Algorithms for computing three-loop Feynman integrals in Higgs plus jet production

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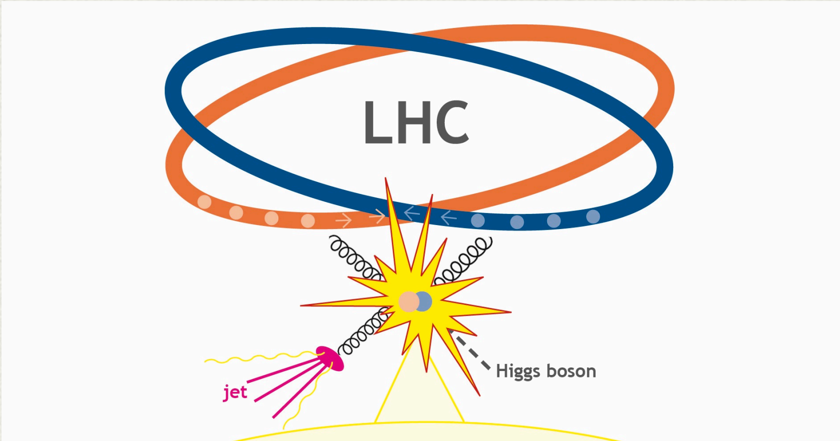
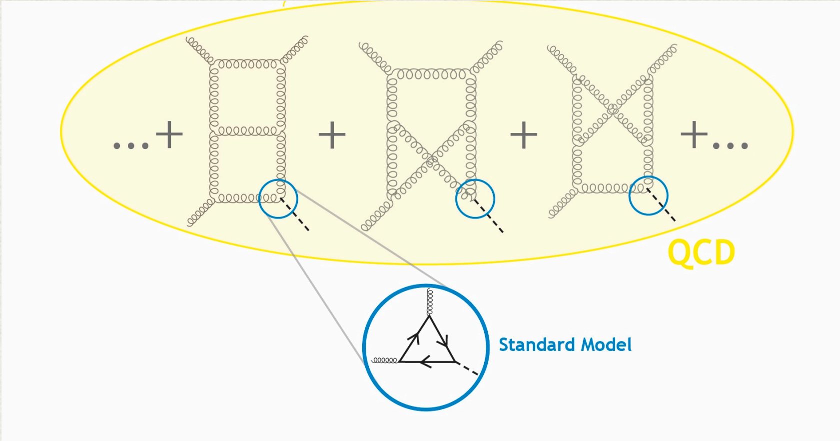
Outline

- Motivation
- Leading and Landau singularities
- Three-loop Feynman integrals for Higgs plus jet production
- Three-loop scattering amplitudes for Higgs plus jet production
- Conclusions/Outlook

Motivation

- Phenomenology :: Higgs/vector + jet production —> N3LO



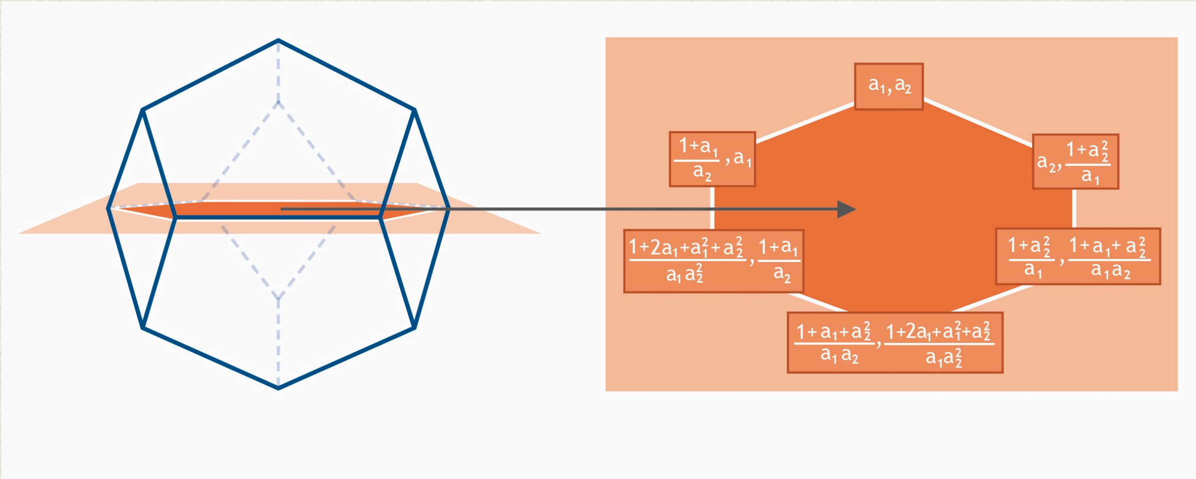


- Mathematically :: Bootstrapping approaches :: N=4 sYM form factors

[Dixon, Gurdogan, McLeod, Wilhelm (2020)]

C_2 Cluster algebras

[Chicherin, Henn, Papathanasiou (2020)]



https://scattering-amplitudes.mpp.mpg.de/updates/surprising-cluster-algebraic-structures

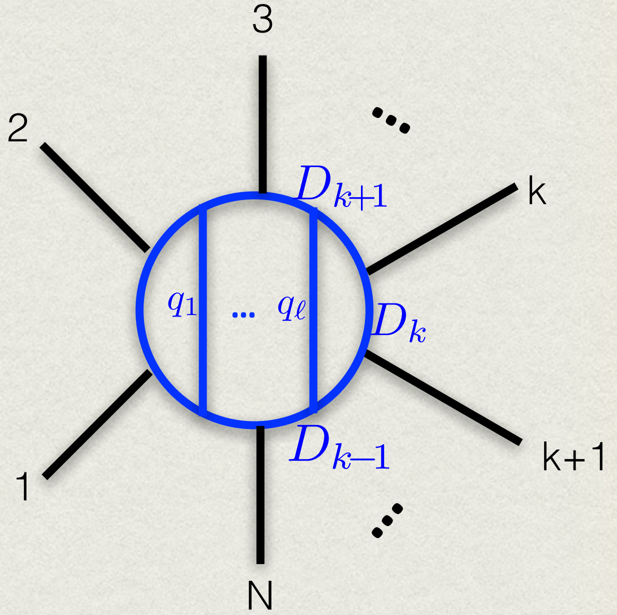
Leading & Landau singularities

Algorithms for computing Feynman integrals

- In loop calculations, one finds

$$J_N^{(L),D}(1,\ldots,n;n+1,\ldots,m) = \int \prod_{i=1}^{L} \frac{d^D\ell_i}{\imath\, \pi^{D/2}} \frac{\prod_{k=n+1}^{m} D_k^{-\nu_k}}{\prod_{j=1}^{n} D_j^{\nu_j}}$$

$$D_i = q_i^2 - m_i^2 + \imath 0$$



- DEQ :: Feynman integrals are not independent

$$\partial_x \vec{J}(x) = A_i(x,\epsilon)\vec{J}(x)$$

Canonical form
Conjecture: there exist a basis of <u>uniform transcendental weight functions</u> **[Henn (2013)]**

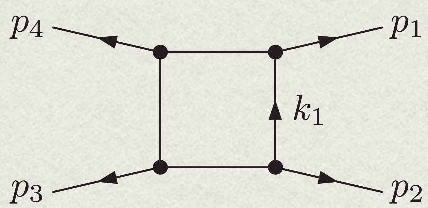
$$\partial_x \vec{g}(x) = \epsilon B(x)\vec{g}(x) \longrightarrow d\vec{g}(x,\epsilon) = \epsilon\left(d\tilde{B}\right)\vec{g}(x;\epsilon)$$

$$\tilde{B} = \sum_k B_k \log \alpha_k(x)$$

Uniform weight function

Dlog representation of Feynman integrals

- Four-point integral family

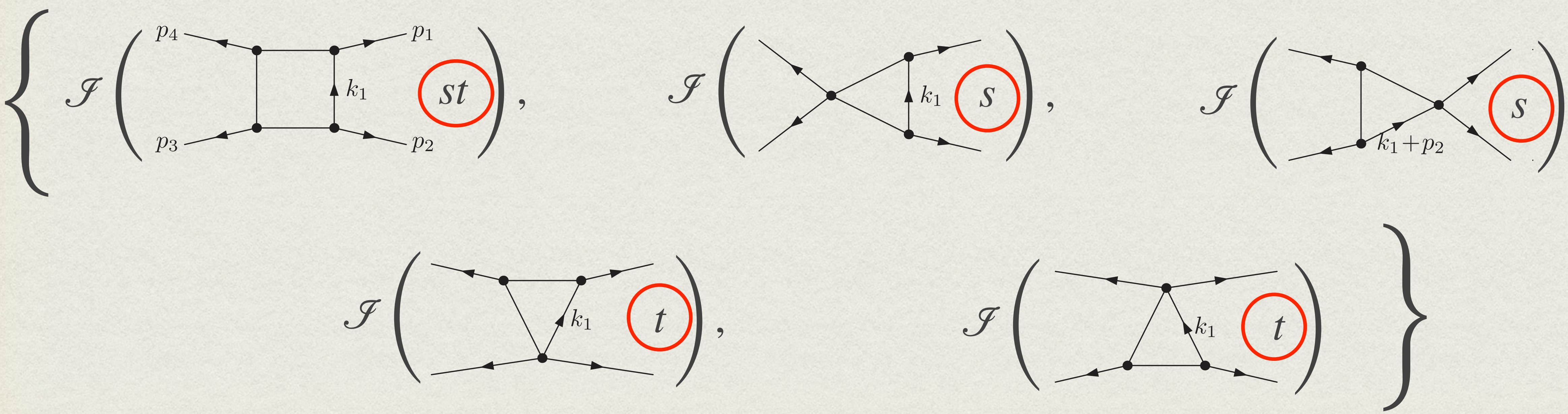


$$\mathcal{I}\left(\cdots \; \mathcal{N} \right) = \frac{d^4k_1\,\mathcal{N}}{(k_1-p_1)^2\,k_1^2\,(k_1+p_2)^2\,(k_1+p_2+p_3)^2} \overset{?}{=} d\log\tau_1\,\ldots d\log\tau_4$$

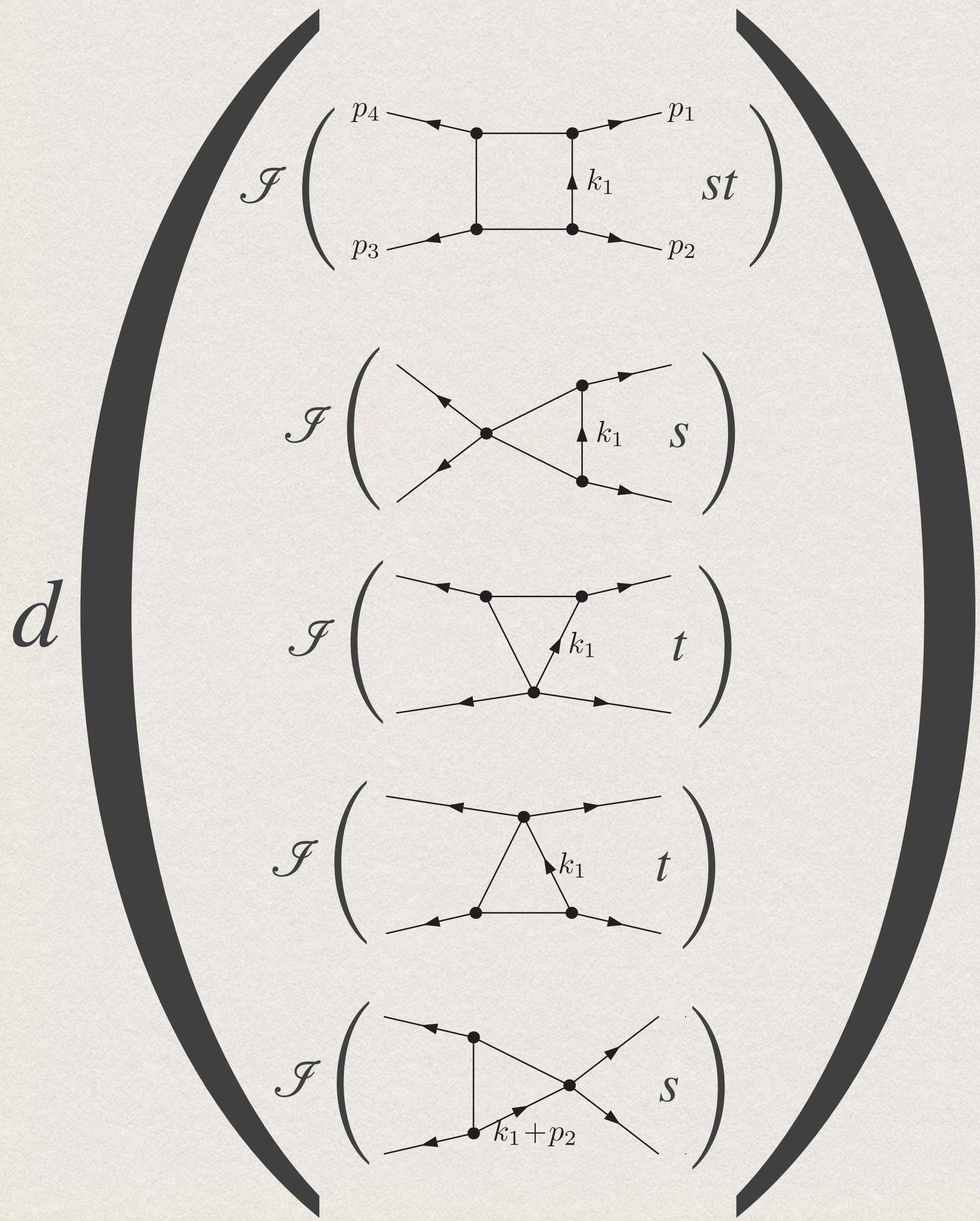
$$s=(p_1+p_2)^2\,,\; t=(p_2+p_3)^2$$

- Obtained with the aid of **[Wasser '18], [Henn++ '20]**

Leading Logarithmic singularities

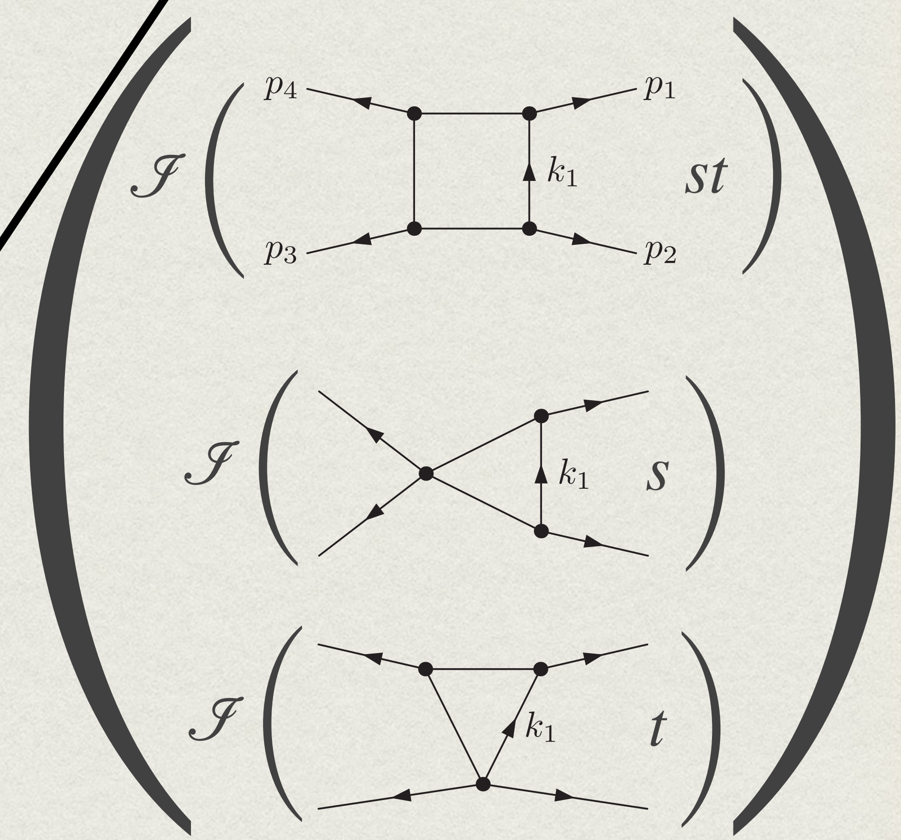


DEQ for Dlog integrals



$$= \epsilon \, d\tilde{A}$$

$$\begin{pmatrix} \mathrm{dlog}(s+t)-\mathrm{dlog}(s)-\mathrm{dlog}(t) & 2\mathrm{dlog}(s)-2\mathrm{dlog}(s+t) & 2\mathrm{dlog}(t)-2\mathrm{dlog}(s+t) \\ 0 & -\mathrm{dlog}(s) & 0 \\ 0 & 0 & -\mathrm{dlog}(t) \\ 0 & -\mathrm{dlog}(s) & 0 \\ 0 & 0 & -\mathrm{dlog}(t) \end{pmatrix}$$



Landau singularities

Feynman integral are many-valued analytic function whose singularities lie on some algebraic varieties — Landau Varieties

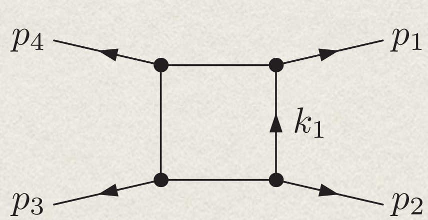
- Landau equations

$$q_i^2 - m_i^2 = 0 \text{ or } \alpha_i = 0$$

$$\sum \alpha_i \frac{\partial D_i}{\partial k_j} = \sum \alpha_i \, q_i = 0$$

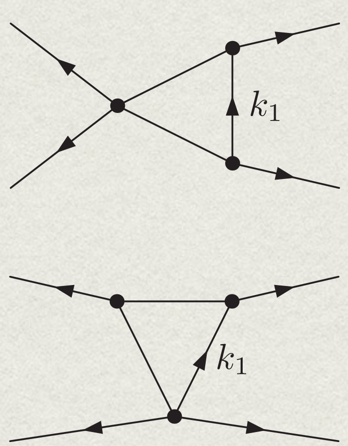
Connection between leading and Landau singularities?

- Landau singularity of a one-loop scalar integrals in D=4



$\to -s^2 t^2$

Landau Singularities of first type



$\to -s^2$

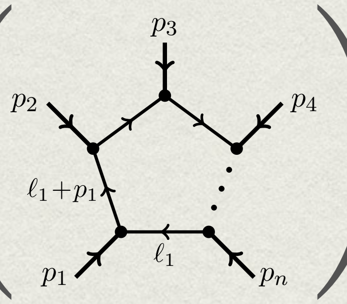
$\to -t^2$

Landau Singularities of second type

Leading & Landau singularities

- Connection between leading & Landau singularities

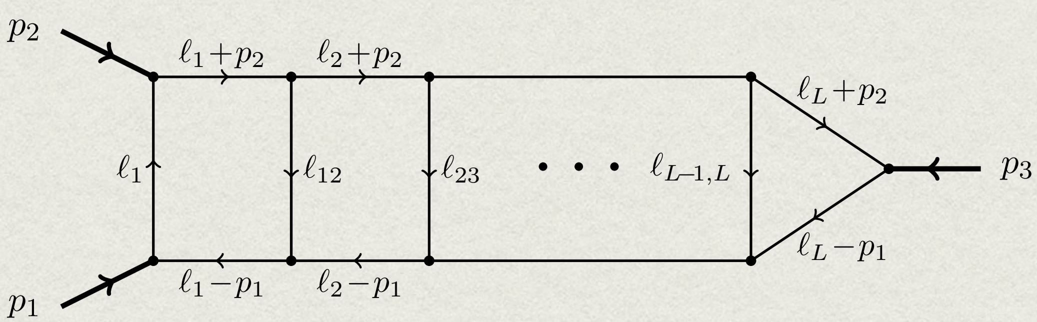
[Flieger, WJT (2022)]

$$\mathrm{LS}\left(\right.$$


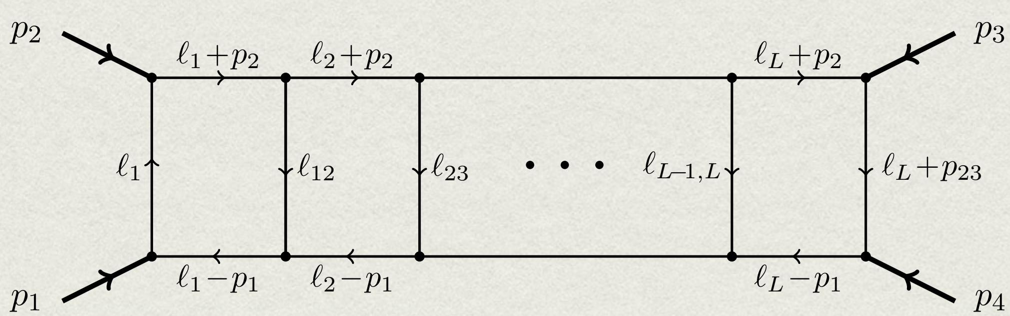
$$\left.\right) \sim \frac{1}{\sqrt{\mathrm{LanS}_n}}$$

D=n :: Landau Singularities of first type
D=n+1 :: Landau Singularities of second type (Gram determinants)

- Loop-by-loop application & use of Leray's residues

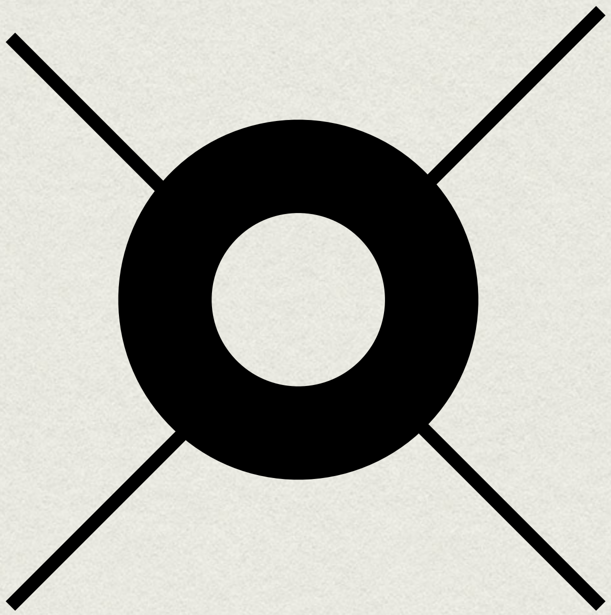


$$\longrightarrow \frac{1}{(p_3^2)^L \sqrt{\lambda_K\left(1, \frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right)}}$$



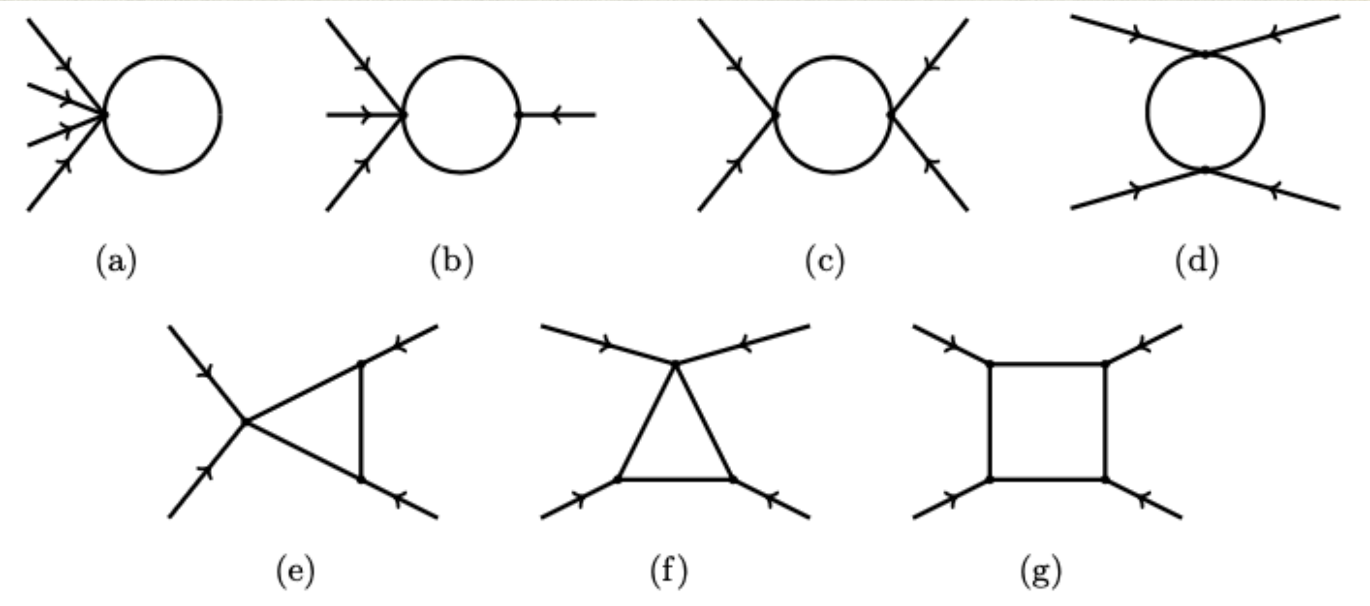
$$\longrightarrow \frac{1}{s^L\, t \sqrt{\lambda_K\left(1, \frac{p_1^2 p_3^2}{st}, \frac{p_2^2 p_4^2}{st}\right)}}$$

Differential equations of one-loop Feynman integrals



$s = (p_1 + p_2)^2\,,\; t = (p_2 + p_3)^2\,,\; p_i^2 = m^2\,.$

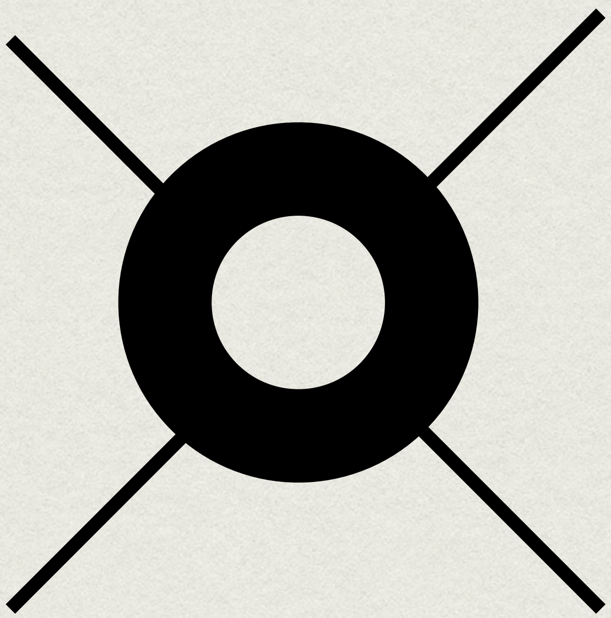
Master integrals (Laporta basis)



- $\partial_x \vec{J} = M(x, \epsilon)\vec{J} \text{ —> } \partial_x \vec{g} = \epsilon A_x \vec{g}$

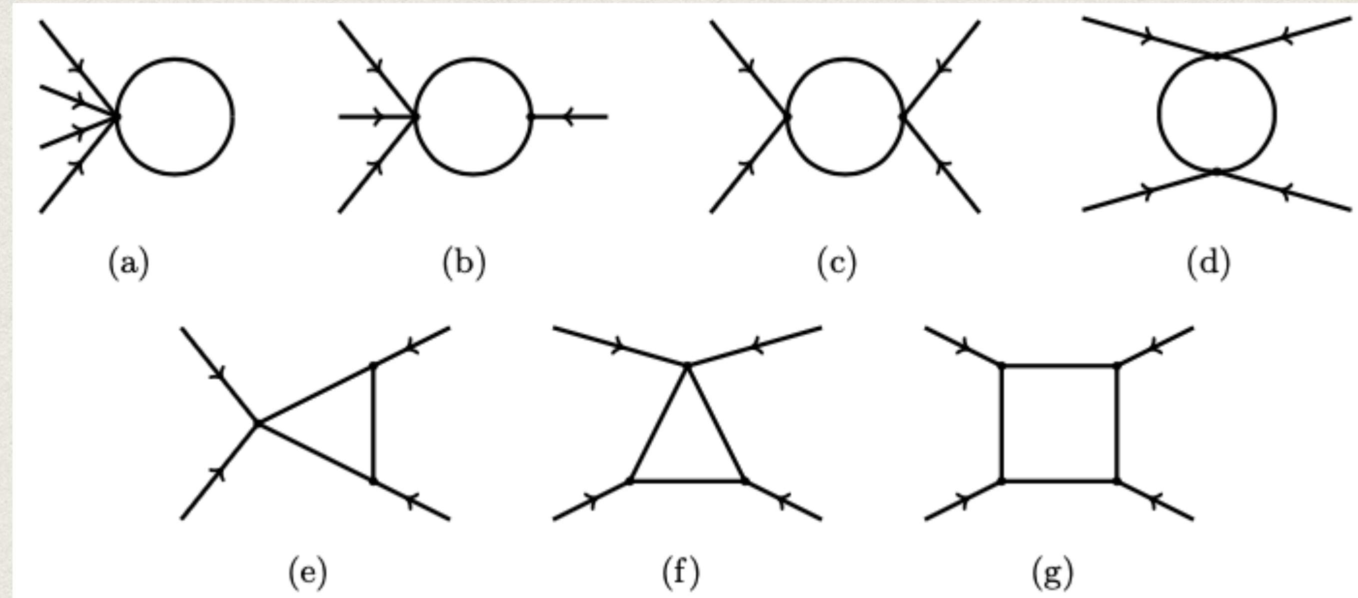
Transform differential equation into *canonical form*

Differential equations of one-loop Feynman integrals



$s = (p_1 + p_2)^2\,,\ t = (p_2 + p_3)^2\,,\ p_i^2 = m^2\,.$

Master integrals (Laporta basis)

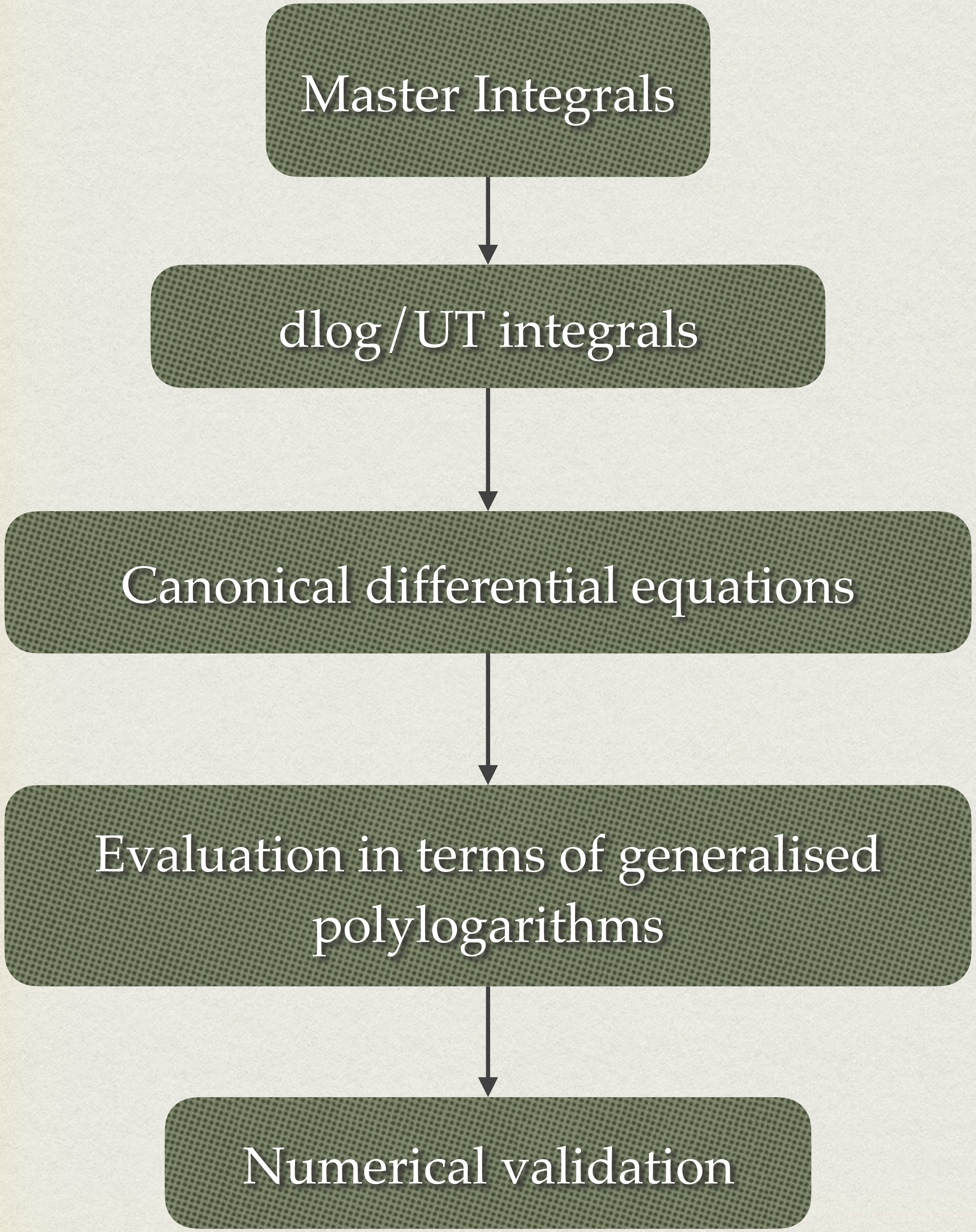


- Canonical differential equations ($\partial_x \vec{g} = \epsilon A_x \vec{g}$) in even & odd dimensions

$$
\begin{aligned}
g_1 &= \epsilon\, J_1^{(1),D=2-2\epsilon}(1)\,,\\
g_2 &= \epsilon\, m^2\, J_2^{(1),D=2-2\epsilon}(1,4)\,,\\
g_3 &= \epsilon\, \sqrt{-s\,(4m^2-s)}\, J_2^{(1),D=2-2\epsilon}(1,3)\,,\\
g_4 &= \epsilon\, \sqrt{-t\,(4m^2-t)}\, J_2^{(1),D=2-2\epsilon}(2,4)\,,\\
g_5 &= \epsilon^2\, \sqrt{-s\,(4m^2-s)}\, J_3^{(1),D=4-2\epsilon}(1,2,4)\,,\\
g_6 &= \epsilon^2\, \sqrt{-t\,(4m^2-t)}\, J_3^{(1),D=4-2\epsilon}(1,2,3)\,,\\
g_7 &= \epsilon^2\, \sqrt{st\,(12m^4-4m^2\,(s+t)+st)}\, J_4^{(1),D=4-2\epsilon}(1,2,3,4)\,.
\end{aligned}
$$

$$
\begin{aligned}
g_1 &= \epsilon\, \sqrt{m^2}\, J_1^{(1),D=1-2\epsilon}(1)\,,\\
g_2 &= \epsilon^2\, \sqrt{m^2}\, J_2^{(1),D=3-2\epsilon}(1,4)\,,\\
g_3 &= \epsilon^2\, \sqrt{-s}\, J_2^{(1),D=3-2\epsilon}(1,3)\,,\\
g_4 &= \epsilon^2\, \sqrt{-t}\, J_2^{(1),D=3-2\epsilon}(2,4)\,,\\
g_5 &= \epsilon^2\, \sqrt{-sm^2\,(3m^2-s)}\, J_3^{(1),D=3-2\epsilon}(1,2,4)\,,\\
g_6 &= \epsilon^2\, \sqrt{-tm^2\,(3m^2-t)}\, J_3^{(1),D=3-2\epsilon}(1,2,3)\,,\\
g_7 &= \epsilon^3\, \sqrt{-st\,(4m^2-s-t)}\, J_4^{(1),D=5-2\epsilon}(1,2,3,4)\,.
\end{aligned}
$$

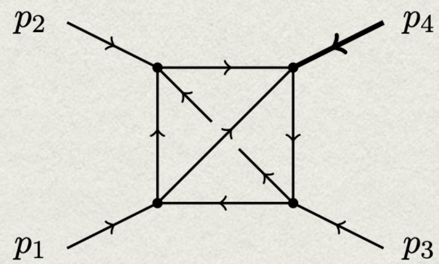
Algorithms for computing Feynman integrals



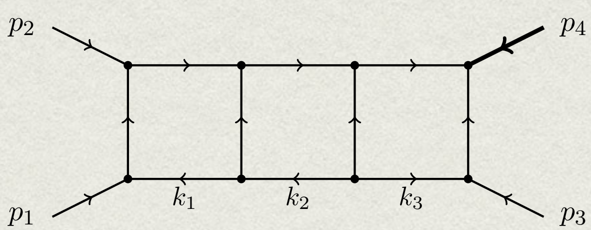
- `DlogBasis` + UT integrals in subsectors. **[Wasser (2020)]**
- IBP reductions w/ `FIRE6+Kira` **[Smirnov, Chuharev (2019)]** **[Klappert, Lange, Maierhöfer, Usovitsch (2020)]**
- Differential equations: in-house implementation+`LiteRed`. **[Lee (2012)]**
- Analytic reconstruction w/ `FiniteFlow`. **[Peraro (2019)]**
- Algebraic manipulation of GPLs with `PolylogTools` **[Duhr, Dulat (2019)]**
- Numerical evaluation w/ `PySecDec` & `FeynTrop` **[Borowka et al (2017)]** **[Borinsky, Munch, Tellander (2023)]**

Three-loop Feynman integrals for Higgs plus jet production

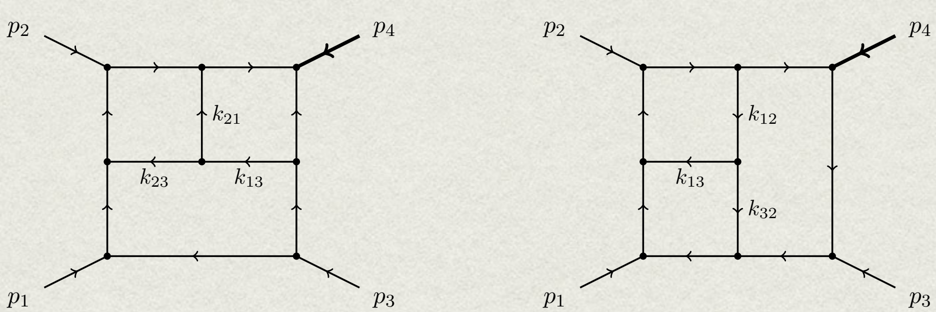
State-of-the-art at three-loop



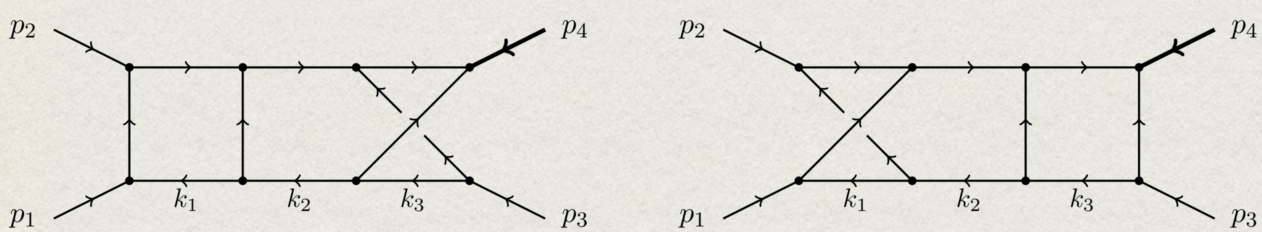
[Henn, Smirnov, Smirnov (2013)]



[di Vita, Mastrolia, Schubert, Yundin (2014)]



[Canko, Syrrakos (2021)]



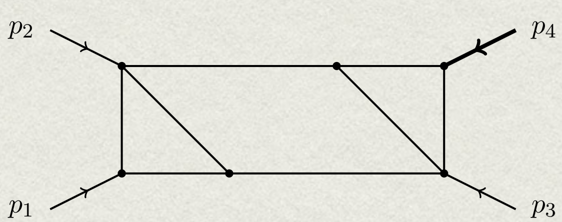
[Henn, Lim, WJT (2023)]

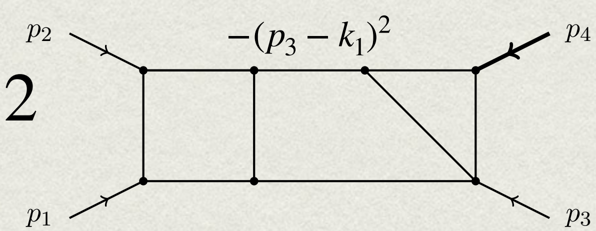
dlog/UT integrals

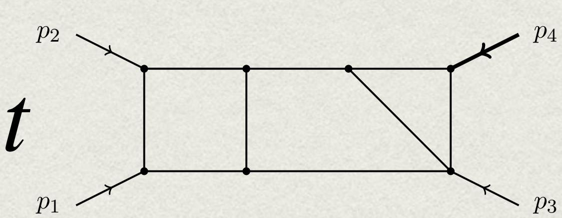
- Dlog basis :: set of integrals admitting a dlog representation

$$\mathcal{I}^{(L)} = \sum_{k=1}^{4L} c_k \, d\log g_1^{(k)} \wedge d\log g_2^{(k)} \wedge \ldots \wedge d\log g_n^{(k)} \longrightarrow \oint_{g_i=0} \mathcal{J} = 1$$

Automated by `DlogBasis` **[Wasser (2020)]**

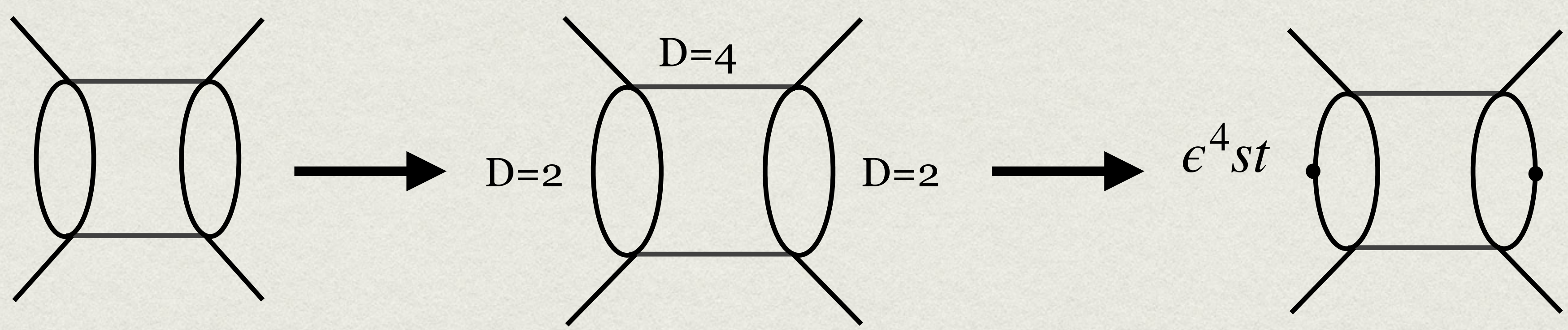
st


$+s^2$


$+s^2 t$


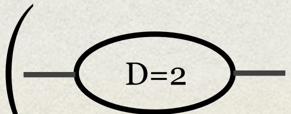
- Uniform transcendental basis :: dlog integrals in particular spacetime dimensions

[Flieger, WJT (2022)]



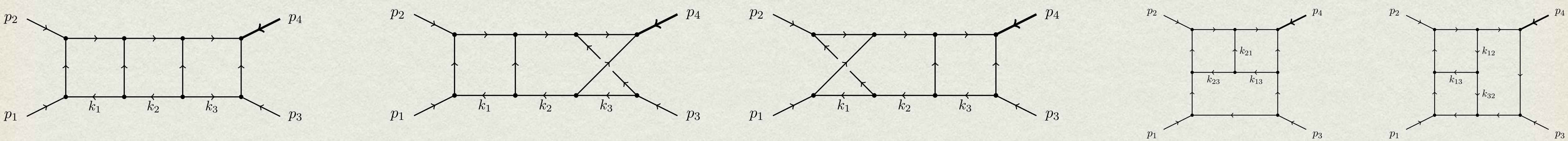
not UT in D=4

UT in D=4

Leading Singularity $\Big($  $\Big) \sim \frac{1}{p^2}$

Differential equations in canonical form

[Henn, Lim, WJT (2023)]



A B1 B2 E1 E2

	A	B1	B2	E1	E2
# master integrals	83	150	114	166	117
# dlog integrals	75	124	106	151	116
# UT integrals	8	26	8	15	1
# Letters	6	8	6	6	6

$$d\vec{f}\left(\vec{x},\epsilon\right) = \epsilon\left(d\tilde{A}\right)\vec{f}\left(\vec{x};\epsilon\right),$$

with $\tilde{A} = \left[\sum_k A_k \log\alpha_k(x)\right]$

- Total derivate in terms of kinematic invariants $s = (p_1+p_2)^2\,, t = (p_1+p_3)^2\,, p_4^2 \neq 0\,.$

- Alphabet

$$\overrightarrow{\alpha} = \left\{\alpha_0, \ldots, \alpha_8\right\} = \left\{p_4^2, s, t, -p_4^2+s+t, -p_4^2+s, -p_4^2+t, s+t, -\left(p_4^2-s\right)^2+p_4^2 t, s^2-p_4^2(s-t)\right\}.$$

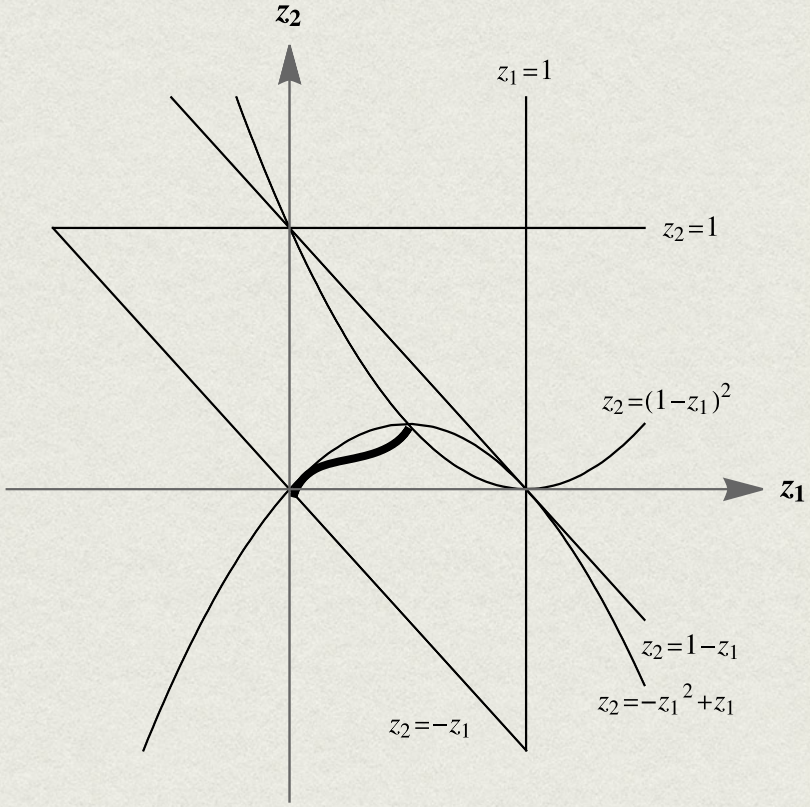
Evaluation of integrals in terms of GPLs

- Solve order-by-order in ϵ in terms of $z_1 = \dfrac{-s}{-p_4^2}$, $z_2 = \dfrac{-t}{-p_4^2}$.

$$d\vec{f}_X = \epsilon \sum_{i=0}^{8} \tilde{A}_{X;i} d \log \alpha_i \vec{f}_X$$

up-to transcendental weight six

$$\vec{f}(z_1, z_2; \epsilon) = \mathbb{P} \exp\left(\epsilon \int_\gamma d\tilde{A}\right) \vec{f}_0(\epsilon)$$



Evaluation of integrals in terms of GPLs

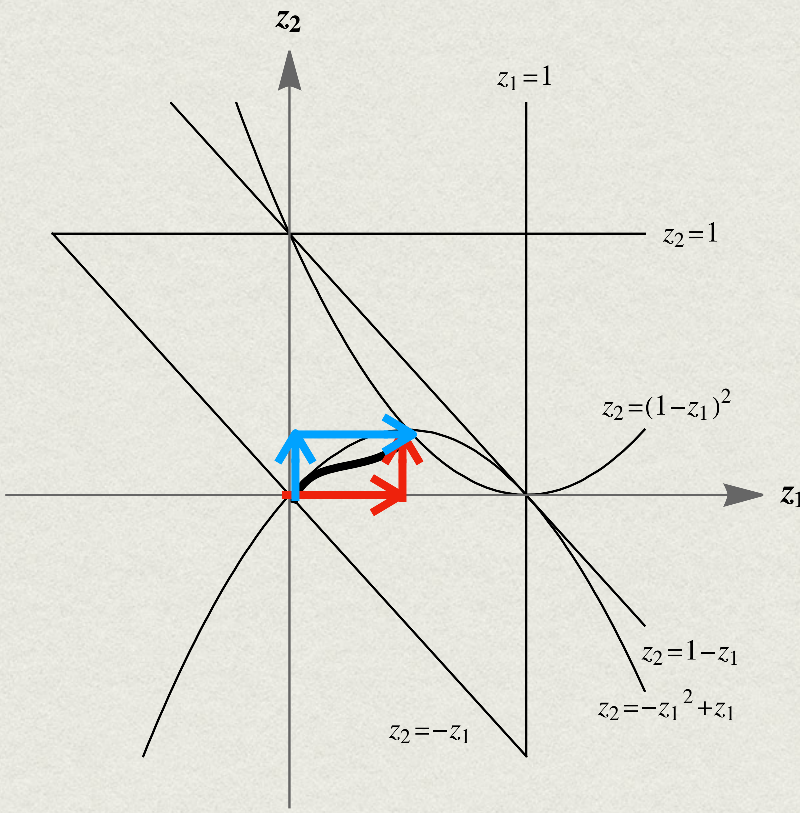
① →
② →

- Solve order-by-order in ϵ in terms of $z_1 = \dfrac{-s}{-p_4^2}$, $z_2 = \dfrac{-t}{-p_4^2}$.

$$d\vec{f}_X = \epsilon \sum_{i=0}^{8} \tilde{A}_{X;i} d \log \alpha_i \vec{f}_X$$

↓ up-to transcendental weight six

$$\vec{f}(z_1, z_2; \epsilon) = \mathbb{P} \exp \left(\epsilon \int_\gamma d\tilde{A} \right) \vec{f}_0(\epsilon)$$



- Integration path γ made of two segments:

$$\vec{g}^{(n)}(z_2) = \vec{f}_0^{(n)} + \int_0^{z_2} d\bar{z}_2 \left[A_{z_2}(z_1, \bar{z}_2) \vec{g}^{(n-1)}(\bar{z}_2) - \partial_{\bar{z}_2} \int_0^{z_1} d\bar{z}_1 A_{z_1}(\bar{z}_1, \bar{z}_2) \vec{f}^{(n-1)}(\bar{z}_1, \bar{z}_2) \right]$$

$$\vec{g}^{(n)}(z_1) = \vec{f}_0^{(n)} + \int_0^{z_1} d\bar{z}_1 \left[A_{z_1}(\bar{z}_1, z_2) \vec{g}^{(n-1)}(\bar{z}_1) - \partial_{\bar{z}_1} \int_0^{z_2} d\bar{z}_2 A_{z_2}(\bar{z}_1, \bar{z}_2) \vec{f}^{(n-1)}(\bar{z}_1, \bar{z}_2) \right]$$

Evaluation of integrals in terms of GPLs

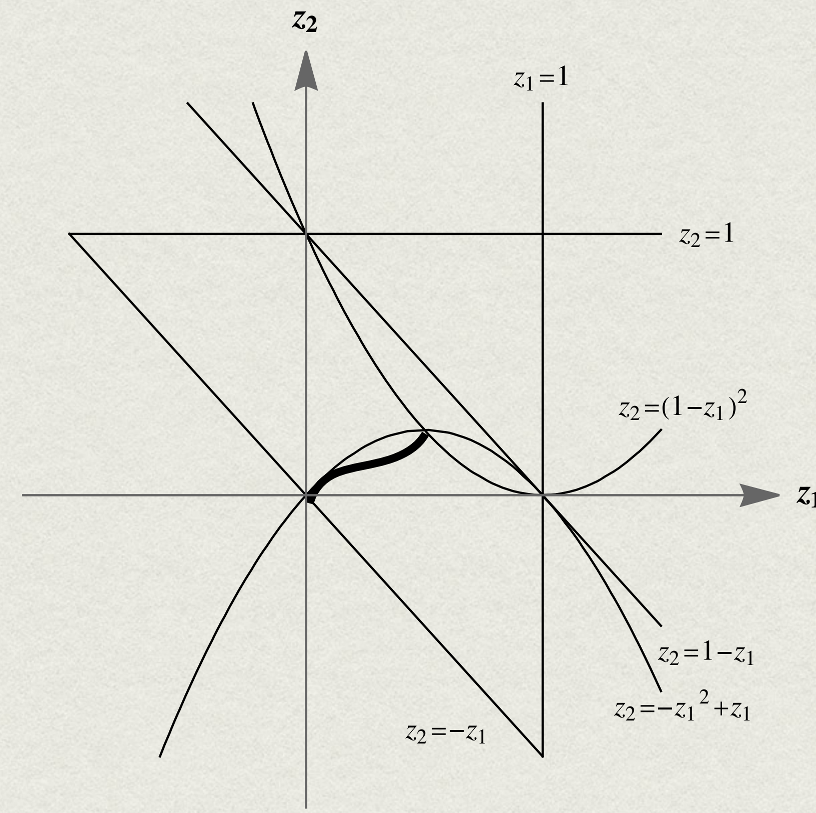
- Solve order-by-order in ϵ in terms of $z_1 = \dfrac{-s}{-p_4^2}\,, z_2 = \dfrac{-t}{-p_4^2}\,.$

$$d\vec{f}_X = \epsilon \sum_{i=0}^{8} \tilde{A}_{X;i} d \log \alpha_i \vec{f}_X$$

$\downarrow$ up-to transcendental weight six

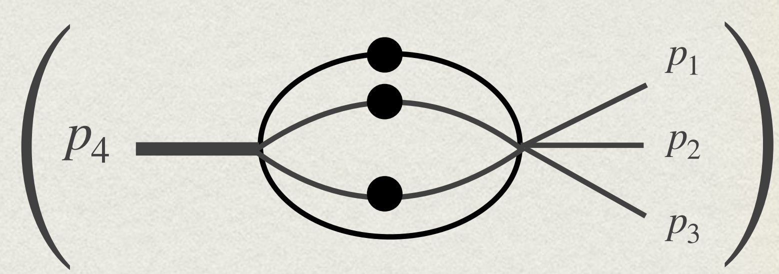
$$\vec{f}(z_1, z_2; \epsilon) = \mathbb{P} \exp\left(\epsilon \int_\gamma d\tilde{A}\right) \vec{f}_0(\epsilon)$$

$\vec{f}_0(\epsilon)$ is a boundary vector



- Boundary conditions :: fixed by studying physical and unphysical thresholds.

$$\lim_{\alpha_i \to 0} \vec{f} = \alpha_i^{\epsilon \tilde{A}_i} \vec{f}\left(\alpha_i = 0\right)$$

$\epsilon^3 p_4^2$


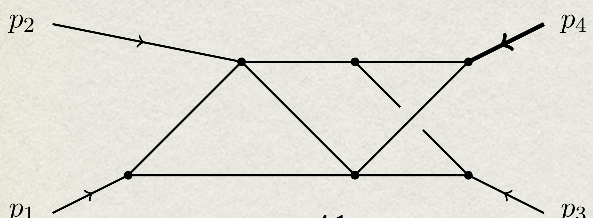
Only one integral needed up-to ϵ^6
(or transcendental weight six)

Numerical validation

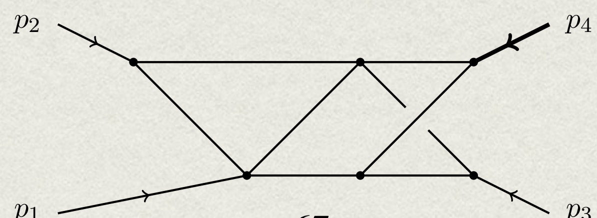
- Analytic expression for master integrals evaluated numerically for sample points in the Euclidean (shaded region) region using `GiNac` through `PolyLogTools`.

- Perfect agreement w/ `PySecDec` and `FeynTrop`.

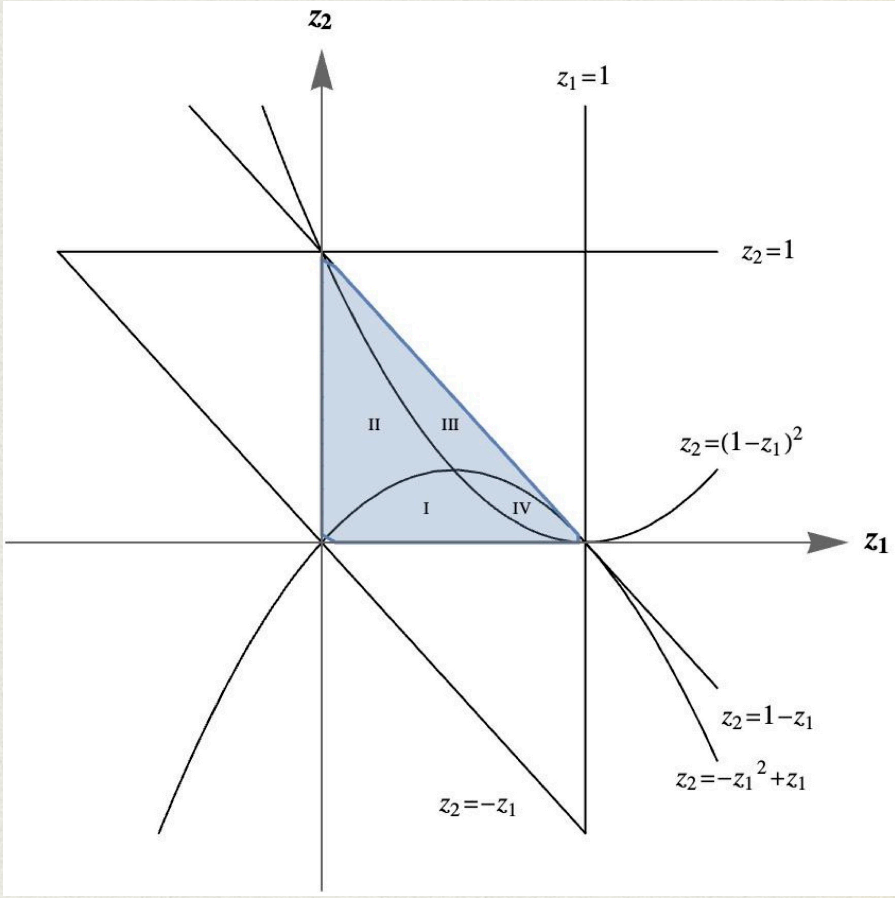
- Integrals of family B1 are manifestly real-valued in region I.



f^{41}_{B1}



f^{67}_{B1}



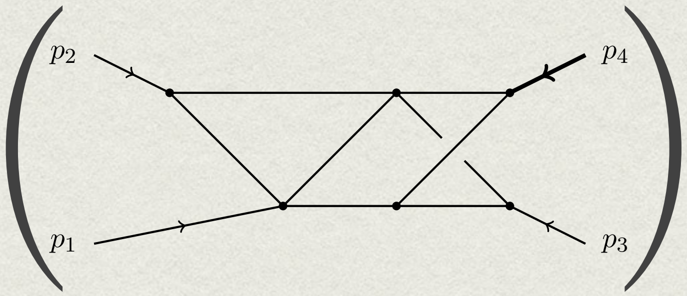
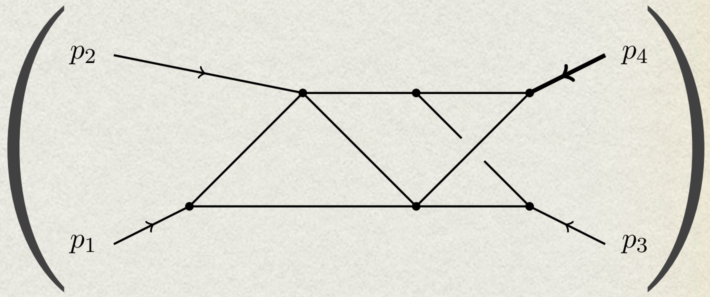
Integral	Evaluation point	ϵ^3		ϵ^4		ϵ^5		ϵ^6	
		Analytic	pySecDec	Analytic	pySecDec	Analytic	pySecDec	Analytic	pySecDec
f^{41}_{B1}	Point 1	0.3768713705	0.37687137(8)	0.2595847621	0.259585(2)	−24.1653497052	−24.1653(2)	−255.4746048147	−255.474(2)
	Point 2	0.0882252953	0.08822531(6)	0.1851070156	0.185107(1)	−3.5650885140	−3.56509(1)	−45.4350139041	−45.4350(2)
f^{67}_{B1}	Point 1	−6.1800769944	−6.1800771(7)	−37.5823284468	−37.58232(7)	−38.4079844011	−38.4080(4)	897.7904682990	897.790(7)
	Point 2	0.3592309958	0.35923099(3)	−1.1083670295	−1.108367(1)	−38.2406764190	−38.2407(1)	−367.9705607540	−367.970(1)

Table 3. Numerical check of integrals f^{41}_{B1} and f^{67}_{B1} against pySecDec at the kinematic points: point 1: $\{s,t,p_4^2\}=\{-0.11,-0.73,-1.00\}$, and point 2: $\{s,t,p_4^2\}=\{-0.18,-0.013,-0.25\}$.

New letters in the alphabet

[Henn, Lim, WJT (2023)]

$$\vec{\alpha} = \{\alpha_0, \ldots, \alpha_8\} = \left\{p_4^2, s, t, -p_4^2+s+t, -p_4^2+s, -p_4^2+t, s+t, -\left(p_4^2-s\right)^2+p_4^2 t, s^2-p_4^2(s-t)\right\}.$$

$\epsilon^6\left(\left(p_4^2-s\right)^2-p_4^2 t\right)$  $\longleftrightarrow$ $\epsilon^6\left(\left(p_4^2-s\right)^2-p_4^2 u\right)$ 

$p_1 \leftrightarrow p_2$ symmetry

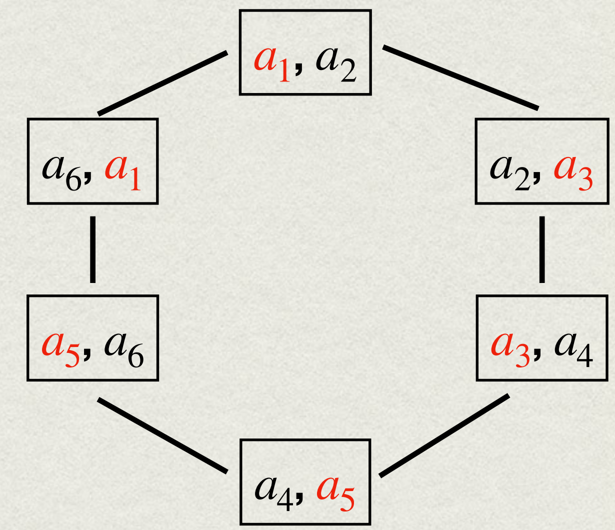
Start appearing at weight 4

$$\mathcal{S}\left(f_{B1}^{41}\right)\Big|_{\epsilon^4} = 6\Big[\alpha_1 \otimes \alpha_1 \otimes \frac{\alpha_2}{\alpha_4} \otimes \alpha_7 - \alpha_1 \otimes \alpha_1 \otimes \alpha_4 \otimes \alpha_7 + \alpha_1 \otimes \frac{\alpha_4}{\alpha_2} \otimes \frac{\alpha_3}{\alpha_1\alpha_4} \otimes \alpha_7 + \alpha_2 \otimes \alpha_1 \otimes \frac{\alpha_1\alpha_4}{\alpha_3} \otimes \alpha_7 + \alpha_2 \otimes \alpha_5 \otimes \frac{\alpha_3}{\alpha_1} - \frac{1}{2}\alpha_2 \otimes \alpha_5 \otimes \alpha_2 \otimes \alpha_7 + \ldots\Big]$$

Adjacency conditions

[Chicherin, Henn, Papathanasiou (2020)]

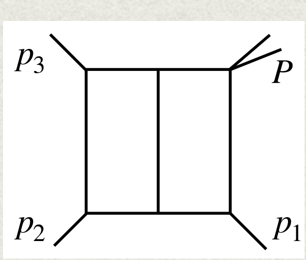
- C_2 cluster algebra

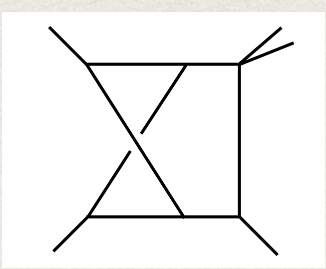


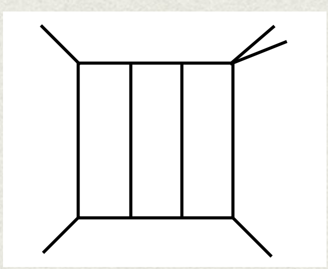
$$\Phi_{C_2} = \{a_1, \ldots, a_6\} = \left\{ a_1, a_2, \frac{1+a_2^2}{a_1}, \frac{1+a_1+a_1^2}{a_1 a_2}, \frac{1+2a_1+a_1^2+a_2^2}{a_1 a_2^2}, \frac{1+a_1}{a_2} \right\}$$

- Connection to loop integrals w/ one massive leg

$$z_1 = -\frac{a_2^2}{1+a_1}, z_2 = -\frac{1+a_1+a_2^2}{a_1(1+a_1)} \longrightarrow \vec{\alpha} = \{z_1, z_2, 1-z_1-z_2, 1-z_1, 1-z_2, z_1+z_2\}$$







Adjacency conditions

[Chicherin, Henn, Papathanasiou (2020)]

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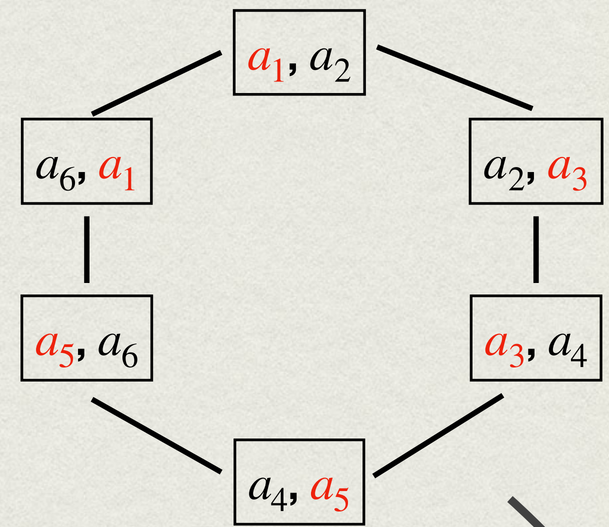
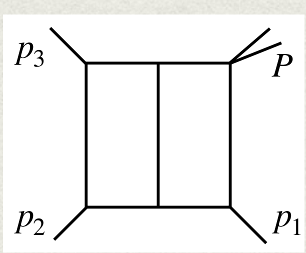
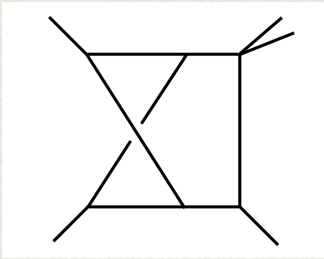
$$\Phi_{C_2} = \{a_1, \ldots, a_6\} = \left\{ a_1, a_2, \frac{1+a_2^2}{a_1}, \frac{1+a_1+a_1^2}{a_1 a_2}, \frac{1+2a_1+a_1^2+a_2^2}{a_1 a_2^2}, \frac{1+a_1}{a_2} \right\}$$

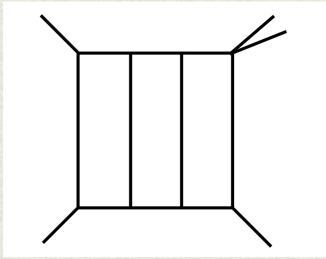


- Connection to loop integrals w/ one massive leg

$$z_1 = -\frac{a_2^2}{1+a_1}, z_2 = -\frac{1+a_1+a_2^2}{a_1(1+a_1)} \longrightarrow \vec{\alpha} = \{z_1, z_2, 1-z_1-z_2, 1-z_1, 1-z_2, z_1+z_2\}$$







$$\tilde{A} = \sum_i \tilde{A}_i \log \alpha_i(\vec{z})$$

$$\tilde{A}_i \cdot \tilde{A}_j = 0 \Longrightarrow \ldots \otimes \alpha_i \otimes \alpha_j \otimes \ldots$$

for $i, j \in \{4,5,6\}$ with $i \neq j$

Partially checked for tennis-court like diagrams (E1 & E2).

[Canko, Syrrakos (2020)]

Adjacency conditions

[Henn, Lim, WJT (2023)]

$$d\vec{f}(\vec{z};\epsilon) = \epsilon \left[\sum_i A_i \, d\log \alpha_i(\vec{z}) \right] \vec{f}_0(\epsilon)$$

$$\vec{\alpha} = \{\alpha_0, \ldots, \alpha_8\} = \left\{ p_4^2, s, t, -p_4^2+s+t, -p_4^2+s, -p_4^2+t, s+t, -\left(p_4^2-s\right)^2 + p_4^2 t, s^2 - p_4^2(s-t) \right\}.$$

- From the C_2 cluster algebra one expects

$$A_i \cdot A_j = 0 \Longrightarrow \ldots \otimes \alpha_i \otimes \alpha_j \otimes \ldots \quad \text{for } i,j \in \{4,5,6\} \text{ with } i \neq j$$

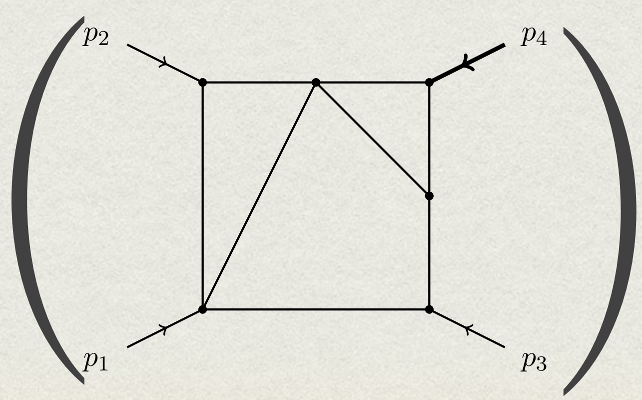
Satisfied for families A, B1, B2, E2.

- Family E1

$$\tilde{A}_4 \cdot \tilde{A}_6 = \tilde{A}_6 \cdot \tilde{A}_4 = \tilde{A}_5 \cdot \tilde{A}_6 = \tilde{A}_6 \cdot \tilde{A}_5 = 0$$

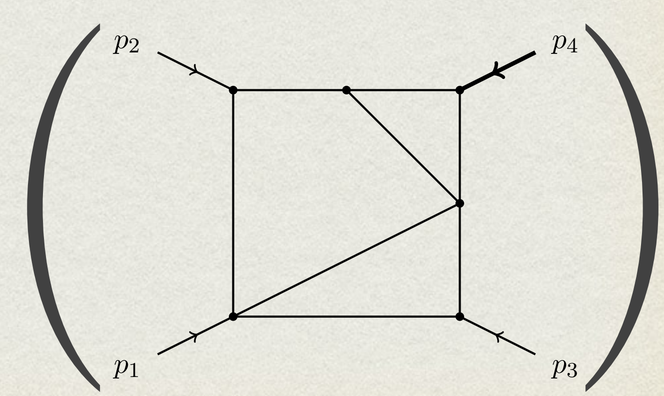
$$\tilde{A}_4 \cdot \tilde{A}_5 \neq 0 \text{ and } \tilde{A}_5 \cdot \tilde{A}_4 \neq 0 \quad \Longrightarrow \ldots \otimes \alpha_4 \otimes \alpha_5 \otimes \ldots$$

Start appearing at weight 5.

$\epsilon^6 \left(p_4^2 - s\right)\left(p_4^2 - t\right)$ 

$\longleftrightarrow$

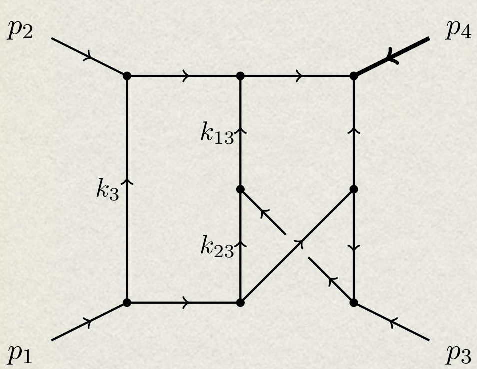
$p_1 \leftrightarrow p_3$ symmetry

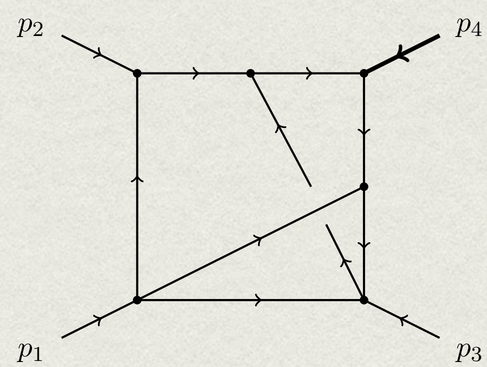
$\epsilon^6 \left(p_4^2 - s\right)\left(p_4^2 - t\right)$ 

Preliminary observations

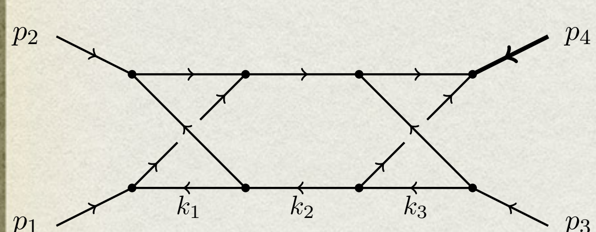
- Appearance of more letters in subsectors

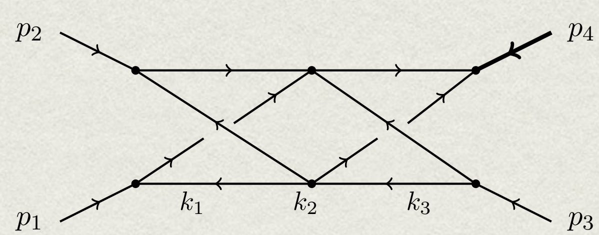
[Henn, Lim, WJT (work in progress)]





$$\left\{-(p_4^2)^2 + (p_4^2 + s)t\right\}$$



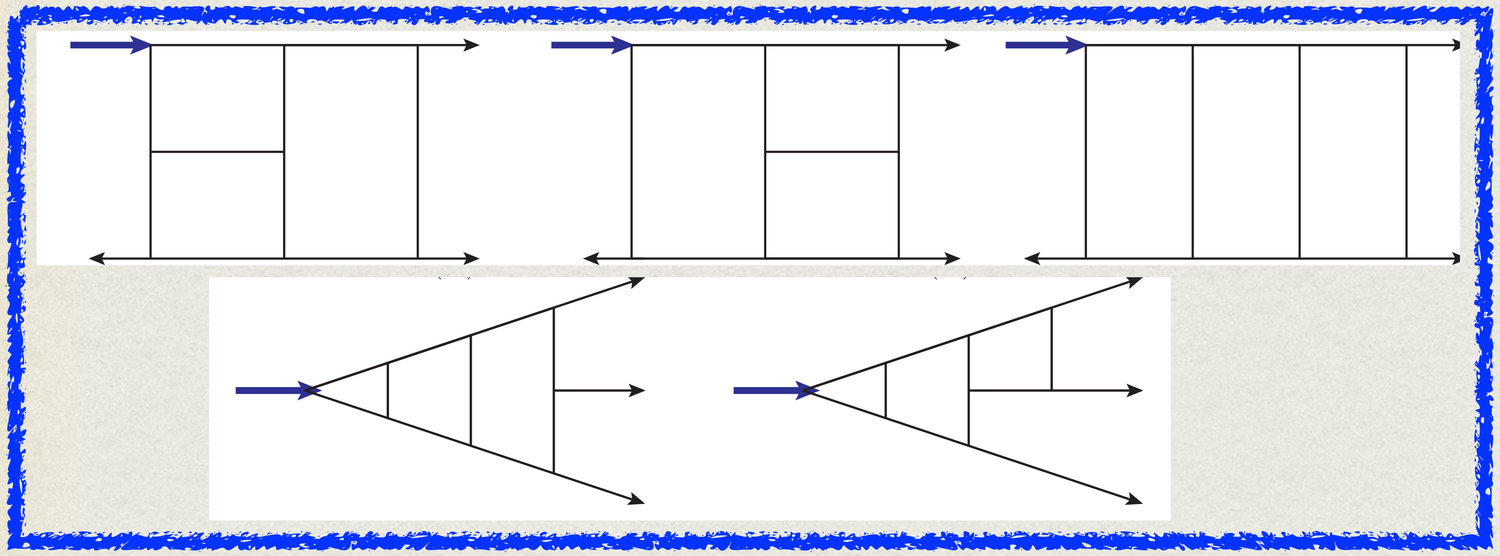


$$\left\{\frac{-p_4^2 t - \sqrt{-p_4^2 st\left(p_4^2 - s - t\right)}}{-p_4^2 t + \sqrt{-p_4^2 st\left(p_4^2 - s - t\right)}}, \frac{st - \sqrt{-p_4^2 st\left(p_4^2 - s - t\right)}}{st + \sqrt{-p_4^2 st\left(p_4^2 - s - t\right)}}\right\}$$

Three-loop amplitudes of Higgs plus jet production

Three-loop amplitudes of Higgs plus jet production

- **Leading-colour approximation**

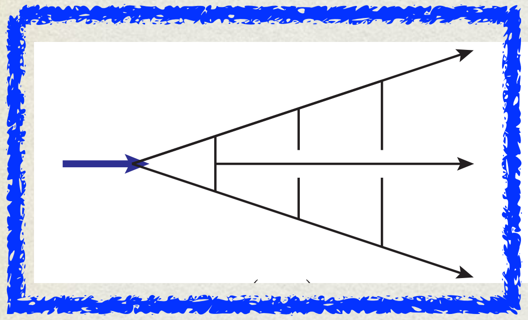


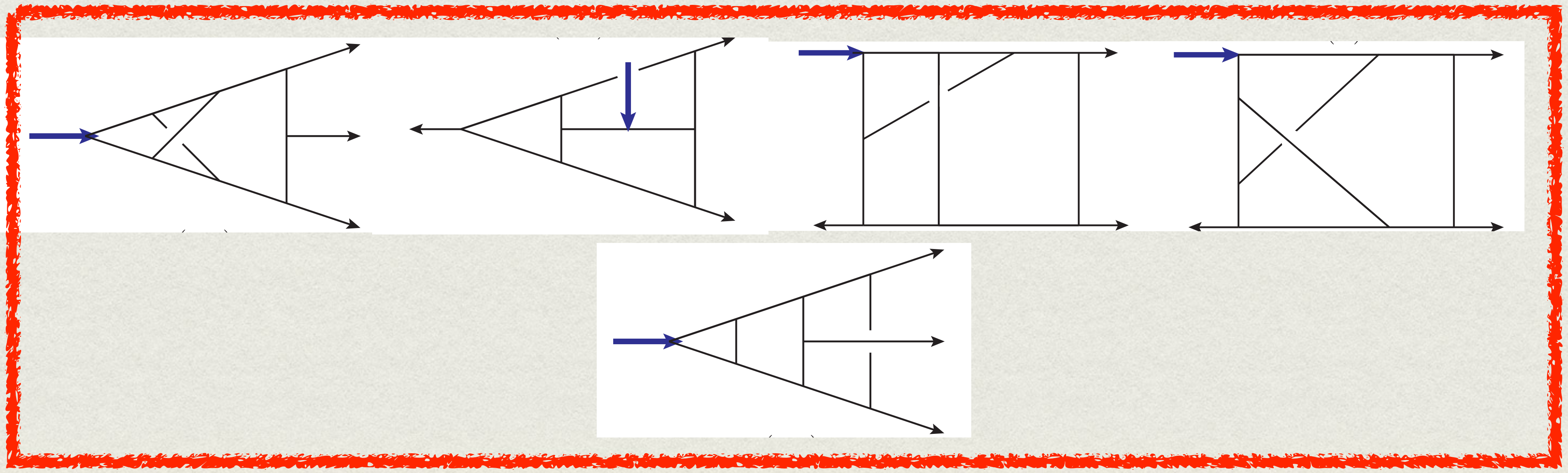
☑ Known MIs

[di Vita, Mastrolia, Schubert, Yundin (2014)]
[Canko, Syrrakos (2021)]
[Henn, Lim, WJT (2023)]
[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi (2023)]

☐ Missing MIs :: work in progress

[Gehrmann, Henn, Jakubčík, Lim, Mella, Syrrakos, Tancredi, WJT]

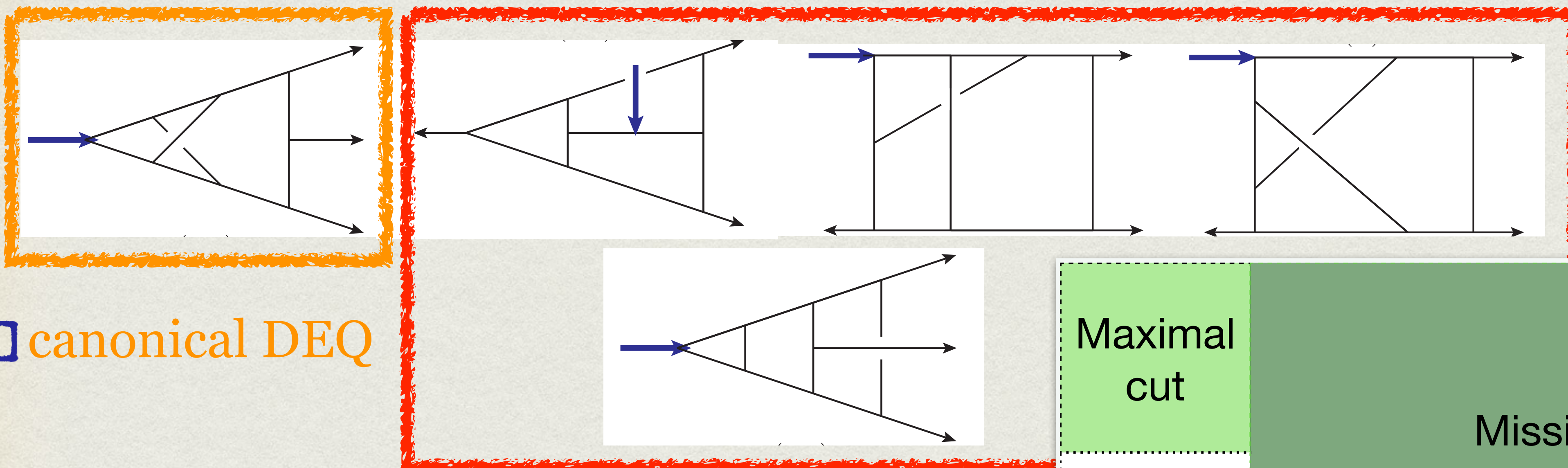




pictures extracted from [2111.03021]

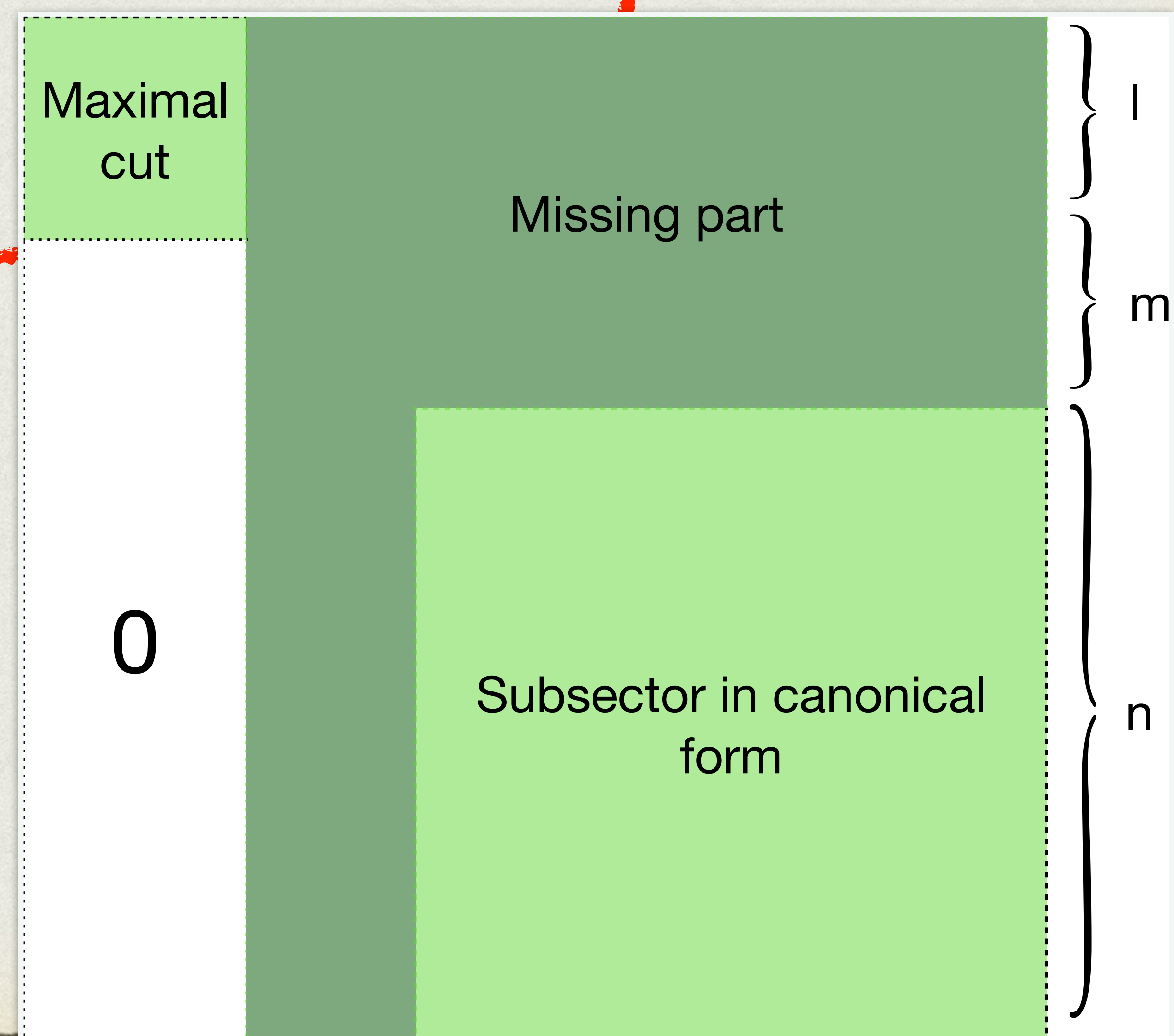
Three-loop amplitudes of Higgs plus jet production

- **Leading-colour approximation**



- canonical DEQ

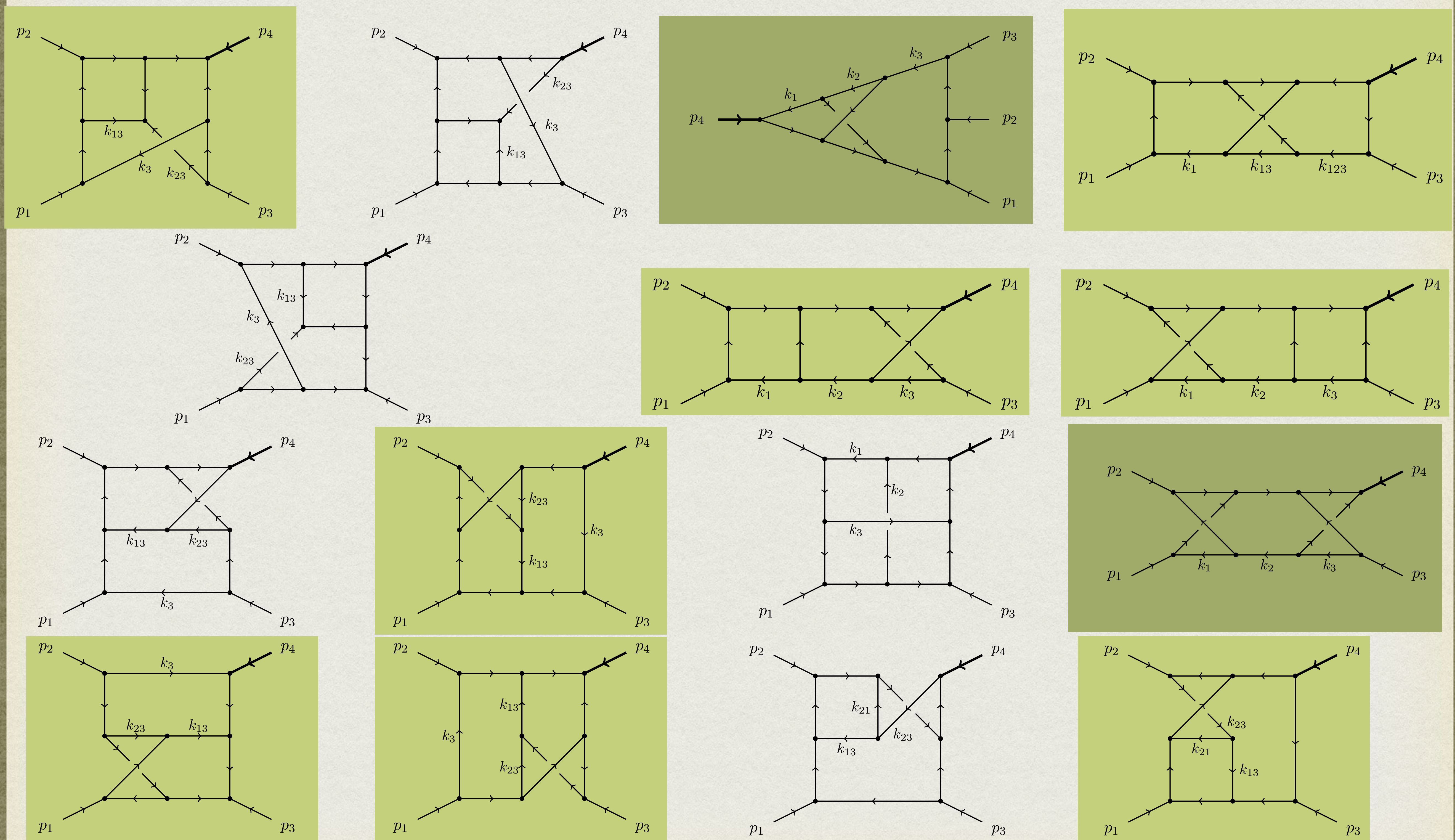
- Where are we?



Three-loop amplitudes of Higgs plus jet production

Full colour

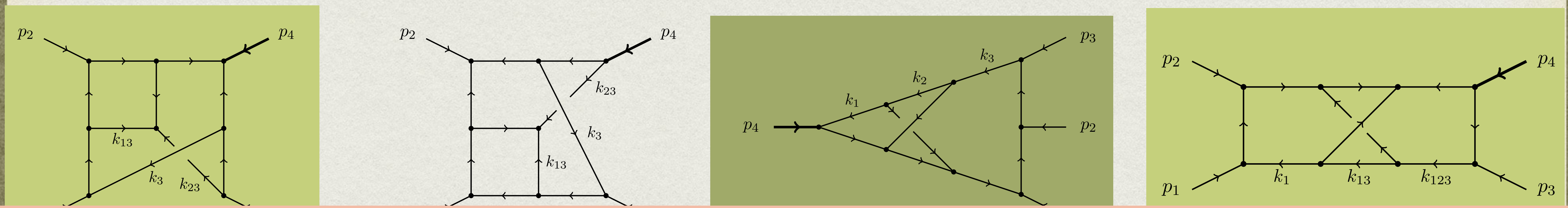
Complete ϵ-expansion
Canonical DEQ

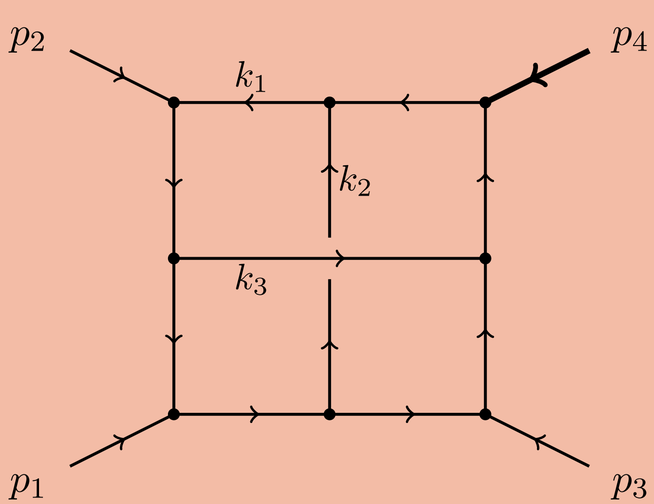


Three-loop amplitudes of Higgs plus jet production

Complete ϵ-expansion
Canonical DEQ

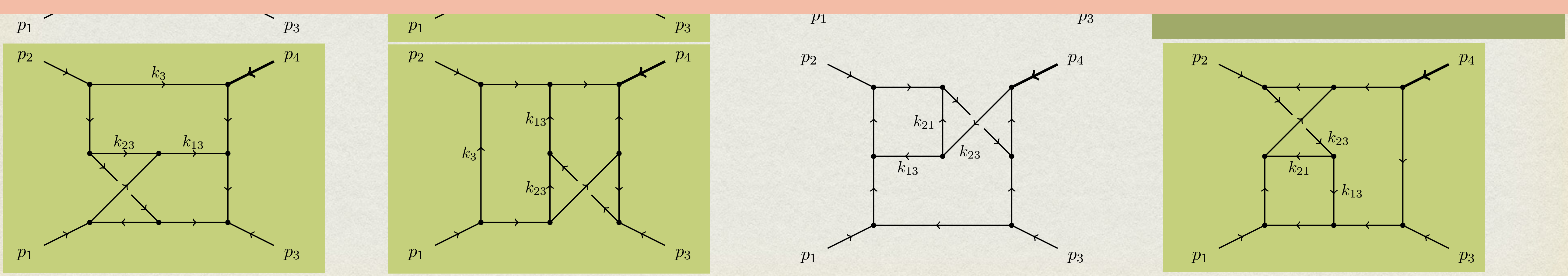
- **Full colour**





- 371 MIs (`FIRE6` & `FiniteFlow`).
- 19 MIs in top sector.
- Canonical DEQ on Maximal cut :: done (`INITIAL`).
- Canonical DEQ?

[Gehrmann, Henn, Jakubčík, Lim, Mella, Syrrakos, Tancredi, WJT]



Conclusions

- We have reached:

- [x] connection between Leading & Landau singularities
- [x] Computed first non-planar families and revisited planar families for three-loop integrals with one massive leg.
- [x] Set stage for calculations of the three-loop scattering amplitudes for Higgs+jet in the leading colour approximation.

- Open questions & future directions

- [ ] Complete three-loop non-planar integral families; unravel function space.
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Thanks