Symbol Alphabets from the Landau Singular Locus

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work with Martin Helmer, Georgios Papathanasiou and Felix Tellander

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Feynman Integrals

- Momentum space representation:

$$\mathcal{I} = \int \prod_{l=1}^{L} \frac{d^D k_l}{i\pi^{D/2}} \int_0^\infty \prod_{e\in E} \frac{1}{(-q_e^2 + m_e^2 - i\epsilon)^{\nu_e}}, \qquad D = D_0 - 2\epsilon$$

- Master integrals and canonical differential equations:

Integration-by-parts (IBP) relations

$$d\vec{f} = dM(\epsilon)\vec{f}, \longrightarrow d\vec{g} = \epsilon\, d\widetilde{M}\vec{g}, \longrightarrow \vec{g} = \sum_{k=0}^{\infty} \epsilon^k \vec{g}^{(k)}$$

$$\vec{g}^{(k)} = \int d\widetilde{M}\vec{g}^{(k-1)}$$

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- Letters and alphabet:

$$\widetilde{M} = \sum_i \tilde{a}_i \log W_i$$

Goal: Find alphabet from integral representation instead of differential equations

Alphabet and Bootstrap

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- Finding canonical basis: INITIAL, CANONICA

[CD, Henn, Yan, '20] [Meyer, '17]

Lee-Pomeransky Representation

- Feynman representation:

$$\mathcal{I} = \Gamma(\omega) \int_0^\infty \prod_{e \in E} \left(\frac{x_e^{\nu_e} dx_e}{x_e \Gamma(\nu_e)} \right) \frac{\delta(1 - H(x))}{\mathcal{U}^{D/2}} \left(\frac{1}{\mathcal{F}/\mathcal{U} - i\epsilon} \right)^\omega, \qquad \omega \equiv \sum_{e \in E} \nu_e - LD/2$$

- Lee-Pomeransky:

$$\mathcal{I} = \frac{\Gamma(D/2)}{\Gamma(D/2 - \omega)} \int_0^\infty \prod_{e \in E} \left(\frac{x_e^{\nu_e} dx_e}{x_e \Gamma(\nu_e)} \right) \frac{1}{\mathcal{G}^{D/2}}, \qquad \mathcal{G} = \mathcal{U} + \mathcal{F}$$

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- Landau equations:

[Klausen, '21]

$$\mathcal{G}_h = \mathcal{U} x_0 + \mathcal{F} = 0, \quad \text{and} \quad \frac{\partial \mathcal{G}_h}{\partial x_i} = 0 \quad \text{or} \quad x_i = 0 \qquad \forall\ i = 0, \ldots, |E|$$

homogenized LP-polynomial

Generic one-loop integrals

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$n = |E|$

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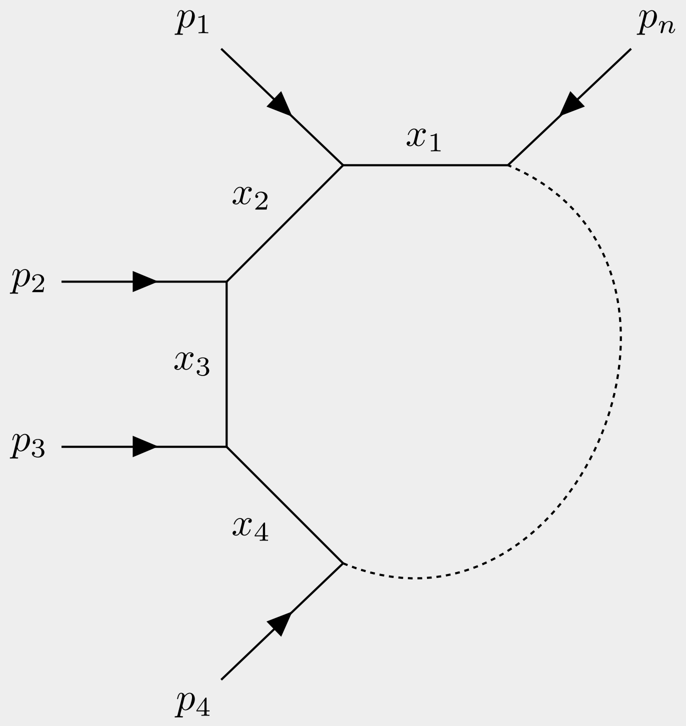

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  - leading Landau singularities (full graph): $x_e \neq 0, \quad \forall\, e \in E$
    - type-I singularity $x_0 = 0 \longrightarrow \mathcal{G}_h|_{x_0=0} = \mathcal{F}$
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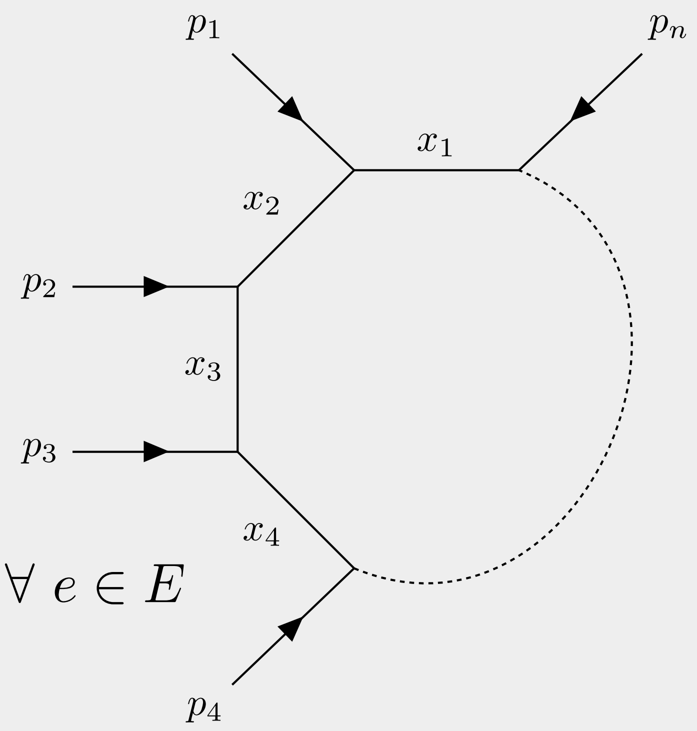

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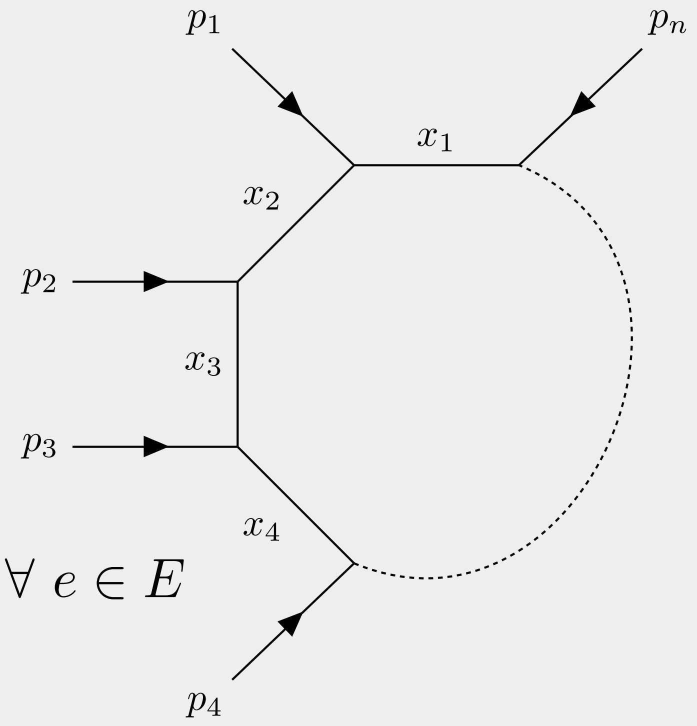

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- sub-graph singularities: $x_e = 0, \quad e \in E$
  - type-I singularity $x_0 = 0 \longrightarrow \mathcal{G}_h\Big|_{\substack{x_0=0\\x_e=0}} = \mathcal{F}|_{x_e=0}$
  - type-II singularity $x_0 \neq 0 \longrightarrow \mathcal{G}_h\Big|_{\substack{x_0=1\\x_e=0}} = \mathcal{G}|_{x_e=0}$

The Landau singular locus at one loop

- Leading type-II singularity: $x_i \neq 0 \quad \forall\, i = 0, \ldots, n$

$$\frac{\partial \mathcal{G}_h}{\partial x_0} = \cdots = \frac{\partial \mathcal{G}_h}{\partial x_n} = 0 \qquad \longrightarrow \qquad \mathcal{G}_h = 0$$

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degree two (→ $\mathcal{G}_h$)

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$$\Rightarrow \begin{pmatrix} \frac{\partial \mathcal{G}_h}{\partial x_0} \\ \vdots \\ \frac{\partial \mathcal{G}_h}{\partial x_n} \end{pmatrix} =: \mathscr{J}(\mathcal{G}_h) \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0},$$

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- Solution space: $\mathbf{V}\left(\frac{\partial \mathcal{G}_h}{\partial x_0}, \ldots, \frac{\partial \mathcal{G}_h}{\partial x_n}\right) := \left\{ x \in \mathbb{C}^n \setminus \{\mathbf{0}\} \middle| \frac{\partial \mathcal{G}_h}{\partial x_0} = \ldots = \frac{\partial \mathcal{G}_h}{\partial x_n} = 0 \right\}$

- Space of kinematic variables for which there is a solution:

$$\overline{\left\{ s_{ij}, m_i^2 \,\middle|\, \mathbf{V}\left(\frac{\partial \mathcal{G}_h}{\partial x_0}, \ldots, \frac{\partial \mathcal{G}_h}{\partial x_n}\right) \neq \emptyset \right\}} \quad \Longleftrightarrow \quad \boxed{\det(\mathscr{J}(\mathcal{G}_h)) = 0}$$

The modified Cayley matrix

- For the LP-polynomial of generic one-loop integrals:

$$\mathscr{I}(\mathcal{G}_h) = \mathcal{Y}$$

$$\mathcal{Y} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & Y_{11} & Y_{12} & \cdots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{pmatrix} \begin{matrix} \leftarrow x_0 \\ \\ \\ \leftarrow x_e \\ \\ \end{matrix}$$

$$Y_{ii} = 2m_i^2, \qquad Y_{ij} = m_i^2 + m_j^2 - s_{ij-1}$$

Cayley matrix

$$s_{ij-1} \equiv (p_i + \ldots + p_{j-1})^2$$

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- Relation to Gram determinants $G(k_1, \ldots, k_m) \equiv \det_{i,j}(k_i \cdot k_j)$
  - type-II singularity: $x_0 \neq 0 \longrightarrow \det(\mathcal{Y}) = -2^{n-1} G(p_1, \ldots, p_n)$ — Gram determinant
  - type-I singularity: $x_0 = 0 \longrightarrow \det(Y) = (-2)^n G(q_1, \ldots, q_n)|_{q_i^2 = m_i^2}$ — Cayley determinant

The principal A-determinant at one loop

- Subgraphs correspond to diagonal minors:

$$\mathcal{Y}\begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{bmatrix}$$

determinant with rows/columns removed

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$\leftarrow x_0$

$\leftarrow x_e$

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  - type-I sub-singularity: $\mathcal{Y}\begin{bmatrix} 1 & e \\ 1 & e \end{bmatrix} = \det(Y|_{E\setminus\{e\}})$
  - type-II sub-singularity: $\mathcal{Y}\begin{bmatrix} e \\ e \end{bmatrix} = \det(\mathcal{Y}_{E\setminus\{e\}})$

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- (reduced) principal A-determinant:

$$\widetilde{E}_A(\mathcal{G}_h) = \mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \prod_{i=1}^{n+1} \mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix} \cdots \prod_{i_{n-1}>\ldots>i_1=1}^{n+1} \mathcal{Y}\begin{bmatrix} i_1 \ldots i_{n-1} \\ i_1 \ldots i_{n-1} \end{bmatrix} \prod_{i=2}^{n+1} \mathcal{Y}_{ii}$$

product of Gram and Cayley determinant of the graph and all subgraphs

Example: Bubble

$$\widetilde{E_A}(\mathcal{G}_h) = m_1^2 m_2^2 \lambda(p^2, m_1^2, m_2^2) p^2, \qquad \lambda(p^2, m_1^2, m_2^2) = p^4 + m_1^4 + m_2^4 - 2p^2 m_1^2 - 2p^2 m_2^2 - 2m_1^2 m_2^2$$

- The factors of the principal A-determinant give all symbol letters!

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  - square-root letters?

$$\frac{-m_1^2 + m_2^2 + p^2 - \sqrt{\lambda(p^2, m_1^2, m_2^2)}}{-m_1^2 + m_2^2 + p^2 + \sqrt{\lambda(p^2, m_1^2, m_2^2)}} \in \{W_i\}$$

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  - come from Jacobi identities:

$$-\mathcal{Y}\begin{bmatrix}\cdot\\ \cdot\end{bmatrix}\mathcal{Y}\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix} = \mathcal{Y}\begin{bmatrix}2\\ 1\end{bmatrix}^2 - \mathcal{Y}\begin{bmatrix}2\\ 2\end{bmatrix}\mathcal{Y}\begin{bmatrix}1\\ 1\end{bmatrix}, \qquad f^2 - g = (f - \sqrt{g})(f + \sqrt{g})$$

Jacobi identities

- For odd n

$$\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \mathcal{Y}\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix} = \mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix} \mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix}^2,$$

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- For even n

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case of Gram and Cayley exchanged

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Jacobi identities

- For odd $n+D_0$

$$\mathcal{Y}\begin{bmatrix}\cdot\\\cdot\end{bmatrix}\mathcal{Y}\begin{bmatrix}1 & i\\1 & i\end{bmatrix}=\mathcal{Y}\begin{bmatrix}i\\i\end{bmatrix}\mathcal{Y}\begin{bmatrix}1\\1\end{bmatrix}-\mathcal{Y}\begin{bmatrix}i\\1\end{bmatrix}^2,$$

$$\mathcal{Y}\begin{bmatrix}\cdot\\\cdot\end{bmatrix}\mathcal{Y}\begin{bmatrix}i & j\\i & j\end{bmatrix}=\mathcal{Y}\begin{bmatrix}i\\i\end{bmatrix}\mathcal{Y}\begin{bmatrix}j\\j\end{bmatrix}-\mathcal{Y}\begin{bmatrix}i\\j\end{bmatrix}^2, \quad i\geq 2$$

- For even $n+D_0$

$$\mathcal{Y}\begin{bmatrix}\cdot\\\cdot\end{bmatrix}\mathcal{Y}\begin{bmatrix}1 & i\\1 & i\end{bmatrix}=\mathcal{Y}\begin{bmatrix}i\\i\end{bmatrix}\mathcal{Y}\begin{bmatrix}1\\1\end{bmatrix}-\mathcal{Y}\begin{bmatrix}i\\1\end{bmatrix}^2,$$

$$\mathcal{Y}\begin{bmatrix}1\\1\end{bmatrix}\mathcal{Y}\begin{bmatrix}1 & i & j\\1 & i & j\end{bmatrix}=\mathcal{Y}\begin{bmatrix}1 & i\\1 & i\end{bmatrix}\mathcal{Y}\begin{bmatrix}1 & j\\1 & j\end{bmatrix}-\mathcal{Y}\begin{bmatrix}1 & i\\1 & j\end{bmatrix}^2$$

case of Gram and Cayley exchanged

Symbol letters

- Case of one edge missing: (next-to-maximal cut)

$$W_{1,\ldots,(i-1),\ldots,n} = \begin{cases} \dfrac{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}}, & D_0 + n \text{ odd}, \\[2em] \dfrac{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{\mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{\mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}, & D_0 + n \text{ even}. \end{cases}$$

Symbol letters

- Case of two edges missing: (next-to-next-to-maximal cut)

$$W_{1,\ldots,(i-1),\ldots,(j-1),\ldots,n} = \begin{cases} \dfrac{\mathcal{Y}\begin{bmatrix} i \\ j \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \mathcal{Y}\begin{bmatrix} i & j \\ i & j \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ j \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \mathcal{Y}\begin{bmatrix} i & j \\ i & j \end{bmatrix}}}, & D_0 + n \text{ odd}, \\ \\ \dfrac{\mathcal{Y}\begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathcal{Y}\begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathcal{Y}\begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}, & D_0 + n \text{ even}, \end{cases}$$

Symbol letters

- Case of no leg missing: (maximal cut)
  - no Jacobi identities
  - only one letter

$$\widetilde{E_A}(\mathcal{G}_h) = \mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \prod_{i=1}^{n+1} \mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix} \cdots \prod_{i_{n-1}>\ldots>i_1=1}^{n+1} \mathcal{Y}\begin{bmatrix} i_1 \ldots i_{n-1} \\ i_1 \ldots i_{n-1} \end{bmatrix} \prod_{i=2}^{n+1} \mathcal{Y}_{ii}$$

$$W_{1,2,\ldots,n} = \frac{\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}}{\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Symbol letters

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$$W_{1,2,\ldots,n} = \frac{\mathcal{Y}\begin{bmatrix}\cdot\\\cdot\end{bmatrix}}{\mathcal{Y}\begin{bmatrix}1\\1\end{bmatrix}}$$

- Letters not all independent
  - triangle in even dimensions:

$$\log W_{(i),j,(k)} = \log W_{(i),(j),k} + \log W_{i,(j),(k)}$$

$$\frac{a-b-c-\sqrt{\lambda}}{a-b-c+\sqrt{\lambda}} = \frac{a-b+c-\sqrt{\lambda}}{a-b+c+\sqrt{\lambda}}\,\frac{a+b-c-\sqrt{\lambda}}{a+b-c+\sqrt{\lambda}}$$

Differential equations:

- For even $n + D_0$

$$
\begin{aligned}
d\mathcal{J}_{1\ldots n} =& \epsilon\, d\log W_{1\ldots n}\ \mathcal{J}_{1\ldots n} \\
&+ \epsilon \sum_{1\leq i\leq n} (-1)^{i+\lfloor \frac{n}{2}\rfloor} d\log W_{1\ldots(i)\ldots n}\ \mathcal{J}_{1\ldots\widehat{i}\ldots n} \\
&+ \epsilon \sum_{1\leq i<j\leq n} (-1)^{i+j+\lfloor \frac{n}{2}\rfloor} d\log W_{1\ldots(i)\ldots(j)\ldots n}\ \mathcal{J}_{1\ldots\widehat{i}\ldots\widehat{j}\ldots n},
\end{aligned}
$$

- For odd $n + D_0$

$$
\begin{aligned}
d\mathcal{J}_{1\ldots n} =& \epsilon\, d\log W_{1\ldots n}\ \mathcal{J}_{1\ldots n} \\
&+ \epsilon \sum_{1\leq i\leq n} (-1)^{i+\lfloor \frac{n+1}{2}\rfloor} d\log W_{1\ldots(i)\ldots n}\ \mathcal{J}_{1\ldots\widehat{i}\ldots n} \\
&+ \epsilon \sum_{1\leq i<j\leq n} (-1)^{i+j+\lfloor \frac{n+1}{2}\rfloor} d\log W_{1\ldots(i)\ldots(j)\ldots n}\ \mathcal{J}_{1\ldots\widehat{i}\ldots\widehat{j}\ldots n},
\end{aligned}
$$

Canonical master integrals

- From literature

dimension of the integral
$\Rightarrow$ DRR to D_0

$$\mathcal{J}_{i_1\ldots i_k} = \begin{cases} \dfrac{\epsilon^{\lfloor \frac{k}{2} \rfloor} \mathcal{I}^{(k)}_{i_1\ldots i_k}}{j_{i_1\ldots i_k}} & \text{for } k + D_0 \text{ even,} \\[2ex] \dfrac{\epsilon^{\lfloor \frac{k+1}{2} \rfloor} \mathcal{I}^{(k+1)}_{i_1\ldots i_k}}{j_{i_1\ldots i_k}} & \text{for } k + D_0 \text{ odd}\,, \end{cases}$$

[Abreu, Britto, Duhr, Gardi, '17 / Chen, Ma, Yang, '22]

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[Abreu, Britto, Duhr, Gardi, '17 / Chen, Ma, Yang, '22]

- Leading singularities

Gram determinant

$$j_{i_1\cdots i_k} = \begin{cases} 2^{-\frac{k}{2}+1}\left[(-1)^{\lfloor \frac{k}{2} \rfloor} \mathcal{Y}\begin{pmatrix} i_1+1 & i_2+1 & \cdots & i_k+1 \\ i_1+1 & i_2+1 & \cdots & i_k+1 \end{pmatrix}\right]^{-1/2}, & \text{for } k + D_0 \text{ even}, \\ 2^{-\frac{k+1}{2}+1}\left[(-1)^{\lfloor \frac{k+1}{2} \rfloor} \mathcal{Y}\begin{pmatrix} 1 & i_1+1 & i_2+1 & \cdots & i_k+1 \\ 1 & i_1+1 & i_2+1 & \cdots & i_k+1 \end{pmatrix}\right]^{-1/2}, & \text{for } k + D_0 \text{ odd}. \end{cases}$$

Cayley determinant

Comparison with the literature

- Symbol alphabet
- Differential equations

1) Diagrammatic coaction:

[Abreu, Britto, Duhr, Gardi, '17] ⟵ only next-to-next-to maximal cut needed

2) Baikov representation:

[Chen, Ma, Yang, '22] ⟵ explicit match

[Jiang, Yang, '23] ⟵ modified Cayley matrix

Limits to non-generic cases

$$\widetilde{E_A}(\mathcal{G}_h) = \mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \prod_{i=1}^{n+1} \mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix} \dots \prod_{i_{n-1}>\dots>i_1=1}^{n+1} \mathcal{Y}\begin{bmatrix} i_1 \dots i_{n-1} \\ i_1 \dots i_{n-1} \end{bmatrix} \prod_{i=2}^{n+1} \mathcal{Y}_{ii}$$

- Consider limits, e.g. $m_i^2, s_{ij}^2 \to 0$ [Klausen, '21]
  - remove vanishing factors
  - ➢ leading term in Tailor expansion

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  - remove vanishing factors
  - ➢ leading term in Tailor expansion
  - multivariate limit for individual factors is not unique, however, limit of principle A-determinant is!
- Limits matches direct computation (omit vanishing factors also here)
  - expect also to work for symbol alphabet [Abreu, Britto, Duhr, Gardi, ’17] [Chen, Ma, Yang, ‘22]

Higher loops

- Principal A-determinant:

$$\overline{\left\{ s_{ij}, m_i^2 \,|\, \mathbf{V}\left(x_0 \frac{\partial \mathcal{G}_h}{\partial x_0}, \ldots, x_n \frac{\partial \mathcal{G}_h}{\partial x_n} \right) \neq \emptyset \right\}} \qquad \Longleftrightarrow \qquad \widetilde{E_A}(\mathcal{G}_h) = 0$$

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  - Prime factorization:

$$\mathcal{G}_h = \sum_{i=0}^{r} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i}$$

$$A = \mathrm{Supp}(\mathcal{G}_h)$$

$$\mathrm{Newt}(\mathcal{G}_h) = \mathrm{conv}\,(A)$$

$$\widetilde{E_A}(\mathcal{G}_h) = \prod_{\Gamma \subseteq \mathrm{Newt}(\mathcal{G}_h)} \Delta_{A \cap \Gamma}(\mathcal{G}_h|_\Gamma),$$

faces of Q

A-discriminants

restriction of $\mathcal{G}_h$ on Γ

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faces of Q — A-discriminants — restriction of $\mathcal{G}_h$ on Γ

  - singularities:

type-I	type-II	mixed
$\Delta_{A \cap \mathrm{Newt}(\mathcal{F})}(\mathcal{F}),$	$\Delta_A(\mathcal{G}),$	$\Delta_{A \cap \Gamma}(\mathcal{G}\|_\Gamma),$

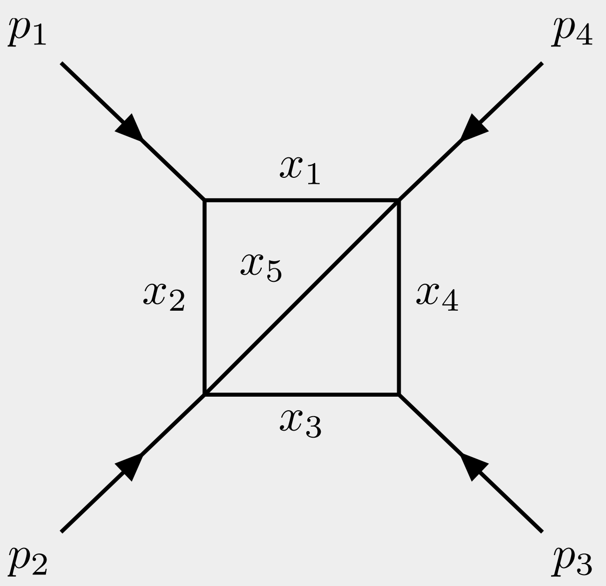
vertices on both $\mathrm{Supp}(\mathcal{U})$ and $\mathrm{Supp}(\mathcal{F})$

Slashed box example

- One-mass configuration

$$p_1^2 \neq 0, \quad m_i^2 = p_2^2 = p_3^2 = p_4^2 = 0$$

$$\widetilde{E_A}(\mathcal{G}_h) = (p_1^2 - t)(p_1^2 - s)(p_1^2 - s - t)(s + t)stp_1^2$$



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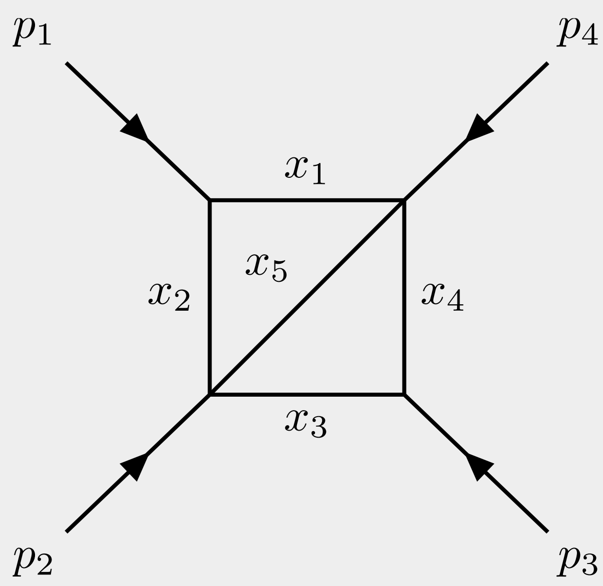
  - two-dimensional harmonic polylogarithms:

$$z_1 = \frac{s}{p_1^2}, z_2 = \frac{t}{p_1^2}, z_3 = 1 - z_1 - z_2$$

$$\widetilde{E_A}(\mathcal{G}_h) \propto (1 - z_2)(1 - z_1)z_3(1 - z_3)z_1 z_2$$



Summary

- Construction of symbol alphabet from the principal A-determinant
  - rational letters
  - square-root letters through re-factorization
- One loop
  - construction based on principal A-determinant
  - re-factorization through Jacobi identities
  - verification through canonical DEs (up to ten legs)
- Unique limits
- Higher loops