Analytic computation of 2-loop scattering amplitudes

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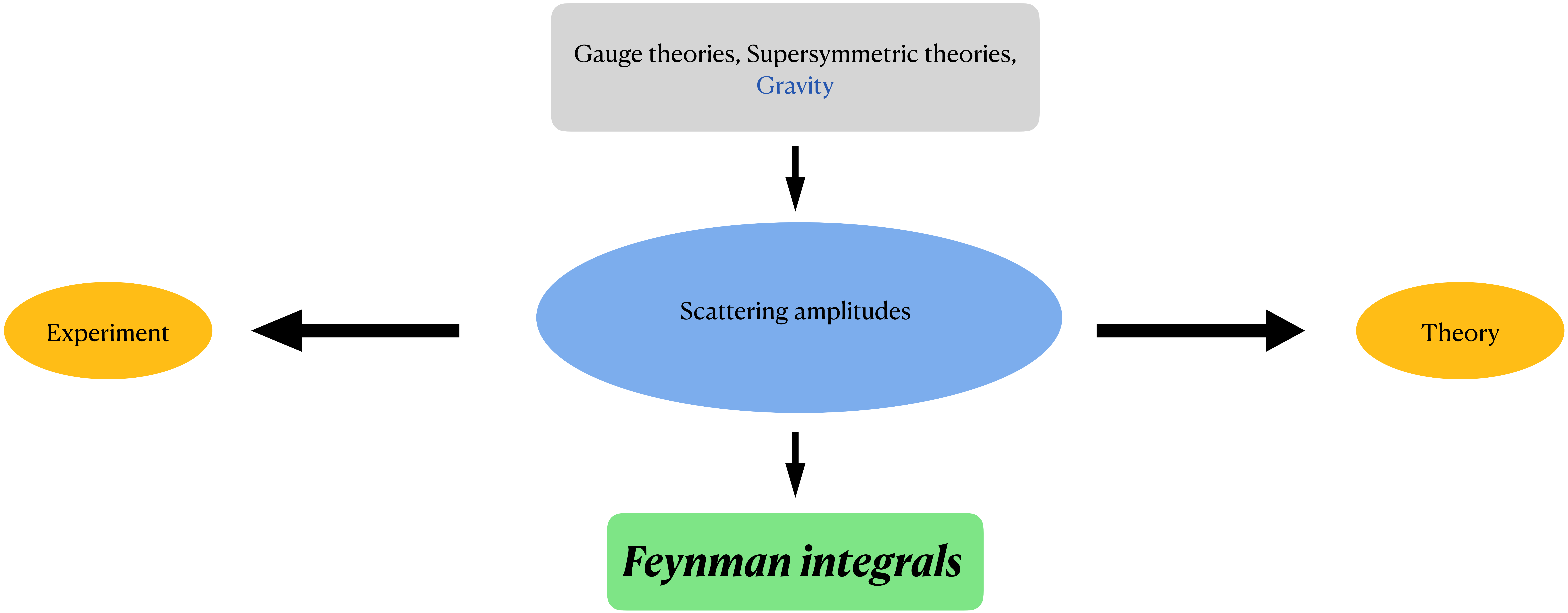


Lance
David
Andreas,
Giuseppe,
Simone

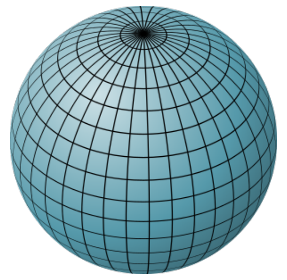
For precision physics, **precise** theoretical predictions are needed
→ computation of **higher-order loop corrections**

Analytic structure

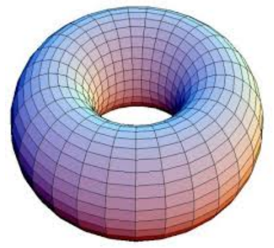
Class of functions?

Rich Mathematical structure

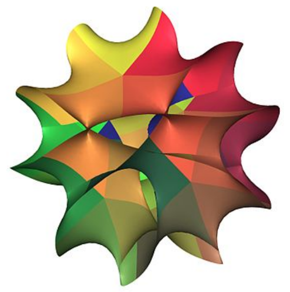
Multiple Polylogarithms



Elliptic Polylogarithms



K3 surfaces



Higher-dimensional Calabi-Yau manifold

Can we identify them?

Can we "use" them?

[Picture Credits- Wikiwand, perso.univ-rennes1.fr, wikipedia.org]

Modern techniques for solving Feynman integrals

Feynman integrals $\longrightarrow$ (IBPs) Master integrals (MI) $\longrightarrow$ $d\vec{I} = A\vec{I}$ — Set up a system of DE wrt kinematics

Eps-form [Henn '13] $\longrightarrow$ Change the basis: $\vec{J} = T\vec{I}$, $d\vec{J} = \epsilon\tilde{A}\vec{J}$ $\longrightarrow$ $J_k = \sum_{j=0} \epsilon^j J_k^{(j)}$

The choice of basis not unique; no general methods fits all

[Lee '14] [Meyer '18] [Wasser '20]
[Adams, **E. Chaubey,** Weinzierl, '17]
[Dlapa, Henn, Wagner, '22]

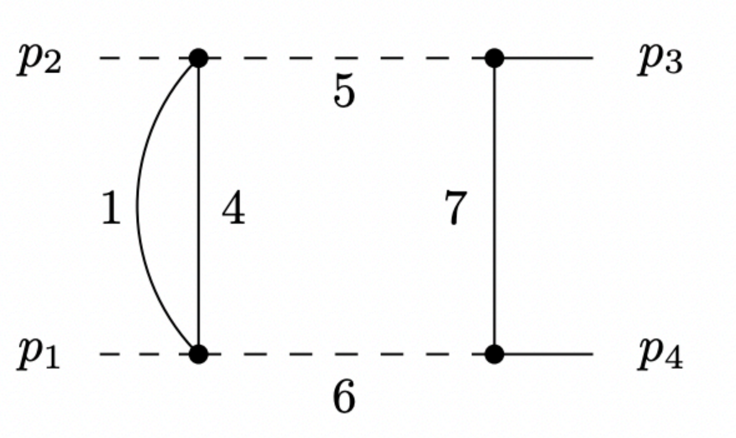
Maximal cuts

[Baikov, '96, '97]
[Frellesvig, Papadopoulos, '17]
[Primo, Tancredi, '17]
[Adams, **E. Chaubey,** Weinzierl, '17, '18]

Christoph, Seva, William, Federico, Jacob

Square roots

Geometry of the Feynman integral often manifested by the square root
[Festi, van Straten, '18] [Adams, **E. Chaubey**, Weinzierl '18]



$$Maxcut_C I_{2001111}(4-2\epsilon) \propto \int_C \frac{dP}{\sqrt{(P-t)(P-t+4m^2)(P^2+2m^2\frac{s+4t}{s-4m^2}P+m^2(m^2-4t)\frac{s}{s-4m^2}-\frac{4m^2t^2}{s-4m^2})}}$$

Functional representation

Chen's definition of iterated integrals: $\gamma : [0,1] \to M$ $\quad x_i = \gamma(0) \quad x_f = \gamma(1)$

[Chen '77]

Martijn

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) ... \int^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k)$$
$$= \int_0 d\lambda_1 f_1(\lambda_1) I_\gamma(\omega_2, ..., \omega_k; \lambda_1),$$

Goncharov's Polylogarithms (MPLs)

Elliptic Polylogarithms (eMPLs)

Numerical evaluation

Numerical evaluation of the functions at various phase-space points

- Multiple polylogarithms
- Elliptic integrals

GINAC PolyLogTools

[Vollinga, Weinzierl, '05] [Duhr, Dulat, '19]

Local series expansion methods

[Moriello, '19] [Hidding, '21]

- Iterated integrals
- Canonical DE

Ya-Qing,
Xiao,
Felix

Ofcourse PySecDec, Fiesta, Feyntrop
Auxiliary mass flow methods
[Liu, Ma, '22]

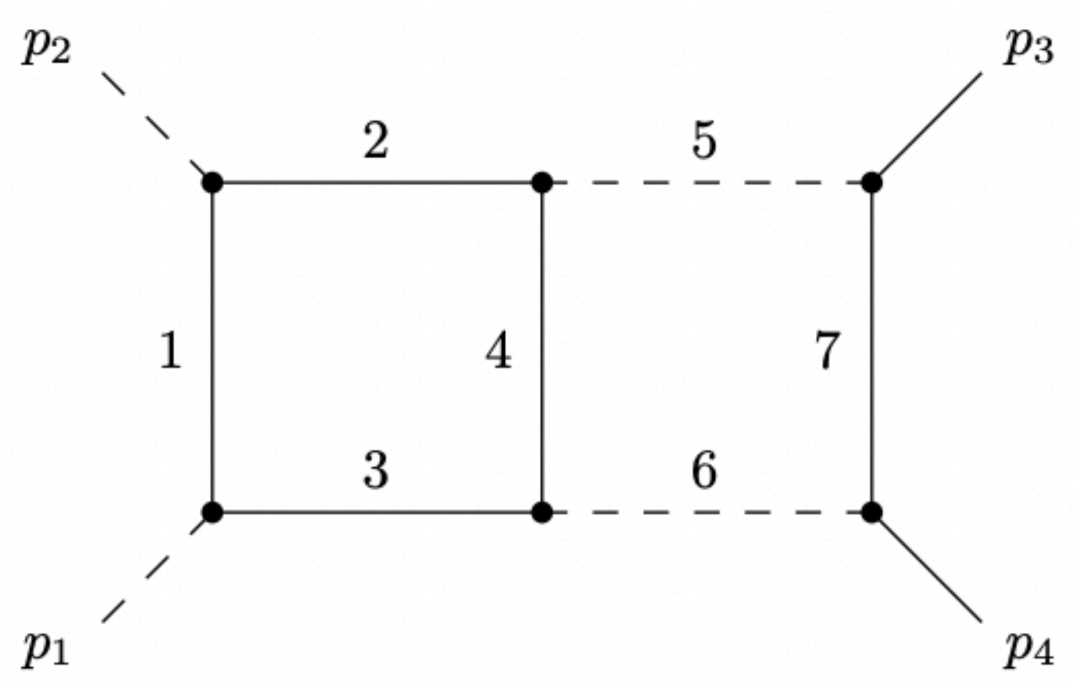
Some selected examples

Scattering amplitudes ($t\bar{t}$)

[Badger, E. Chaubey, Hartanto, Marzucca, ’21]

Needed for precise understanding of top quarks

- Computation of helicity amplitudes for top-quark pair production involves elliptic integral contributions
- Topbox evaluates to 3 different elliptic curves
- Faster numerical evaluation of elliptic functions needs optimisation of the basis [E. Chaubey’ 21]

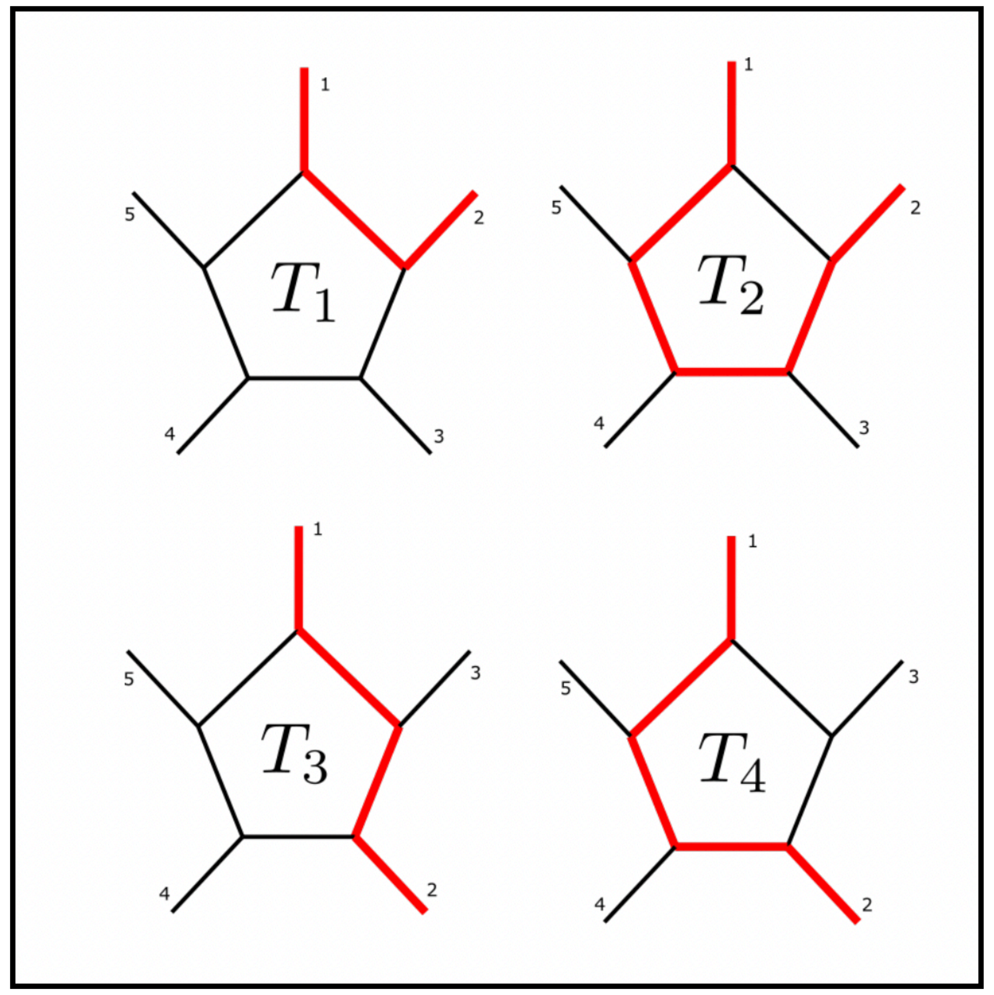


[Adams, E. Chaubey, Weinzierl, ’18]

1-loop amplitude for tt-jet

[Badger, Becchetti, E. Chaubey, Marzucca, Sarandrea, '22]

- Analytic helicity amplitudes for 1-loop QCD corrections, Previously missing ingredient for NNLO, expansion of the 1-loop helicity amplitudes up to $O(\epsilon^2)$



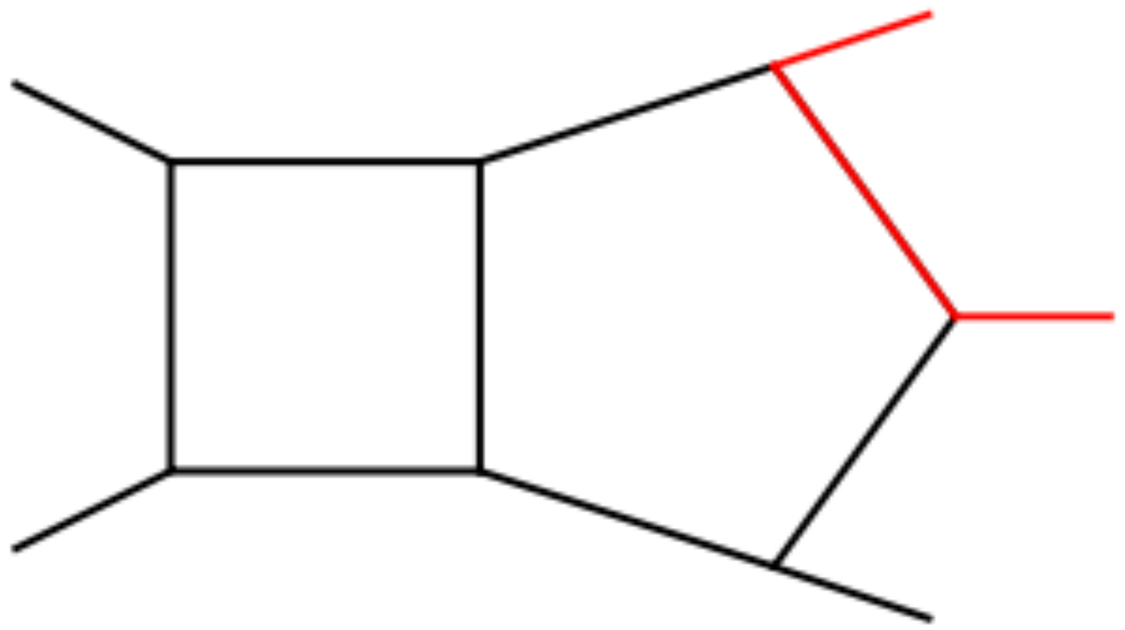
- Canonical form DE for all 130 MIs across 4 pentagon topologies
- Numerical solution using generalised power series expansion in DiffExp
- Analytic result of boundary constants

2-loop 5-point integrals

[Badger, Becchetti, E. Chaubey, Marzucca, '22]

- One of the topologies that appear in the NNLO corrections for tt-jet production
- 5 Mandelstam variables and mass dependence through top-quarks
- 88 master integrals
- Well-thought basis important to reconstruct analytically

$$\mu_{ij} = -k_i^{(-2\epsilon)} \cdot k_j^{(-2\epsilon)}$$



$$pp \to t\bar{t}j$$

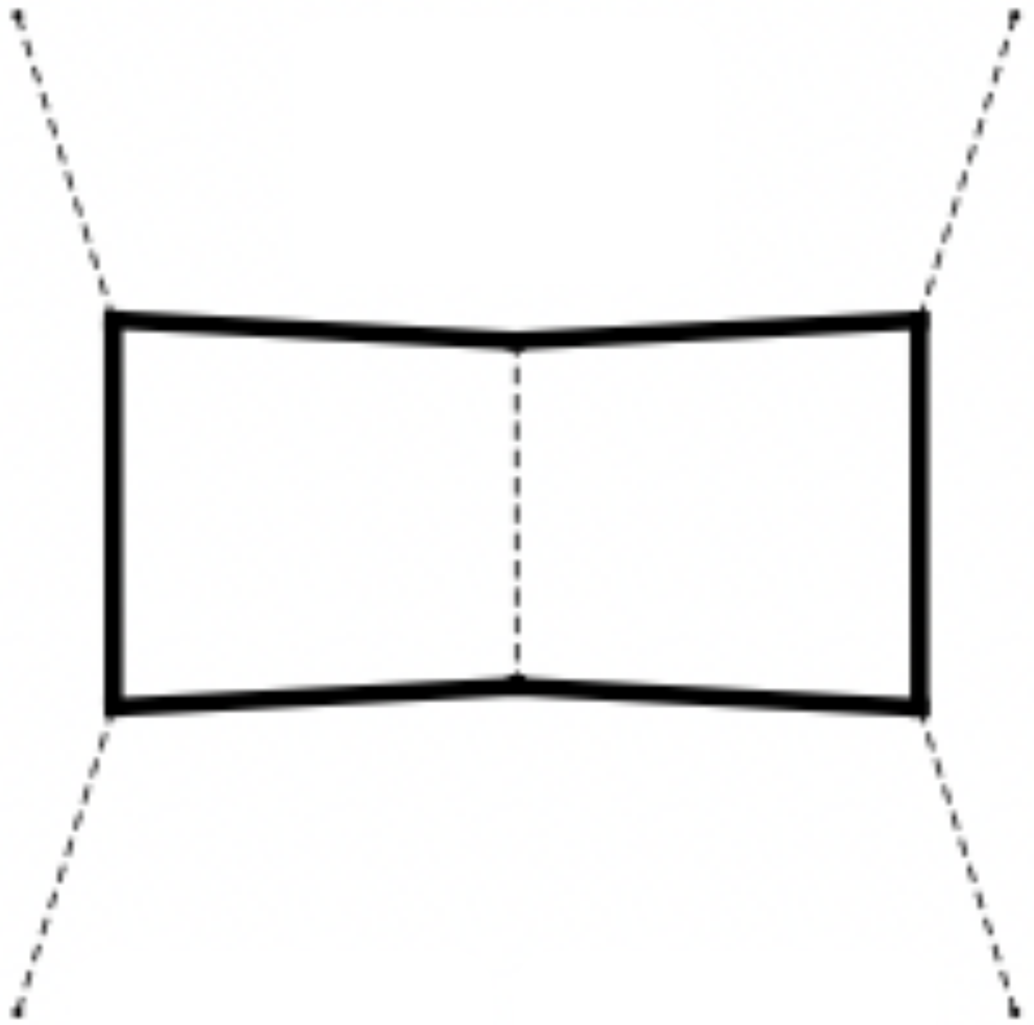
- Evaluation at 1 boundary point with a precision of $O(100)$ using AMFlow
- Integration of analytic differential equations using generalised power series expansion as implemented in DiffExp

Light-by-light scattering

[AH, E. Chaubey, Shao, '23]

$$\gamma(p_1, \lambda_1) + \gamma(p_2, \lambda_2) + \gamma(p_3, \lambda_3) + \gamma(p_4, \lambda_4) \to 0$$

- 2-loop QCD & QED corrections in the ultra-relativistic limit ($s, t, u >> m^2$, massless internal lines)
  [Bern, De Freitas, Dixon, Ghinculov & Wong, 2001]
- Two-loop corrections to light-by-light scattering in supersymmetric QED
  [Binoth, Glover, Marquard, & van Der Bij, 2002]



Amplitude computation

$$\mathcal{M} = \varepsilon_{1,\mu_1}\varepsilon_{2,\mu_2}\varepsilon_{3,\mu_3}\varepsilon_{4,\mu_4}\mathcal{M}^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4)$$

$$\begin{aligned}\mathcal{M}^{\mu_1\mu_2\mu_3\mu_4} =& A_1 g^{\mu_1\mu_2}g^{\mu_3\mu_4} + A_2 g^{\mu_1\mu_3}g^{\mu_2\mu_4} + A_3 g^{\mu_1\mu_4}g^{\mu_2\mu_3} + \sum_{j_1,j_2=1}^{3}\Big(B^1_{j_1j_2}g^{\mu_1\mu_2}p^{\mu_3}_{j_1}p^{\mu_4}_{j_2}\\ &+ B^2{}_{j_1j_2}g^{\mu_1\mu_3}p^{\mu_2}_{j_1}p^{\mu_4}_{j_2} + B^3{}_{j_1j_2}g^{\mu_1\mu_4}p^{\mu_2}_{j_1}p^{\mu_3}_{j_2} + B^4{}_{j_1j_2}g^{\mu_2\mu_3}p^{\mu_2}_{j_1}p^{\mu_4}_{j_2}\\ &+ B^5{}_{j_1j_2}g^{\mu_2\mu_4}p^{\mu_1}_{j_1}p^{\mu_3}_{j_2} + B^6{}_{j_1j_2}g^{\mu_3\mu_4}p^{\mu_1}_{j_1}p^{\mu_2}_{j_2}\Big)\\ &+ \sum_{j_1,j_2,j_3,j_4=1}^{3} C_{j_1j_2j_3j_4}p^{\mu_1}_{j_1}p^{\mu_2}_{j_2}p^{\mu_3}_{j_3}p^{\mu_4}_{j_4}.\end{aligned}$$

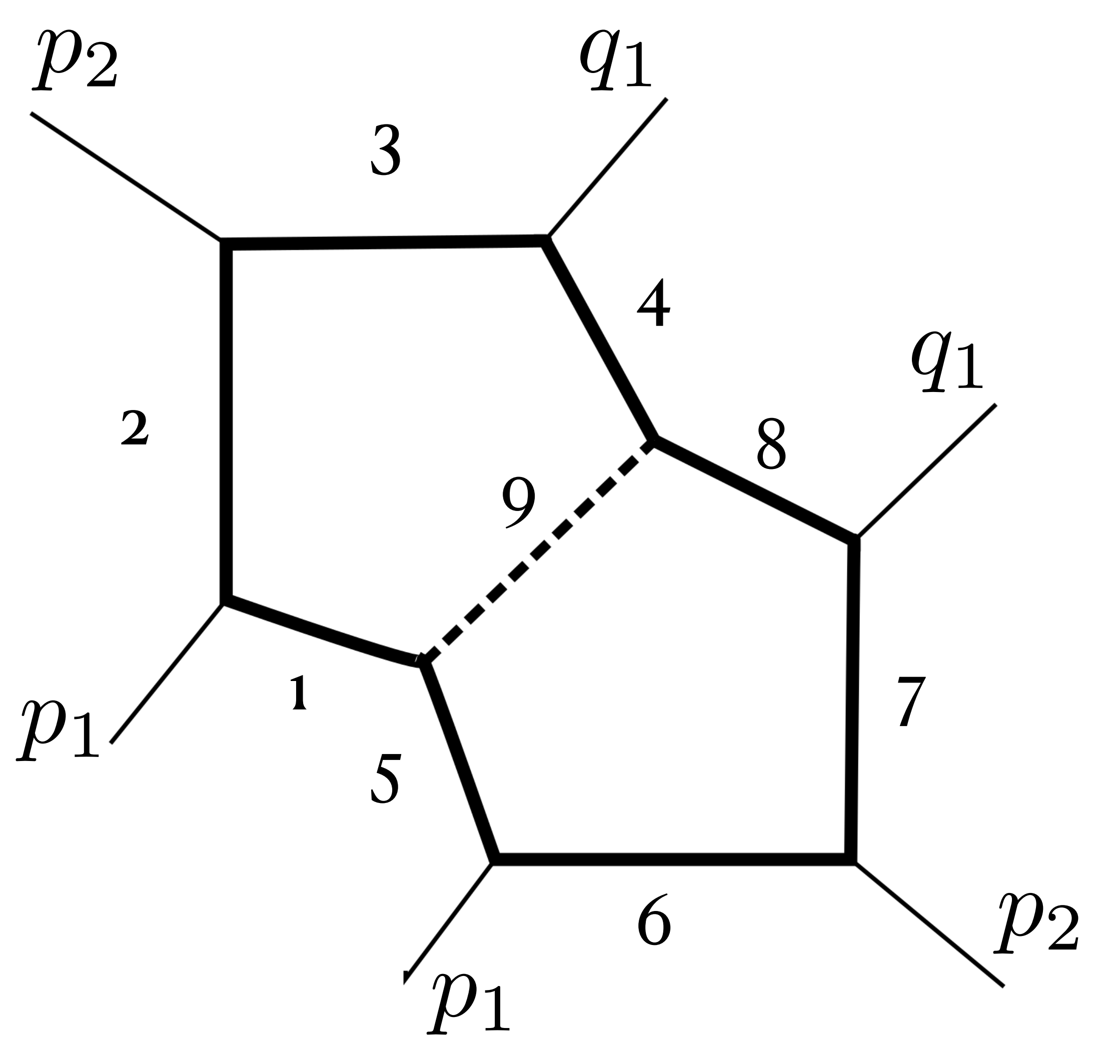
Number of independent functions

$A_1(s,t,u)$ $\quad$ $B^1_{11}(s,t,u)$ $\quad$ $C_{2111}(s,t,u)$

$\varepsilon_j \cdot p_j = 0$
Bose symmetry
Gauge symmetry

[Binoth, Glover, Marquard, & van Der Bij, 2002]

Reduction to MIs & Simplification



Johann, Yang, Yao, Andreas

$$I_{a_1,\cdots,a_9} = \left(\frac{e^{\epsilon\gamma_E}}{i\pi^{\frac{d}{2}}}\right)^2 \int \prod_{i=1}^{2} d^d k_i \frac{D_4^{a_4} D_6^{a_6}}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_5^{a_5} D_7^{a_7} D_8^{a_8} D_9^{a_9}},$$

$k_1^2 - m_t^2, (k_1+p_1)^2 - m_t^2, (k_1+p_1+p_2)^2 - m_t^2, (k_1+p_1+p_2+p_3)^2 - m_t^2,$
$(k_2)^2 - m_t^2, (k_2+p_1)^2 - m_t^2, (k_2+p_1+p_2)^2 - m_t^2, (k_2+p_1+p_2+p_3)^2 - m_t^2,$
$(k_2-k_1)^2$

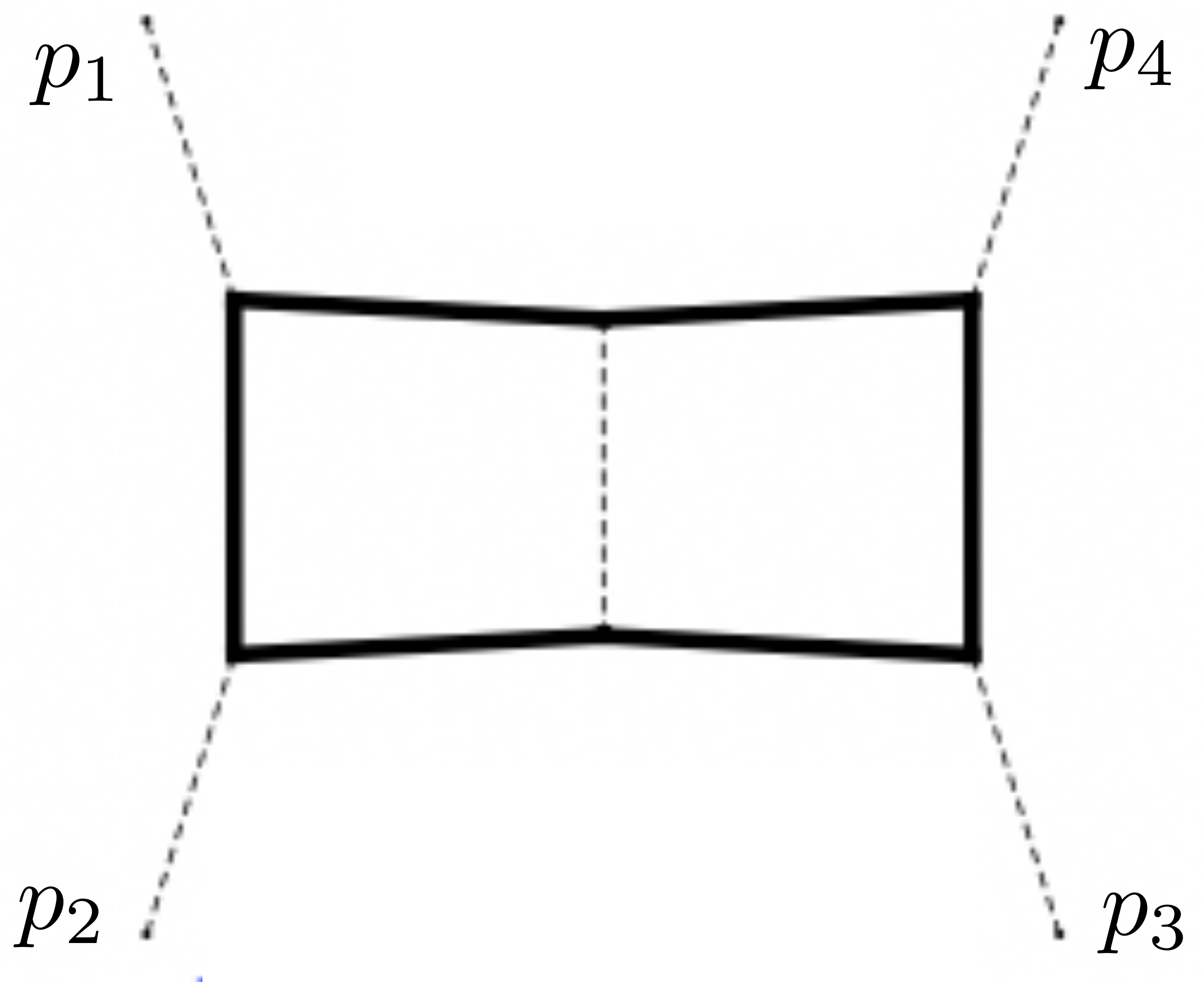
LiteRed (FiniteFlow), KIRA
[Lee, `13] [Peraro, `19] [Klappert, Lange, Maierhöfer, Usovitsch,

60 diagrams in total

7798 integrals before IBP

18 top-level sectors

Can be mapped into the 2-loop diagram shown on the right



Analytic computation of the MIs

- 29 MIs; use of differential equations;
- Choice of a canonical basis [Caron-huot, Henn, 14]

$$\partial_s \vec{f} = \epsilon A_s \vec{f}$$
$$\partial_t \vec{f} = \epsilon A_t \vec{f}$$

- Square roots:

$$\sqrt{s(s-4m^2)} \qquad \sqrt{t(t-4m^2)} \qquad \sqrt{st(st-4m^2(s+t))} \qquad \sqrt{s(m^4 s - 2m^2 t(s+2t) + st^2)}$$

- Choice of variables

$$s = -\frac{4(w-z)^2}{(1-w^2)(1-z^2)} \qquad t = -\frac{(w-z)^2}{wz} \qquad m^2 = 1$$

$$\sqrt{-2wz + z^2 + w^4 z^2 - 2w^3 z^3 + w^2(1+z^2+z^4)}$$ *Non-rationalizable??..*

The alphabet

$$sq = \sqrt{-2wz + z^2 + w^4 z^2 - 2w^3 z^3 + w^2(1 + z^2 + z^4)}$$ [Caron-huot, Henn, 14]

$$1-w, 1+w, 1-wz, w-z, w, w+z, 1+w-z+wz, 1-w+z+wz, 1+wz, 1-z, 1+z,$$
$$z, \frac{-sq+w-z-wz+w^2z-wz^2}{sq+w-z-wz+w^2z-wz^2}, \frac{-sq+w^2-3wz+z^2}{sq+w^2-3wz+z^2}, \frac{-1-sq+w^2z-wz^2}{-1+sq+w^2z-wz^2},$$
$$\frac{1-sq+w^2z-wz^2}{1+sq+w^2z-wz^2}$$

6 master integrals containing all 4 square roots in the integrand at weight-4.

Solving the canonical master integrals

We keep the analytic result in terms of iterated integrals with dog one-forms constructed using

[Heller, von Manteuffel, Schabinger, `20]

We convert all the integral in terms of 1-dimensional integrations

[Caron-huot, Henn, 14] [Chicherin, Sotnikov `21] [Chicherin, Sotnikov, Zoia `22]

We also express the first two orders in terms of logs and classical polylogs by matching symbols

[Duhr, Gangl, Rhodes, `11]

$$\int_\gamma I(\omega_1 \ldots \omega_4;\lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \underbrace{\int_0^{\lambda_2} d\lambda_3 f_3(\lambda_3) \int_0^{\lambda_3} d\lambda_4 f_4(\lambda_4)}_{\{\mathrm{Log}^2(z),\, \mathrm{Li}_2(z)\}}$$

Numerical evaluation

Physical phase-space regions of interest

$$0 < w < 1 \;\; \& \;\; (0 < z < w \,|\, w < z < 1)$$
$$0 < w < 1 \;\; \& \;\; (1 < z < \frac{1}{w} \,|\, z > \frac{1}{w})$$
$$0 < w < 1 \;\; \& \;\; (-1 < z < -w \,|\, -w < z < 0)$$

We obtain different analytic representations of the results valid in different regions

Outputs →

- *We obtained a completely analytic representation for the squared matrix element at 2-loop*
- *NLO cross section with massive contributions for light-by-light scattering at UPC within reach!*

Take away

- Computation of higher order perturbative corrections important
- Analytic solution of multi-loop integrals requires understanding the mathematical structure
- With the inclusion of masses in the loop, the analytic structure starts becoming more complicated
- With the inclusion of more legs, often one needs go beyond current mathematical understand; often also desirable to combine analytic as well as numerical techniques