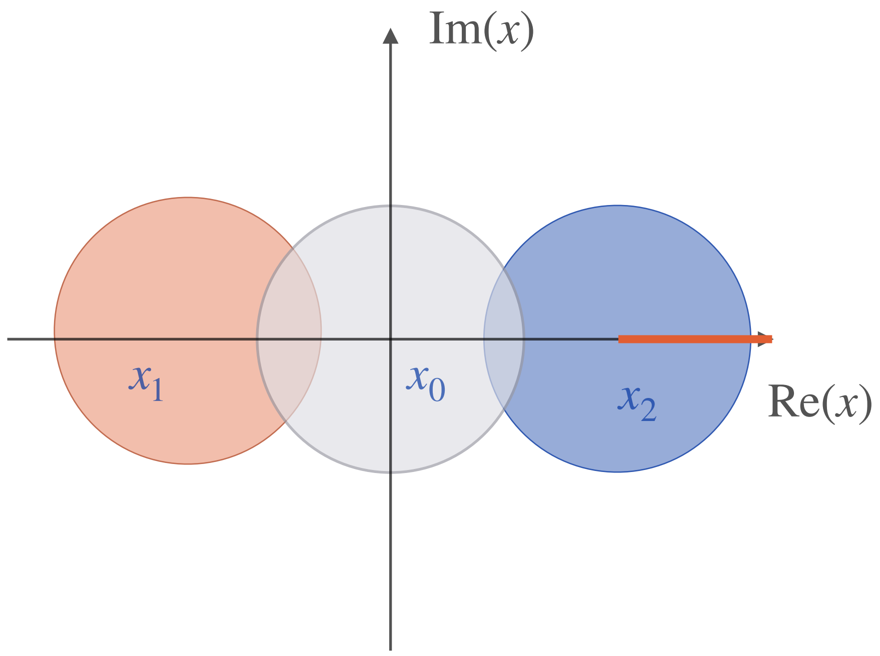
- Inserting the ansatz into the differential equation

$$\underbrace{\sum_m \sum_{n=1} n c_{a,mn} \epsilon^m (x - x_0)^{n-1}}_{\partial j_a / \partial x} = \sum_b M_{ab}(x, \epsilon) \underbrace{\sum_m \sum_{n=0} c_{b,mn} \epsilon^m (x - x_0)^n}_{j_b}$$

- Establish a **linear system of equations** for the expansion coefficients $c_{k,mn}$
- Solve the liner system in term of a **minimal set of coefficients** $\tilde{c}_{a,mn}$
- The minimal set of undetermined coefficients are **fixed from boundary conditions**



- Proceeds with a **new expansion around $x = x_1$**
- Match new expansion to the previous one (with finite accuracy)
- **Iterate** until all range of x is covered



- Proceeds with a new expansion around $x = x_1$
- Match new expansion to the previous one (with finite accuracy)
- Iterate until all range of x is covered

- Expansion around **singular points** (poles and thresholds)

$$j_a(x,\epsilon) = \sum_{m=m_{\min}}^{m_{\max}} \sum_{n=0}^{n_{\max}} \sum_{l\geq 0} c_{a,mnl}\, \epsilon^m\, (x - x_2)^{\alpha n - \beta} \log^l(x - x_2)$$

Features

Fael, Lange, Schönwald, Steinhauser *JHEP* 09 (2021) 152

- **GOAL**: cover physical range of x with series expansions.
- **No special form** of the differential equations
- Well suited for fast numerical evaluation
- Precision systematic improvable:
  - more expansion points
  - deeper expansion in x
  - variable transformation

  (Möbius transformation)
- **Bottleneck**
  - Problems with $O(10^2)$ masters
  - Solve linear system with $O(10^6)$ equations
  - Match expansion in numerically stable way

Similar approaches

- SYS

  Laporta, Int.J.Mod.Phys.A 15 (2000) 5087
- SolveCoupledSystems.m

  Blümlein, Schneider, Phys.Lett.B 771 (2017) 31
- DESS

  Lee, Smirnov, Smirnov, JHEP 03 (2018) 008
- DiffExp

  Hidding, Comput.Phys.Commun. 269 (2021) 108125
- SeaSide

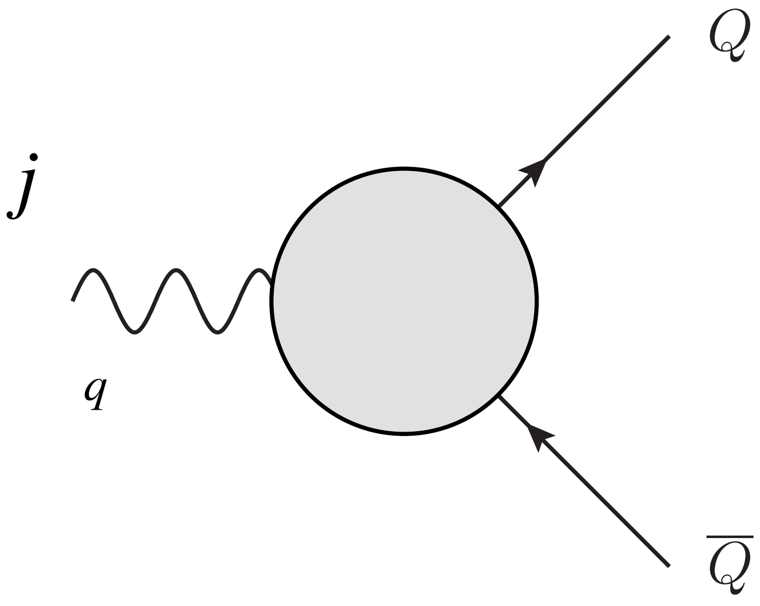
  Armadillo, Bonciani, Devoto, Rana, Vicini, Comput.Phys.Commun. 282 (2023) 108545
- AMFlow

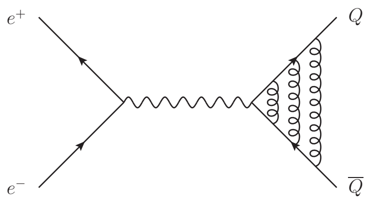
  Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565

Application I:
QCD massive form factors at 3 loops

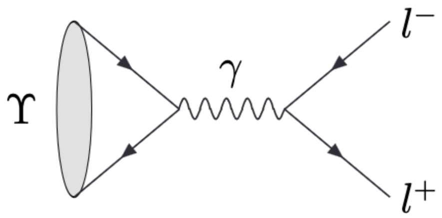
Phys.Rev.Lett. 128 (2022), Phys.Rev.D 106 (2023), Phys.Rev.D 107 (2023)

Massive form factors

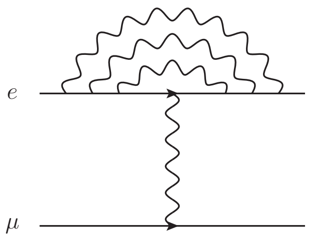




$t\bar{t}$ at e^+e^- collider



Quarkonium decay



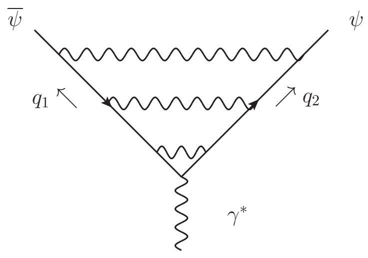
μe scattering

$$V(q_1, q_2) = \bar{u}(q_2)\Gamma(s)v(q_1)$$

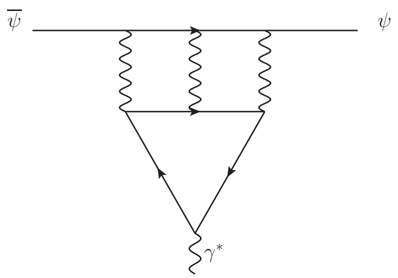
with $q_1^2 = q_2^2 = m^2$ and $s = (q_1 + q_2)^2$

- 2 loop QED
  Mastrolia, Remiddi, Nucl. Phys. B 664 (2003)
  Bonciani, Mastrolia, Remiddi, Nucl. Phys. B 676 (2004) 399
- 2 loop QCD
  Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi, Nucl. Phys. B 706 (2005) 245
  Gluza, Mitov, Moch, Riemann, JHEP 07 (2009), 001
  Ahmed, Henn, Steinhauser, JHEP 06 (2017), 125. Ablinger, et al, Phys.Rev. D 97 (2018), 094022
- 3 loop planar
  Henn, Smirnov, Smirnov, Steinhauser, JHEP 01 (2017), 074.
  Ablinger, Bl¨umlein, Marquard, Rana, Schneider, Phys. Lett. B 782 (2018), 528
- 3 loop fermions
  Lee, Smirnov, Smirnov and M. Steinhauser, JHEP 03 (2018), 136.
  Blümlein, Marquard, Rana, Schneider, Nucl. Phys. B 949 (2019), 114751, hep-ph 2307.02983

	non singlet	n_h singlet	n_l singlet
diagrams	271	66	66
families	34	17	13
masters	422	316	158



Non-singlet



Singlet

- Before complete IBP reduction, we search for a **good basis** of master integrals

In the coefficients in front of the master integrals, ϵ and the kinematic variables factories in the denominators

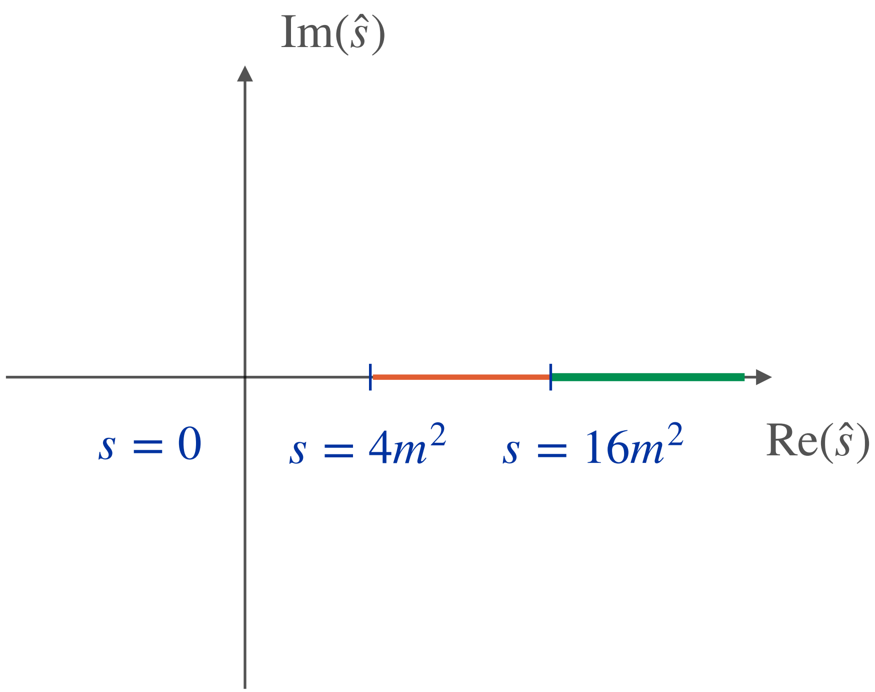
ImproveMasters.m A. & V. Smirnov, Nucl.Phys.B 960 (2020) 115213, Usovitsch, hep-ph/2002.08173.

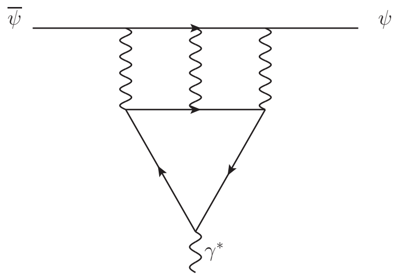
- Reduce each integral family to good basis
- Reduce masters across families to minimal set

Kira A. & V. Smirnov, Nucl.Phys.B 960 (2020) 115213, Usovitsch, hep-ph/2002.08173.

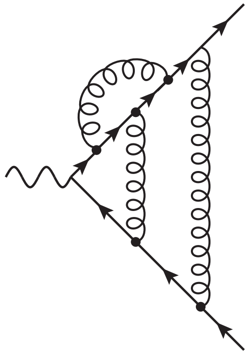
	Current	Form factors
vector	$j^v_\mu = \overline{\psi}\gamma_\mu\psi$	$\Gamma^v_\mu(s) = F^v_1(s)\gamma_\mu - \frac{i}{2m}F^v_2(s)\sigma_{\mu\nu}q^\nu$
axial-vector	$j^a_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi$	$\Gamma^a_\mu(s) = F^a_1(s)\gamma_\mu\gamma_5 - \frac{1}{2m}F^a_2(s)\gamma_5 q_\mu$
scalar	$j_s = m\overline{\psi}\psi$	$\Gamma^s(s) = mF^s(s)$
pseudo-scalar	$j_p = im\overline{\psi}\gamma_5\psi$	$\Gamma^p(s) = imF^p(s)$

$\hat{s} = s/m^2$





Singlet



$\hat{s} = 4$: two-particle threshold

$$j_n = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{50} \sum_{k=0}^{i+3} c_{n,ijk} \epsilon^i \left[4 - \hat{s}\right]^{j/2} \log^k(4 - \hat{s})$$

$\hat{s} = 16$: four-particle threshold

$$j_n = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{50} \sum_{k=0}^{i+3} c_{n,ijk} \epsilon^i \left[16 - \hat{s}\right]^{j/2} \log^k(16 - \hat{s})$$

$\hat{s} = \infty$: high energy/massless limit

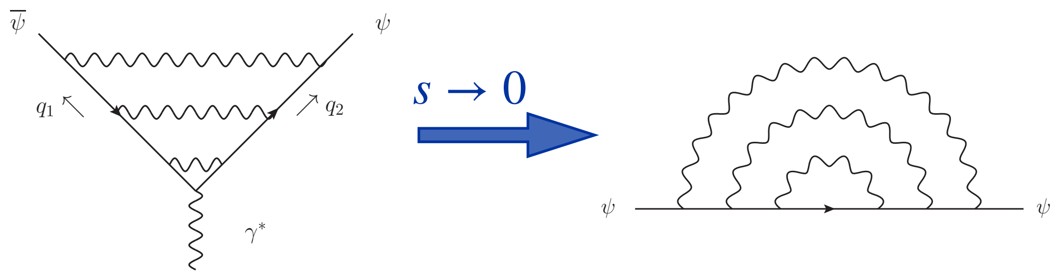
$$j_n = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{50} \sum_{k=0}^{i+6} c_{n,ijk} \left(\frac{1}{\hat{s}}\right)^j \log^k(-\hat{s})$$

also $\hat{s} = 0$ for singlet diagrams

Boundary conditions

Non-singlet

- **Analytic results at $s = 0$**
- Simple Taylor expansion



Melnikov, van Ritbergen, Phys.Lett.B 482 (2000) 99
Lee, Smirnov, JHEP 02 (2011) 102

- However we need higher orders in ϵ (up to weight 9)
- Use `SummerTime.m` and PSLQ

Lee, Mingulov, Comput.Phys.Commun. 203 (2016) 255

Singlet

- Asymptotic expansion for singlet at $s = 0$
- n_h-singlet we have **analytic boundary cond**.
  - `asy.m` Jantzen, Smirnov, Smirnov, Eur. Phys. J. C 72 (2012), 2139
  - `HyperInt` Panzer, Comput. Phys. Commun. 188 (2015), 148
- n_l-singlet: **numerical boundary conditions**
  - `AMFlow`, high-precision evaluation at $\hat{s} = -1$

Liu, Ma, Comput.Phys.Commun. 283 (2023) 108565

Computational challenges

- Generation linear equations with Mathematica — ~ few hours
- Interface to Kira and solution via reduce_user_defined_system — ~ 1d per expansion point
- Singular points: finite field methods and rational reconstruction: Kira+FireFly — ~ 15 d

von Manteuffel, Schabinger, Phys.Lett.B 744 (2015) 101
Peraro, *JHEP* 12 (2016) 030
Klappert, Klein, Lange, Comput. Phys. Commun. 264 (2021), 107968

- Better more matching points than deeper expansions

e.g. for the non-singlet

$$\hat{s}_0 = \{\infty, -32, -28, -24, -16, -12, -8, -4, 0, 1, 2, 5/2, 3, 7/2, 4, 9/2, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 22, 28, 40, 52\}$$

- Numerical instabilities in the matching

Renormalization and IR subtraction

- UV renormalisation in the on-shell scheme

Melnikov, van Ritbergen, Phys.Lett.B 482 (2000) 99;
Chetyrkin, Steinhauser, Nucl.Phys.B 573 (2000) 617-651

- Structure of IR poles is universal
- Minimal subtraction

$$F_i^{\mathrm{UV\,ren}}(s) = Z_{\mathrm{IR}} F_i^f(s)$$

with Z_{IR} given by Γ_{cusp}

$$\log Z_{\mathrm{IR}} = -\frac{1}{2\epsilon}\frac{\alpha_s}{\pi}\Gamma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{\beta_0\Gamma^{(1)}}{16\epsilon^2} - \frac{\Gamma^{(2)}}{4\epsilon}\right] + \left(\frac{\alpha_s}{\pi}\right)^3 \left[-\frac{\beta_0^2\Gamma^{(1)}}{96\epsilon^3} + \frac{\beta_1\Gamma^{(1)} + 4\beta_0\Gamma^{(2)}}{96\epsilon^2} - \frac{\Gamma^{(3)}}{6\epsilon}\right]$$

Grozin, Henn, Korchemsky, Marquard, Phys.Rev.Lett. 114 (2015) 6, 062006; JHEP 01 (2016) 140.

Results

- Fortran library for Monte Carlo implementation

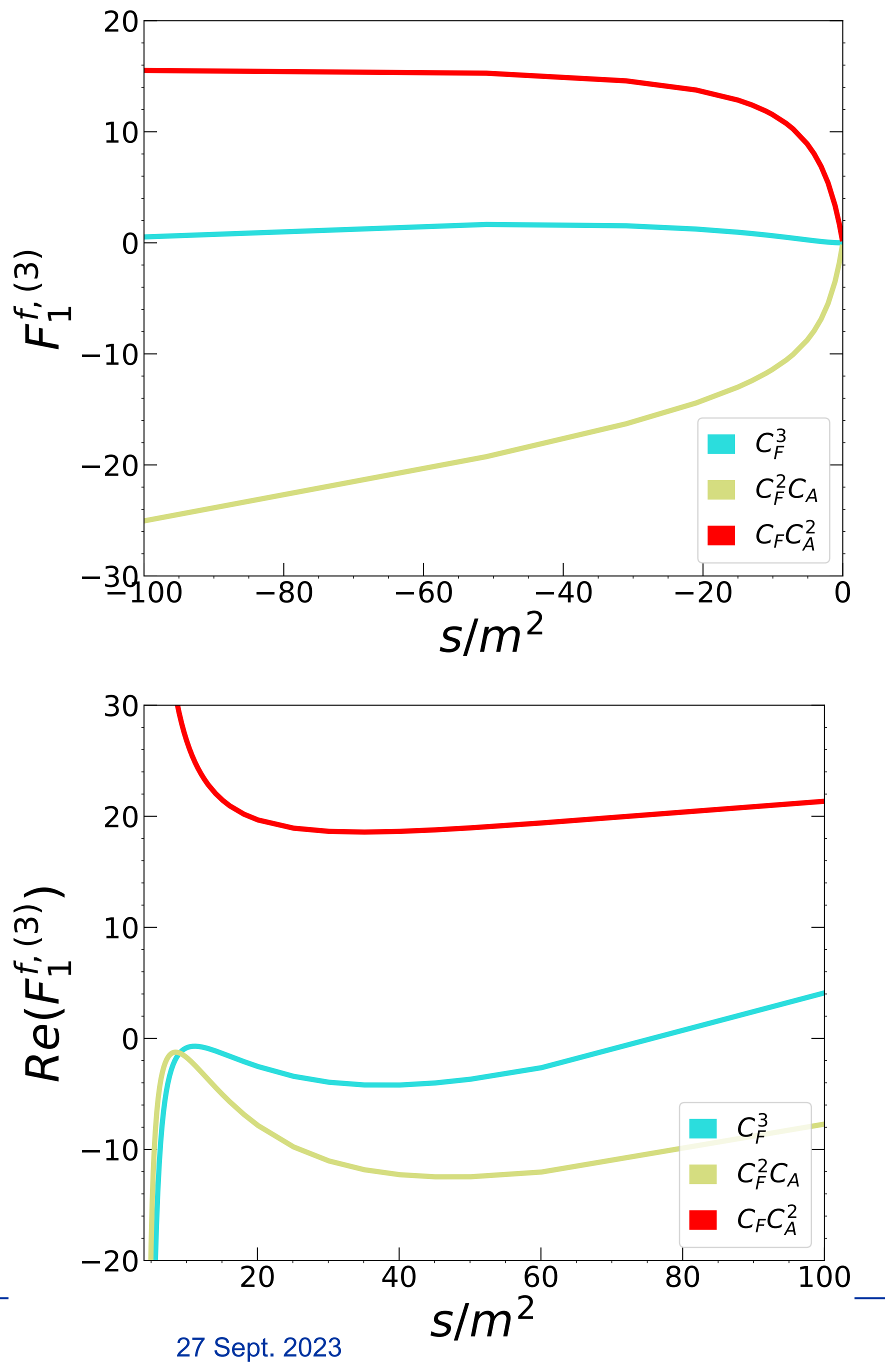
https://gitlab.com/formfactors3l/FF3l

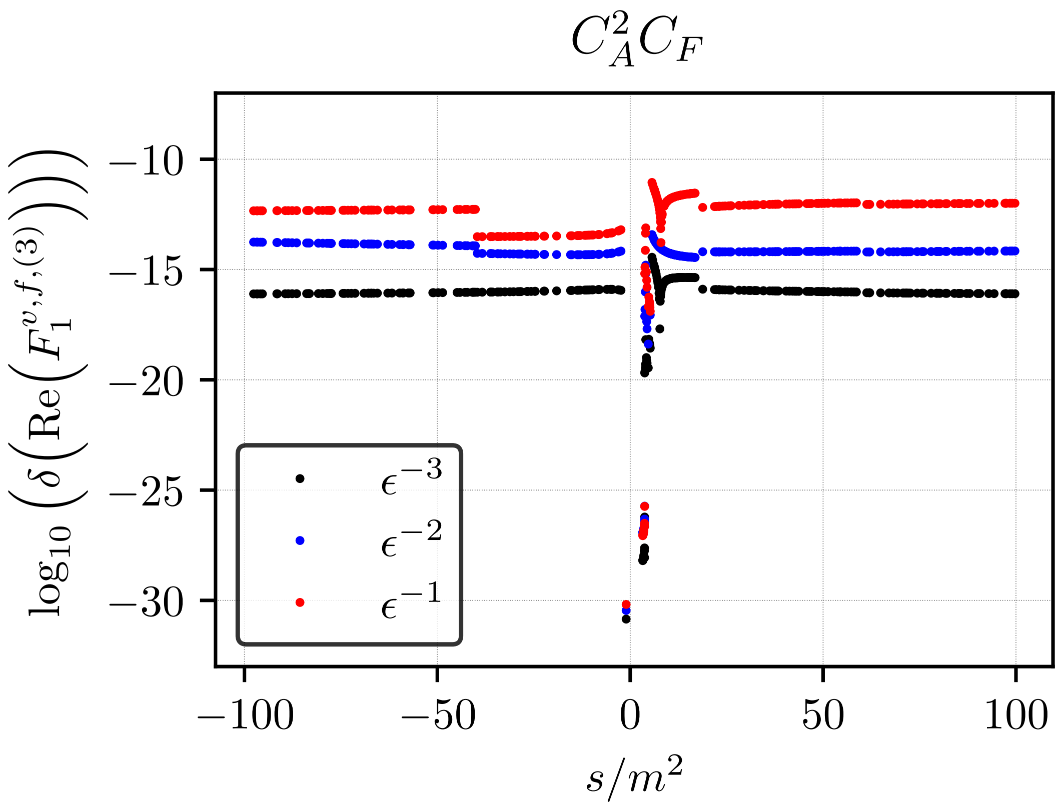
- UV renormalised
- No IR subtraction, other schemes beyond minimal can be applied
- Chebyschev interpolation grids for

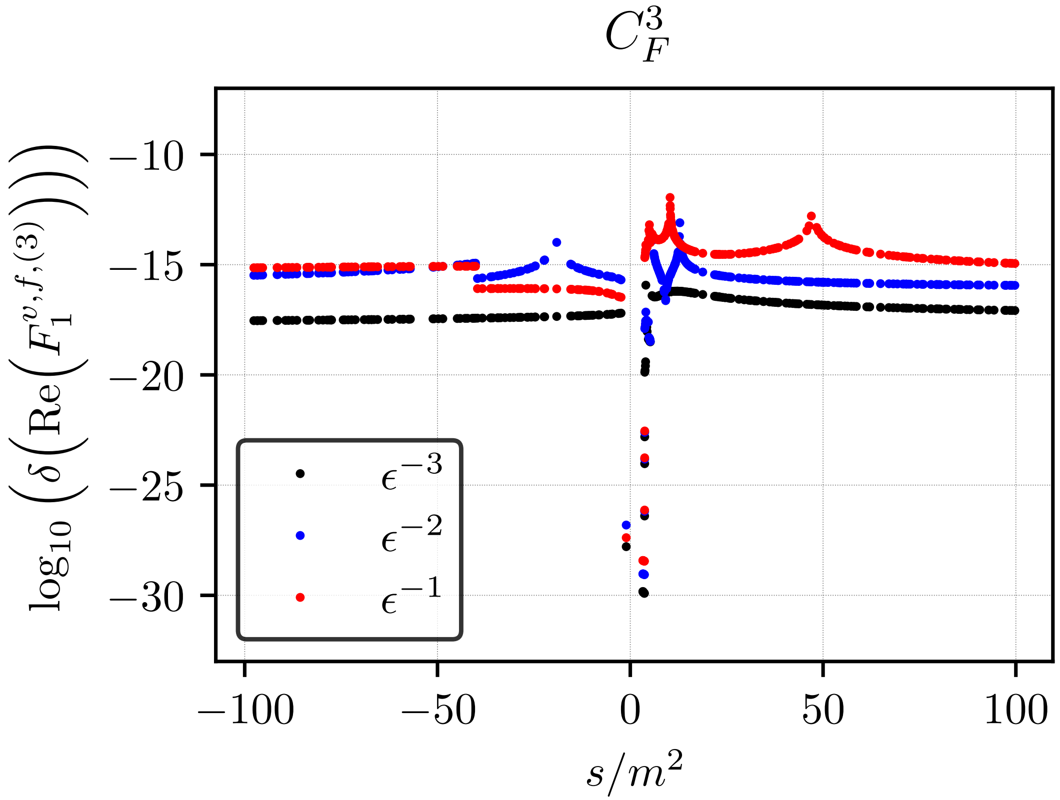
$$-40 < \hat{s} < 3.75 \quad \text{and} \quad 4.25 < \hat{s} < 60$$

- Series expansions for

$s = \pm\infty, s = 4m^2$ (also $s = 0$ for singlet)





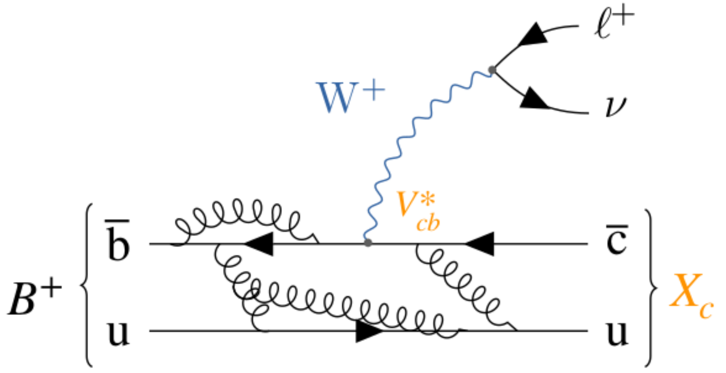


$$\delta\left(F^{f,(3)}\big|_{\epsilon^i}\right) = \frac{F^{(3)}\big|_{\epsilon^i} + F^{(\mathrm{CT+Z})}\big|_{\epsilon^i}}{F^{(\mathrm{CT+Z})}\big|_{\epsilon^i}}$$

Application II: B-meson decays

JHEP 09 (2023) 112, hep-ph 2309.14706

Inclusive decays of B mesons

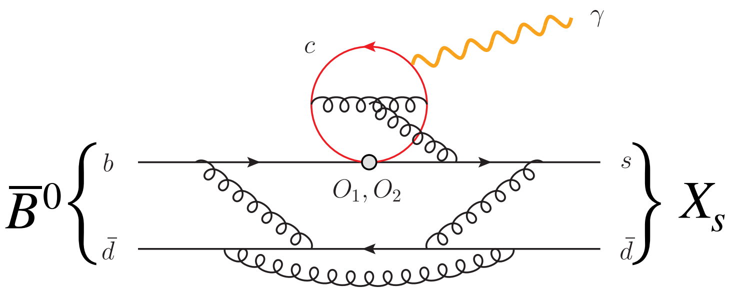


- Inclusive B-meson decays admit an **OPE**
- The ratio $m_c/m_b \simeq 0.25$ all over the place
- Several short-distance mass schemes are used

$$m_b^{\rm OS} : m_c^{\rm OS} \qquad \Gamma(B \to X_c \ell \bar\nu_\ell) \sim 1 - 1.78\left(\frac{\alpha_s}{\pi}\right) - 13.1\left(\frac{\alpha_s}{\pi}\right)^2 - 163.3\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm kin}(1\text{ GeV}) : \overline{m}_c(2\text{ GeV}) \qquad \Gamma(B \to X_c \ell \bar\nu_\ell) \sim 1 - 1.24\left(\frac{\alpha_s}{\pi}\right) - 3.65\left(\frac{\alpha_s}{\pi}\right)^2 - 1.0\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm 1S} : m_c \text{ via HQET} \qquad \Gamma(B \to X_c \ell \bar\nu_\ell) \sim 1 - 1.38\left(\frac{\alpha_s}{\pi}\right) - 6.32\left(\frac{\alpha_s}{\pi}\right)^2 - 33.1\left(\frac{\alpha_s}{\pi}\right)^3$$



- Estimate **theoretical uncertainties** with scale variations

$$\overline{m}_c(\mu_c), \overline{m}_b(\mu_b), m_b^{\rm kin}(\mu_{WC}), \ldots$$

$B \to X_s\gamma$

$$\mathrm{Br}^{\mathrm{exp}}(B \to X_s\gamma, E_\gamma > 1.6\,\mathrm{GeV}) = (3.49 \pm 0.19) \times 10^{-4}$$

HFLAV, Phys. Rev. D 107 (2023), 052008

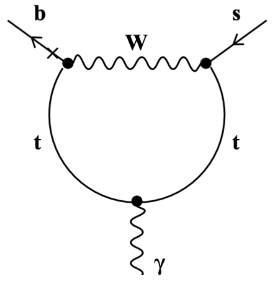
$$\mathrm{Br}^{\mathrm{th}}(B \to X_s\gamma, E_\gamma > 1.6\,\mathrm{GeV}) = (3.40 \pm 0.17) \times 10^{-4}$$

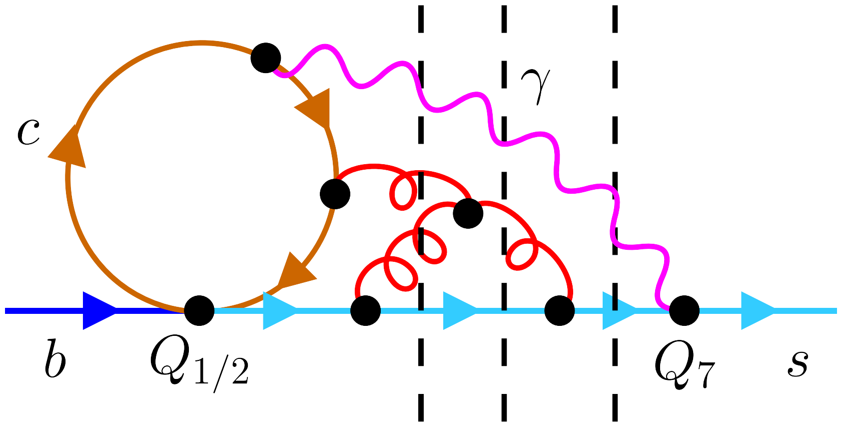
Misiak et al, Phys. Rev. Lett. 114
Misiak, Rehman, Steinhauser, JHEP 06 (2020), 175

- Includes NNLO QCD corrections
- Charm mass interpolation responsible for 3% uncertainty
- Unknown higher-order correction (3%)
- Input and non-perturbative parameters (2.5%)

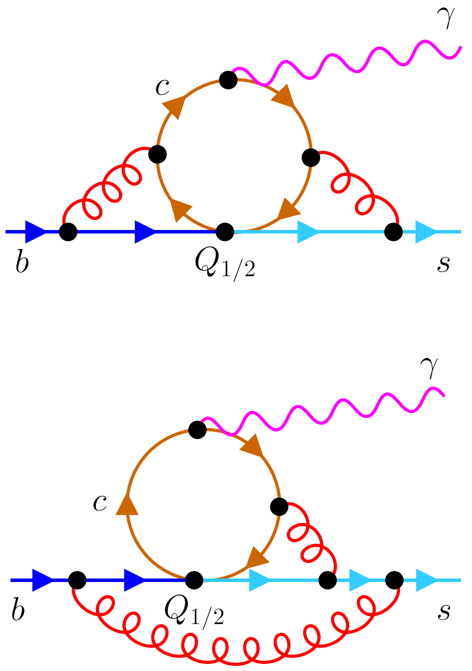
$$\mathcal{H}_{\mathrm{eff}} = -\frac{4G_F}{\sqrt{2}} V_{ts}^\star V_{tb} \sum_i C(\mu_b) Q_i$$

$$Q_1 = (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L)$$
$$Q_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L)$$
$$Q_7 = \frac{e m_b}{16\pi^2} (\bar{s}_L \sigma^{\mu\nu} b_R) F_{\mu\nu}$$





Three-loop corrections to $b \to s\gamma$ vertex



$$\mathcal{M}(b \to s\gamma) = \frac{4G_F m_b^2}{\sqrt{2}} V_{ts}^\star V_{tb} \varepsilon_\mu(q_\gamma)\, \bar{u}_s(p_s) P_R \left(t_1 \frac{q_\gamma}{m_b} + t_2 \frac{q_b^\mu}{m_b} + t_3 \gamma^\mu \right) u_b(p_b)$$

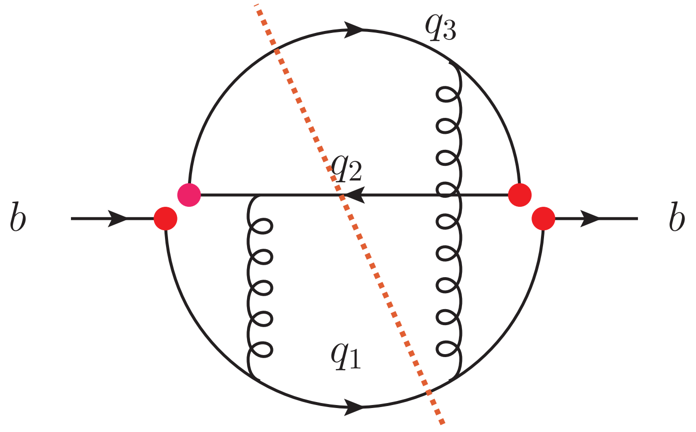
- Differential equations for 479 master integrals w.r.t. $x = m_c/m_b$
- Apply semi-analytic method
- Boundary conditions at $x_0 = m_c/m_b = 1/5$ with AMFlow
- Taylor expansions at $x_0 = 1/5, 1/10$ and power-log expansion at $x_0 = 0$

$$\mathrm{Re}(t_2^{Q_1}) = n_l \Bigg\{ -\frac{0.643804}{\epsilon^2} - \frac{6.31123}{\epsilon} - 27.9137$$
$$+ x^2 \frac{1}{\epsilon}\Big(2.107 \log^3(x) + 3.16049 \log^2(x) - 27.8263 \log(x)\Big) - 11.7523) + \ldots \Bigg\}$$

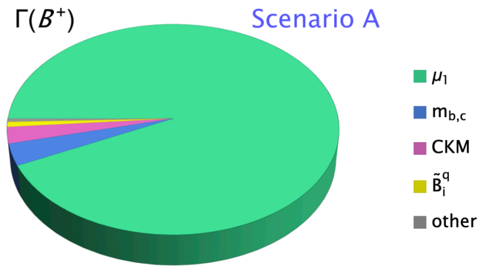
Fael, Lange, Schönwald, Steinhauser, 2309.14706
Misiak et al, 2309.14707

Lifetime of B mesons

$$\Gamma(B_q) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \ldots$$



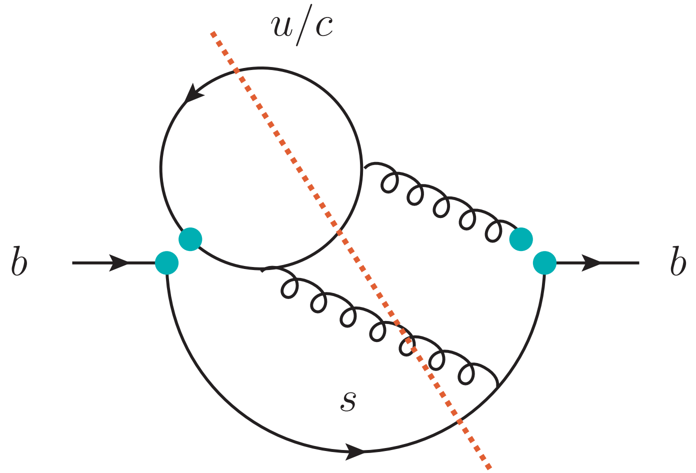
$$\Gamma(B^+) = (0.59^{+0.11}_{-0.07})\,\mathrm{ps}^{-1}$$
$$\Gamma(B_d) = (0.63^{+0.11}_{-0.07})\,\mathrm{ps}^{-1}$$
$$\Gamma(B_s) = (0.63^{+0.11}_{-0.07})\,\mathrm{ps}^{-1}$$



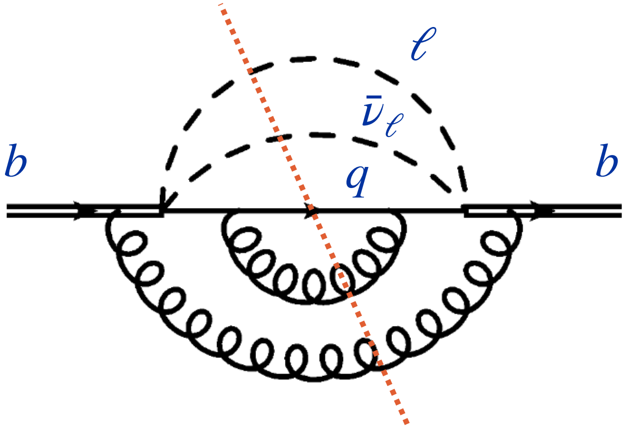
Lenz, Piscopo, Rusov, JHEP 01 (2023) 004

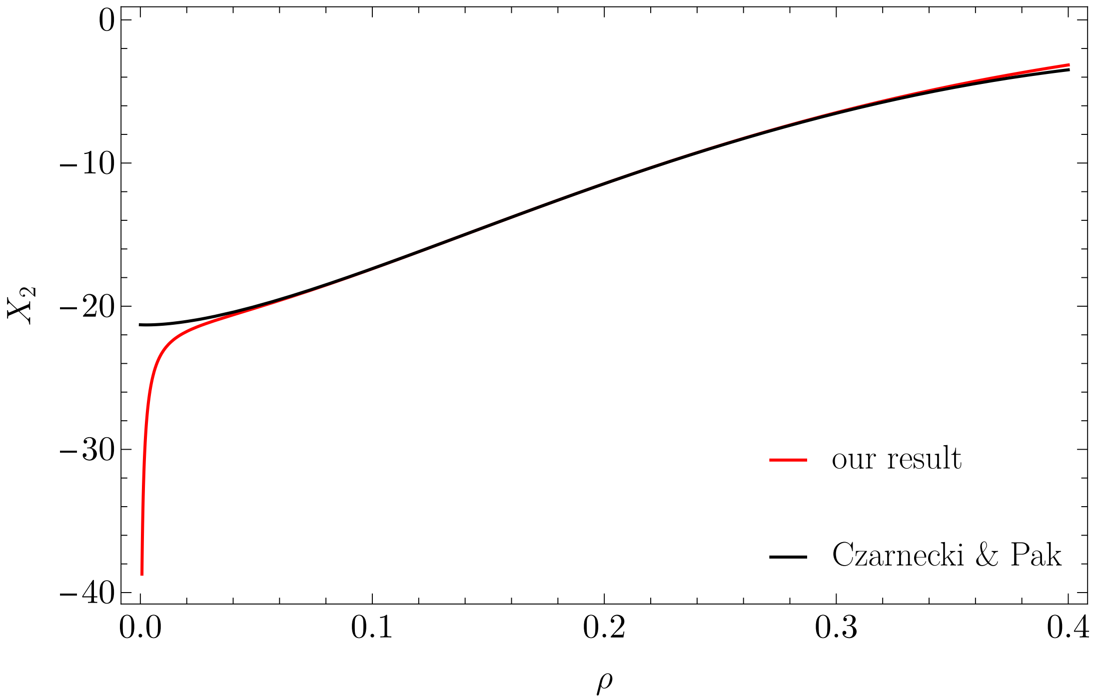
- NLO QCD corrections to non-leptonic decay
  Bagan, Patricia Ball, Braun, Gosdzinsky, Nucl.Phys.B 432 (1994) 3
  Krinner, Lenz, Rauh, Nucl.Phys.B 876 (2013) 31

- $\Gamma(B) = \Gamma(b \to c\ell\bar{\nu}_\ell) + \Gamma(b \to c\bar{u}d) + \Gamma(b \to c\bar{c}s) + \ldots$



$$\Gamma(B \to X_q \ell \bar{\nu}_\ell) = \frac{G_F^2 m_b^5 |V_{qb}|^2}{192\pi^3} \left[X_0(\rho) + \frac{\alpha_s}{\pi} X_1(\rho) + \left(\frac{\alpha_s}{\pi}\right)^2 X_2(\rho) + \dots \right]$$

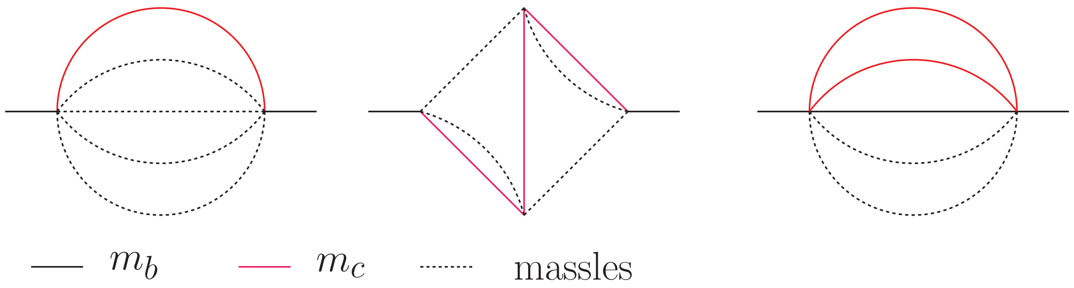




Egner, Fael, Schönwald, Steinhauser, JHEP 09 (2023) 112

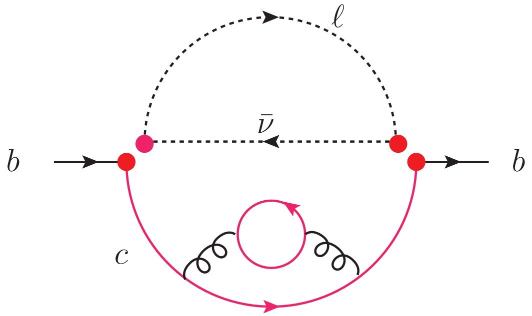
asymptotic expansion for $\rho = 0$ Czarnecki, Pak, Phys.Rev.D 78 (2008) 114015

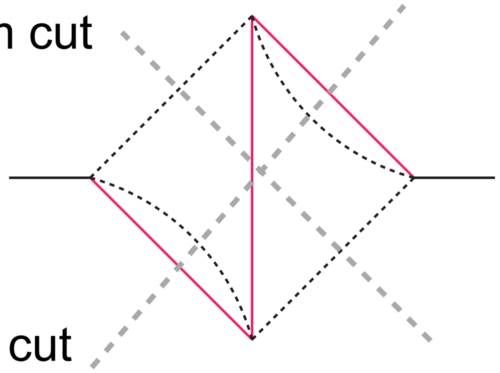
- Establish differential equations w.r.t. $\rho = m_c/m_b$
- Consider only the imaginary part of masters

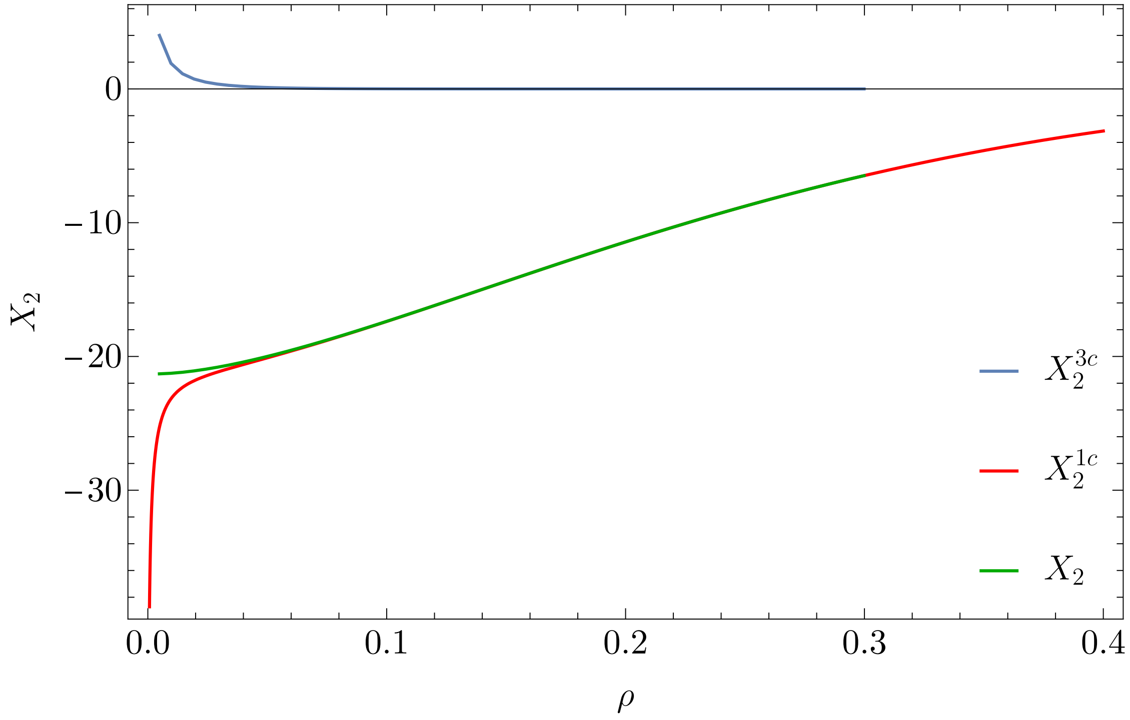


- Analytic boundary conditions in the limit $\rho \to 1$
- Continue the solution to $\rho = 0$

- Semi-analytic method should be applied at the level of Feynman integrals
- Boundary conditions at $\rho = 1$ contains final states with **only one charm quark**
- New threshold at $\rho = 1/3$







- Solving for complete real + imaginary part correctly reproduce the massless limit

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Conclusions

- Numerical and semi-analytic methods offer very powerful tools for phenomenology
- Our method is capable of dealing with difficult problems with one-scale Feynman integrals
- We can study also interesting singular limits, e.g. threshold production or high-energy limits
- Extend to two-scale problems via construction of interpolation grids

Facts of life with γ_5

- For singlet diagrams we use the Larin prescription

$$\gamma^\mu \gamma_5 \to \frac{1}{12} \varepsilon^{\mu\nu\rho\sigma} (\gamma^\nu \gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho \gamma^\nu)$$

Larin, Phys.Lett.B 303 (1993) 113

Larin, Vermaseren, Phys. Lett. B 259 (1991), 345

- **Finite renormalization** constants for $j_a^\mu(x)$ and $j_p(x)$
- Only the sum of singlet and non-singlet diagrams renormalizes multiplicative
- Non-singlet must to be calculated in the Larin scheme as well
- We check the **Chiral Ward Identity**

$$F^{a,f}_{1,\text{sing}} + \frac{s}{4m^2} F^{a,f}_{2,\text{sing}} = F^{p,f}_{\text{sing}} + \frac{\alpha_s}{4\pi} T_F F^f_{G\tilde{G}}$$