Intersection Numbers, Polynomial Division and Relative Cohomology

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A collaboration with
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Master Integrals and IBPs

Modern multiloop calculations often require the computation of very large sets of integrals

Can be cut down by orders of magnitude by reducing to master integrals

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$\mathcal{O}(10^{4-5})$ integrals → **Integral Reduction** → $\mathcal{O}(10^{1-2})$ master integrals

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Integration by Parts Identities (IBPs)

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{i,\mu}} \left\{ v_\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} \right\} = 0$$

Requires solving a very large linear system

Systematised by Laporta's Algorithm

[Chetyrkin, Tkachov, 1981]

[Laporta, 2001]

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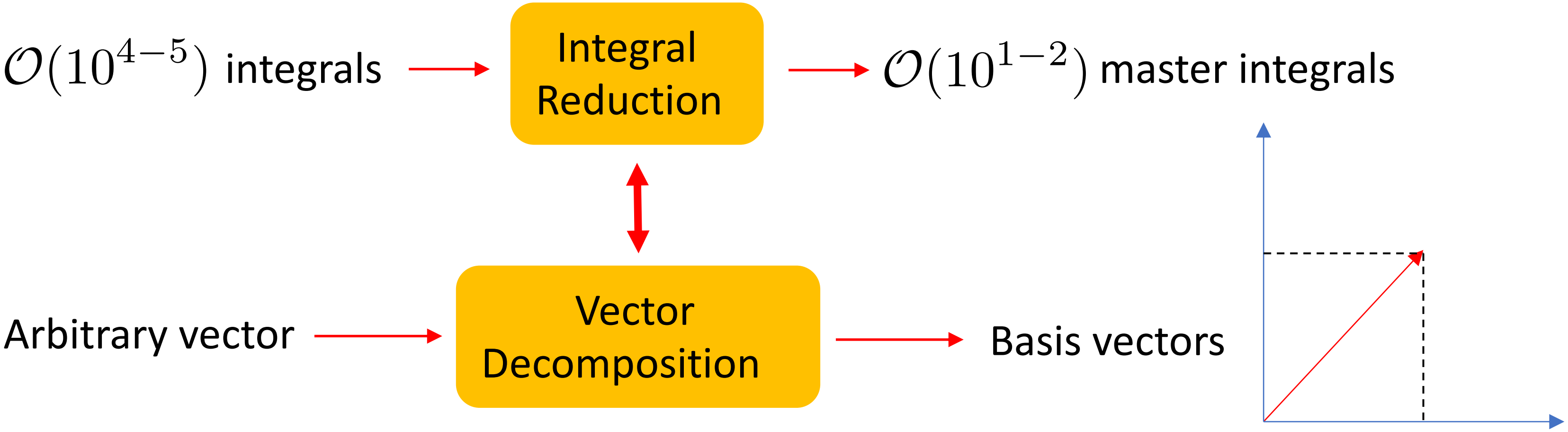

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Arbitrary vector → Vector Decomposition → Basis vectors

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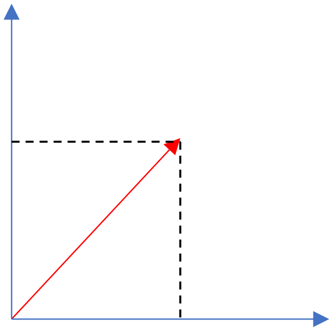

A given family of Feynman Integrals forms a (finite dimensional) vector space

Feynman Integrals

Master Feynman Integrals

$$J = \sum_{i=1}^{n} c_i I_i$$

IBP Coefficients

[Smirnov, Petukhov, 2011]
[Frellesvig, Gasparotto, Mandal, Mastrolia, Matiazzi, Mizera 2019]

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Vector Spaces do not need inner products to exist. But what if we endow our vector space with an inner product $\langle - | - \rangle$?

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Yes we can! [Mastrolia, Mizera, 2018]

To see how we must first switch our representation for Feynman Integrals

Twisted Feynman Integrals

Momentum Representation

$$I_{\alpha_1 \cdots \alpha_m} \sim \int \left(\prod_i d^d k_i \right) \frac{1}{D_1^{\alpha_1} \cdots D_m^{\alpha_m}}$$

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Baikov Representation

$$I_{\alpha_1\cdots\alpha_m} \sim \int \left(\prod_i dz_i\right) \frac{p(z_1,\cdots,z_m)^\gamma}{z_1^{\alpha_1}\dots z_m^{\alpha_m}}$$

[Baikov, 1996]

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Does not change for a given family

Is different for each Feynman Integral

Intersection Number

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For more than one variable the intersection number can be computed via a recursive formula, or via a direct higher order partial differential equation

[Chestnov, Frellesvig, Gasparotto, Mandal, Mastrolia, 2022]

Intersection Theory for Reducing Feynman Integrals in Padua

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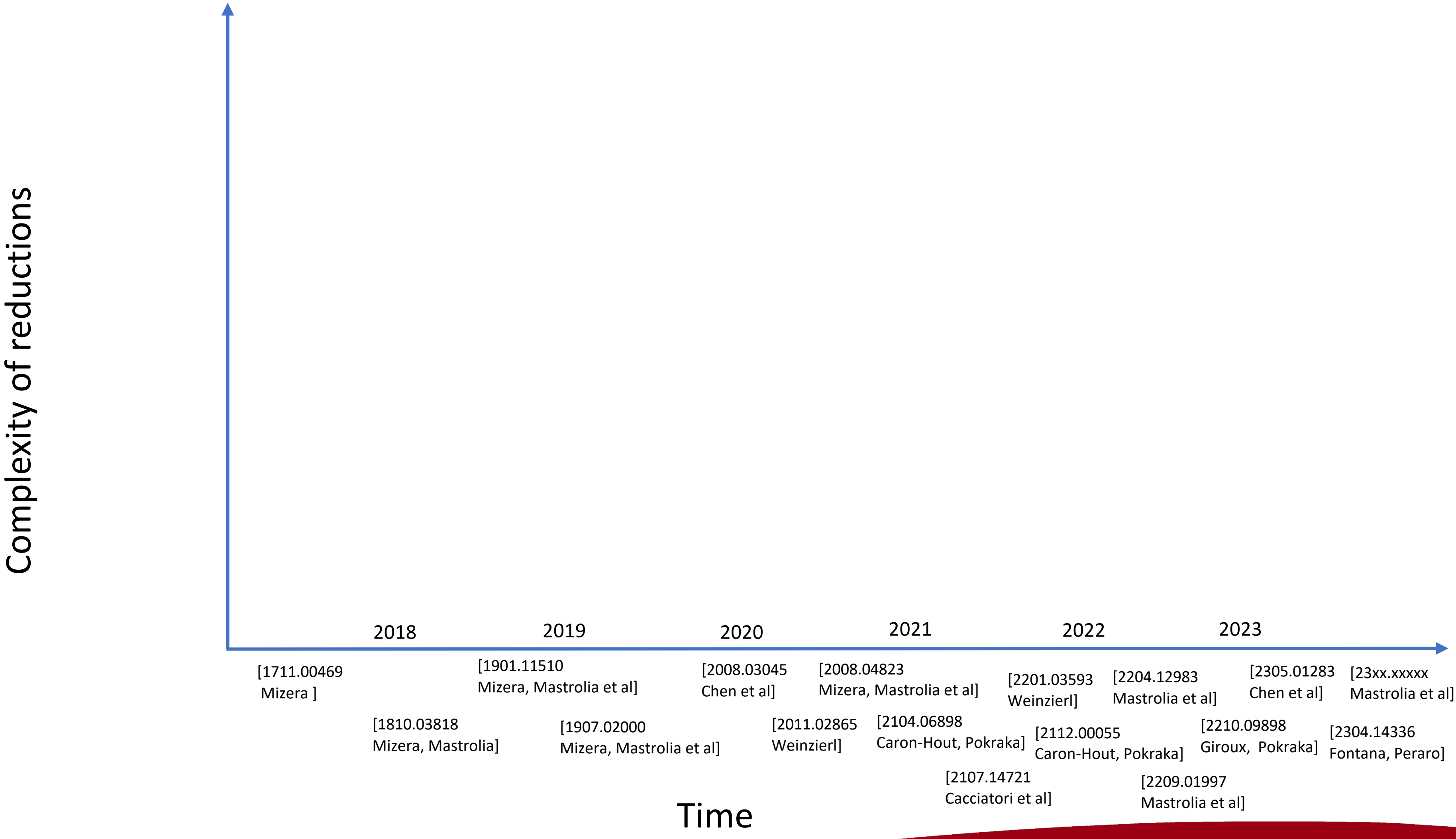

Complexity of reduction

Time

2018

2019

2020

2021

2022

2023

[1711.00469 Mizera]

[1810.03818 Mizera, Mastrolia]

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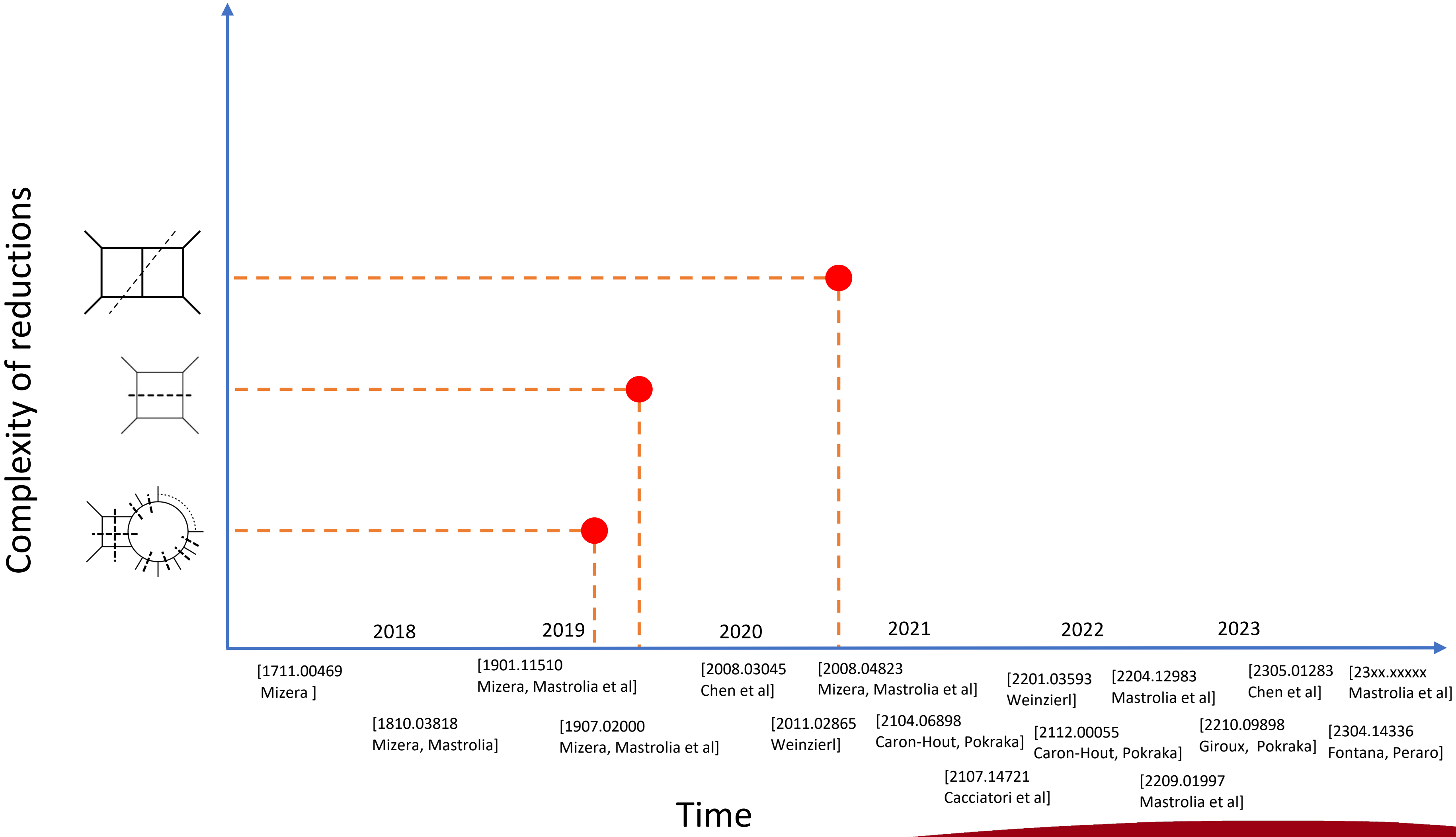

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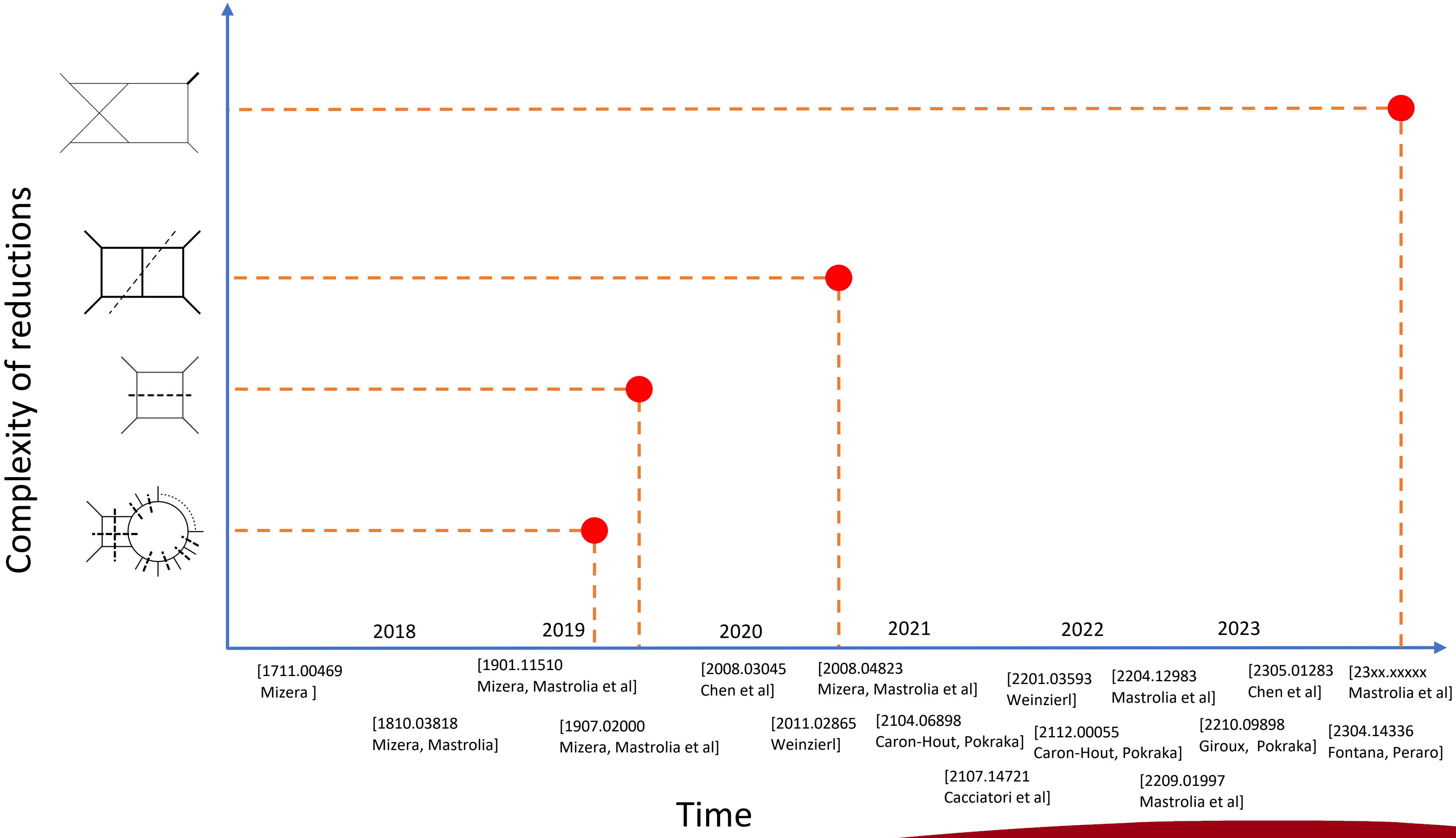

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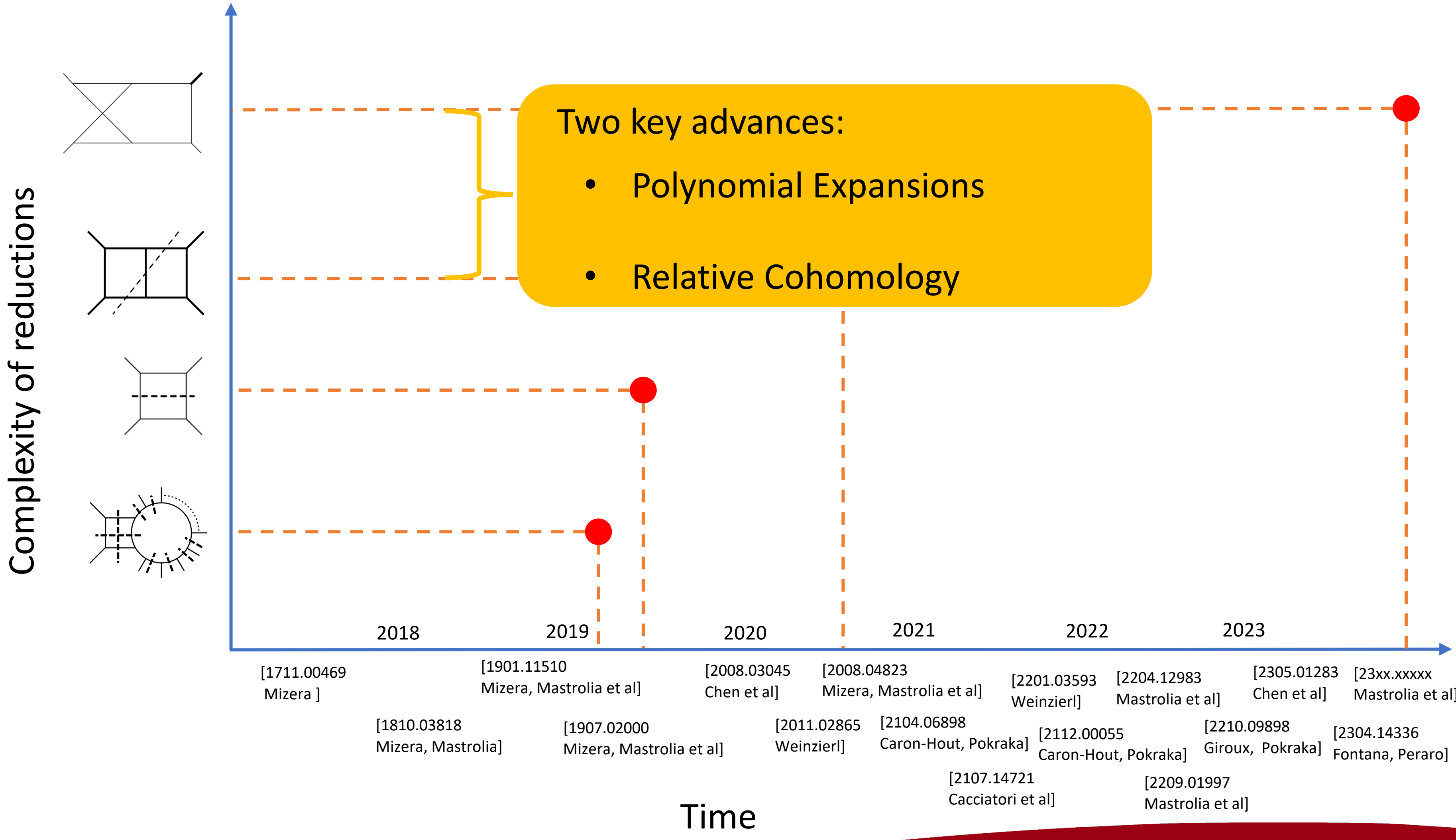

Two key advances:

- Polynomial Expansions
- Relative Cohomology

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ψ is usually solved by ansatz around each zero of p

$$\psi_p = \sum_{i=\min}^{\max} c_i \left(z - p \right)^i$$

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$$\psi_p = \sum_{i=\min}^{\max} c_i \left(z-p\right)^i \longrightarrow \psi = \sum_{i=\min}^{\max} c_i(z)\, b(z)^i \qquad c_i(z) = \sum_{j=0}^{\deg(b)-1} c_{ij} z^j$$

no longer a constant but another polynomial

[Fontana,Peraro,2023]

Roughly speaking the series will converge for when b is small

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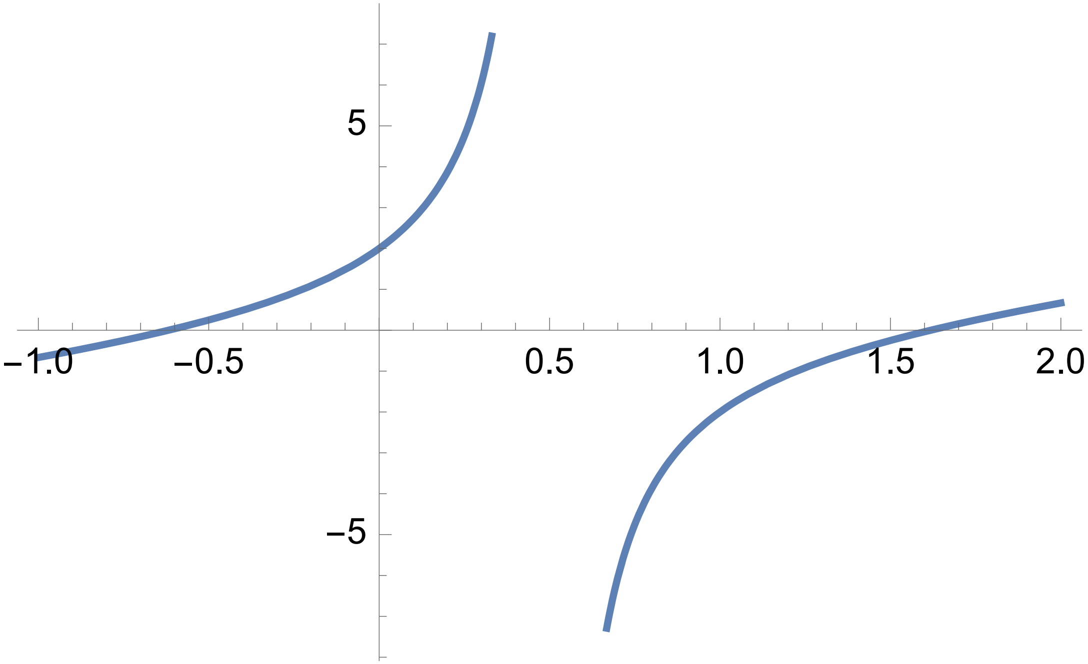
Allows us to construct series expansions around multiple points simultaneously!

Reduces to a “normal” series expansion when $b(z) = z$

Polynomial Expansion Example

$$f(z) = \frac{2(z-1)z-2}{2z-1}$$



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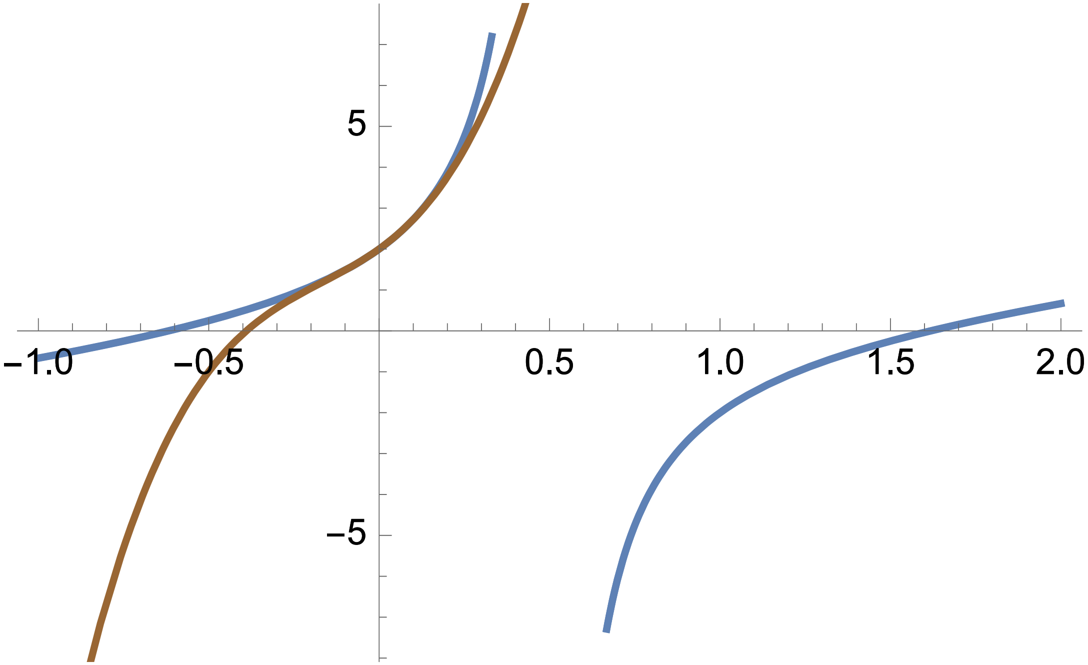

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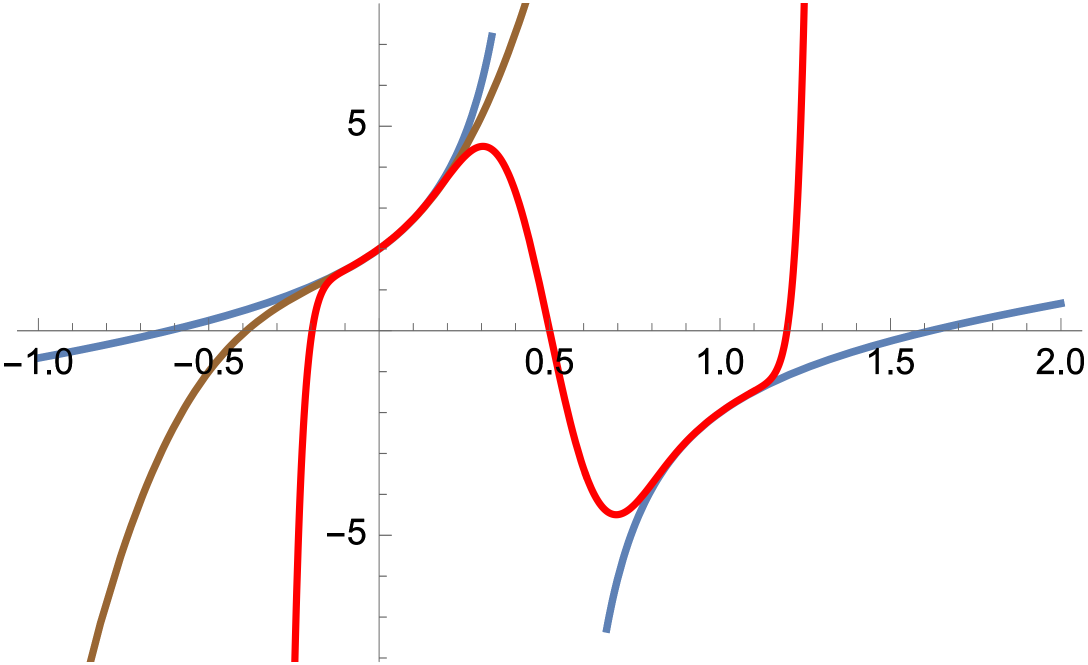

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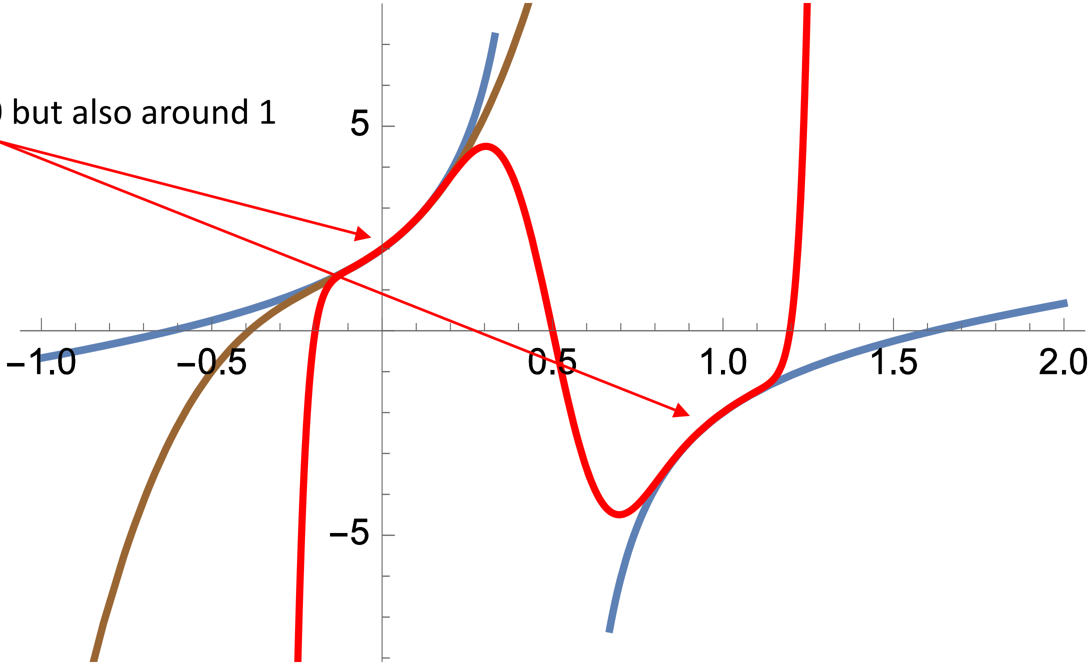

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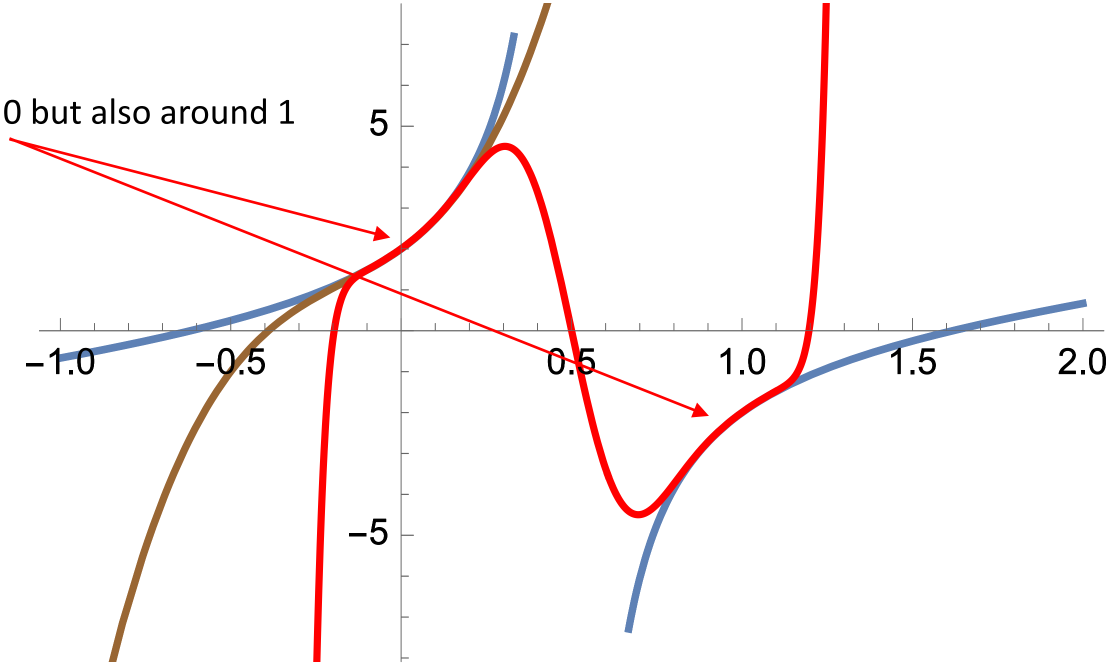

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Can use this method to solve the ansatz around all poles simultaneously and find the sum over the residues in one operation. Avoids intermediate irrational expressions.

See Gaia’s talk for many more details!

Relative Cohomology (1/2)

There is an important condition that the differential forms must satisfy for the intersection number computation to be well defined

$$I_{\alpha_1 \cdots \alpha_m} \sim \int u \, \varphi_{\alpha_1 \cdots \alpha_m} \qquad \varphi_{\alpha_1 \cdots \alpha_m} = \frac{dz_1 \wedge \cdots \wedge dz_m}{z_1^{\alpha_1} \cdots z_m^{\alpha_m}}$$

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In practice unfortunately this condition is rarely satisfied

One solution: $u \to u_\rho = z_1^\rho \cdots z_m^\rho u$ $\qquad \rho \to 0$ at the end of the calculation

Extra parameter weighs down the computation

Relative Cohomology (2/2)

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$$\delta_z(z) \equiv \frac{u(0)}{u(z)} d\theta_{z,0} \longrightarrow \langle \delta_z | \varphi_R \rangle = \mathrm{Res}_{z=0}\left(\varphi_R \frac{u(0)}{u(z)} \right)$$

[Matsumoto,2018]
[Caron-Huot,Pokraka,2021]
[Caron-Huot,Pokraka,2022]

Relative Cohomology (2/2)

Another solution: We work with elements of a relative twisted cohomology group instead of the "ordinary" cohomology group.

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[Matsumoto,2018]
[Caron-Huot,Pokraka,2021]
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Relative intersection numbers are easier to compute and do not need the introduction of a regulator ρ to the twist

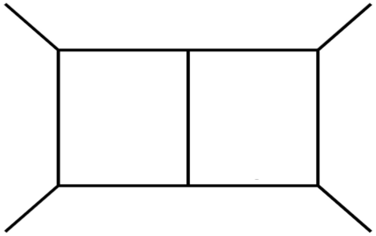
The C-matrix in this setup becomes block triangular

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_{1,1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{C}_{2,1} & \boldsymbol{C}_{2,2} & \ddots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_{n,1} & \boldsymbol{C}_{n,2} & \cdots & \boldsymbol{C}_{n,n} \end{bmatrix}$$

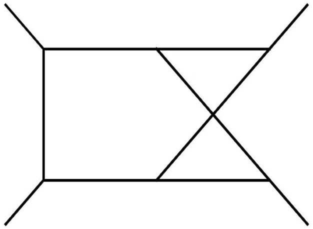
Many more details in Pokraka's talk?

Applications and Outlook

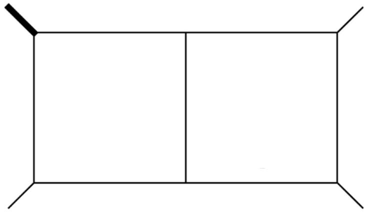
We have applied the techniques of polynomial expansion and relative cohomology to several examples presented in detail in our upcoming work



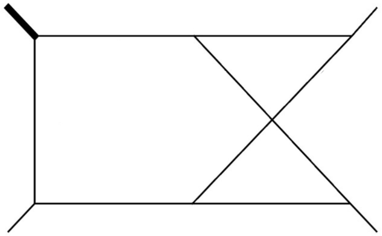
- 12 MIs
- 9 variables
- 3 triple cuts
- 3 quadruple cuts



- 16 MIs
- 9 variables
- 4 triple cuts
- 2 quadruple cuts



- 19 MIs
- 9 variables
- 4 triple cuts
- 3 quadruple cuts



- 24 MIs
- 9 variables
- 5 triple cuts
- 2 quadruple cuts

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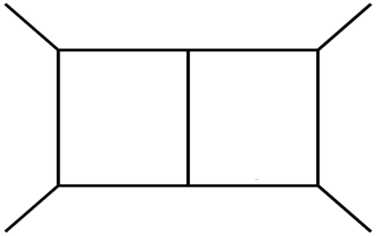

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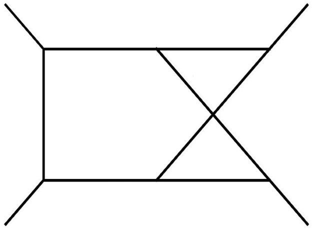

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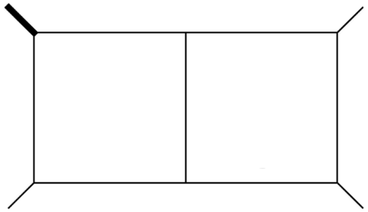

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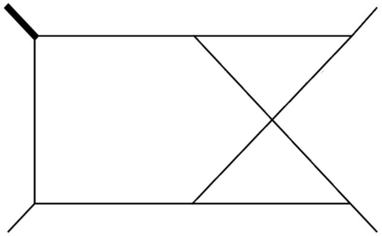

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Applications of Intersection Theory go beyond just Feynman Integrals

- String Theory/Double Copy [Mizera, 2017]
- Path Integrals [Cacciatori,Mastrolia, 2022]
- Cosmological Correlators [Shounak De,Pokraka, 2023]
- QFT Lattice computations [Gasparotto, Weinzierl, 2022]
- Post Minkowskian Integrals
- And much more!

Thank you for listening!

Backup

Relative Cohomology (3/2)

Can also view relative cohomology intersection numbers as the leading order contributions to regulated intersection numbers.

$$\nabla_u \psi \equiv \left(d + \frac{du}{u}\right) \psi = \varphi_L \xrightarrow{\text{Formal solution}} \psi_p(z) = \frac{1}{(e^{2\pi i \alpha_p} - 1)\, u_\rho(z)} \int_{C_p(z)} u_\rho(t)\, \varphi_L(t)$$

Monodromy factor around the pole p

Regulated twist

Unregulated twist

$$u_\rho(z) = z^\rho\, u(z)$$

$$\psi_0(z) = \frac{1}{(e^{2\pi i \rho} - 1)\, z^\rho\, u(z)} \int_{C_j(z)} t^\rho\, u(t)\, \varphi_L(t)\, \mathrm{d}t \;=\; \frac{1}{2\pi i \rho} \frac{1}{u(z)} \oint_\epsilon u(t)\, \varphi_L(t)\, \mathrm{d}t + \mathcal{O}(\rho^0)$$

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i \rho} \mathrm{Res}_{z=0} \left(\varphi_R \frac{1}{u(z)} \oint_\epsilon u(t)\, \varphi_L(t)\, \mathrm{d}t \right) + \mathcal{O}(\rho^0)$$

Same action as the $\delta_z(z)$ form

$$\langle 1/z | \varphi_R \rangle = \frac{1}{2\pi i \rho} \mathrm{Res}_{z=0} \left(\varphi_R \frac{1}{u(z)} \oint_\epsilon u(t)\, \frac{1}{t}\, \mathrm{d}t \right) + \mathcal{O}(\rho^0) = \frac{1}{\rho} \mathrm{Res}_{z=0} \left(\varphi_R \frac{u(0)}{u(z)} \right) + \mathcal{O}(\rho^0)$$

Will not contribute to the integral reduction coefficient