Cosmology meets cohomology

Andrzej Pokraka with Shounak De (2308.03753)

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Outline

1. Amplitudes $\cap$ cosmology
    - Cosmological correlators for toy models of FRW spacetimes
    - FRW from flat space
2. Mathematics of FRW integrals
    - FRW cohomology and DEQs
    - Dual (relative twisted) cohomology and DEQs
    - Integrated results
3. Conclusions and future directions

Amplitudes ∩ cosmology: intro

What happened in the early universe? What is the origin of the universe?
("the Ultimate Question of Life, the Universe, and Everything")

Infers a cosmological history by studying late time spatial correlations compatible with time evolution

Time-evolution/causality presents unique challenges for developing quantum theories of cosmologies

It is not well understood how causal time evolution is reflected in the boundary observables at infinity

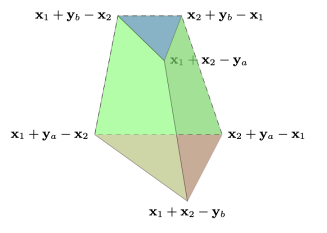
Question [Nima]: Are cosmological correlators the answer to a different question? Can time evolution be an emergent phenomena?

Amplitudes ∩ cosmology: cosmological polytopes

Similar story in scattering amplitudes: positive geometry provides new question for which planar integrands of $\mathcal{N} = 4$ SYM are the answer [Arkani-Hamed, He, Trnka, Henn, + many more]

Planar $\mathcal{N} = 4$ integrands = canonical form of Amplituhedron facets

Same story for flat space cosmological correlators: cosmological polytope [Arkani-Hamed, Benincasa, Postnikov, Baumann, Pimentel + many more]



Goal: geometrize the method of passing from flat space correlators to FRW correlators

The toy model (1)

Conformally-coupled scalar field in a general FRW cosmology with non-conformal polynomial interactions in $(d+1)$-dimensional spacetime

[Arkani-Hamed&Maldacena '15, AH&Benincasa '17, AH&Hillman '19, + many more],

$$\mathcal{S}[\phi] = \int \mathrm{d}^d x \, \mathrm{d}\eta \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{d-1}{4d} R \phi^2 - \sum_{k \geq 3} \frac{\lambda_k}{k!} \phi^k \right]$$

$$\mathrm{d}s^2 \equiv g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\eta) \left[-\mathrm{d}\eta^2 + \mathrm{d}x_i \mathrm{d}x^i \right]$$

$$a(\eta) = \frac{1}{\eta^{1+\varepsilon}} \qquad \begin{cases} \varepsilon = 0 & (\text{dS}) \\ \varepsilon = -1 & (\text{flat}) \\ \varepsilon = -2 & (\text{RD}) \\ \varepsilon = -3 & (\text{MD}) \end{cases}$$

The toy model (2)

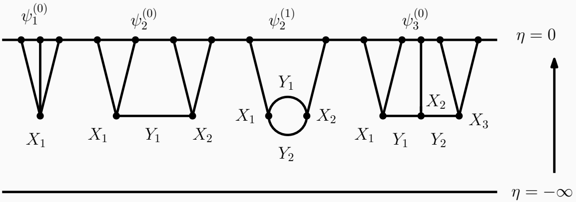
Conformally equivalent to flat space action w/ time-dependent couplings (set $\lambda_k = 0 \; \forall \; k \geq 4$ in practice)

$$S[\phi] = \int \mathrm{d}^d x \, \mathrm{d}\eta \left[\frac{1}{2}(\partial\phi)^2 - \sum_{k\geq 3} \frac{\lambda_k(\eta)}{k!} \phi^k \right]$$

$$\lambda_k(\eta) = \lambda_k \big(a(\eta)\big)^{(2-k)\frac{d-1}{2}+2}$$

$$\lambda_3(\eta)\Big|_{d=3} = \frac{\lambda_3}{\eta^{1+\varepsilon}}$$

From flat space to FRW correlators



X_i: sum of external energies; Y_i energy of sums of momenta

Flat space correlators: simple poles only!

$$\psi_{n,\text{flat}}^{(L)}(\mathbf{X},\mathbf{Y}) = \frac{\mathcal{N}}{\prod_i S_i} \quad \text{where } S_i \text{ are linear in } X_i \text{ and } Y_i$$

FRW-correlators by recycling flat space correlators

$$\boxed{\psi_{n,\text{FRW}}^{(L)}(\mathbf{X},\mathbf{Y}) = \int_0^\infty \left(\bigwedge_{v=1} \mathrm{d}x_v \; x_v^{\varepsilon}\right) \psi_{n,\text{flat}}^{(L)}(\mathbf{x}+\mathbf{X},\mathbf{Y})}$$

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FRW 2-site/4-point correlator

$$\psi^{(0)}_{2,\mathsf{FRW}} = \int_0^\infty \mathrm{d}x_1 \wedge \mathrm{d}x_2 \ (x_1 x_2)^\varepsilon \ \psi^{(0)}_{2,\mathsf{flat}}(x_1 + X_1, x_2 + X_2; Y)$$

$$\psi^{(0)}_{2,\mathsf{flat}} = \frac{1}{(X_1+X_2)(X_1+Y_1)(X_2+Y_1)} \equiv \frac{1}{\prod_{i=1}^3 S_i}$$

In general, too hard to do directly $\implies$ define family of integrals $\{I_a\} \supset \psi^{(0)}_{2,\mathsf{FRW}}$ and find canonical DEQ

$$\partial_z I_a = \varepsilon A^{(z)}_{ab} I_b \qquad z \in \{X_1, X_2, Y\}$$

Sol in terms of iterated integrals [Chen '77, Henn '13]

$$\mathbf{I} = \mathbf{I}_0 + \varepsilon \int \underline{\boldsymbol{A}} \cdot \mathbf{I}_0 + \varepsilon^2 \int \underline{\boldsymbol{A}} \cdot \int \underline{\boldsymbol{A}} \cdot \mathbf{I}_0 + \cdots$$

Family of integrals $\{I_a\}$ fixed by geometry!

The 2-site/4-point family of integrals

$$\psi^{(0)}_{2,\text{FRW}}(\boldsymbol{\mu}|\boldsymbol{\nu}) = \int u \, \varphi_{(\boldsymbol{\mu}|\boldsymbol{\nu})}$$

- u → multi-valued function called twist: $u = \prod_{i=1}^{2} T_i^{\varepsilon}$
- $\varphi_{(\boldsymbol{\mu}|\boldsymbol{\nu})}$ → single-valued differential form

$$\varphi_{(\boldsymbol{\mu}|\boldsymbol{\nu})} = \frac{N(\mathbf{x})}{\prod_{j=1}^{2} T_j^{\mu_j} \prod_{k=1}^{3} S_k^{\nu_k}} \mathrm{d}^2\mathbf{x} \quad (\nu_i, \mu_i \in \{0, 1, 2, \dots\}),$$

$N(\mathbf{x})$ → arb. num. (often 1)

Physical FRW correlator: $\mu = \{0, 0\}$ and $\nu = \{1, 1, 1\}$

Twisted sing. (mild)	Untwisted sing. (dangerous)
$T_1 = x_1$	$S_1 = x_1 + X_1 + Y_1$
$T_2 = x_2$	$S_2 = x_2 + X_2 + Y_1$
	$S_3 = x_1 + x_2 + X_1 + X_2$

FRW-cohomology

total (covariant) derivatives integrates to zero

Integrands not unique: $\int u\,\varphi = \int \big(u\,\varphi + \mathrm{d}(u\,\psi)\big) = \int u\big(\varphi + \nabla\psi\big)$

Introduce covariant derivative $\Longrightarrow$ work w/ single-valued forms

$$\nabla = \mathrm{d} + \omega\wedge, \qquad \omega = \mathrm{d}\log u, \qquad \nabla^2 = 0$$

Enter twisted cohomology [Matsumoto, Aomoto, Kita, . . . , Mizera, Mastrolia, Frellesvig, Chestnov, Gasparotto, Mandal, Mattiazzi, Cacciatori, AP, Caron-Huot, Chen, Jiang, . . .]
[Talks by Crisanti, Fontana]

$$|\varphi\rangle \in H^p(X, \nabla) = \frac{\text{covariant closed forms (eqiv FRW-forms)}}{\text{covariant exact forms (equiv IBPs)}}$$

covariant closed forms (eqiv FRW-forms) $\longrightarrow \nabla\varphi = 0$

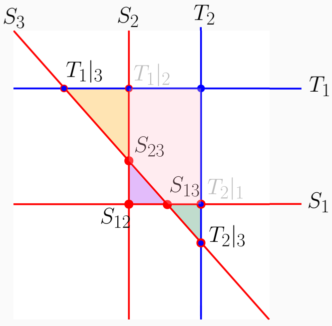
covariant exact forms (equiv IBPs) $\longrightarrow \varphi \neq \nabla\psi$

X: $\mathbb{C}^2 \setminus \{T_i = 0, \infty\} \cup \{S_i = 0\}$

Access powerful mathematical theorems and connection to geometry:

- Basis size = Euler characteristic: $|\chi(X)| = 4$
  [Bitoun, Bogner, Klausen, Panzer '18] [Talks by Fevola&Telen]
- Intersection theory (inner product on space of integrands)

Basis of FRW cohomology



$\mathbb{C}^2 \setminus \{T_i = 0, \infty\} \cup \{S_i = 0\}$

Physical 2-site/4-point FRW form:

$$\psi^{(0)}_{2,\mathsf{FRW}} = \int u\ \vartheta_1$$

$$\vartheta_1 = \mathrm{d}\log\frac{S_1}{S_2} \wedge \mathrm{d}\log\frac{S_2}{S_3}$$

Choose remaining canonical forms for other basis elements: ϑ_2, ϑ_3, ϑ_4

DEQs for basis of canonical forms

$$\nabla_{\mathsf{kin}}\, \vartheta_a = (A_\vartheta)_{ab} \wedge \vartheta_b$$

$$\underline{\boldsymbol{A}}_\vartheta = \varepsilon \, \mathrm{d}\log \begin{pmatrix} (X_1+Y)(X_2+Y) & \frac{Y-X_1}{X_1+Y} & \frac{X_2+Y}{Y-X_2} & 0 \\ 0 & (X_1-Y)(X_1+X_2) & \frac{X_2+Y}{X_1+X_2} & \frac{X_2+Y}{X_1+X_2} \\ 0 & \frac{X_1+Y}{X_1+X_2} & (X_2-Y)(X_1+X_2) & \frac{X_1+X_2}{X_1+Y} \\ 0 & \frac{Y-X_1}{X_1+Y} & \frac{X_2+Y}{Y-X_2} & (X_1+Y)(X_2+Y) \end{pmatrix}$$

gauge redundancy in choice of basis

$$\boldsymbol{\vartheta} = \underline{\boldsymbol{U}} \cdot \boldsymbol{\varphi} \implies \begin{cases} \nabla_{\mathsf{kin}} \boldsymbol{\varphi} = \underline{\boldsymbol{A}}_\varphi \cdot \boldsymbol{\varphi} \\ \underline{\boldsymbol{A}}_\varphi = \underline{\boldsymbol{U}}^{-1} \cdot \underline{\boldsymbol{A}}_\vartheta \cdot \underline{\boldsymbol{U}} - \underline{\boldsymbol{U}} \cdot \mathrm{d}_{\mathsf{kin}} \underline{\boldsymbol{U}} \end{cases}$$

How to choose a good basis with simple DEQ? ($\underline{\boldsymbol{A}}_\vartheta$ quite dense)

Intersection theory and the dual cohomology

Dual cohomology: $\check{H}^p(\check{X}, \nabla) = ??$

Intersection number is inner product on vector space of integrands:

compact support; very simple for FRW forms!

$$\langle \check{\varphi} | \varphi \rangle \propto \underbrace{\mathrm{Reg}[\check{\varphi}]}_{\text{compact support}} \wedge \varphi =^* \sum_{a,b} \mathrm{Res}_{T_a=0}\mathrm{Res}_{S_b=0}[\bullet\varphi]$$

always regulate $\check{\varphi}$ at twisted singularities

untwisted singularities
twisted singularities

$\check{\varphi}$ must be regular at all $S_i = 0$ to regulate $\implies$ boundaries at $S_i = 0$

can have twisted singularities because of Reg[]

Boundaries $\implies$ dual cohomology is a **relative** twisted cohomology

$$\check{H}^p(X, \nabla) = H^p(\check{X}, \check{X} \cap \{S_i = 0\}, \check{\nabla})$$ [Caron-Huot&AP '21, '22]

$\check{X} \to \mathbb{C}^2 \setminus \{T_i = 0\}$

$\check{\nabla} \to \mathrm{d} - \omega\wedge$

Relative cohomology

Relative: fancy name for keeping track of boundary terms from Stokes' theorem on a manifold with boundary

FRW cohomology: did not allow boundaries $\implies \int \mathrm{d}(\text{anything}) = 0$

Dual cohomology: the untwisted singular surfaces become boundaries
$\implies \int \mathrm{d}(\text{anything}) = 0 +$ boundary terms

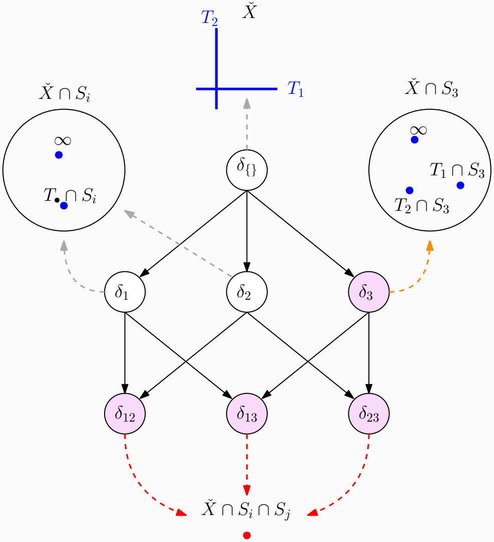
Boundary terms: integrals over lower-degree forms of boundary surfaces

$\implies H^n(\check{X}, \check{X} \cap S; \nabla) =$ direct sum of each boundary cohomology

$$= H^n\left(\check{X}; \nabla\right) \bigoplus_i H^{n-1}\left(\check{X} \cap S_i; \nabla\right) \bigoplus_{ij} H^{n-2}\left(\check{X} \cap S_{ij}; \nabla\right) \bigoplus \cdots$$

0-bondary ($\check{X}$), 1-bondaries ($\check{X} \cap S_i$), 2-bondaries ($\check{X} \cap S_{ij}$)

Basis of dual-forms



Dual basis:

$$\check{\varphi}_1 = \varepsilon\ \delta_3 \left(\mathrm{d}\log \frac{T_1|_3}{T_2|_3} \right)$$

$$\check{\varphi}_2 = \delta_{12}(1)$$

$$\check{\varphi}_3 = \delta_{23}(1)$$

$$\check{\varphi}_4 = \delta_{13}(1)$$

Dual DEQ more sparse!

$$\check{\nabla}_{\mathsf{kin}}\ \check{\varphi}_a = (\check{A}_\vartheta)_{ab} \wedge \check{\varphi}_b$$

$$\underline{\check{\boldsymbol{A}}} = \varepsilon\ \mathrm{d}\log \begin{pmatrix} (X_1+X_2)^2 & 0 & \dfrac{Y+X_2}{Y-X_1} & \dfrac{Y+X_1}{Y-X_2} \\ 0 & (Y+X_1) & 0 & 0 \\ 0 & 0 & \dfrac{Y-X_1}{Y+X_2} & 0 \\ 0 & 0 & 0 & \dfrac{Y-X_2}{Y+X_1} \end{pmatrix}$$

Dual DEQ always lower block triangular! $\implies$ Much easier to build a dual basis with a simple DEQ

Choose FRW basis φ_a orthonormal to $\check{\varphi}_a$: $(C_\varphi)_{ab} = \langle \check{\varphi}_a | \varphi_b \rangle = \delta_{ab}$

$$\underline{\boldsymbol{A}}_\varphi = -\left(\underline{\check{\boldsymbol{A}}}\right)^\top$$

2-site/4-point correlator integrated

dS result [Arkani-Hamed, Maldacena '15]:

$$\psi^{(0)}_{2,\mathrm{dS}} = \mathrm{Li}_2\left(\frac{X_1 - Y}{X_1 + Y}\right) + \mathrm{Li}_2\left(\frac{X_2 - Y}{X_2 + Y}\right) - \mathrm{Li}_2\left(\frac{X_1 - Y}{X_1 + Y}\frac{X_2 - Y}{X_2 + Y}\right) - \frac{\pi^2}{6}$$

1st correction (weight 3):

$$\begin{aligned}
\psi^{(0)}_{2,\mathrm{FRW}} = &- \mathrm{Li}_3\left(\frac{Y + X_1}{Y - X_2}\right) + \mathrm{Li}_3\left(-\frac{X_1 - Y}{Y + X_2}\right) + 2\mathrm{Li}_3\left(\frac{X_1 - Y}{X_1 + X_2}\right) - 2\mathrm{Li}_3\left(\frac{Y + X_1}{X_1 + X_2}\right) \\
&+ 2\log\left(X_1 + X_2\right)\mathrm{Li}_2\left(-\frac{Y + X_2}{X_1 - Y}\right) - 2\log\left(X_1 + X_2\right)\mathrm{Li}_2\left(\frac{Y - X_2}{Y + X_1}\right) \\
&+ \frac{1}{6}\log^3\left(X_2 - Y\right) - \frac{1}{6}\log^3\left(X_2 + Y\right) - \frac{1}{2}\log\left(X_1 + Y\right)\log^2\left(X_2 - Y\right) \\
&- \frac{1}{2}\log^2\left(X_1 + Y\right)\log\left(X_2 - Y\right) - \log^2\left(X_1 + X_2\right)\log\left(X_2 - Y\right) \\
&+ \frac{1}{2}\log\left(X_1 + Y\right)\log^2\left(X_2 + Y\right) + \log^2\left(X_1 + X_2\right)\log\left(X_2 + Y\right) \\
&+ \frac{1}{2}\log^2\left(X_1 + Y\right)\log\left(X_2 + Y\right) + \log\left(X_1 + X_2\right)\log^2\left(X_1 - Y\right) \\
&- \log\left(X_1 + X_2\right)\log^2\left(X_1 + Y\right) + 2\log\left(X_1 + X_2\right)\log\left(X_1 + Y\right)\log\left(X_2 - Y\right) \\
&+ \frac{1}{6}\pi^2\left(\log\left(X_2 - Y\right) - \log\left(X_2 + Y\right)\right) \\
&- 2\log\left(X_1 + X_2\right)\log\left(X_1 - Y\right)\log\left(X_2 + Y\right)
\end{aligned}$$

Summary and Conclusions

Geometric framework for constructing the DEQs of FRW cosmological correlators from flat space correlators

Using dual IBPs/intersection numbers significantly simplifies the computation of the DEQs

Dual cohomology helps choose good basis with sparse DEQ

Computed the DEQs for a variety of examples

- 2-site/4-point
- 3-site/5-point
- 2-site/2-point 1-loop

Integrated first correction to the de Sitter 2-site/4-point correlator

Future Directions

DEQs for 6-point FRW correlators: 2-topologies, basis $\sim$ 350 each

- Expect algorithm to scale well since each boundary cohomology has small dim
- Biggest challenge will be combinatorics of boundary cross talk

Add mass (pheno) and include spinning particles (similar to conformal scalars)? Other theories?

Construction of the de Sitter cohomology

- Surprisingly, completely removing ε makes things more complicated
- Such a construction would find use describing finite $\mathcal{N} = 4$ integrals and the $m = 1$ amplituhedron

What do the other entries of the period matrix mean?

Extra slides

Wavefunctional of the Universe ($\sim$ partition function)

$$\Psi[\Phi] = \int_{\phi(\mathbf{x},-\infty(1-i\epsilon))}^{\phi(\mathbf{x},0)=\Phi} \mathcal{D}\phi\, e^{iS[\phi]}$$

$\phi(\mathbf{x},\eta)$ field config w/ Dirichlet bdc on future horizon $\phi(\mathbf{x},\eta{=}0){=}\Phi(\mathbf{x})$

$i\epsilon$ selects BD/HH vacuum at early-time boundary $\eta \to -\infty$

Expand in momentum space:

$$\Psi[\Phi] = \exp\left\{ i \sum_n \frac{1}{n!} \int \left[\prod_i \mathrm{d}^d\mathbf{k}_i\, \Phi(\mathbf{k}_i) \right] \underbrace{\delta^{(d)}\left(\sum_{i=1}^n \mathbf{k}^{(i)} \right)}_{\text{translation inv}} \psi_n(\{\mathbf{k}_i\}) \right\}$$

wavefn coeff/cosmological correlator

bulk-boundary: $e^{iE\eta}$

bulk-bulk: $\sim \dfrac{e^{-iE(\eta-\eta')}}{2E}\theta\left(\eta{-}\eta'\right) + \cdots$

The coboundary δ_J

bd form: $\check{\phi}_J \in H^{n-|J|}(\check{X} \cap_{a\in J} S_a; \check{\nabla})$

Dual forms: $\check{\varphi} = \delta_J \left(\check{\phi}_J \right)$

coboundary symbol: keeps track of bd's J and Stokes'theorem

Acts like a form:

- Anti-commutes with other coboundaries:
  $\delta_{ij} = \delta_i \wedge \delta_j = -\delta_j \wedge \delta_i = -\delta_{ji}$
- Anti-commutes with $\check{\nabla}$ (eqiv d) up to boundary terms

$$\check{\nabla}\delta_J(\check{\phi}) = \underbrace{(-1)^{|J|}\delta_J\left(\check{\nabla}|_J\ \check{\phi}\right)}_{\|\ \text{transports } \check{\nabla} \text{ to } \check{\nabla}|_J} + \underbrace{(-1)^{|J|}\sum_{k\notin J}\delta_{Jk}\left(\check{\phi}|_k\right)}_{\text{boundary terms/Stokes'theorem}}$$

Takes a residue on the boundary in the intersection number

$$\langle \delta_J(\check{\phi}_J)|\varphi\rangle \sim \langle \check{\phi}_J|\text{Res}_J[\varphi]\rangle \implies \delta_J \text{ dual to Res}_J \text{ or } \bigwedge_{a\in J} \mathrm{d}\log S_a \text{ (to LO)}$$

Orthonormal basis

Choose FRW basis orthonormal to dual basis $\check{\varphi}_a$

$$\varphi_1 = -\frac{1}{2}\mathrm{d}\log(S_3) \wedge \mathrm{d}\log\left(\frac{T_1}{T_2}\right)$$

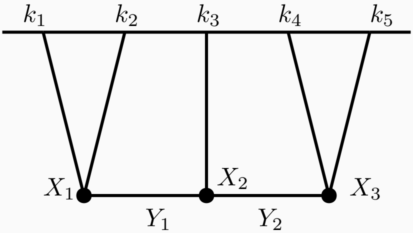
$$\varphi_2 = \mathrm{d}\log S_1 \wedge \mathrm{d}\log S_2$$

$$\varphi_3 = \mathrm{d}\log S_2 \wedge \mathrm{d}\log S_3$$

$$\varphi_4 = \mathrm{d}\log S_1 \wedge \mathrm{d}\log S_3$$

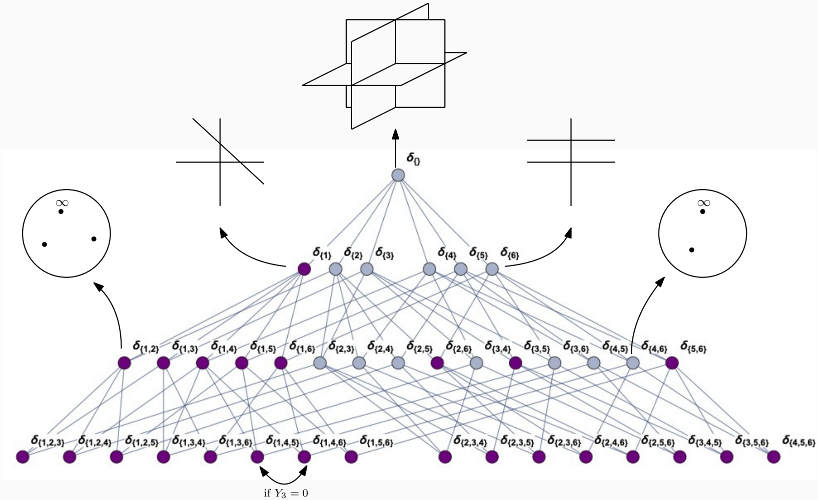
$$(C_\varphi)_{ab} = \langle \check{\varphi}_a | \varphi_b \rangle = \delta_{ab}$$

3-site/5-point correlator

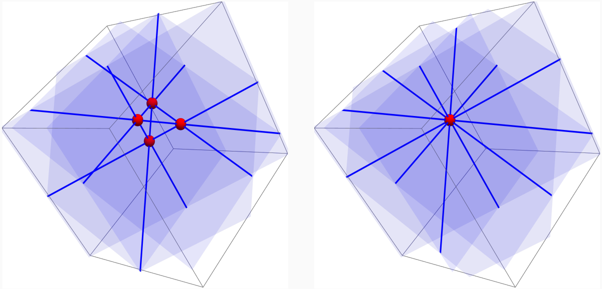


3 twisted coordinate planes ($T_{i=1,2,3}$) and 6 boundary planes ($S_{i=1,2,\ldots,6}$)
$\implies$ basis size of 25 $\implies$ too big to show in this talk

3-site/5-point boundary structure



Degenerate boundaries

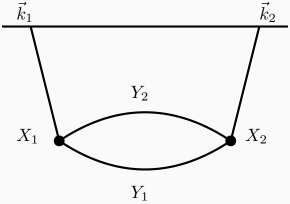


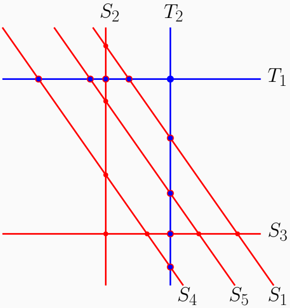
FRW hyperplane arrangements often degenerate with 4 boundaries intersecting at a point $\implies$ identification of boundaries

Intersection number detects the relations between degenerate boundaries (can also use blow-ups)

Complicates computing basis size a bit but cannot be avoided (there on the non-dual side too)

Loops: 2-site/2-point diagram





10-dimensional basis

Similar to the tree-level 2-site/4-point example except we still need to integrate over the spatial loop-momentum