MATHEMATICAL AND PHYSICAL STRUCTURES OF RATIONAL FUNCTIONS IN SCATTERING AMPLITUDES

Title for Mathematicians: Vector Spaces over Fraction Fields of Polynomial Quotient Rings

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(GDL, B. Page)

arXiv:2305.17056
(S. Abreu, GDL, H. Ita, M. Klinkert, B. Page, V. Sotnikov)

MathemAmplitudes 2023 - Padova

Find these slides at gdelaurentis.github.io/slides/mathemamplitudes_sept2023

INTRODUCTION

Scattering Amplitudes

- Amplitude (integrands) can be written as

$$A(\lambda,\tilde\lambda,\ell) = \sum_{\substack{\Gamma,\\ i\in M_\Gamma\cup S_\Gamma}} c_{\Gamma,i}(\lambda,\tilde\lambda,\epsilon)\,\frac{m_{\Gamma,i}(\lambda\tilde\lambda,\ell)}{\prod_j \rho_{\Gamma,j}(\lambda\tilde\lambda,\ell)} \xrightarrow{\int d^D\ell} \sum_{\substack{\Gamma,\\ i\in M_\Gamma}} c_{\Gamma,i}(\lambda,\tilde\lambda,\epsilon)\, I_{\Gamma,i}(\lambda\tilde\lambda,\epsilon)$$

- For a suitable choice of integrands, we get:

$$c_{\Gamma,i}(\lambda,\tilde\lambda,\epsilon) = \frac{\sum_{k=0}^{\text{finite}} c^{(k)}_{\Gamma,i}(\lambda,\tilde\lambda)\,\epsilon^k}{\prod_j(\epsilon - a_{ij})}\,, \quad \text{with} \quad a_{ij}\in\mathbb{Q}$$

Some notation:

- Γ: topologies
- M_Γ: masters
- S_Γ: surface terms
- Spinors: $\lambda_i = |i\rangle, \tilde\lambda_i = [i|$
- 4-momenta: $\lambda\tilde\lambda = \not{p}$
- Loop D-momenta: ℓ

Outline

Disclaimer: I will focus on the $c_{\Gamma,i}^{(k)}(\lambda, \tilde{\lambda})$ – call them $c_i(\lambda, \tilde{\lambda})$ for short, nevertheless some concepts can be extended to whole amplitudes.

We will discuss:

1. Where are the poles? What is their order?
2. Relation between pole structure, and analytic constraints (e.g. partial fraction decomposition)
3. Reconstructing sets of functions: Why is it easier than reconstructing individual ones?
4. Example process @ 2-loop: $pp \to \gamma\gamma\gamma$ and $pp \to Wjj$ (preliminary)

Key take away:

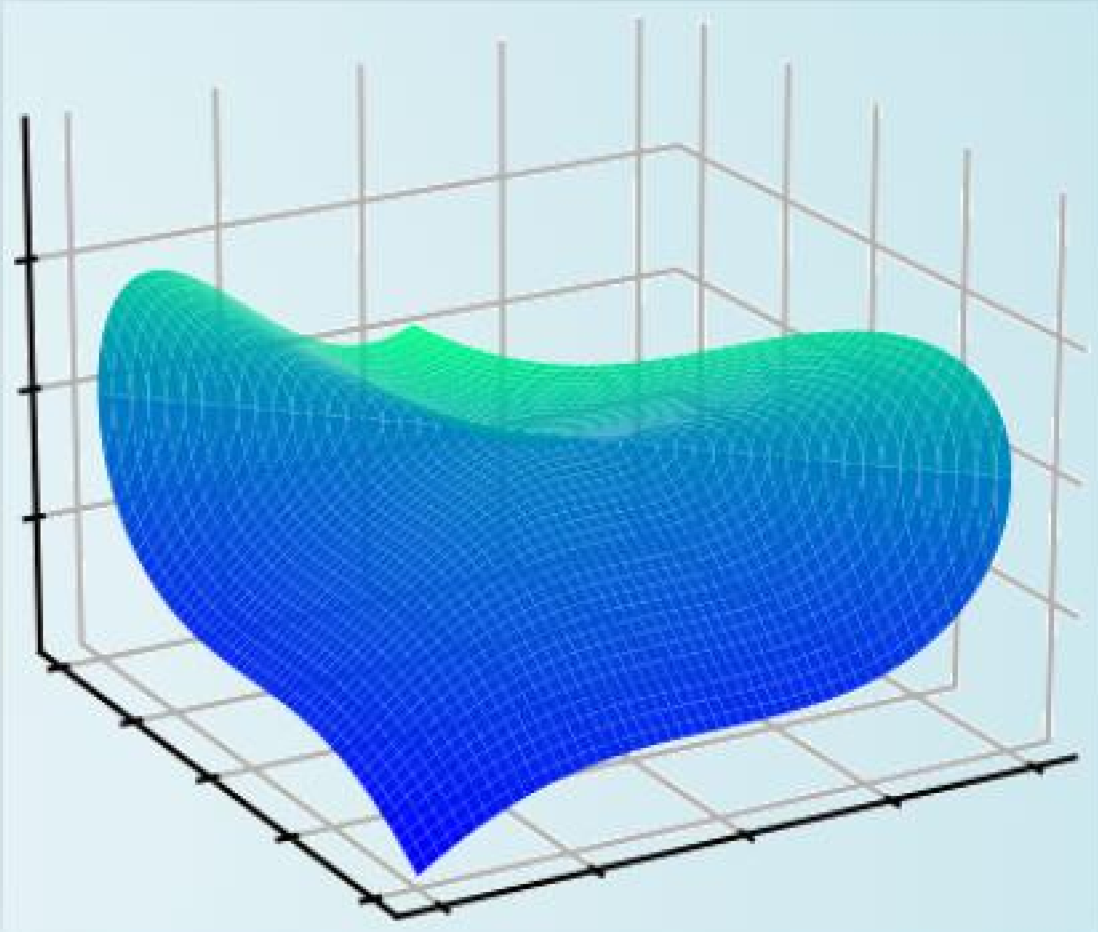
What is important is not the number of variables, but the size of the parametrization (a.k.a. ansatz), and our ability to constrain it.

Polynomial Quotient Rings

- Let us start from the polynomial ring of spinor components

$$S_n = \mathbb{F}\left[|1\rangle, [1|, \ldots, |n\rangle, [n|\right]$$

the field $\mathbb{F}$ can be any of $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p, \mathbb{Q}_p, \ldots$



Artist's Impression of $V(J_{\Lambda_n})$
I can't draw in $4n$ dims!

- Define the momentum-conservation ideal as

$$J_{\Lambda_n} = \left\langle \sum_i |i\rangle[i| \right\rangle_{S_n}$$

physically, two polynomials p and q are equivalent if $p - q \in J_{\Lambda_n}$

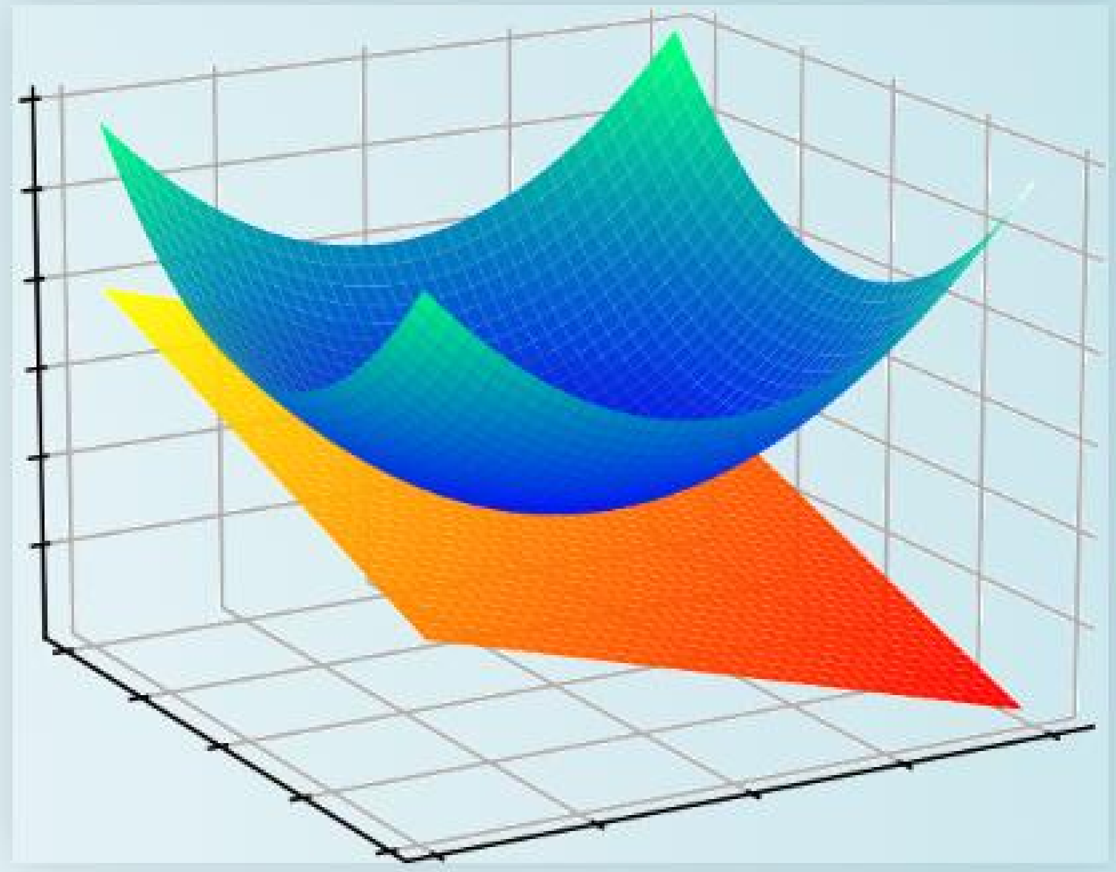
- This defines the needed polynomial **quotient** ring*: $R_n = S_n / J_{\Lambda_n}$

$c_i(\lambda, \tilde{\lambda})$ at n-point belong to the Field of Fractions† of R_n

$^\star R_4$ is "weird" (not a UFD), but it proves that polynomial rings are insufficient; † The field of fractions of R_3 does not exist.

THE POLE STRUCTURE

Prime Ideals & Irreducible Varieties



Artist's Impression of $V(\langle\langle 23\rangle\rangle_{R_4})$ as the union of two irreducibles

- Let us consider a very simple example

$$iA^{\text{tree}}_{g^-g^-g^+g^+} = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \frac{[34]^3}{[12][23][41]}$$

  is, say, $\langle 23\rangle$ a pole of this amplitude?

- The question is ill posed!

  $\langle 23\rangle$ does not identify an irreducible variety in R_4.

  Compute primary decompositions, such as

$$\big\langle\langle 23\rangle\big\rangle_{R_4} = \big\langle\langle 23\rangle, [14]\big\rangle_{R_4} \cap \big\langle\langle 12\rangle, \langle 13\rangle, \langle 14\rangle, \langle 23\rangle, \langle 24\rangle, \langle 34\rangle\big\rangle_{R_4}$$

  On the **first branch** there is a simple pole, on the **latter branch** the amplitude is regular.

Poles & Zeros	$\Leftrightarrow$	Irreducible Varieties	$\Leftrightarrow$	Prime Ideals
Physics		*Geometry*		*Algebra*

Constraints from Poles

Bootstrapping trees (?)

- The degree of divergence / vanishing on various surfaces imposes strong constraints, e.g.

$$A^{\text{tree}}_{q^+g^+g^+\bar{q}^-g^-g^-} = \frac{\mathcal{N}(m.d.{=}6\ ,\ p.w.{=}[-1{,}0{,}0{,}1{,}0{,}0])}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}$$

- Pretend this is un unknown integral coefficient, $\mathcal{N}$ has 143 free parameters.
- List the various prime ideal, such as

$$\big\langle\langle 12\rangle, \langle 23\rangle, \langle 13\rangle\big\rangle,\ \big\langle|1\rangle\big\rangle,\ \big\langle\langle 12\rangle, |1+2|3]\big\rangle, \dots$$

and impose that $\mathcal{N}$ vanishes to the correct order. We determine it up to an overall constant.

GDL, Page ('22)

- Likewise, the ansatz for $A^{\text{tree}}_{g^+g^+g^+g^-g^-g^-}$ shrinks $1326 \to 1$, etc..

*Effectively, we can **compute** trees, just from their poles orders.*
Note: compared to BCFW there is no information about residues.

Partial Fraction Decompositions

- For true integral coefficients, we can't rely on the Ansatz to shrinks to an overall constant.
- Partial fraction decompositions (PFDs) are a popular method to tame algebraic complexity.
- In my opinion, a PFD algorithm needs

1. to say if two poles W_a and W_b are separable into different fractions;
2. ideally, to answer $(1.)$ without having access to an analytic expression.

- Hilbert's nullstellensatz: if $\mathcal{N}$ vanishes on all branches of $\langle W_a, W_b \rangle$, then the PFD is possible†.
- Generalizing to powers >1 can be done via symbolic powers and the Zariski-Nagata Theorem.
- Similarly, generalizing to non-radical ideals requires ring extensions.

GDL, Page ('22)

Campbell, GDL, Ellis ('22)

Issue: evaluations on singular surfaces are expensive, but are not always needed!

Opportunity: we get more than partial fraction decompositions.

† $\langle W_a, W_b \rangle$ needs to be radical.

Beyond Partial Fractions

- Case 0: the ideal does not involve denominator factors.

E.g. a 6-point function c_i has a pole at $\langle 1|2+3|4]$ but not at $\langle 4|2+3|1]$, yet it is regular on the irreducible surface $V(\langle\langle 1|2+3|4], \langle 4|2+3|1]\rangle)$. Then

$$c_i \sim \frac{\langle 4|2+3|1]}{\langle 1|2+3|4]} + \mathcal{O}(\langle 1|2+3|4]^0) \;\text{ instead of }\; c_i \sim \frac{1}{\langle 1|2+3|4]} + \mathcal{O}(\langle 1|2+3|4]^0)$$

- Case 1: the degree of vanishing is non-uniform across branches, for example:

$$\frac{s_{14}-s_{23}}{\langle 1|3+4|2]\langle 3|1+2|4]}$$

has a double pole on the first branch, and a simple pole on the second branch of

$$\big\langle \langle 1|3+4|2], \langle 3|1+2|4] \big\rangle_{R_6} = \big\langle \langle 13\rangle, [24] \big\rangle_{R_6} \cap \big\langle \langle 1|3+4|2], \langle 3|1+2|4], (s_{14}-s_{23}) \big\rangle_{R_6}$$

- Case 2: ideal is non-radical (example on last slide)

$$\sqrt{\Big\langle \langle 3|1+4|2], \Delta_{23|14|56} \Big\rangle_{R_6}} = \Big\langle \langle 3|1+4|2], s_{124}-s_{134} \Big\rangle_{R_6}$$

ANALYTIC RECONSTRUCTION

Choosing Independent Functions

- The set c_i can be very large, so pick a set of independent ones, and write:

$$c_i = \tilde{c}_j M_{ji} \quad \text{with} \quad M_{ji} \in \mathbb{Q}$$

  with $\tilde{c}$ an independent subset of c.
  $\Longrightarrow M_{ji}$ is, up to a permutation of columns, in row reduced echelon form.

- We might as well use a set $\tilde{c}$ which is not a subset of c, at the trivial cost of changing M_{ji}.

- Consider a PFD of one of the $\tilde{c}$'s:

$$\tilde{c}_i(\lambda, \tilde{\lambda}) = \frac{\mathcal{N}_i(\lambda, \tilde{\lambda})}{\prod_j W_j^{q_{ij}}(\lambda, \tilde{\lambda})} = \sum_k \frac{\mathcal{N}_{ik}(\lambda, \tilde{\lambda})}{\prod_j W_j^{q_{ijk}}(\lambda, \tilde{\lambda})} = \sum_k \tilde{c}_{ik}(\lambda, \tilde{\lambda})$$

We cannot have $\tilde{c}_i \in \text{span}(\tilde{c}_{j\neq i})$, but we can have $\tilde{c}_{ik} \in \text{span}(\tilde{c}_{j\neq i})$, for some, but not all, k.

Least Least-Common-Denominator

- In other words, the $\tilde{c}$ span a vector space, and we should consider one modulo the others

$$\tilde{c}_i = \sum_{j \neq i} q_j \tilde{c}_j + \tilde{c}'_i$$

  the basis function $\tilde{c}_i$ can be replaced by $\tilde{c}'_i$ without changing the vector space.

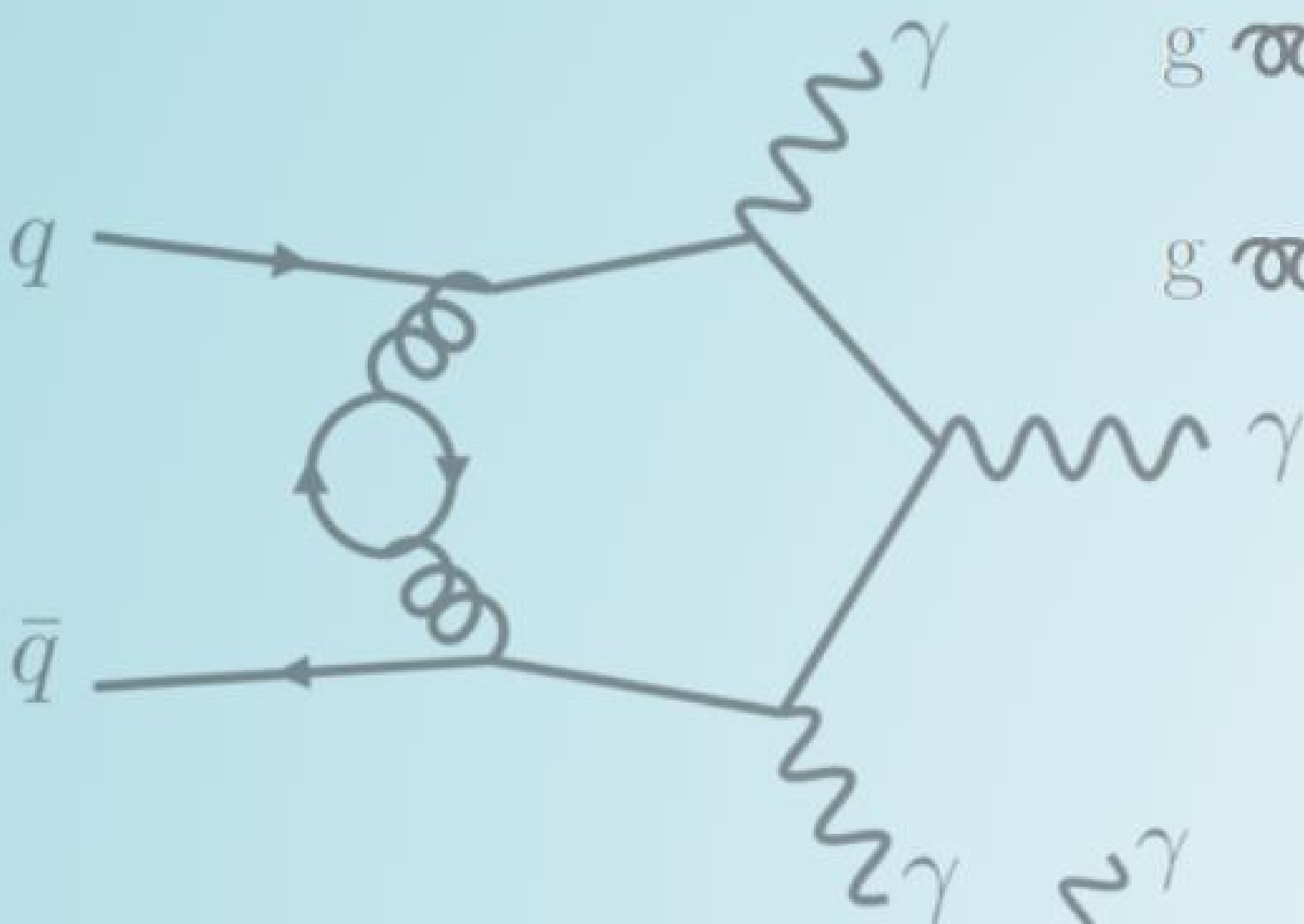
- In particular, $\tilde{c}'_i$ needs not have all the poles of $\tilde{c}_i$, thus it can be much simpler.
  In other words, the LCD of $\tilde{c}'_i$ can be smaller than that of $\tilde{c}_i$.

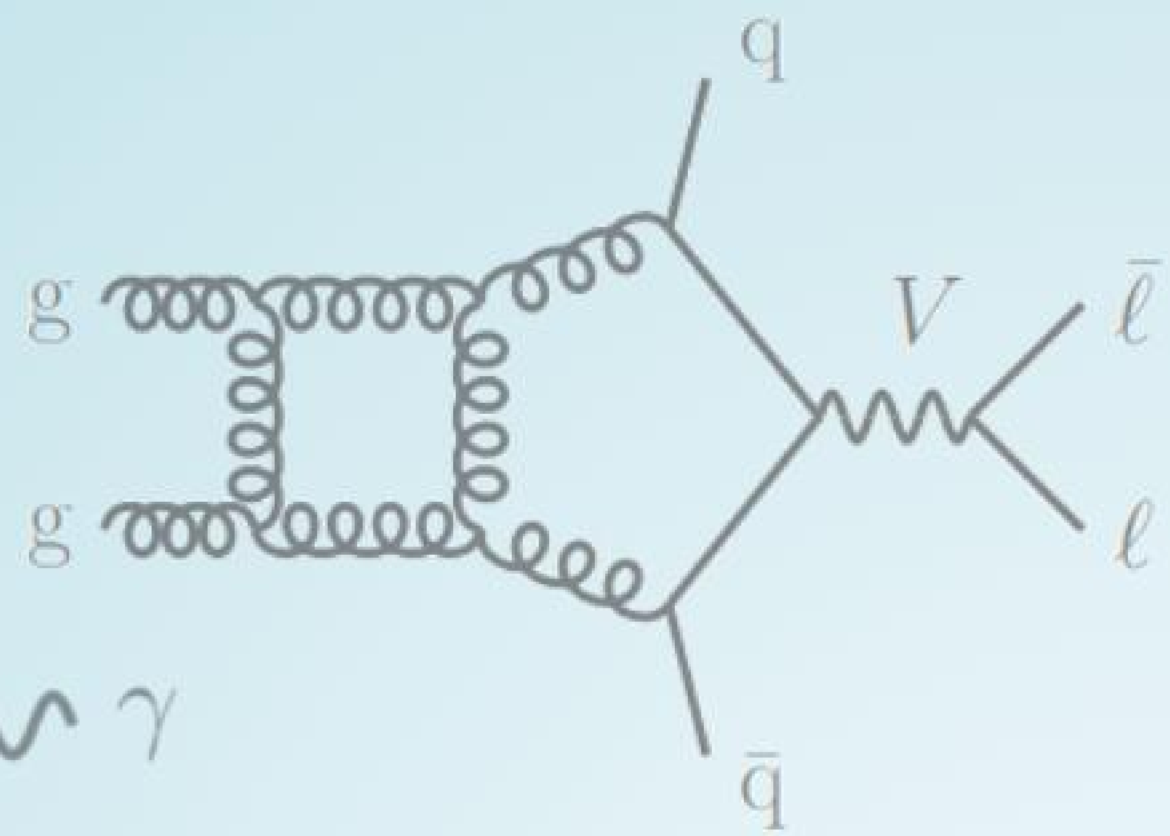
- Brute-force search works well when an analytic expressions is available.

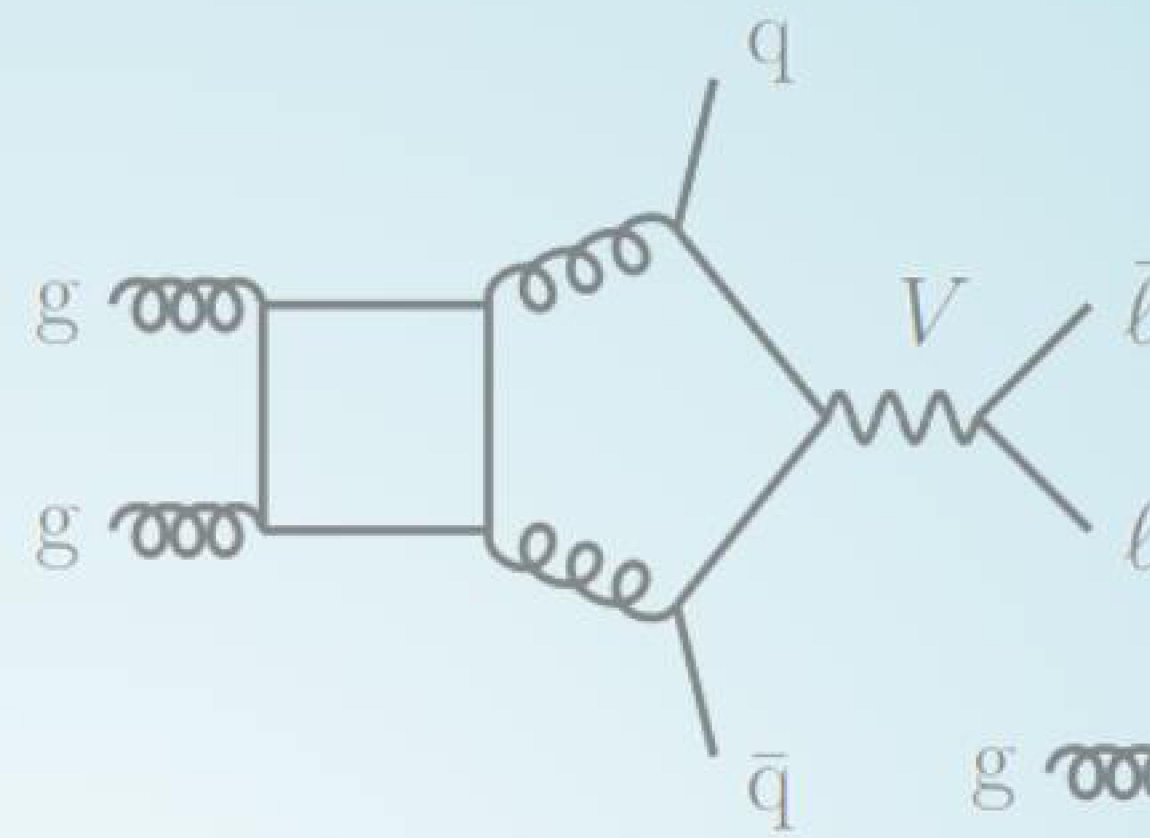
- In a future publication we will provide an algorithm based on finite field evaluations.

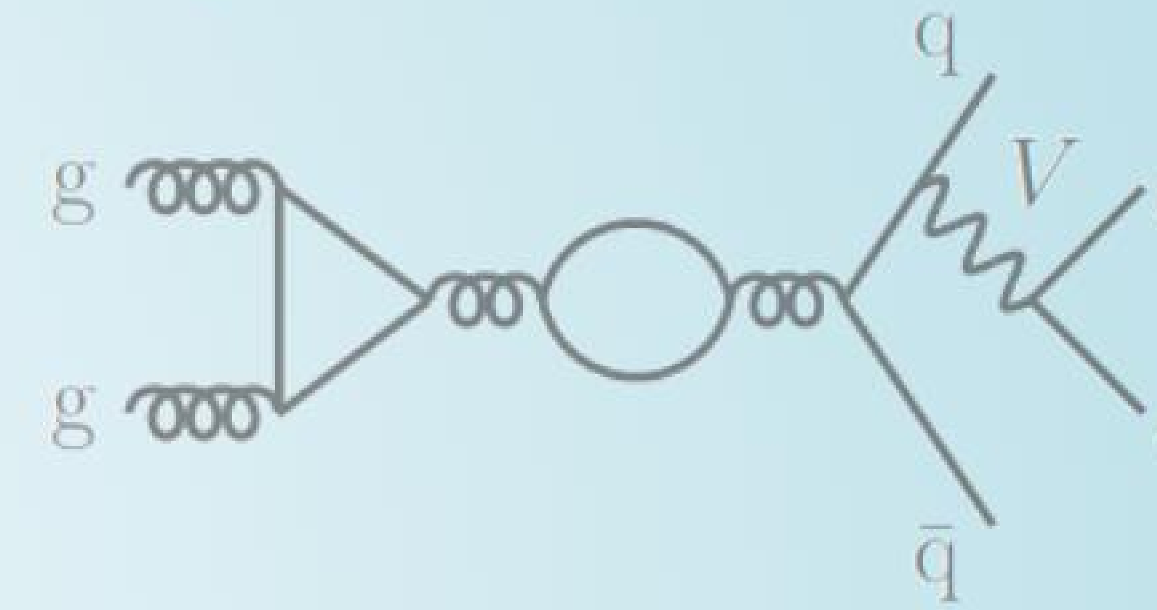
Reconstructing a set of c_i is not as bad as reconstructing the most complex function in the set.

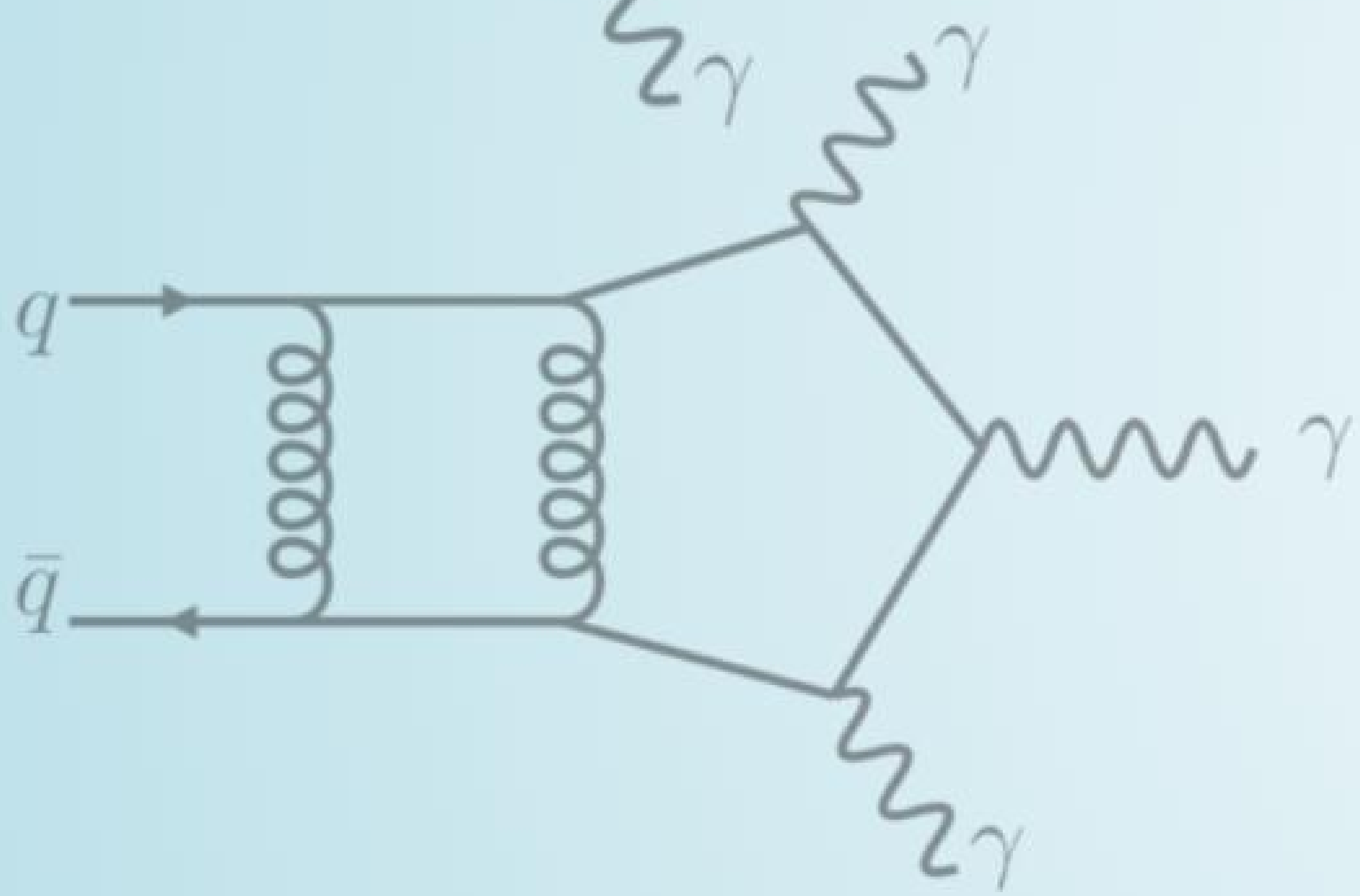
EXAMPLE PROCESSES

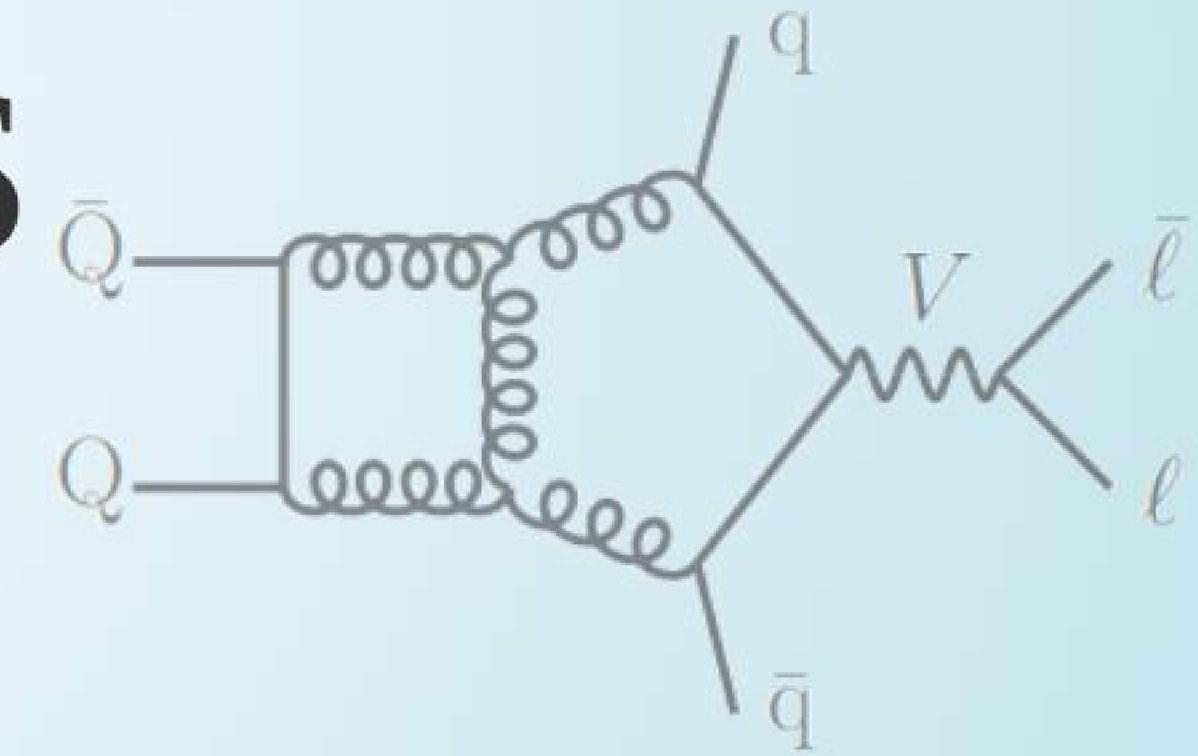


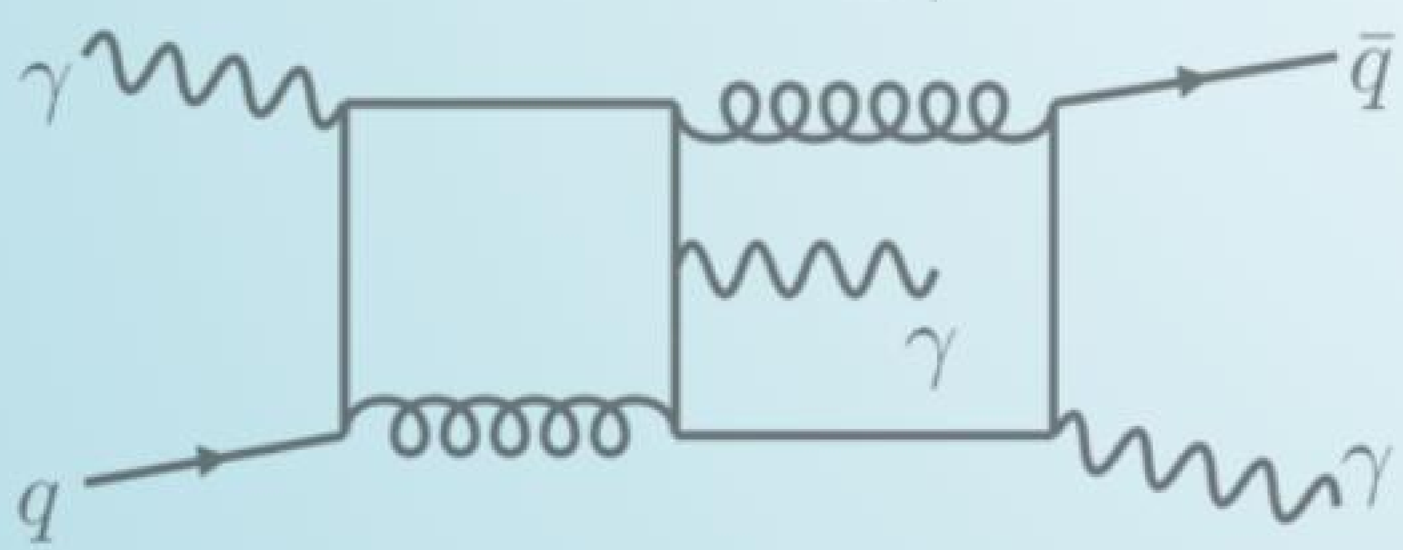


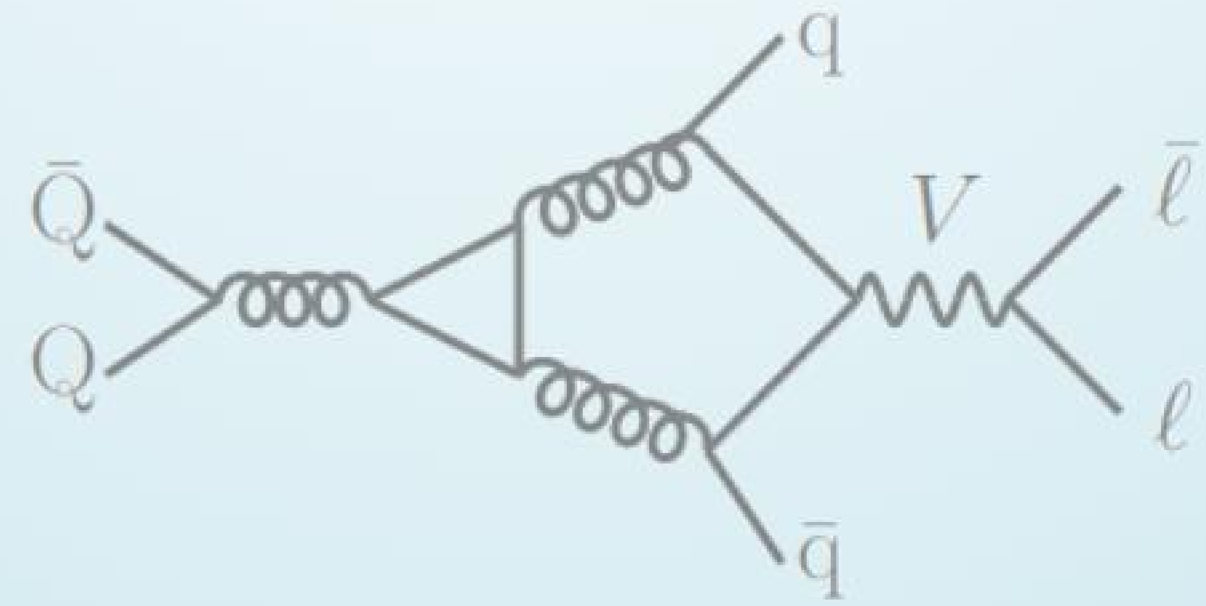


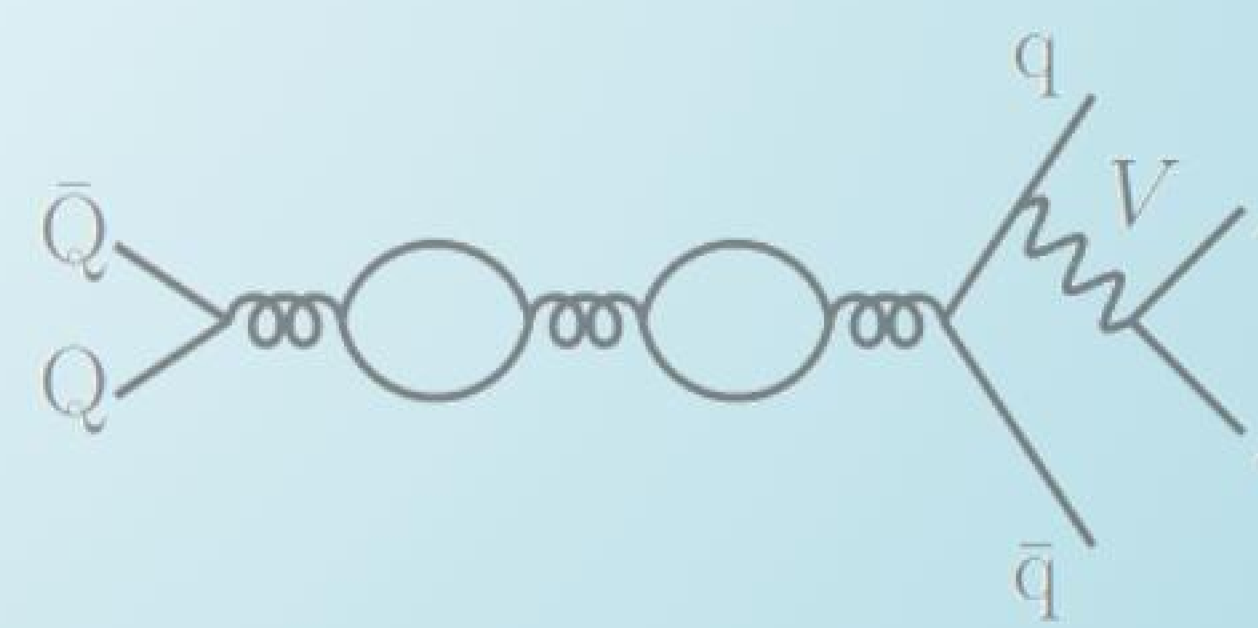












Three-photon production at two loops

- The denominator factors W_j are conjectured to be restricted to the letters of the symbol alphabet

Abreu, Dormans, Febres Cordero, Ita, Page ('18)

$$\{W_j\} = \bigcup_{\sigma \,\in\, \mathrm{Aut}(R_5)} \sigma \circ \big\{\langle 12\rangle, \langle 1|2+3|1]\big\} \quad \text{Identical to 1-loop!}$$

- Advantage of spinor variables due to:

  1. little group covariant LCD (no spurious poles); 2. avoiding parity even/odd split;

$\Rightarrow$ in LCD form we would need 29 059 evaluations instead of 117 810 (with s_{ij}) for $\mathcal{R}^{(2)}_{2q3\gamma}$.

- To avoid evaluations on singular surfaces, use insights from physics (locality).
  E.g. conjecture that no 5-point denominator has pairs of $\langle i|j+k|i]$, like at 1 loop.

- Remove some overlap with other $\tilde{c}$'s, obtain the $\tilde{c}_i$ with higher degree LCD with 4 003 points.
- A posteriori, we find that for the c_i with highest degree LCD the following would have sufficied

$$\tilde{c}_i = \frac{\langle 13\rangle[14]^2\langle 24\rangle\langle 34\rangle[45]}{\langle 45\rangle\langle 4|1+3|4]^3} - \frac{[14]\langle 25\rangle\langle 34\rangle^2[45]}{\langle 45\rangle^2\langle 4|1+3|4]^2} - \frac{[14]\langle 24\rangle\langle 34\rangle\langle 35\rangle}{\langle 45\rangle^3\langle 4|1+3|4]}$$

W+2-JETS: SIMPLIFICATION STRATEGY

0. Start from analytics of Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov ('21) - 1.2GB of C++ source code.

1. Script to split up the expressions, and compile them (~ 20GB binaries) for evaluation over $\mathbb{F}_p$;

2. Recombine the 3 projections $p_V \parallel p_1, p_V \parallel p_2, p_V \parallel p_3$ and reintroduce the little group factors to build 6-point spinor-helicity amplitudes (subject to degree bounds on $|5\rangle$, $[5|$, $|6\rangle$, $[6|$);

3. Perform (rough) PFDs based on expected structures and fit the Ansatze.

Comparison of $q\bar{q} \to \gamma\gamma\gamma$ (in full color) to $pp \to Wjj$ (at leading color):

Kinematics	**# Poles (W)**	**LCD Ansatz**	**Partial-Fraction Ansatz**	**Rational Functions**
5-point massless	30	29k	4k	∼300 KB
5-point 1-mass	>200	>5M	∼40k	∼25 MB

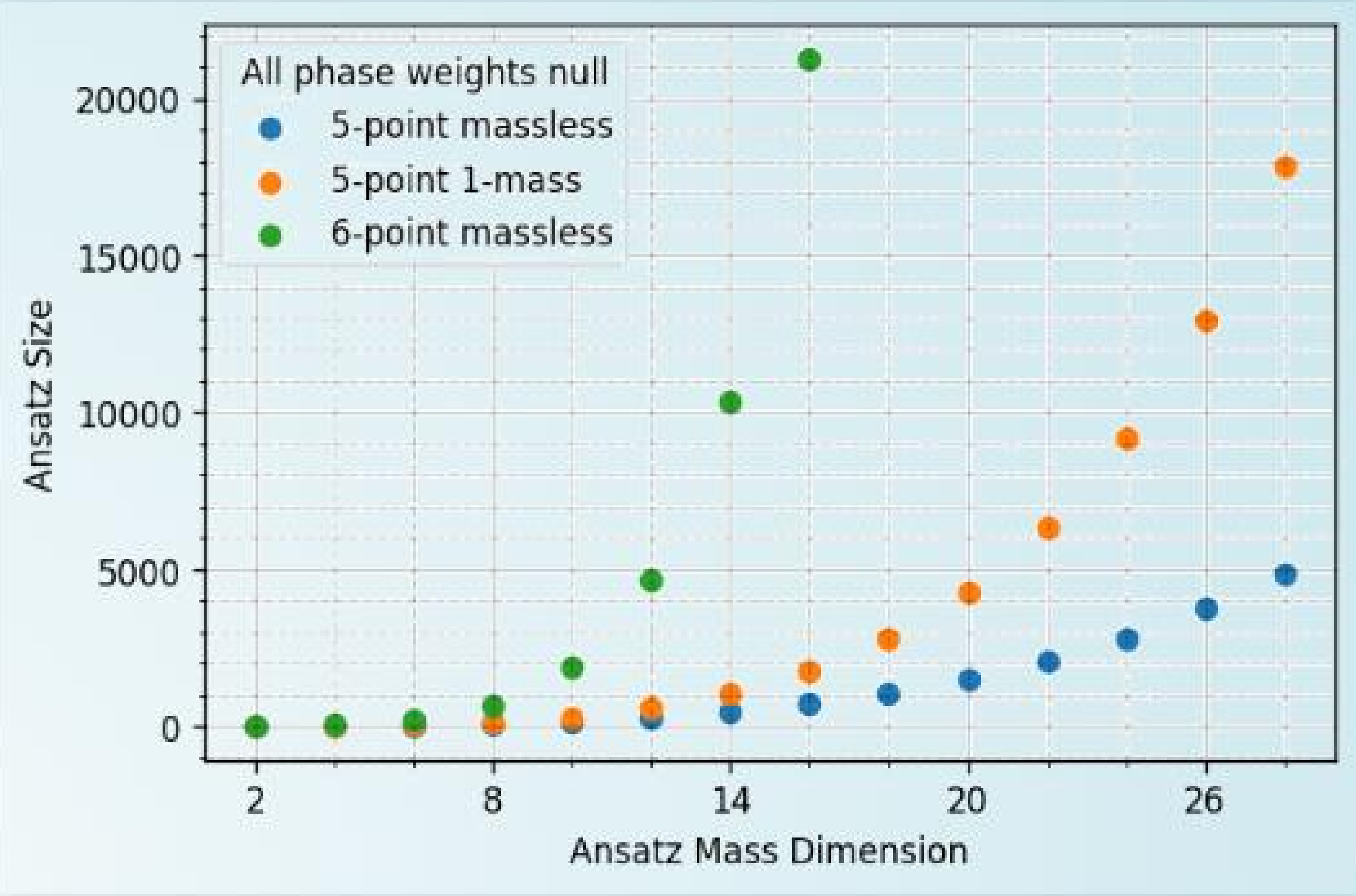
$$\{W_j\} = \bigcup_{\sigma \in \mathrm{Aut}(R_6)} \sigma \circ \big\{\langle 12\rangle, \langle 1|2+3|1], \langle 1|2+3|4], s_{123}, \Delta_{12|34|56}, \langle 3|2|5+6|4|3] - \langle 2|1|5+6|4|2]\big\}$$

Analytic Structures of 2-loop 5-point 1-mass Amplitudes

- The Ansatz size grows quickly with multiplicity (m) and mass dimension (d):

$$\left(\!\!\binom{m(m-3)/2}{d/2}\!\!\right)$$

is a lower bound. GDL, Maître ('20)



- Compact residues for the new 2-loop (spurious?) pole, $\langle k|j|\not{p}_V|l|k] - \langle j|i|\not{p}_V|l|j]$, e.g.:

$$r^{(5\text{ of }54)}_{\bar{u}^+g^+g^+d^-(V\to\ell^+\ell^-)} = \frac{[12][23]\langle 24\rangle\langle 46\rangle^2\langle 1|2+3|4]\langle 2|1+3|4]}{\langle 12\rangle\langle 23\rangle\langle 56\rangle(\langle 3|2|5+6|4|3] - \langle 2|1|5+6|4|2])^2}$$

- The three mass Grams, $\Delta_{12|34|p_V}$, $\Delta_{14|23|p_V}$, behave analogously to one-loop amplitudes, e.g.:

$$r^{(73\text{ of }120)}_{\bar{u}^+g^-g^+d^-(V\to\ell^+\ell^-)} = \frac{105}{128}\frac{\langle 2|1+4|3]\langle 4|2+3|1]\langle 6|1+4|5]s_{14}s_{23}s_{56}(s_{124}-s_{134})(s_{123}-s_{234})(s_{25}+s_{26}+s_{35}+s_{36})}{\langle 3|1+4|2]\Delta^4_{23|14|56}} +$$

$$\left[-6\frac{[12]^2\langle 13\rangle[25]\langle 34\rangle\langle 36\rangle\langle 56\rangle[56](s_{124}-s_{134})}{\langle 3|1+4|2]^5}\right] + \Big[\Big]_{1234\to\overline{4321}} + \mathcal{O}\left(\frac{1}{\langle 3|1+4|2]^4\Delta^3_{23|14|56}}\right)$$

Thank you for your attention!

These slides are powered by:

markdown, html, revealjs, hugo, mathjax, github

BACKUP SLIDES

Finite remainders & the Rational / Transcendental split

- In general, in $D = 4 - 2\epsilon$, with *pure* master integrals $I_{\Gamma,i}$ we have

$$A_n^{\ell-loop}(\lambda, \tilde{\lambda}) = \sum_{\Gamma} \sum_{i \in M_\Gamma} \frac{c_{\Gamma,i}(\lambda, \tilde{\lambda}, \epsilon)\, I_{\Gamma,i}(\lambda\tilde{\lambda}, \epsilon)}{\prod_j (\epsilon - a_{ij})} \,, \quad \text{with} \quad a_{ij} \in \mathbb{Q}$$

- For NNLO applications, we are interested in the *finite remainder*

$$\mathcal{A}_R^{(2)} = \underbrace{\mathcal{R}}_{\text{finite remainder}} + \underbrace{I^{(1)}\mathcal{A}_R^{(1)} + I^{(2)}\mathcal{A}_R^{(0)}}_{\text{divergent + convention-dependent finite part}} + \mathcal{O}(\epsilon)$$

- Finite remainder as a weighted sum of *pentagon functions*

Chicherin, Sotnikov ('20);

$$\mathcal{R}(\lambda, \tilde{\lambda}) = \sum_i r_i(\lambda, \tilde{\lambda})\, h_i(\lambda\tilde{\lambda})$$

Reconstruct $r_i(\lambda, \tilde{\lambda})$ from $\mathbb{F}_p$ samples

von Manteuffel, Schabinger ('14)
Peraro ('16)

The Numerator Ansatz

- The numerator Ansatz takes the form

GDL, Maître ('19)

$$\text{Num. poly}(\lambda, \tilde{\lambda}) = \sum_{\vec{\alpha},\vec{\beta}} c_{(\vec{\alpha},\vec{\beta})} \prod_{j=1}^{n} \prod_{i=1}^{j-1} \langle ij \rangle^{\alpha_{ij}} [ij]^{\beta_{ij}}$$

subject to constraints on $\vec{\alpha}, \vec{\beta}$ due to: 1) mass dimension; 2) little group; 3) linear independence.

- Construct the Ansatz via the algorithm from Section 2.2 of GDL, Page ('22)

Linear independence = irreducibility by the Gröbner basis of a specific ideal.

- Efficient implementation using open-source software only

SINGULAR

Gröbner bases $\rightarrow$ constrain $\vec{\alpha}, \vec{\beta}$

Decker, Greuel, Pfister, Schönemann

Google OR-Tools

Integer programming $\rightarrow$ enumerate sols. $\vec{\alpha}, \vec{\beta}$

Perron and Furnon (Google optimization team)

- All linear systems solved with CUDA over $\mathbb{F}_{p\leq 2^{31}-1}$ on a laptop ($t_{\text{solving}} \ll t_{\text{sampling}}$)

Polynomial Quotient Rings
vs.
Polynomial Rings

- Can we get rid of equivalence relations (redundancies) by changing variables?
  In other words, can we solve the redundancies and turn the quotient ring in a ring?

With four-point massless kinematics, you cannot. In R_4 we have

$$\langle 12\rangle[12] = \langle 34\rangle[34]$$

this means that R_4 is <u>not</u> a unique factorization domain (UFD).
All polynomial rings are UFDs, so R_4 cannot be isomorphic to one.

THE FIELD OF FRACTIONS OF R_3 DOES NOT EXISTS

- R_3 is not an integral domain, i.e. there exists products of non-zero elements which are zero.

$$\langle 12\rangle[23] = \langle 1|2|3] = -\langle 1|1+3|3] = 0 \text{ but } \langle 12\rangle \neq 0 \text{ and } [23] \neq 0$$

Hence, one cannot define a field of fractions, as this would not be closed under multiplication.