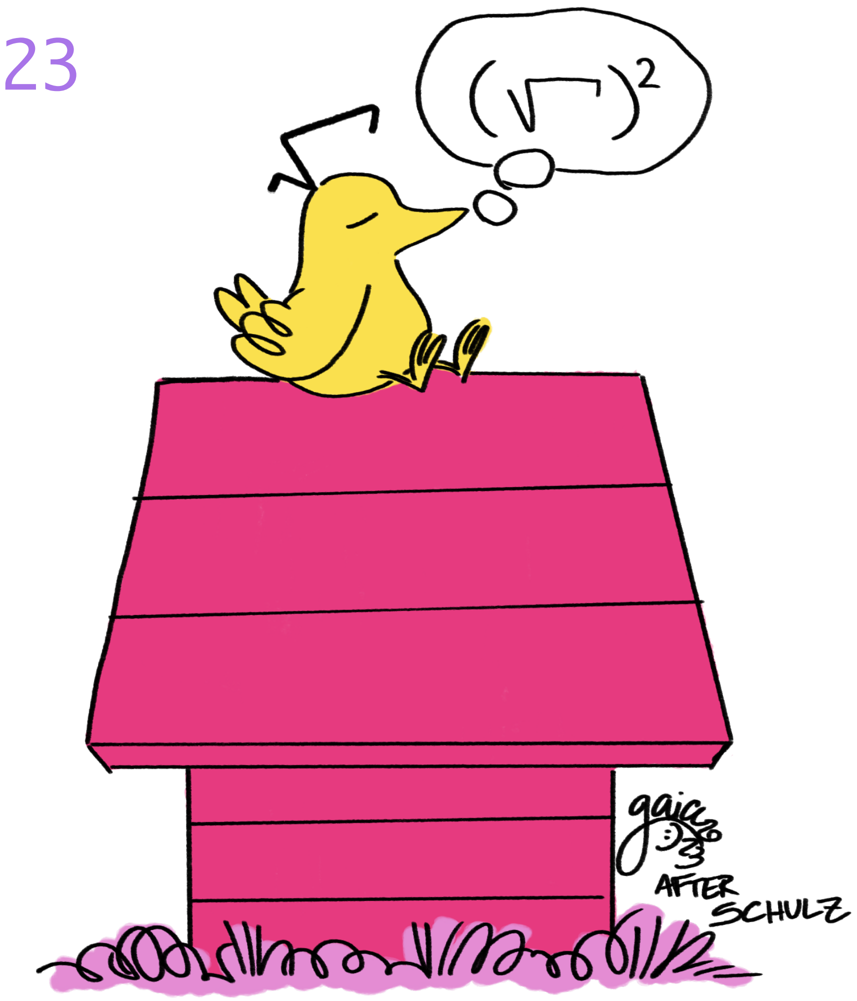
# Intersection numbers
# &
# rational algorithms

MathemAmplitudes 2023
Padova

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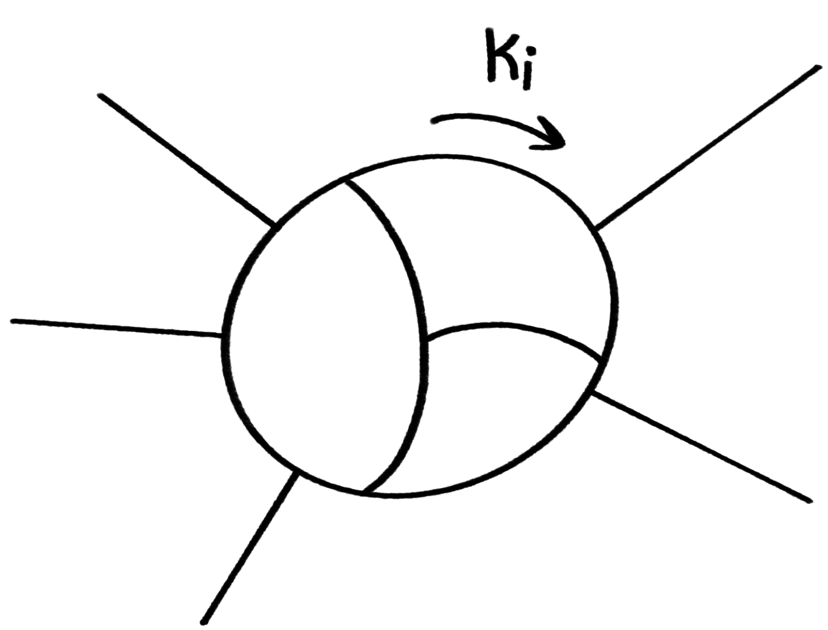
Based on arXiv:2304.14336
in collaboration with Tiziano Peraro

# Loop integrals

* LEGO® blocks of perturbative QFT beyond tree level
* Key ingredient of phenomenological predictions
* Rich and interesting mathematical structures

$$I_i[\alpha_1, \ldots, \alpha_n] = \int_{-\infty}^{+\infty} \left( \prod_{i=1}^{\ell} d^d k_i \right) \frac{1}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}}$$




inverse propagators

$z_i = k_i^2 - m_i^2$ + ISPs

Not all are linearly independent!

# Reduction to master integrals

## why?

* Extremely large number of integrals contributing to an amplitude
* Properties/symmetries of an amplitude manifest only after the reduction
* Important for the calculation of the integrals

Reduction into a basis of linearly independent master integrals $\{G_j\} \subset \{I_j\}$

$$I_j = \sum c_{jk} G_k$$

$\{G_j\} =$ minimal linearly independent set

# Laporta algorithm

Feynman integrals in dimensional regularization obey linear relations, e.g. Integration By Parts identities

Chetyrkin, Tkachov (1981), Laporta (2000)

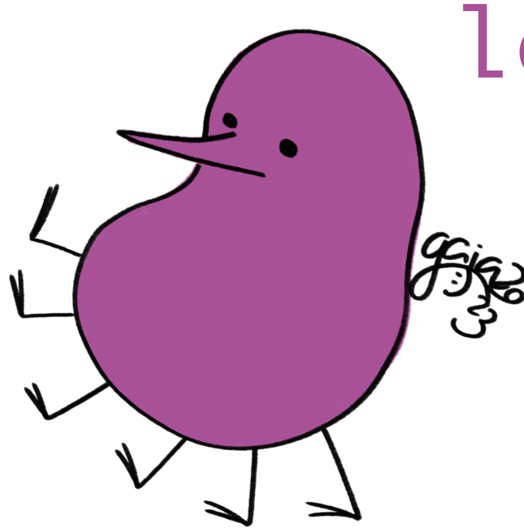
$$\int \left( \prod_{i=1}^{\ell} d^d k_i \right) \frac{\partial}{\partial k_i^\mu} \left( \frac{v_j^\mu}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}} \right) = 0, \qquad v^\mu = \begin{cases} p_i^\mu = \text{external} \\ k_i^\mu = \text{loop} \end{cases}$$

\+ Lorentz Invariance ids, symmetry relations, …

reduction as solution of a large and sparse system of identities

# Algebraic complexity

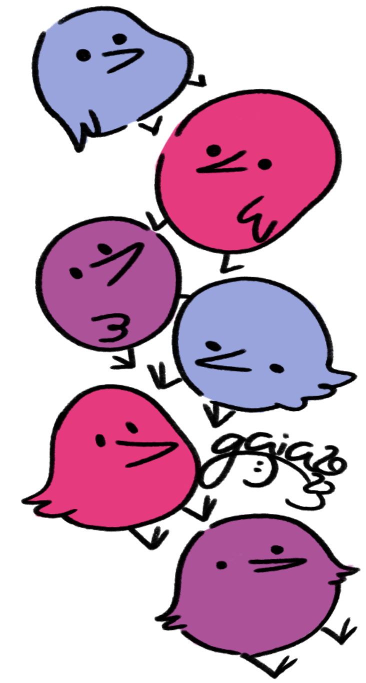
Processes with many



legs

&

loops



give rise to HUGE intermediate expressions

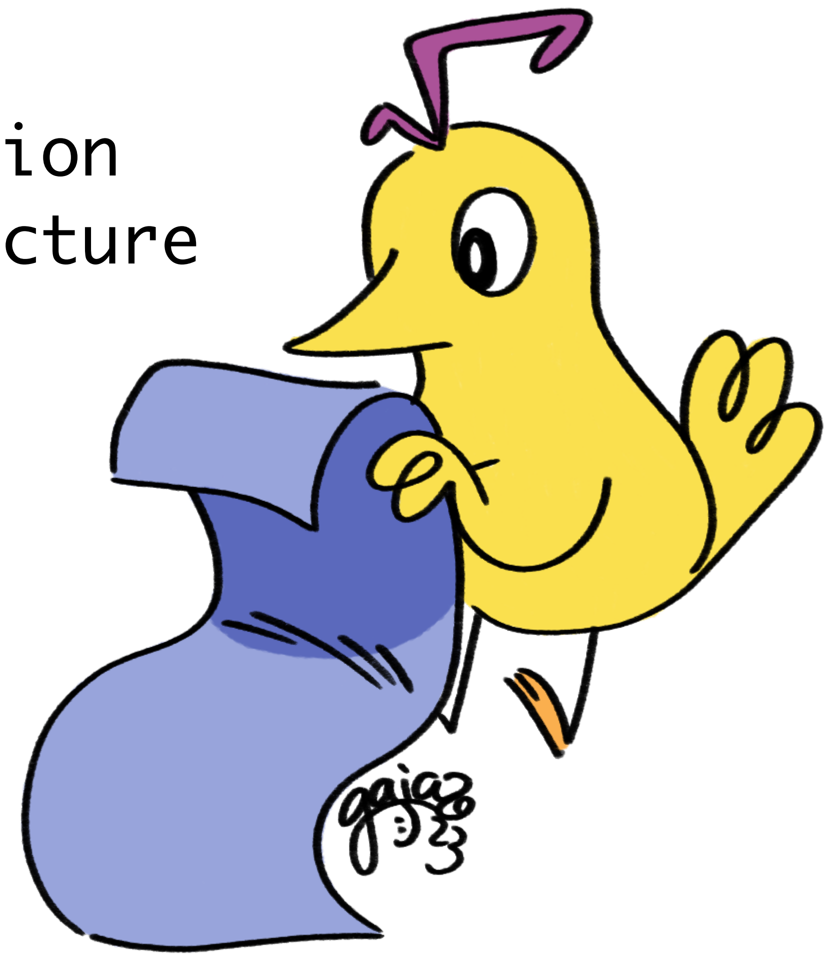
## Drawbacks of Laporta procedure

* Very large system → computational bottleneck
* Algebraic structure of FI not manifest

# Looking for other ways…

Wishlist:

* Allows for a direct decomposition
* Exploits the vector space structure obeyed by Feynman integrals



One option:

# Intersection theory

# Framework

Vector space of $n$-folds integrals in $\mathbf{z} = (z_1, \ldots, z_n)$

"right" integrals

$$|\varphi_R\rangle = \int dz_1 \ldots dz_n \frac{1}{u(\mathbf{z})} \varphi_R(\mathbf{z})$$

"dual" or "left" integrals

$$\langle \varphi_L| = \int dz_1 \ldots dz_n u(\mathbf{z}) \varphi_L(\mathbf{z})$$

with

- $\varphi_L / \varphi_R$ rational functions
- $u(\mathbf{z}) = \prod_j B(\mathbf{z})_j^{\gamma_j}$, $\begin{cases} \gamma_j \text{ generic exponents} \\ B_j \text{ polynomials} \end{cases}$

Intersection numbers:
scalar products between left and right integrals $\langle \varphi_L | \varphi_R \rangle$ Mastrolia, Mizera (2018)

Vector space:

- * Dimension $\nu$
- * Basis $|e_i^{(R)}\rangle$ and dual basis $\langle e_i^{(L)}|$
- * Scalar product: intersection number

# Change of representation

Baikov change of vars

$$k_j \to z_j$$

Baikov (1996)

$$I[\alpha_1, \ldots, \alpha_n] = \int \left( \prod_{i=1}^{\ell} d^d k_i \right) \prod_{j=1}^{n} \frac{1}{z_j^{\alpha_j}} \to \int dz_1 \ldots dz_n \, B^\gamma \prod_{j=1}^{n} \frac{1}{z_j^{\alpha_j}} = | \varphi_R \rangle$$

$$| \varphi_R \rangle = \int dz_1 \ldots dz_n \frac{1}{u(\mathbf{z})} \varphi_R(\mathbf{z})$$

with

$$u(\mathbf{z}) = B^{-\gamma} \prod z_j^{\rho_j}, \qquad \gamma = \frac{d - E - L - 1}{2}$$

$$\varphi_R(\mathbf{z}) = \frac{1}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}}$$

analytic regulators of $z_j \to 0$ singularities

# Identifications

- $|\varphi\rangle$ generic vector
  $\rightarrow$ Feynman integral to reduce
- $\{|e_i^{(R)}\rangle\}_{i=1}^{\nu}$ basis vectors
  $\rightarrow$ master integrals

decomposition of integrals as

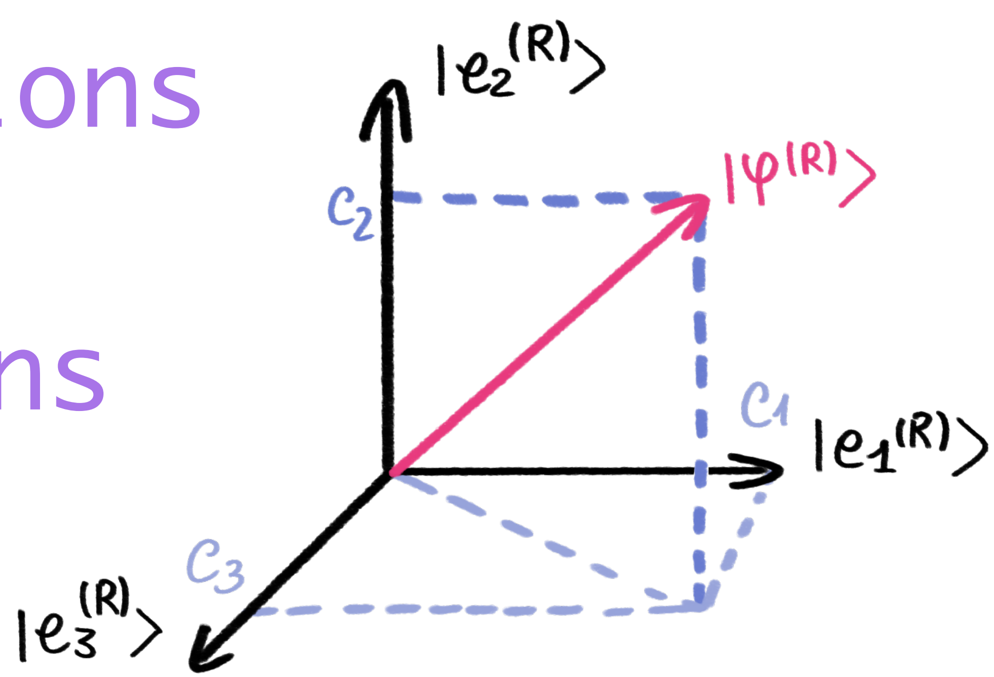
$$|\varphi_R\rangle = \sum_{i=1}^{\nu} c_i^{(R)} |e_i^{(R)}\rangle \qquad c_i^{(R)} = \sum_{j=1}^{\nu} (\mathbf{C}^{-1})_{ij} \langle e_j^{(L)} | \varphi_R \rangle$$

where we introduce the metric

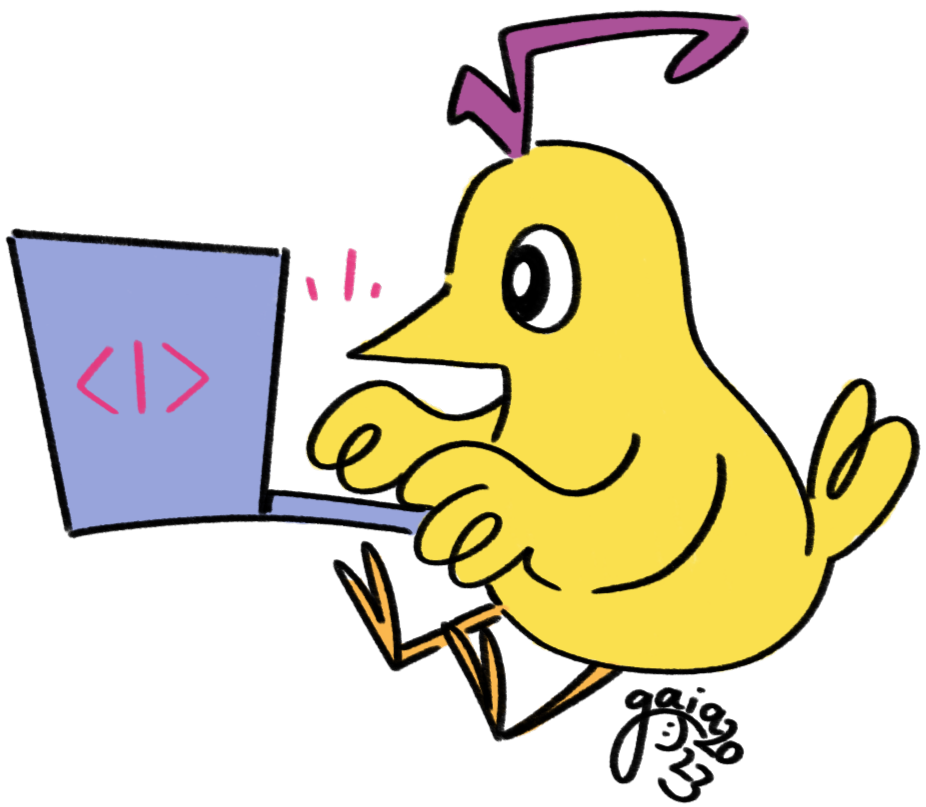
$$\mathbf{C}_{ij} = \langle e_i^{(L)} | e_j^{(R)} \rangle$$

- similar formulae for dual integrals

Decompositions = projections



# Computation of intersection numbers



# Univariate algorithm

Frellesvig et al. (2019)

We have 1-fold integrals in the variable $z$

$$|\varphi_R\rangle = \int dz \frac{1}{u(z)} \varphi_R(z)$$

- univariate intersection numbers

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathscr{P}_\omega} \text{Res}_{z=p}(\psi \, \varphi_R)$$

- $\psi$ is the local solution of

$$(\partial_z + \omega)\psi = \varphi_L, \quad \omega \equiv \frac{\partial_z u}{u}$$

- around each $p \in \mathscr{P}_\omega$

$$\mathscr{P}_\omega = \{z \,|\, z \text{ is a pole of } \omega\} \bigcup \{\infty\}$$

- ansatz around $p$

$$\psi = \sum_{i=min}^{max} c_i (z-p)^i + O\Big((z-p)^{max+1}\Big)$$

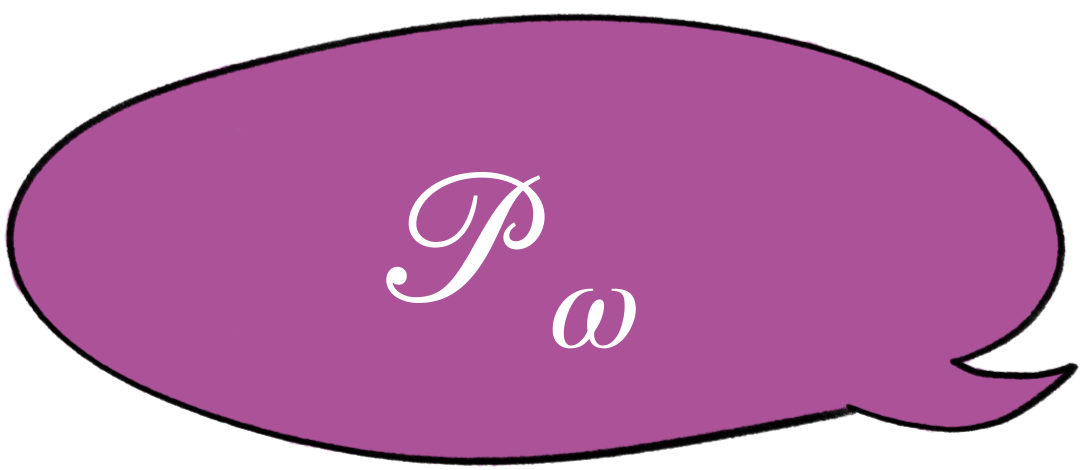
* plug in the DE
* solve for the $c_i$

$$\langle \varphi_L | \varphi_R \rangle = \text{rational}$$

BUT

* Non-rational contributions in intermediate stages
* Cancellations after sum over all residues

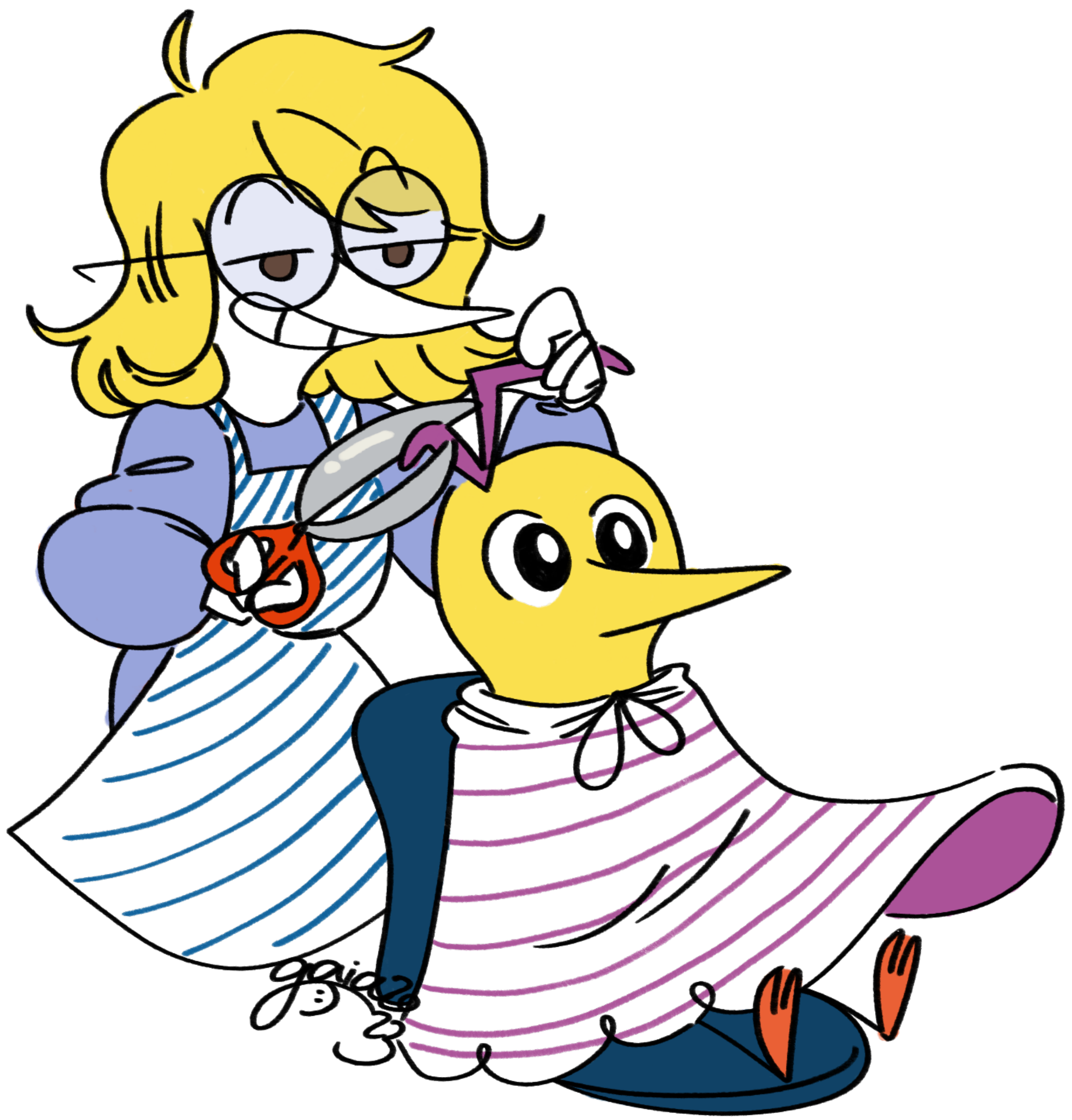
non-rational terms in the poles of $\omega$




* Computational bottleneck
* Non-suitable for applications with finite-fields

- similar for multivariate case

# $p(z)$–adic expansion



# $p(z)$-adic series expansion

Expansion around all the roots of polynomials $p(z)$ at once

$$f(z) = \sum_{i=min}^{max} c_i(z)\, p^i(z) + \mathcal{O}\left(p(z)^{max+1}\right)$$

rational function

(prime) polynomial over $\mathbb{Q}$

polynomial coefficients $c_i(z)$

$$c_i(z) = \sum_{j=0}^{deg\ p-1} c_{ij}\, z^j$$

Obtained via repeated polynomial divisions

* NO irrational operations
* NO knowledge of explicit location

# Example: univariate algorithm

## Before:

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathscr{P}_\omega} \mathrm{Res}_{z=p}(\psi \varphi_R)$$

summing over all $p \in \mathscr{P}_\omega$

$$\mathscr{P}_\omega = \{z \,|\, z \text{ is a pole of } \omega\} \bigcup \{\infty\}$$

# Example: univariate algorithm

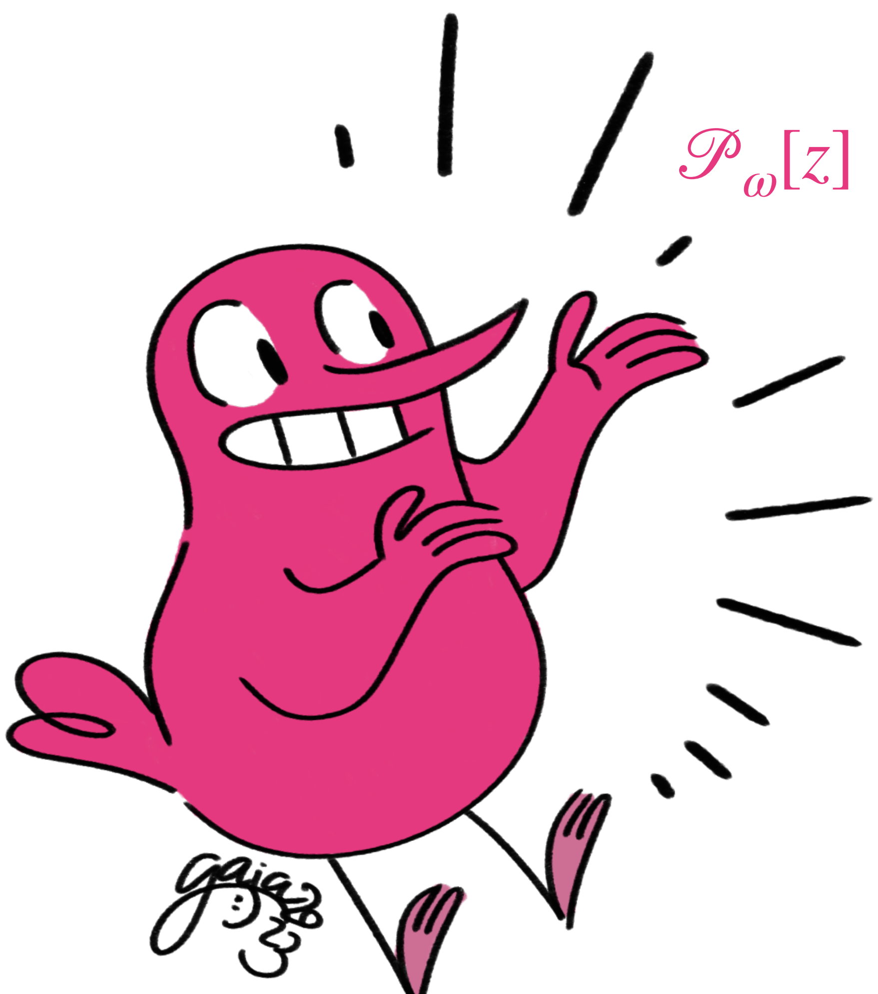
## Now:

GF, T. Peraro (2022)

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p(z) \in \mathscr{P}_\omega[z]} \langle \varphi_L | \varphi_R \rangle_{p(z)}$$

summing over all $p(z) \in \mathscr{P}_\omega[z]$

$\mathscr{P}_\omega[z] = \{$ factors of the denominator of $\omega\} \bigcup \{\infty\}$



# Example: univariate algorithm

GF, T. Peraro (2022)

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p(z)\in\mathcal{P}_\omega[z]} \langle \varphi_L | \varphi_R \rangle_{p(z)}$$

summing over all $p(z) \in \mathcal{P}_\omega[z]$

$\mathcal{P}_\omega[z] = \{$ factors of the denominator of $\omega\} \bigcup \{\infty\}$

* Each addend of the form $\langle \varphi_L | \varphi_R \rangle_{p(z)}$ is the sum of all contributions to the intersection number coming from the roots of $p(z)$
* $\langle \varphi_L | \varphi_R \rangle_\infty$ is computed as the contribution at $p = \infty$ with the "standard" algorithm

- similar for multivariate case

# In practice:

to solve $(\partial_z + \omega)\psi = \varphi_L$

- we make an ansatz of the form

$$\psi = \sum_{i=min}^{max} \sum_{j=0}^{degp-1} c_{ij} z^j p(z)^i + O\left(p(z)^{max+1}\right)$$

- we multiply the solution by $\varphi_R$

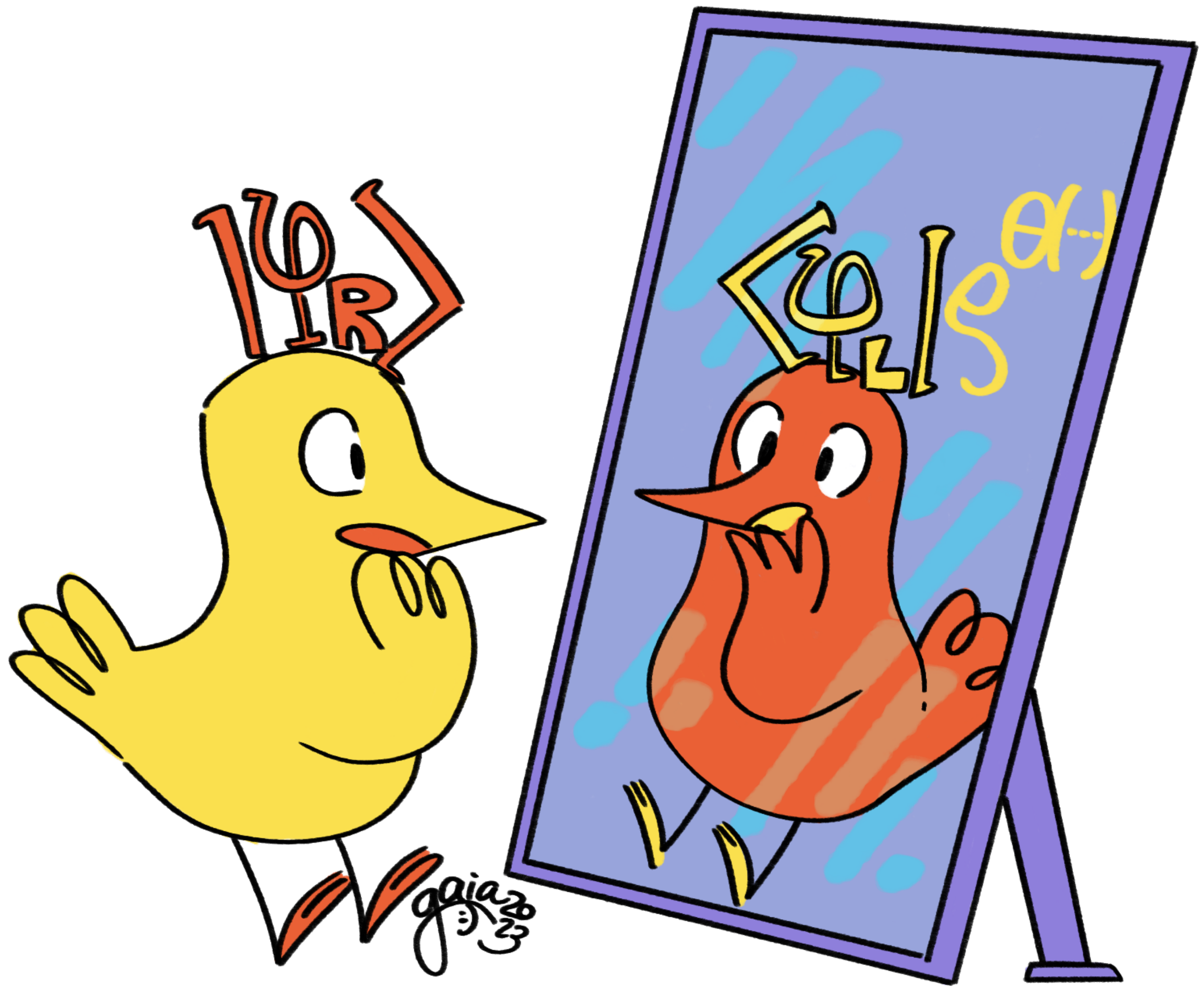
$$\psi\varphi_R = \sum_{i}^{-1} \sum_{j=0}^{degp-1} \tilde{c}_{ij} z^j p(z)^i + O\left(p(z)^0\right)$$

- by the univariate global residue theorem

Weinzierl (2021)

$$\langle \varphi_L | \varphi_R \rangle = \frac{\tilde{c}_{-1,degp-1}}{l_c}$$

# Dual integrals



# Analytic regulators: $\rho_j$

Are needed to regulate $\varphi_L \sim \frac{1}{z_j}$

$$u = B^{-\gamma} \prod_{j=1}^{n} z_j^{\rho_j}$$

* otherwise no solution in DE for $\psi$
* if $\varphi_L \sim 1/z^{\nu_j}$ $(\nu_j > 0)$ then $\psi \sim 1/\rho_j$

Then limit $\rho_j \to 0$ in the coefficients of the decomposition

BUT :(

* Additional variables in intermediate steps
* No block-triangular structure of decompositions
* More MIs in intermediate steps

# Dual Integrals…

Remember the right-integrals decomposition:

$$|\varphi_R\rangle = \sum_{i=1}^{\nu} c_i^{(R)} |e_i^{(R)}\rangle, \qquad c_i^{(R)} = \sum_{j=1}^{\nu} (\mathbf{C}^{-1})_{ij} \langle e_j^{(L)} | \varphi_R \rangle$$

Observation

Coefficients $c_i^{(R)}$ are independent of the choice of the dual basis $\{\langle e_j^{(L)}|\}_{j=1}^{\nu}$

⇒Idea

Exploit the freedom of choice of the dual basis to simplify the calculation

# …& how to choose them

Two approaches (different formalism but similar outcomes)

- ∗Dual space of loop integrals [Caron-Huot, Pokraka (2021)]
- ∗Simple choice [GF, T. Peraro (2023)]

Choose dual integrals of the form

$$\varphi_L(\mathbf{z}) = \rho_1^{\Theta(\alpha_1-\frac{1}{2})}\cdots\rho_n^{\Theta(\alpha_n-\frac{1}{2})}\frac{1}{z_1^{\alpha_1}\ldots z_n^{\alpha_n}}$$

∗If there's a denominator factor $z_j^{\alpha_j}$ (with $\alpha_j > 0$), multiply by $\rho_j$

∗Systematically work in the limit $\rho_j \to 0$ (only leading terms in a $\rho_j \to 0$ expansion in each step)

:)

* No dependance on $\rho_j$ in calculations (work on leading coeff.s of $\rho_j \to 0$ expansion, never sample or reconstruct $\rho_j$ dependence over FF )

* Simpler intermediate expressions

* Block triangular metric and reduction tables (blocks~sectors)

$$\mathbf{C} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & 0 & 0 & * & * \\ * & * & 0 & 0 & * & * \end{pmatrix}$$

top sector

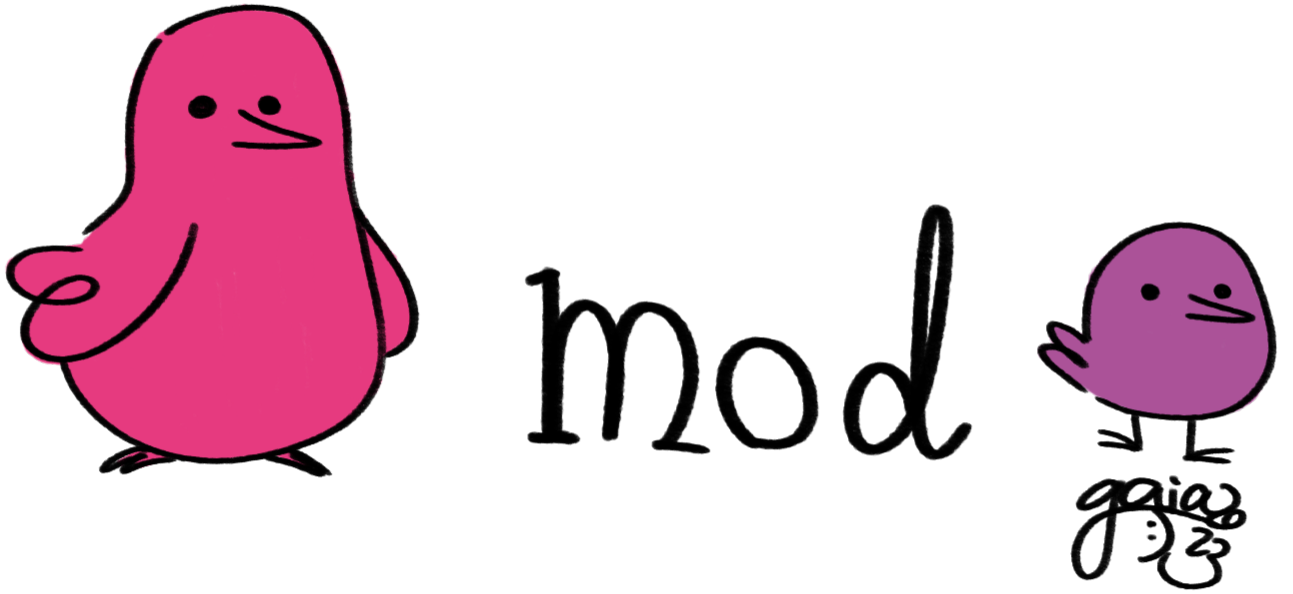
subsector 1

subsector 2

* Many intersection numbers and contributions of poles to them vanish

* Fewer MIs in intermediate steps!

# Finite fields implementation



# Implementation on FiniteFlow of the multivariate recursive rational algorithm

GF, Peraro (2023)

## Input

list of $n$-variate intersection numbers to compute $\{\langle e_j^{(L)} | \varphi_R \rangle, \langle e_j^{(L)} | e_i^{(R)} \rangle\}$

## Preliminary step

recursively deduce the intersection numbers needed for each step

* $\langle \varphi_L | e_j^{(R)} \rangle_{n-1}$
* $\left(\partial_{z_n} \langle e_j^{(L)} |_{n-1} \right) | e_j^{(R)} \rangle_{n-1}$
* $\langle e_i^{(L)} | e_j^{(R)} \rangle_{n-1}$
* $\langle e_j^{(L)} | \varphi_R \rangle$

## Univariate algorithm

analytic input: $u(\mathbf{z})$

## Multivariate algorithm

inputs

* denominator factors $p_i(z_n)$
* $(n-1)$-variate intersection numbers reconstructed in $z_n$ only

## Dealing with poles

* $p = 0, \infty \rightarrow$ Laurent expansion
* all other factors $\rightarrow p(z)$-adic expansion

our implementation is an iteration:

$1-$forms $\rightarrow$ $n-$forms

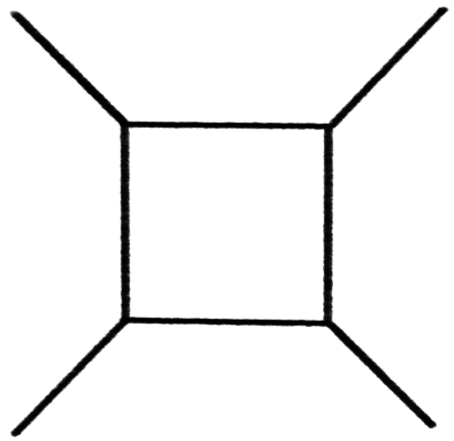
## Input for the $n^{th}$- step

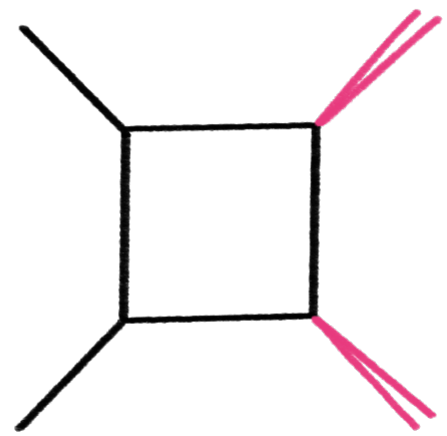
$\mathcal{X}_n =$ list of $(n-1)-$variate intersection numbers and reduction coeff.s

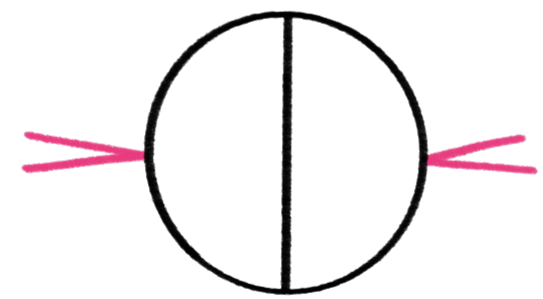
## Between two steps:

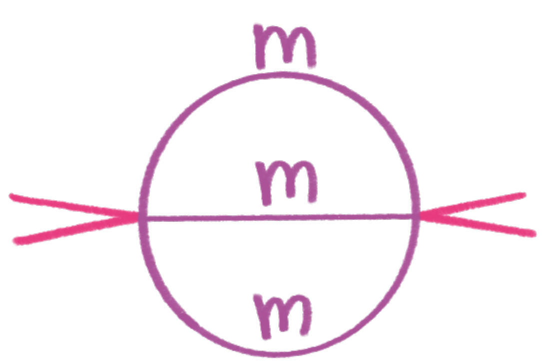
* rational reconstruction of $\mathcal{X}_n$ only in $z_n$, with everything else set to a number mod $p$
* identify denominator factors of $\mathcal{X}_n$ in $z_n$, fully reconstruct them from a simple subset of $\mathcal{X}_n$

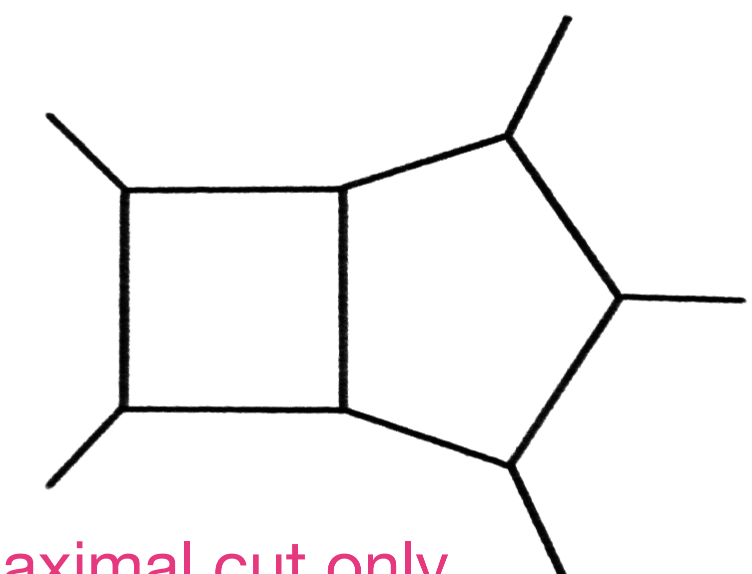
# Examples









maximal cut only

# Conclusions…

* Intersection theory: new mathematical structures, direct integral reduction
* $p(z)$-adic expansion: simplify study of functions close to roots of polynomials

# …& Outlook

* Simplifications/optimizations
* Application to different integral representations (loop-by-loop Baikov, Lee-Pomeransky)
* Non-recursive multivariate generalization (based on Chestnov, Frellesvig, Gasparotto, Mandal, Mastrolia (2022))
* New applications of $p(z)$-adic expansion

# Thank you for your attention!