Lattice Correlation Functions from Differential Equations

Federico Gasparotto

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Based on joined work w/ A. Rapakoulias & S. Weinzierl arXiv: 2210.16052

S. Weinzierl & X. Xu arXiv: 2305.05447

MathemAmplitudes 2023 – Padova, Italy – 26 September 2023

Correlation functions are key objects in QFT

$$G_n(x_1,\dots,x_n) = \frac{\int \mathcal{D}\phi\, \phi(x_1)\dots\phi(x_n)\ \exp(iS)}{\int \mathcal{D}\phi\ \exp(iS)}$$

We focus on scalar action$-\lambda\phi^4$ model

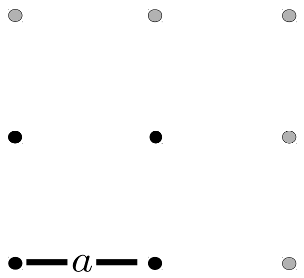
$$S = \int d^D x\, \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{m^2}{2}\phi^2(x) - \lambda\phi^4(x)$$

Problem tough $\rightsquigarrow$ Lattice regularization

lattice Λ, spacing a, L_μ pts $\mu-$direction + periodicity

$$N = \prod_{\mu=0}^{D-1} L_\mu = \#\,\text{lattice pts}$$

$D{=}2 \quad L_0{=}L_1{=}2 \quad N{=}4$



$$\int d^D x \rightsquigarrow a^D \sum_{x\in\Lambda}$$

$$\partial_\mu\phi(x) \rightsquigarrow \frac{\phi(x+a\hat{e}_\mu)-\phi(x)}{a}$$

Minkowskian signature

$$S_M = i \sum_{x\in\Lambda} \left[\phi(x)\,\phi(x+a\hat{e}_0) - \sum_{\mu=1}^{D-1} \phi(x)\,\phi(x+a\hat{e}_\mu) + \left(D + \frac{m^2}{2} - 2 \right) \phi^2(x) + \lambda\phi^4(x) \right]$$

Euclidean signature

$$S_E = \sum_{x\in\Lambda} \left[- \sum_{\mu=0}^{D-1} \phi(x)\,\phi(x+a\hat{e}_\mu) + \left(D + \frac{m^2}{2} \right) \phi^2(x) + \lambda\phi^4(x) \right]$$

Schematically

$$\begin{aligned} S_\bullet &= \text{“polynomial in fields } \phi(x_i)\text{”} \\ &= S_\bullet^{\text{next neigh.}} + S_\bullet^{(2)} + \lambda S_\bullet^{(4)} \end{aligned} \qquad \bullet = M, E$$

Everything boils down to computation of finite dimensional integrals

$$I_{\nu_1 \ldots \nu_N} = \int_{\mathbb{R}^N} \exp(-S_\bullet)\, \phi^{\nu_1}(x_1) \ldots \phi^{\nu_N}(x_N)\, d^N\phi$$

$\nu_i \in \mathbb{N}$
$\bullet = M, E$
λ not small

Correlation functions recovered as

$$G_{\nu_1 \ldots \nu_N} = \frac{I_{\nu_1 \ldots \nu_N}}{I_{0 \ldots 0}}$$

$\bullet = E$ traditionally via Monte Carlo, $\bullet = M$ harder

In this talk, apply methods from perturbation theory

integral reduction $\oplus$ differential eqs

in order to describe non-perturbative physics

In Feynman integrals: [Chetyrkin & Tkachov, Laporta; Kotikov, Remiddi, Gehrmann & Remiddi]

Talks by: [Chaubey, Chestnov, Fael, Huber, Ma, Torres, Usovitch, von Manteuffel, Zeng, Zhang, Zoia]

Integral family

$$I_{\nu_1 \dots \nu_N} = \int_{\mathbb{R}^N} \exp(-S_\bullet)\, \phi^{\nu_1}(x_1) \dots \phi^{\nu_N}(x_N)\, d^N\phi$$

$\nu_i \in \mathbb{N}$

$\bullet = M, E$

λ not small

Framework of Twisted Co–Homology

[Aomoto]

Talks by: [Crisanti, Fontana, Pokraka]

$$\int_{\mathcal{C}} u\, \Phi$$

$u = \exp(-S_\bullet)$

$\Phi = \phi^{\nu_1}(x_1) \dots \phi^{\nu_N}(x_N)\, d^N\phi$

$\mathcal{C} = \mathbb{R}^N$ $\mathcal{C}$ st $u|_{\partial\mathcal{C}} = 0$

Introduce "auxiliary flow" t

(eventually $t = 1$)

$$S_\bullet \rightsquigarrow S_\bullet(t) = t\, S_\bullet^{\text{next neigh.}} + S_\bullet^{(2)} + S_\bullet^{(4)}$$

$\bullet = M, E$

Integration by parts (twisted Co-Homology POV)

[Mastrolia & Mizera]

$$0 = \int_{\mathcal{C}} d\,(u\xi) = \int_{\mathcal{C}} u\,(d + d\log u \wedge)\,\xi$$

$$= \int_{\mathcal{C}} u \nabla_\omega \xi$$

$$\nabla_\omega(\bullet) = d(\bullet) + \omega \wedge \bullet$$

$$\omega = d\log u = -dS_\bullet$$

- Φ & $\Phi + \nabla_\omega \xi$ integrate same result $\int u\bullet$

$\langle \Phi |\ :\ \Phi \sim \Phi + \nabla_\omega \xi$

- $\nabla_\omega \circ \nabla_\omega = 0$

$$\langle \Phi | \in \mathrm{H}^N = \frac{\mathrm{Ker}\ \nabla_\omega}{\mathrm{Im}\ \nabla_\omega} \rightsquigarrow \text{twisted Co}-\text{Homology group}$$

$$\text{Study } \int_{\mathcal{C}} u\Phi \rightsquigarrow \text{Study } \mathrm{H}^N$$

- Dimension Co-Homology group [Weinzierl]

$$\dim \mathrm{H}^N = 3^N \qquad N = \#\ \text{lattice pts}$$

- Basis

$$\phi^{\nu_1}(x_1)\dots\phi^{\nu_N}(x_N)d^N\phi, \qquad \nu_1,\dots,\nu_N < 3$$

- “Proof”

$$S_\bullet \propto \sum_{x\in\Lambda}\phi^4(x) + \text{“lower degree”} \rightsquigarrow \omega = -dS_\bullet \propto -\sum_{x\in\Lambda}\phi^3(x)d\phi + \text{“lower degree”}$$

Given: $\Phi = \phi^{\nu_1}(x_1)\dots\phi^{\nu_k}(x_k)\dots\phi^{\nu_N}(x_N)d^N\phi, \qquad \nu_k \geq 3$

Choose: $\xi_\Phi \propto \phi^{\nu_1}(x_1)\dots\phi^{\nu_k-3}(x_k)\dots\phi^{\nu_N}(x_N)d^{N-1}\phi$

$$\Phi \sim \Phi + \nabla_\omega\xi_\Phi = \Phi - \Phi + \sum \text{“lower degree”}$$

- Dimension Co-Homology group [Weinzierl]

$$\dim \mathrm{H}^N = 3^N \qquad N = \#\text{ lattice pts}$$

- Basis

$$\phi^{\nu_1}(x_1)\ldots\phi^{\nu_N}(x_N)d^N\phi, \qquad \nu_1,\ldots,\nu_N < 3$$

- "Proof"

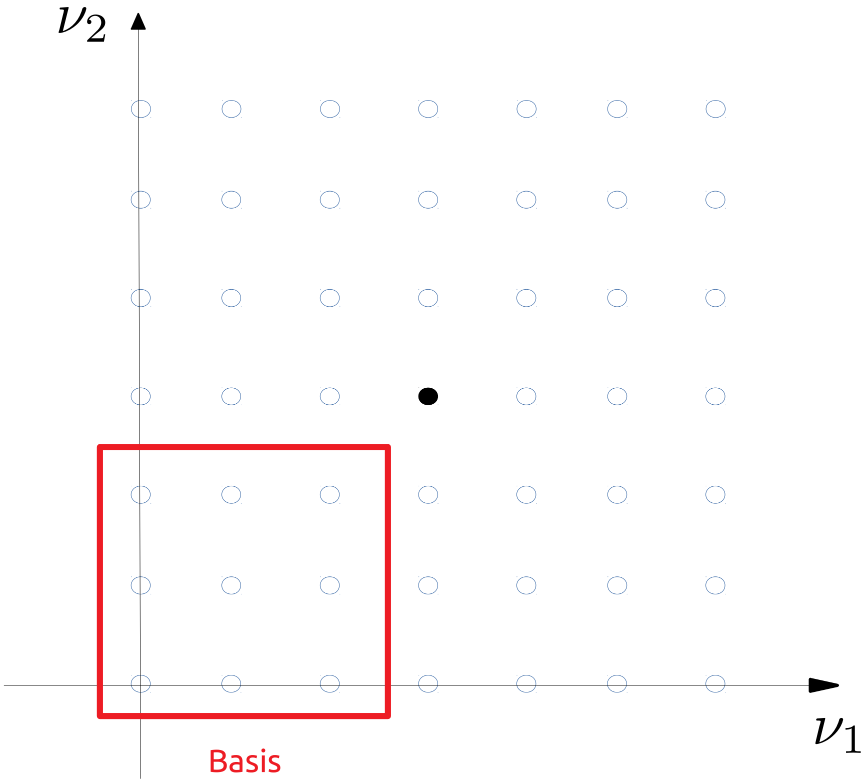
$$S_\bullet \propto \sum_{x\in\Lambda}\phi^4(x) + \text{"lower degree"} \rightsquigarrow \omega = -dS_\bullet \propto -\sum_{x\in\Lambda}\phi^3(x)d\phi + \text{"lower degree"}$$

Given: $\Phi = \phi^{\nu_1}(x_1)\ldots\phi^{\nu_k}(x_k)\ldots\phi^{\nu_N}(x_N)d^N\phi, \qquad \nu_k \geq 3$

Choose: $\xi_\Phi \propto \phi^{\nu_1}(x_1)\ldots\phi^{\nu_k-3}(x_k)\ldots\phi^{\nu_N}(x_N)d^{N-1}\phi$

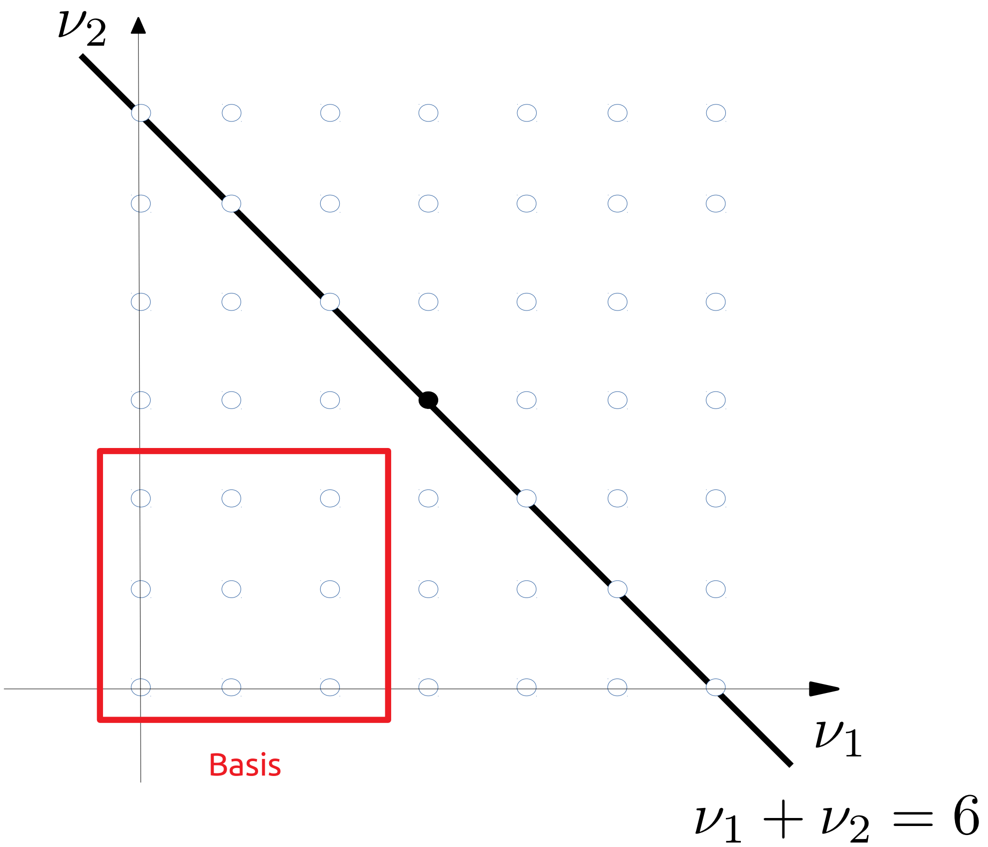
$$\nabla_\omega\xi_\Phi = 0 \implies \Phi = \sum \text{"lower degree"}$$

Example $D = 1$ Euclidean with $L = N = 2$



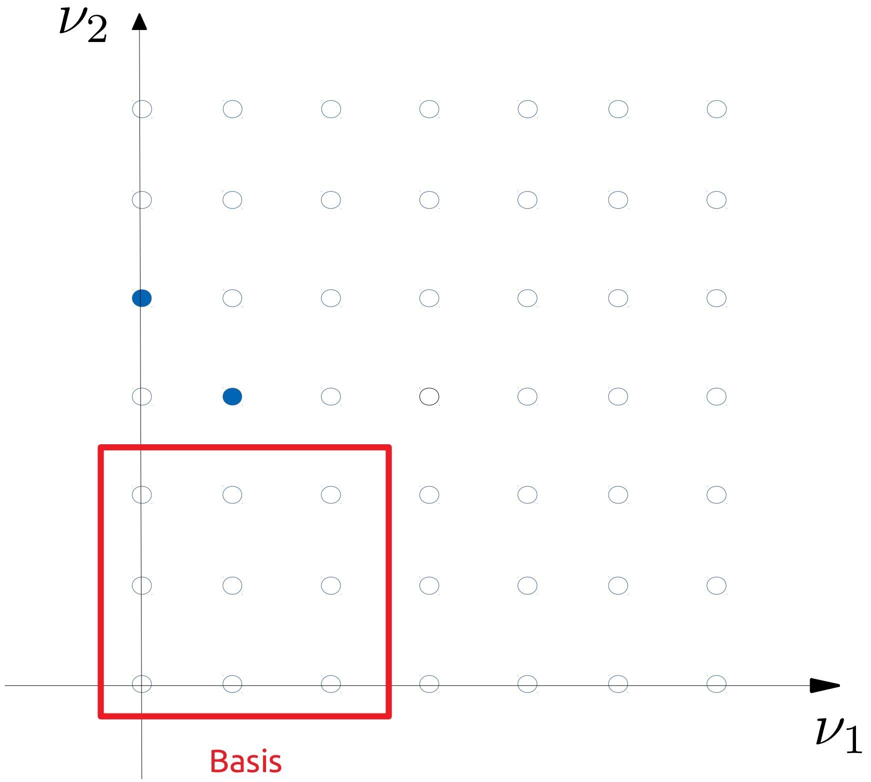
$$\phi^3(x_1)\phi^3(x_2)$$

Example $D = 1$ Euclidean with $L = N = 2$



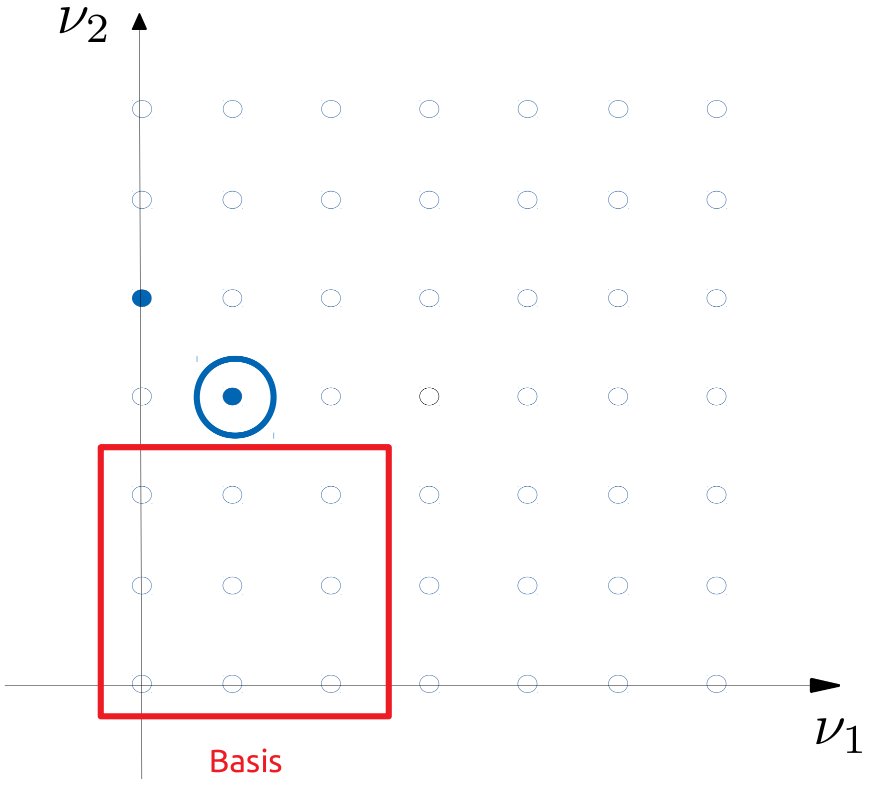
$$\phi^3(x_1)\phi^3(x_2)$$

Example $D=1$ Euclidean with $L=N=2$



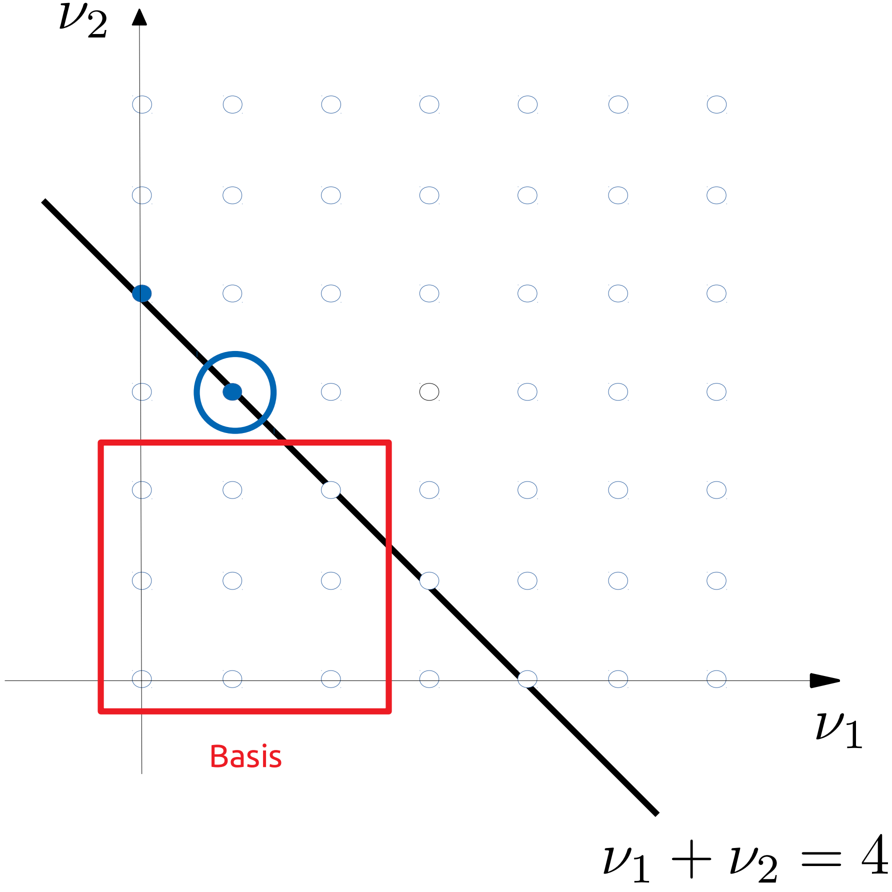
$$-\frac{(2+m^2)}{4\lambda}\phi(x_1)\phi^3(x_2)+\frac{t}{2\lambda}\phi^4(x_2)$$

Example $D = 1$ Euclidean with $L = N = 2$



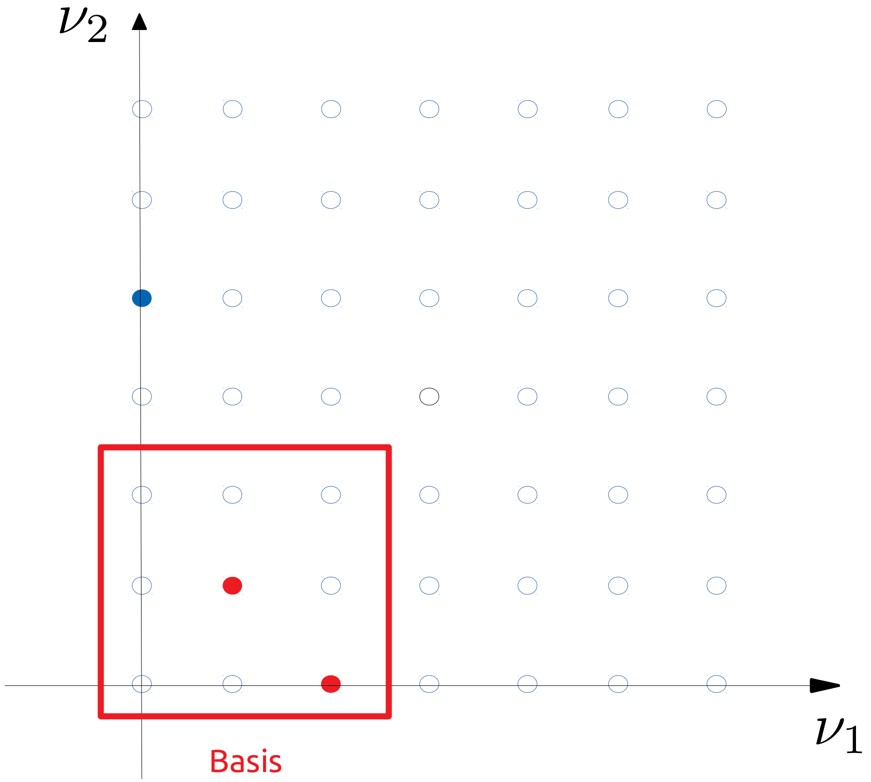
$$-\frac{(2+m^2)}{4\lambda}\phi(x_1)\phi^3(x_2) + \frac{t}{2\lambda}\phi^4(x_2)$$

Example $D = 1$ Euclidean with $L = N = 2$



$$-\frac{(2+m^2)}{4\lambda}\phi(x_1)\phi^3(x_2) + \frac{t}{2\lambda}\phi^4(x_2)$$

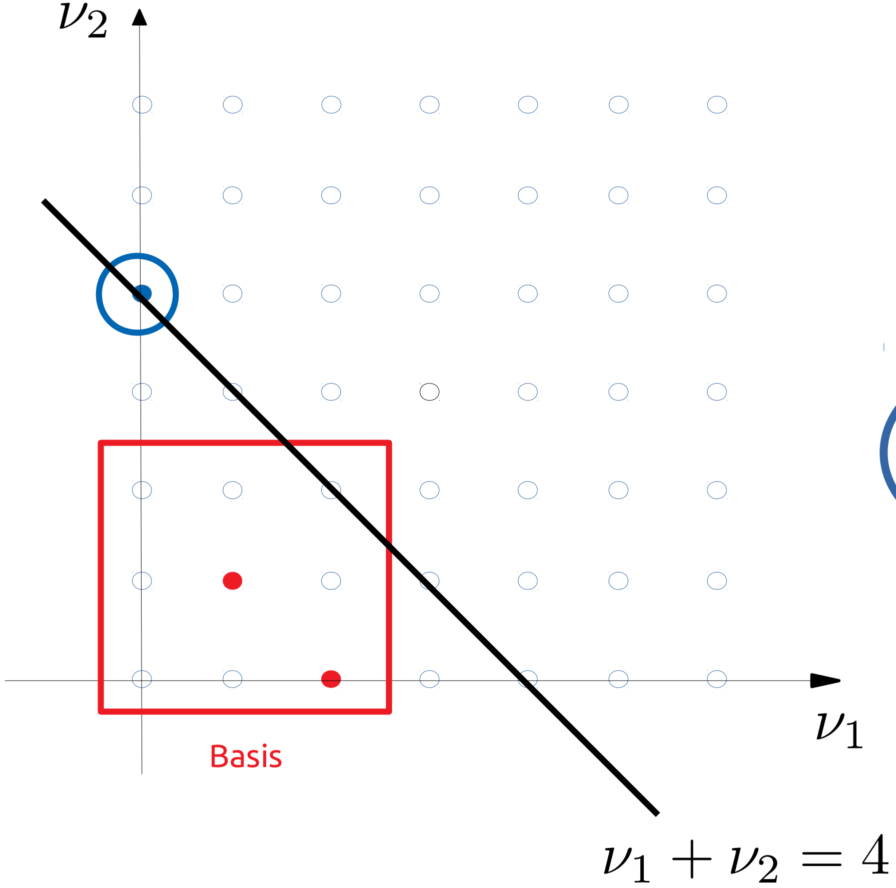
Example $D = 1$ Euclidean with $L = N = 2$



$$+\frac{(2+m^2)}{16\lambda^2}\left(-2t\varphi^2(x_1) + (2+m^2)\varphi(x_1)\varphi(x_2)\right)$$

$$+\frac{t}{2\lambda}\varphi^4(x_2)$$

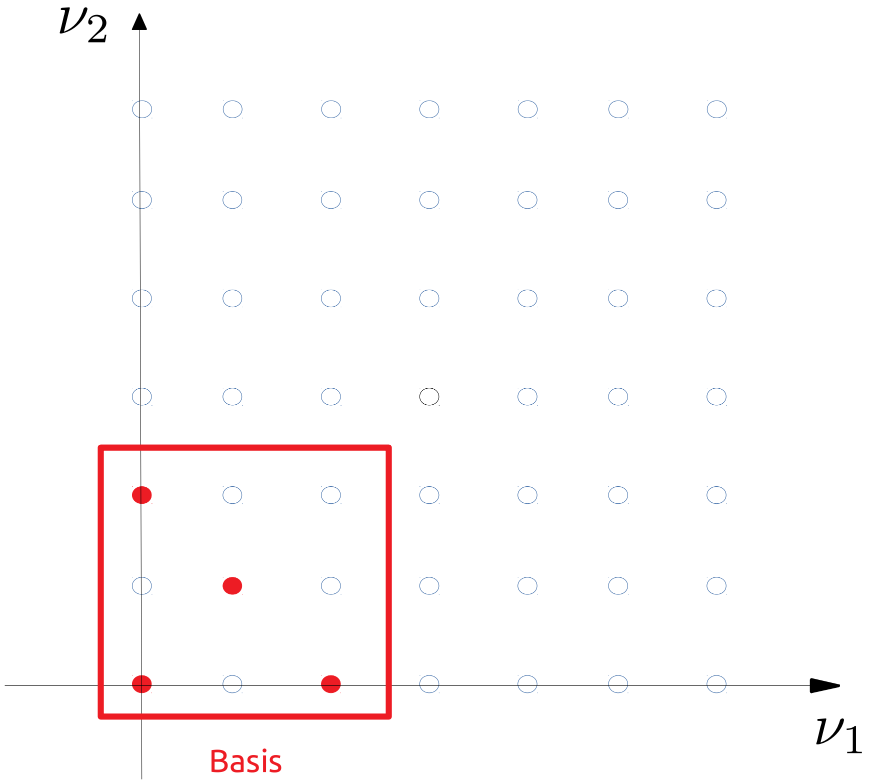
Example $D = 1$ Euclidean with $L = N = 2$



$$+\frac{(2+m^2)}{16\lambda^2}\left(-2t\varphi^2(x_1) + (2+m^2)\varphi(x_1)\varphi(x_2)\right)$$

$$+\frac{t}{2\lambda}\varphi^4(x_2)$$

Example $D = 1$ Euclidean with $L = N = 2$



$$+\frac{(2+m^2)}{8\lambda^2}\left(-t\varphi^2(x_1) - t\varphi^2(x_2)\right)$$

$$+\frac{1}{16\lambda^2}\left(2t\,1 + ((2+m^2)^2 + 4t^2)\varphi(x_1)\varphi(x_2)\right)$$

Comment: never produce powers of t in the denominator

($L = 2$ each direction)

<u>D vs dim H</u>

D	1	2	3	4
# lattice pts	2	4	8	16
# indep. forms	9	81	6 561	43 046 721

($L = 2$ each direction)

D vs dim H

D	1	2	3	4
# lattice pts	2	4	8	16
# indep. forms	9	81	6 561	43 046 721

Another source of redundancy: symmetry relations!

(not seen by diff. forms, seen by ints.)

Example $D = 1$ Euclidean with $L = N = 2$

$$S_E = -2t\phi(x_1)\phi(x_2) + \left(1 + \frac{m^2}{2}\right)(\phi^2(x_1) + \phi^2(x_2)) + \lambda(\phi^4(x_1) + \phi^4(x_2))$$

$\mathbb{Z}_2$

$$\phi(x_i) \to -\phi(x_i)$$

Permutation

$$\phi(x_1) \leftrightharpoons \phi(x_2)$$

($L = 2$ each direction)

$\underline{D \text{ vs dim H}}$

D	1	2	3	4
# lattice pts	2	4	8	16
# indep. forms	9	81	6 561	43 046 721

Another source of redundancy: symmetry relations!

(not seen by diff. forms, seen by ints.)

Example $D = 1$ Euclidean with $L = N = 2$

$$S_E = -2t\phi(x_1)\phi(x_2) + \left(1 + \frac{m^2}{2}\right)(\phi^2(x_1) + \phi^2(x_2)) + \lambda(\phi^4(x_1) + \phi^4(x_2))$$

$\mathbb{Z}_2$

$$\phi(x_i) \to -\phi(x_i)$$

$$I_{\nu_1\nu_2} = -I_{\nu_1\nu_2} = 0$$

$\nu_1 + \nu_2$ odd

Permutation

$$\phi(x_1) \leftrightharpoons \phi(x_2)$$

$$I_{\nu_1\nu_2} = I_{\nu_2\nu_1}$$

($L = 2$ each direction)

D vs dim H

D	1	2	3	4
# lattice pts	2	4	8	16
# indep. forms	9	81	6 561	43 046 721

Another source of redundancy: symmetry relations!

(not seen by diff. forms, seen by ints.)

D vs indep. ints

D	1	2	3	4
# lattice pts	2	4	8	16
# indep. ints	4	13	147	66 524

Embed integrals into a vector

$$\mathbf{I} = \left(I_1, \ldots, I_{\#\,\text{indep. ints}}\right)^\top$$

Use reductions & symmetries to obtain system 1^{st} order DEQ

$$\frac{d\,\mathbf{I}}{d\,(\bullet)} \; = \; A_\bullet(m^2, \lambda, t)\,\mathbf{I} \qquad\qquad \bullet = m^2, \lambda, t$$

Face (at least) 2 problems:

i) Singularities along integration path

ii) Boundary constants: $\mathbf{I}_0$

Auxiliary parameter t solves both problems! $\quad \bullet = t$

By construction: A_t polynomial in t (# finite$= 2^D$ for $L=2$ each direction)

$$A_t(m^2,\lambda,t) = \sum_{j=0}^{\#\,\text{finite}} \mathcal{A}_j(m^2,\lambda)\, t^j$$

Singularities only at $t=\infty$!

Integrate $t \in [0,1]$ (no singularities)

Boundary constants $\mathbf{I}_0$ @ $t=0$: product $1-$fold ints! $\left(\exp\left(\sum\right) = \prod \exp\right)$

$$S_\bullet(t=0) = \cancel{0\cdot S_\bullet^{\text{next neigh.}}} + S_\bullet^{(2)} + \lambda\, S_\bullet^{(4)}$$

The strategy

(inspired by "auxiliary mass flow") [Liu & Ma & Wang]

- Fix values for $(m^2, \lambda) \rightsquigarrow$ compute $\mathbf{I}_0$

- $A_t(t)$ holomorphic on $\mathbb{C}$ $\Rightarrow$ $\mathbf{I}(t)$ holomorphic on $\mathbb{C}$ [Wasow]

$$\mathbf{I}(t) = \sum_{k=0}^{\infty} \mathbf{I}_k \, t^k$$

- System 1^{st} order DEQ gives recursion

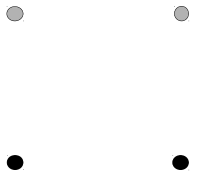
$$\mathbf{I}_k = \frac{1}{k} \sum_{j=0}^{\#\,\text{finite}} \mathcal{A}_j \, \mathbf{I}_{k-j-1} \qquad \text{with } A_t(t) = \sum_{j=0}^{\#\,\text{finite}} \mathcal{A}_j t^j$$

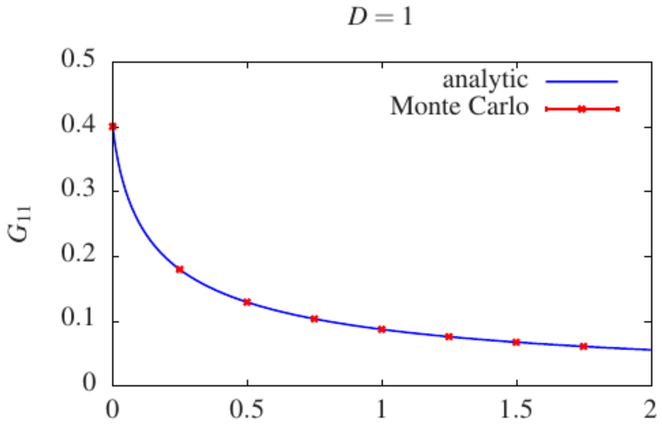
- Stop recursion at desired k, evaluate $\mathbf{I}(t=1)$

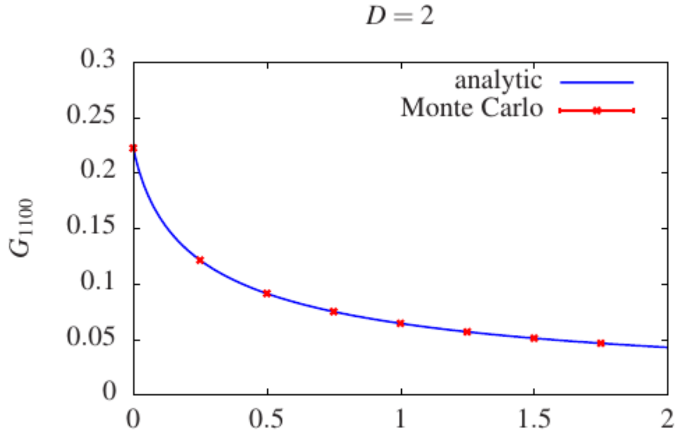
- Change values for (m^2, λ) and iterate

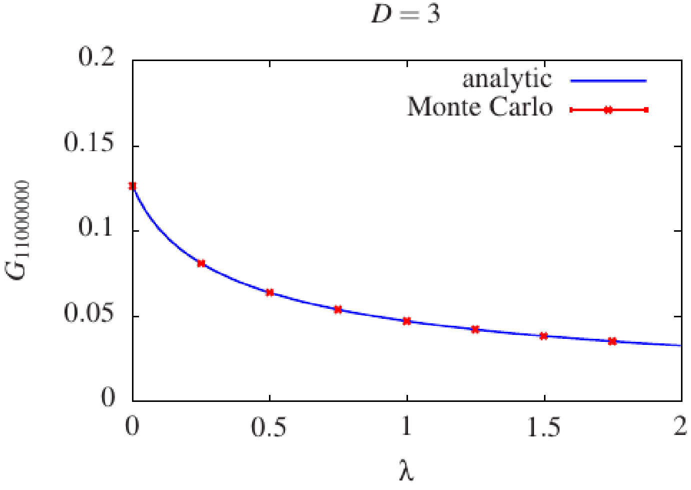
$(L = 2 \text{ each direction}, m^2 = 1)$

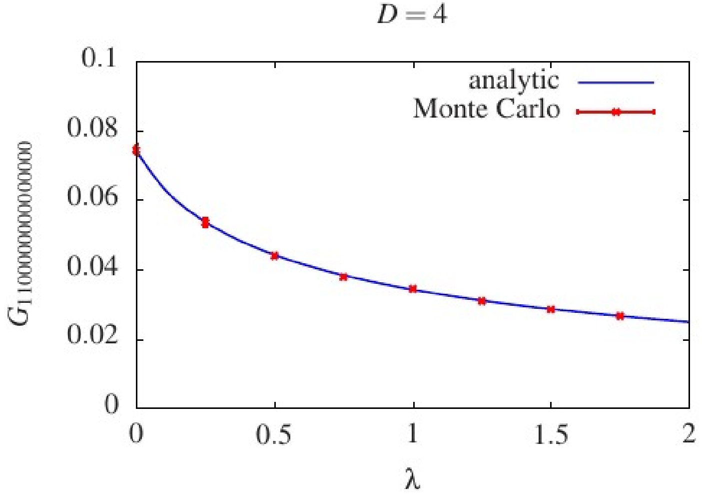
$$G_{110\ldots0} = \frac{I_{110\ldots0}}{I_{000\ldots0}} \quad \text{Euclidean}$$





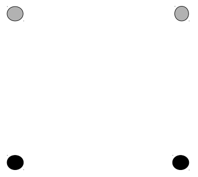




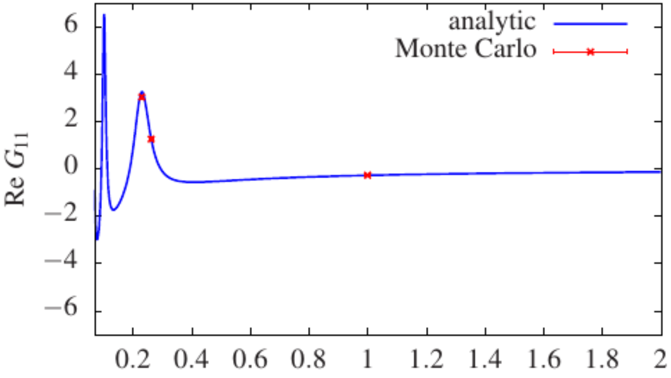


$(L = 2 \text{ each direction}, m^2 = 1)$

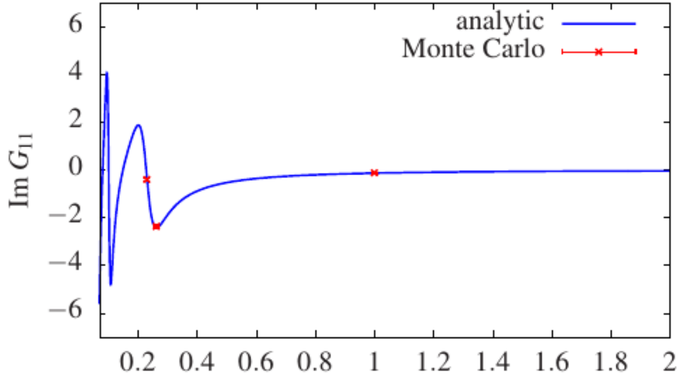
$$G_{110\ldots0} = \frac{I_{110\ldots0}}{I_{000\ldots0}} \quad \text{Minkowskian}$$



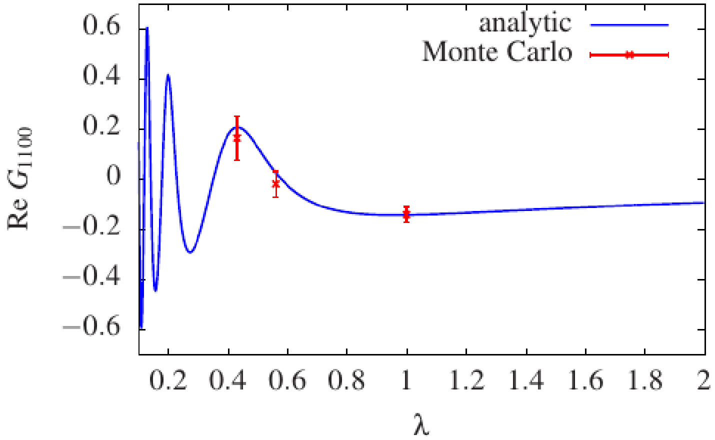
$D = 1$



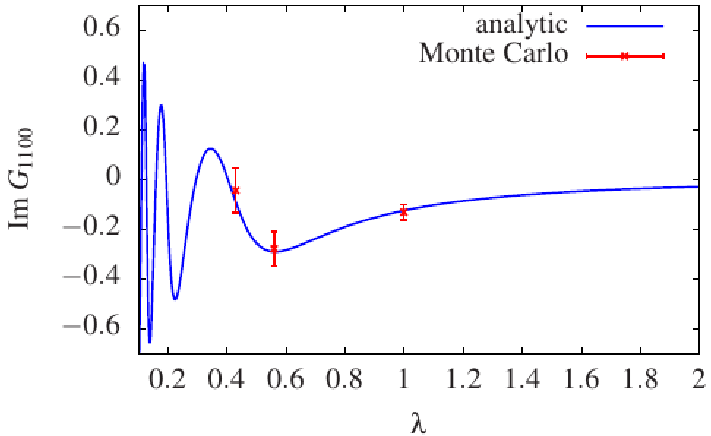
$D = 1$



$D = 2$

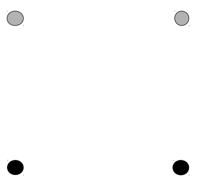


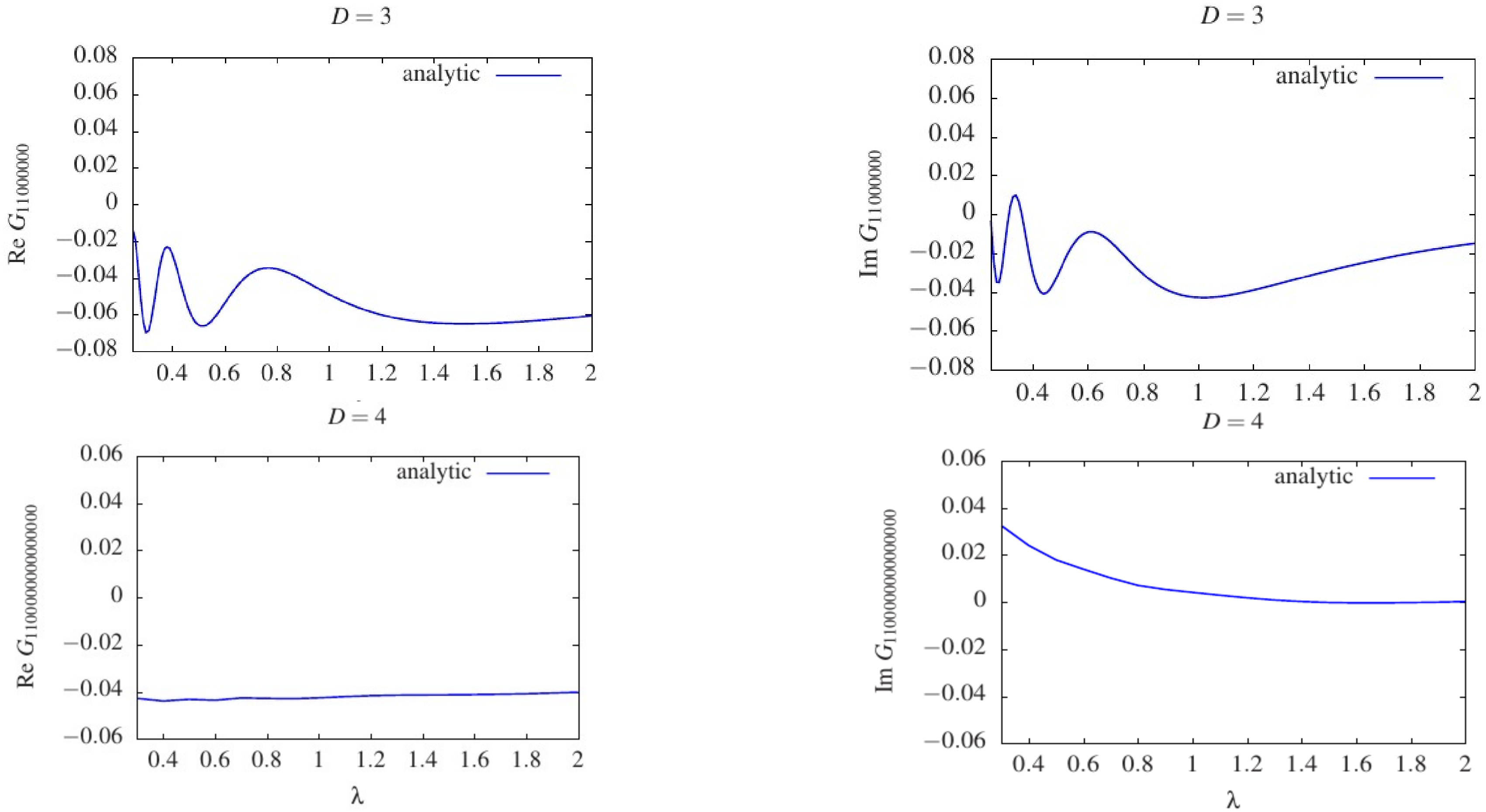
$D = 2$



($L = 2$ each direction, $m^2 = 1$)

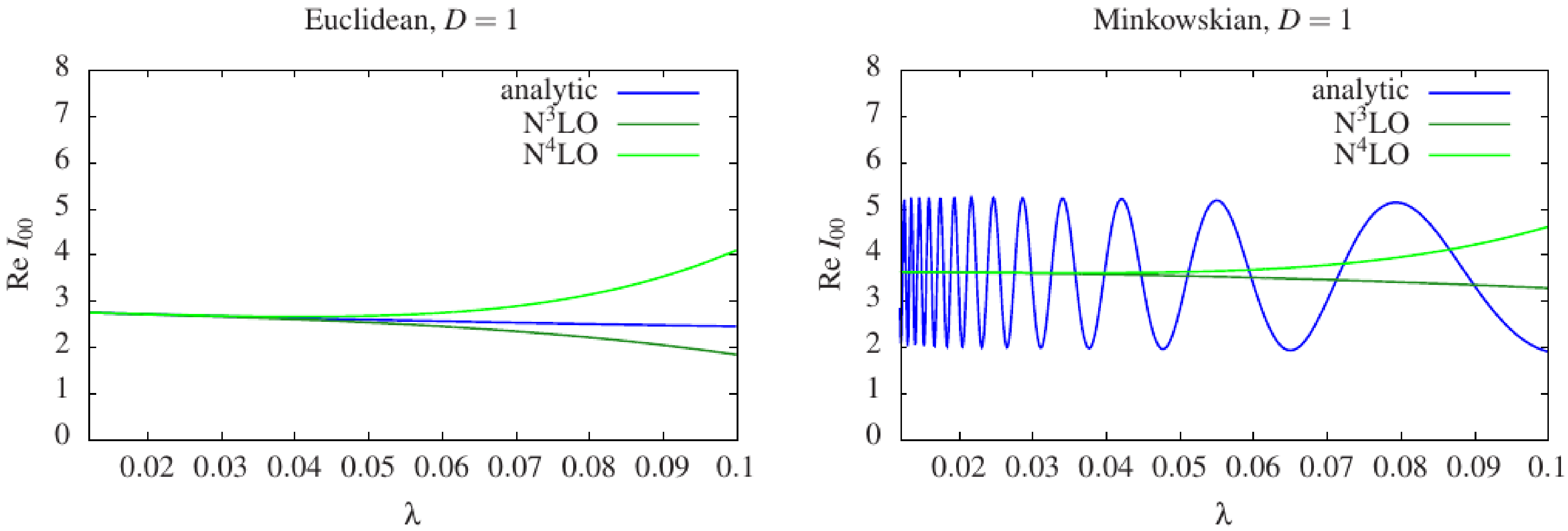
$$G_{110\ldots0} = \frac{I_{110\ldots0}}{I_{000\ldots0}} \quad \text{Minkowskian}$$





$(L = 2 \text{ each direction}, \ m^2 = 1)$

Small Coupling $\rightsquigarrow$ comparison with perturbation theory



Qualitative behaviour

(A from perturbation theory)

$$A + Be^{-\frac{c}{\lambda}} \quad (c > 0)$$

$$A + Be^{i\frac{c}{\lambda}} \quad (c > 0)$$

Conclusions

- Lattice correlation functions studied within twisted Co-Homology
- Methods from perturbation theory transferred to non-perturbative physics

  reduction to integral basis $\oplus$ system 1^{st} order DEQ

  "auxiliary flow" t

- Applicable to both Euclidean *and* Minkowskian signature

Future directions

- Organize calculation more efficiently for bigger lattices (this talk: 16 GB RAM)
- Apply to $\lambda\phi^4 \oplus$ chemical potential, Yang-Mills theory

Extra slides

- Ideal $$\mathcal{I} = \langle \omega_1, \ldots, \omega_N \rangle = \langle -\partial_1 S, \ldots, -\partial_N S \rangle \qquad \partial_i \bullet := \frac{\partial \bullet}{\partial \phi(x_i)}$$

<u>Groebner basis for free w.r.t. deglex or degrevlex</u>

- Basis $\mathbb{C}[\phi(x_1), \ldots, \phi(x_N)]/\mathcal{I} \rightsquigarrow$ basis H^N

$$\phi^{\nu_1}(x_1) \ldots \phi^{\nu_N}(x_N), \qquad \nu_1, \ldots, \nu_N < 3$$

- Reduction $$\int e^{-S} \Phi \overset{\spadesuit}{=} \int e^{-S} q_i(-\partial_i S) + r \qquad \int := \int d^N \phi$$

$$\overset{\clubsuit}{=} \cancel{\int \partial_i (e^{-S} q_i)} - \int e^{-S} \partial_i q_i + \int e^{-S} r$$

$\spadesuit$ multivariate pol. div.

$\clubsuit$ naive IBPs

$$\Phi \to -\partial_i q_i + r \text{ and Iterate}$$