Tropical Feynman integration in the physical region

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MathemAmplitudes, 26 Sep. 2023

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

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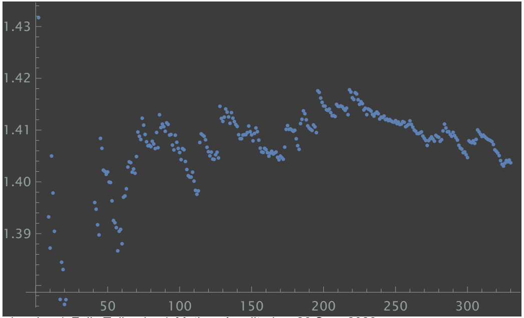

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This type of integrable boundary singularities are ubiquitous in Feynman integrals.

Tropical Approximation

The tropical approximation of a polynomial $p(\boldsymbol{x}) = \sum_{\alpha \in \text{supp}(p)} c_\alpha \boldsymbol{x}^\alpha$:

$$p^{\text{tr}}(\boldsymbol{x}) = \max_{\alpha \in \text{supp}(p)} \{\boldsymbol{x}^\alpha\}.$$

Example

$p(\boldsymbol{x}) = x_1 + x_2 + (5+2i)x_1x_2 - 3x_1^2 + \pi x_2^2$
$p^{\text{tr}}(\boldsymbol{x}) = \max\{x_1, x_2, x_1x_1, x_1^2, x_2^2\}$.

Theorem (Borinsky)

For a homogeneous polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ that is completely non-vanishing in $\mathbb{P}_+^n$ there exists constants $C_1, C_2 > 0$ s.t.

$$C_1 \leq \frac{|p(\boldsymbol{x})|}{p^{\text{tr}}(\boldsymbol{x})} \leq C_2 \quad \textit{for all } \boldsymbol{x} \in \mathbb{P}_+^n$$

$$\mathcal{I} = \int_{\mathbb{P}^n_+} p(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\mathbb{P}^n_+} \frac{p(\boldsymbol{x})}{p^{\text{tr}}(\boldsymbol{x})} p^{\text{tr}}(\boldsymbol{x}) d\boldsymbol{x}$$

Q: Can we efficiently sample from $p^{\text{tr}}(\boldsymbol{x})d\boldsymbol{x}$?
A: Yes, if the Newton polytope of $p(\boldsymbol{x})$ is a generalized permutohedra.

First, what is a Feynman integral and how do we deal with $i\varepsilon$?

The Feynman Integral

- Feynman graph $G := (V, E)$ is 1PI (and 1VI).
- Loop number $L := |E| - |V| + 1$
- Every edge $e \in E$ is assigned a direction:

$$\mathcal{E}_{ve} := \begin{cases} 1 & e \text{ ends at } v, \\ -1 & e \text{ starts at } v, \end{cases} \qquad 0 \text{ otherwise.}$$

- $V := V_{\text{ext}} \sqcup V_{\text{int}}$, all $v \in V_{\text{ext}}$ is assigned external incoming momenta $p_v \in \mathbb{R}^{1,D_0-1}$ and $p_v = 0$ for all $v \in V_{\text{int}}$.

Feynman integral:

$$\mathcal{I} = \lim_{\varepsilon \to 0^+} \int \prod_{e \in E} \frac{d^D q_e}{i\pi^{D/2}} \left(\frac{-1}{q_e^2 - m_e^2 + i\varepsilon} \right)^{\nu_e} \prod_{v \in V \setminus \{v_0\}} i\pi^{D/2} \cdot \delta^{(D)} \left(p_v + \sum_{e \in E} \mathcal{E}_{ve} q_e \right)$$

$D := D_0 - 2\epsilon$, q_e total momentum and m_e the mass of edge $e \in E$.

Introducing Schwinger parameters:

$$\left(\frac{i}{q_e^2 - m_e^2 + i\varepsilon}\right)^{\nu_e} = \frac{1}{\Gamma(\nu_e)} \int_0^\infty \frac{dx_e}{x_e}\, x_e^{\nu_e} \exp\left[i x_e (q_e^2 - m_e^2 + i\varepsilon)\right]$$

$$(SP):\ \mathcal{I} = (i)^\omega \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}_+^E} \prod_{e\in E} \left(\frac{x^{\nu_e} dx_e}{\Gamma(\nu_e) x_e}\right) \mathcal{U}^{-D/2} \exp\left[i\left(-\mathcal{V} + i\varepsilon \sum_{e\in E} x_e\right)\right]$$

$$(FP):\ \mathcal{I} = \lim_{\varepsilon\to 0^+} \Gamma(\omega) \int_{\mathbb{R}_+^E} \prod_{e\in E} \left(\frac{x^{\nu_e} dx_e}{\Gamma(\nu_e) x_e}\right) \mathcal{U}^{-D/2} \frac{\delta(1 - x_1 - \cdots - x_E)}{\left(\mathcal{V} - i\varepsilon \sum_{e\in E} x_e\right)^\omega}$$

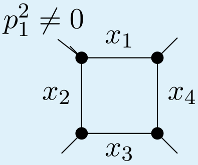
with the superficial degree of divergence: $\omega := \sum_{e\in E} \nu_e - LD/2$ and $\mathcal{V} = \mathcal{F}/\mathcal{U}$ with homogeneous graph/Symanzik polynomials

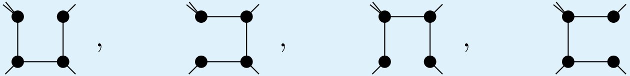
$$\mathcal{U} = \sum_{T \text{ a spanning tree of } G} \prod_{e\notin T} x_e, \qquad \deg(\mathcal{U}) = L$$

$$\mathcal{F} = \mathcal{F}_m + \mathcal{F}_0 = \mathcal{U} \sum_{e\in E} m_e^2 x_e - \sum_{F \text{ a spanning } 2\text{–forest of } G} p(F)^2 \prod_{e\notin F} x_e, \qquad \deg(\mathcal{F}) = L + 1$$

Example

First Symanzik polynomial: $\mathcal{U} = \sum_{\text{spanning trees } T} \prod_{e \notin T} x_e$

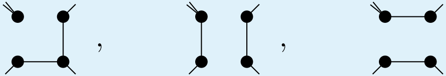




$$\mathcal{U} = x_1 + x_2 + x_3 + x_4$$

Second Symanzik polynomial:

$$\mathcal{F} = -\sum_{\text{spanning two-forests } F} p(F)^2 \prod_{e \notin F} x_e + \mathcal{U} \sum_{e=1}^{|E|} m_e^2 x_e$$



$$\mathcal{F} = -p_1^2 x_1 x_2 - (p_1 + p_2)^2 x_1 x_3 - (p_2 + p_3)^2 x_2 x_4$$

In practice:

Spanning trees and two-forests suffer from combinatorial blowup.

Express $\mathcal{U}$ and $\mathcal{F}$ in terms of matrices and use fast linear algebra to evaluate the matrices.

For $u, v \in V \setminus \{v_0\}$:

Reduced Laplacian :
$$\mathcal{L}(\boldsymbol{x})_{u,v} = \sum_{e \in E} \frac{\mathcal{E}_{u,e}\mathcal{E}_{v,e}}{x_e} = \begin{cases} -1/x_e, & e \text{ connects } u, v \\ \sum_{e \to v} 1/x_e, & u = v \\ 0, & \text{otherwise} \end{cases}$$

Gram matrix :
$$\mathcal{P}_{u,v} = p_u \cdot p_v,$$

$$\mathcal{U}(\boldsymbol{x}) = \det(\mathcal{L}(\boldsymbol{x})) \cdot \left(\prod_{e \in E} x_e \right),$$

$$\mathcal{F}(\boldsymbol{x}) = \mathcal{U} \left(- \sum_{u,v \in V \setminus \{v_0\}} \mathcal{P}^{u,v} \mathcal{L}^{-1}(\boldsymbol{x})_{u,v} + \sum_{e \in E} m_e^2 x_e \right)$$

Projective integrals:

$$(FP):\ \mathcal{I} = \lim_{\varepsilon\to 0^+} \Gamma(\omega) \int_{\mathbb{R}_+^E} \prod_{e\in E} \left(\frac{x^{\nu_e} dx_e}{\Gamma(\nu_e) x_e} \right) \mathcal{U}^{-D/2} \frac{\delta(1 - H(\boldsymbol{x}))}{\left(\mathcal{V} - i\varepsilon \sum_{e\in E} x_e\right)^{\omega}}$$

with $H(\boldsymbol{x}) : \mathbb{R}^{|E|} \to \mathbb{R}_+$ homogeneous of degree 1.
The role of $\delta(1 - H(\boldsymbol{x}))$ is to pick an affine chart of a projective integral.

Projective space:

$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\sim}, \quad (x_0, \cdots, x_n) \sim (y_0 : \cdots : y_n) \Leftrightarrow (x_0, \ldots, x_n) = \lambda(y_0, \ldots, y_n),\ \lambda \neq 0.$$

$\mathcal{I}$ is lifted to an integral over the positive projective simplex
$\mathbb{P}_+^E = \{\boldsymbol{x} = [x_1 : \cdots : x_{|E|}] \in \mathbb{RP}^{|E|-1} | x_e > 0\}$

$$\mathcal{I} = \Gamma(\omega) \int_{\mathbb{P}_+^E} \phi \quad \text{with} \quad \phi = \left(\prod_{e\in E} \frac{x_e^{\nu_e}}{\Gamma(\nu_e)} \right) \frac{1}{\mathcal{U}(\boldsymbol{x})^{D/2}} \left(\frac{1}{\mathcal{V}(\boldsymbol{x}) - i\varepsilon \sum_{e\in E} x_e} \right)^{\omega} \Omega$$

and $\Omega = \sum_{e=1}^{|E|} (-1)^{|E|-e} \frac{dx_1}{x_1} \wedge \cdots \wedge \widehat{\frac{dx_e}{x_e}} \wedge \cdots \wedge \frac{dx_{|E|}}{x_{|E|}}$.

Contour Deformation

Why $i\varepsilon$?
Chooses the causal branch and ensures the convergence.

Why <u>not</u> $i\varepsilon$?

- Modifies the analytic structure by displacing branch points and introducing spurious branch cuts.
- Numerics is hard, as $\varepsilon \to 0$ poles can get arbitrarily close to the integration contour.

$$(SP):\ \mathcal{I} = (i)^{\omega} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}_+^E} \prod_{e\in E} \left(\frac{x^{\nu_e} dx_e}{\Gamma(\nu_e) x_e} \right) \mathcal{U}^{-D/2} \exp\left[i \left(-\mathcal{V} + i\varepsilon \sum_{e\in E} x_e \right) \right]$$

$i\varepsilon$ really only dampens the integral for large x_e. For convergence we also need $\mathrm{Im}(-\mathcal{V}) > 0$ which identifies the causal branch.

Introduce a phase to the Schwinger parameters

$$X_e = x_e \exp\left(-i\lambda \frac{\partial \mathcal{V}}{\partial x_e}\right)$$

Taylor expand

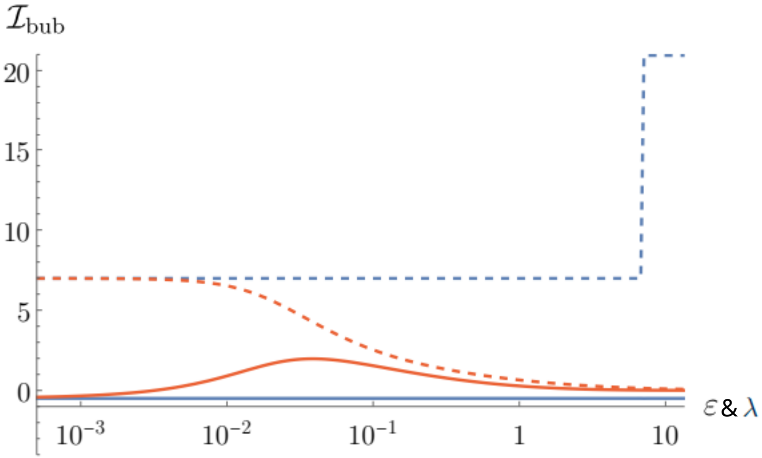
$$-\mathcal{V}(\boldsymbol{X}) = -\mathcal{V}(\boldsymbol{x}) + i\lambda \sum_{e \in E} x_e \left(\frac{\partial \mathcal{V}}{\partial x_e}\right)^2 + \mathcal{O}(\lambda^2)$$

Then $\text{Im}(-\mathcal{V}(\boldsymbol{X})) \geq 0$ for sufficiently small λ as long as

$$x_e \frac{\partial \mathcal{V}}{\partial x_e} \neq 0 \quad \forall e \in E$$

i.e. the Landau equations have no solutions.

Comparison with direct numerics on the Feynman parameterization with $i\varepsilon$ and with deformation:



[Hannesdottir,Mizera], for too large λ we get a jump.

In general: Embedding $\iota_\lambda : \mathbb{P}^E_+ \hookrightarrow \mathbb{CP}^{|E|-1}$:

$$\iota_\lambda : x_e \mapsto x_e \exp\left(-i\lambda \frac{\partial \mathcal{V}}{\partial x_e}(\boldsymbol{x})\right)$$

Boundary is preserved since it is characterized by $x_e = 0$.

$$\mathcal{I} = \Gamma(\omega) \int_{\iota_\lambda\left(\mathbb{P}^E_+\right)} \phi = \Gamma(\omega) \int_{\mathbb{P}^E_+} \iota^*_\lambda \phi$$

where $\iota^*_\lambda \phi$ is the pull-back.
We obtain $\iota^*_\lambda \Omega = \det(\mathcal{J}_\lambda(\boldsymbol{x}))\Omega$ with

$$\mathcal{J}_\lambda(\boldsymbol{x})^{e,h} = \delta_{e,h} - i\lambda x_e \frac{\partial^2 \mathcal{V}}{\partial x_e \partial x_h}(\boldsymbol{x}) \text{ for all } e, h \in E$$

Deforemed parametric Feynman integral:

$$\mathcal{I} = \Gamma(\omega) \int_{\mathbb{P}^E_+} \iota^*_\lambda \phi = \Gamma(\omega) \int_{\mathbb{P}^E_+} \left(\prod_{e\in E} \frac{X_e^{\nu_e}}{\Gamma(\nu_e)}\right) \frac{\det \mathcal{J}_\lambda(\boldsymbol{x})}{\mathcal{U}\left(\boldsymbol{X}\right)^{D/2} \cdot \mathcal{V}\left(\boldsymbol{X}\right)^{\omega}} \Omega$$

where $\boldsymbol{X} = \iota_\lambda(\boldsymbol{x})$.

Expanding in ϵ:

Assuming that the only potential divergence comes from $\Gamma(\omega)$ we have:

$$\mathcal{I} = \Gamma(\omega_0 + \epsilon L) \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \int_{\mathbb{P}_+^E} \left(\prod_{e \in E} \frac{X_e^{\nu_e}}{\Gamma(\nu_e)} \right) \frac{\det \mathcal{J}_\lambda(\boldsymbol{x})}{\mathcal{U}(\boldsymbol{X})^{D_0/2} \cdot \mathcal{V}(\boldsymbol{X})^{\omega_0}} \log^k \left(\frac{\mathcal{U}(\boldsymbol{X})}{\mathcal{V}(\boldsymbol{X})^L} \right) \Omega$$

where $\omega_0 = \sum_{e \in E} \nu_e - D_0 L/2$.

Writing the integral with these fractions in the integrand:

$$\mathcal{I} = \frac{\Gamma(\omega_0 + \epsilon L)}{\prod_{e \in E} \Gamma(\nu_e)} \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \mathcal{I}_k$$

with

$$\mathcal{I}_k = I^{\text{tr}} \int_{\mathbb{P}_+^E} \frac{\left(\prod_{e \in E} (X_e/x_e)^{\nu_e}\right) \det \mathcal{J}_\lambda(\boldsymbol{x})}{\left(\mathcal{U}(\boldsymbol{X}) / \mathcal{U}^{\text{tr}}(\boldsymbol{x})\right)^{D_0/2} \cdot \left(\mathcal{V}(\boldsymbol{X}) / \mathcal{V}^{\text{tr}}(\boldsymbol{x})\right)^{\omega_0}} \log^k \left(\frac{\mathcal{U}(\boldsymbol{X})}{\mathcal{V}(\boldsymbol{X})^L} \right) \mu^{\text{tr}}$$

and

$$\mu^{\text{tr}} = \frac{1}{I^{\text{tr}}} \frac{\prod_{e \in E} x_e^{\nu_e}}{\mathcal{U}^{\text{tr}}(\boldsymbol{x})^{D_0/2} \, \mathcal{V}^{\text{tr}}(\boldsymbol{x})^{\omega_0}} \Omega, \qquad \int_{\mathbb{P}_+^E} \mu^{\text{tr}} = 1.$$

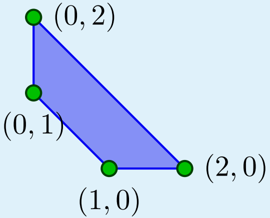
(Newton) polytopes

A polytope $P \subset \mathbb{R}^n$ is the convex hull of a finite set of points $\iff$ compact solution to a linear system of equalities and inequalities.

Example

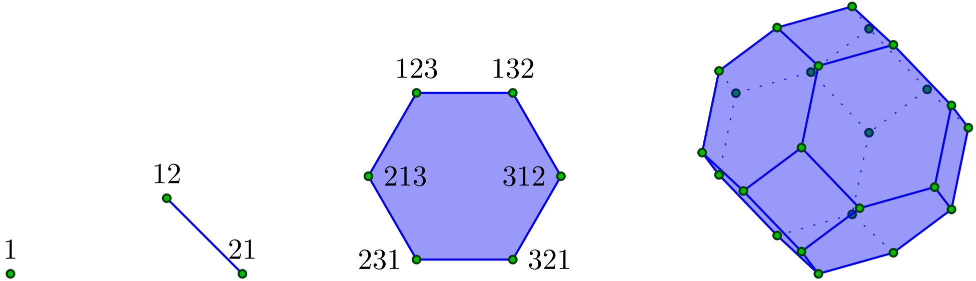
$$p(\boldsymbol{x}) = x_1 + x_2 + (5+2i)x_1x_2 - 3x_1^2 + \pi x_2^2$$

$$\mathbf{N}[p] = \text{conv}\begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix}, \quad \mathbf{N}[p] = \text{All } (t_1, t_2) \in \mathbb{R}^2 \ s.t. \begin{cases} t_1 \geq 0, & t_1 + t_2 \geq 1 \\ t_2 \geq 0, & -t_1 - t_2 \geq -2 \end{cases}$$



Permutohedra

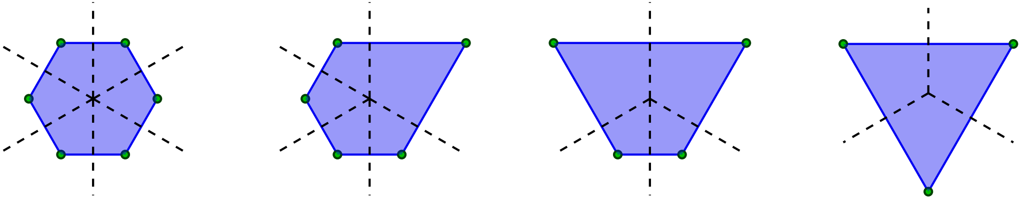
The permutohedron is a polytopal model of permutations of a finite set:



Note: The difference between two connected vertices contains two non-zero elements: 1 and -1, with the rest being zero.

Generalized Permutohedra

Polytope whose normal fan is a coarsening of a permutohedron's normal fan



Theorem (   )

A polytope $P \subset \mathbb{R}^n$ is a GP $\Longleftrightarrow$ every edge is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for $i \neq j$.

Example

Every matroid polytope is a GP, especially $\mathbf{N}[\mathcal{U}] := \mathrm{Newt}(\mathcal{U})$ is a GP.

For calculations the following classification is more convenient:

Theorem (Postnikov)

A polytope $P \subset \mathbb{R}^n$ is a GP $\iff$ it can be written as

$$P_n(\{z(I)\}) = \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} t_i = z([n]), \ \sum_{i \in I} t_i \geq z(I) \ \forall I \subseteq [n] \right\}$$

where $z(I)$ is a real number for each $I \subseteq [n] := \{1, \ldots, n\}$, $z(\emptyset) = 0$ and z is supermodular:

$$z(I) + z(J) \leq z(I \cup J) + z(I \cap J) \ \forall I, J \subseteq [n].$$

$G = (V, E)$, identify a subgraph with its edge set $\gamma \subset E$, we have $\mathbf{N}[\mathcal{U}] = P_{|E|}(z)$ for

$$z(\gamma) = L_\gamma$$

When is $\mathbf{N}[\mathcal{F}]$ a GP?

Highly non-trivial, the full answer is not known.
Following sufficient conditions are known/conjectured:

- All internal masses are non-zero, i.e. $m_e \neq 0$ for all $e \in E$.
- All internal edges are massless, $m_e = 0$ for all $e \in E$, every vertex is external, i.e. $V = V_{\text{ext}}$ and $p(V')^2 \neq 0$ for all $V' \subset V$.
- All internal edges are massless, $m_e = 0$ for all $e \in E$ and $p(V')^2 \neq 0$ for all $V' \subset V_{\text{ext}}$.

If

$$\operatorname{supp}\left(\sum_{\substack{F \text{ a spanning} \\ 2-\text{forest of } G}} p(F)^2 \prod_{e \notin F} x_e \right) \subset \operatorname{supp}\left(\mathcal{U} \sum_{e \in E} m_e^2 x_e \right)$$

and no cancellation between them occur, then $\mathbf{N}[\mathcal{F}]$ is a GP.

When every internal vertex is connected to an external vertex with a massive path.

When is $\mathbf{N}[\mathcal{F}]$ not a GP?

To the best of my knowledge, no necessary condition for $\mathbf{N}[\mathcal{F}]$ to be GP is known.

Example

Fully massless box (only s, t as kinematic variables):

$$\mathbf{N}[\mathcal{F}] = \text{conv}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}^T, \quad edge = (1, -1, 1, -1).$$

Similarly one-mass box and two-mass hard.

More guessing:

- Fully massless one-loop graphs are not GP. (Very well-tested guess.)
- Every fully massless is not a GP.

A related unrelated question:

How many solutions does a GKZ system have?

- GKZ: for generic β, $\text{rank}(H_A(\beta)) = \text{vol}(\text{conv}(\mathcal{G}))$
- $$\text{rank}(H_A(\beta)) = \text{vol}(\text{conv}(\mathcal{G}))\ \forall\, \beta \iff I_A \text{ is Cohen} - \text{Macaulay}$$
- If A is normal then I_A is Cohen-Macaulay [Hochster]

$$\mathbb{N}A = \mathbb{Z}A \cap \mathbb{R}_+ A \ \Rightarrow\ I_A \text{ is Cohen} - \text{Macaulay.}$$

Normality was of Feynman integrals have been analysed in [Helmer, FT], [Walther] and [Dlapa,Helmer,Papathanasiou, FT].

In some cases, polytopal properties like GP implies Cohen-Macaulayness.

Matroid Polytope

$\Downarrow$

Generalized Permutohedra (GP) $\Longrightarrow$ Edge Unimodular

$\Downarrow$

Initial Ideal of $I_{P \cap \mathbb{Z}^n}$ is Radical $\Longleftrightarrow$ P has a Regular Unimodular Triangulation

$\Downarrow$

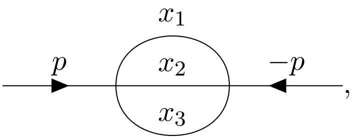
Normality $\Longleftrightarrow$ (if and only if $A_- = P \cap \mathbb{Z}^n$) IDP

$\Downarrow$

Cohen-Macaulay $\Longleftrightarrow$ $\mathrm{rank}(H_A(\beta)) = \mathrm{vol}(\mathrm{Newt}(\mathcal{G})) \ \forall \beta$

Factorization: feature or bug?

Yesterday (talk by Claudia and Simon), we saw that



$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix} \Rightarrow \mathcal{G}|_\Gamma = (x_1 + x_2)(1 + m_3^2 x_3)x_3 \Rightarrow \Delta_{A\cap\Gamma}(\mathcal{G}) = 0$$

$$\mathcal{G}|_\Gamma = \mathcal{U}_\gamma(\mathcal{U}_{G/\gamma} + \mathcal{F}_{G/\gamma}), \qquad \gamma = \{1, 2\}$$

Lemma

$\gamma \subset E,\ \sigma_e = 1$ *if* $e \in \gamma$, *0 otherwise. For* $e \in \gamma,\ x_e \to \rho x_e$, *then*

$$\mathcal{U}_G \to \rho^{L_\gamma}(\mathcal{U}_\gamma \mathcal{U}_{G/\gamma} + \mathcal{O}(\rho^{>0}))$$
$$\mathcal{F}_G \to \rho^{L_\gamma}(\mathcal{U}_\gamma \mathcal{F}_{G/\gamma} + \mathcal{O}(\rho^{>0}))$$

If γ *contains all kinematics*

$$\mathcal{F}_G \to \rho^{L_\gamma+1}(\mathcal{F}_\gamma \mathcal{U}_{G/\gamma} + \mathcal{O}(\rho^{>0}))$$

For generalized permutohedra $P \subset \mathbb{R}^{|E|}$ and $\gamma \subset E$:

$$P|_{\sigma_\gamma} = P|_\gamma \times P/\gamma$$

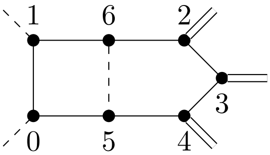
GP factorizes on their faces.

The program feyntrop

Tabletop evaluation of quasi-finite Feynman integral with many scales, preferably with $\mathbf{N}[\mathcal{F}]$ GP.

Available at https://github.com/michibo/feyntrop
A C++ program with Python and JSON interface.

Example:




```
edges = [((0,1), 1, 'mm_top'), ((1,6), 1, 'mm_top'),
         ((5,6), 1, '0'), ((6,2), 1, 'mm_top'),
         ((2,3), 1, 'mm_top'), ((3,4), 1, 'mm_top'),
         ((4,5), 1, 'mm_top'), ((5,0), 1, 'mm_top')]
```

Phase space point:

$$m_t^2 = 1.8995\,, \quad m_H^2 = 1\,,$$

$$s_{02} = -4.4\,, \quad s_{03} = -0.5\,, \quad s_{12} = -0.6\,, \quad s_{13} = -0.7\,, \quad s_{23} = 1.8\,,$$

Setting $\lambda = 0.64$ and $N = 10^8$, we get:

```
Prefactor: gamma(2*eps + 4).
(Effective) kinematic regime: Minkowski (generic).
Finished in 8.12 seconds.
-- eps^0: [-0.0114757 +/- 0.0000082]
           + i * [0.0035991 +/- 0.0000068]
-- eps^1: [ 0.003250  +/- 0.000031 ]
           + i * [-0.035808 +/- 0.000041 ]
-- eps^2: [ 0.046575  +/- 0.000098 ]
           + i * [0.016143  +/- 0.000088 ]
-- eps^3: [ -0.01637  +/-  0.00017 ]
           + i * [ 0.03969  +/-  0.00016 ]
-- eps^4: [ -0.02831  +/-  0.00023 ]
           + i * [-0.00823  +/-  0.00024 ]
```

Summary

- Lorentz invariant contour deformation with finite parameter.
- Classified some cases when $\mathbf{N}[\mathcal{F}]$ is a GP.
- Introduced the idea of tropical Monte Carlo methods.
- Numerical evaluation of complicated integrals needed for precision physics

Improvements:

- Integrals that are not quasi-finite
- Special kinematic setups
- Canonical choice of deformation parameter

Thank you!

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