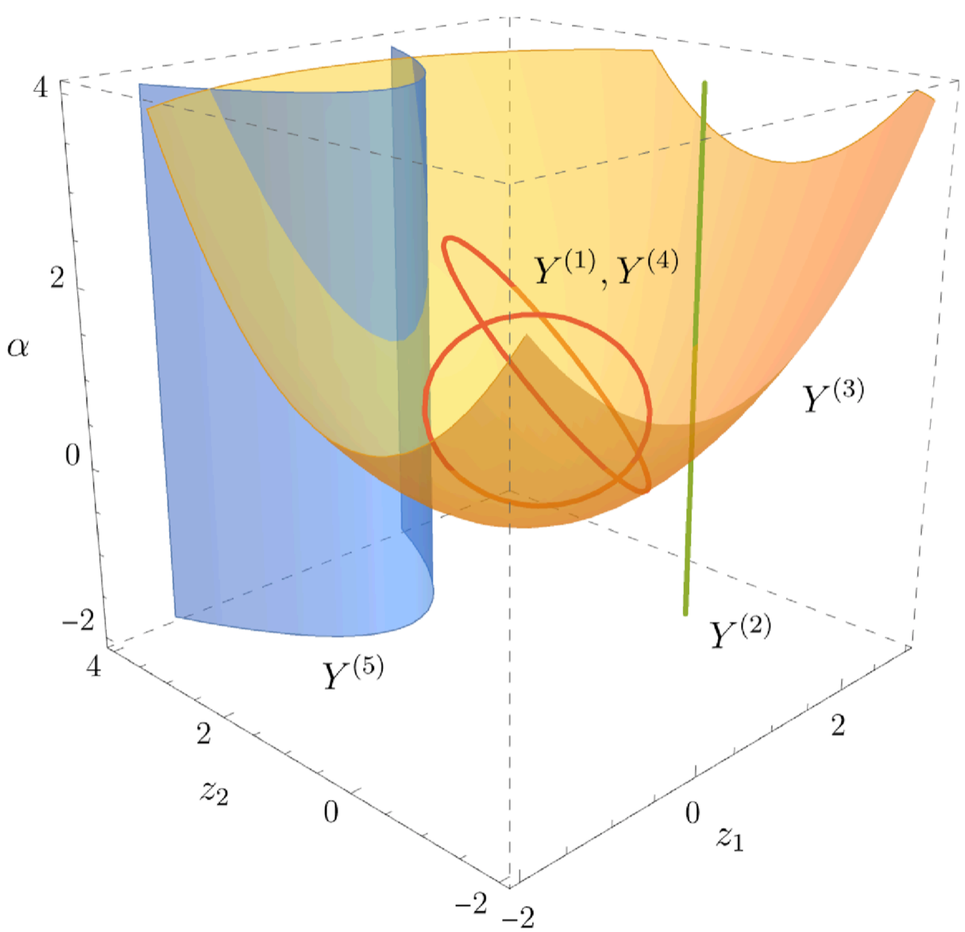
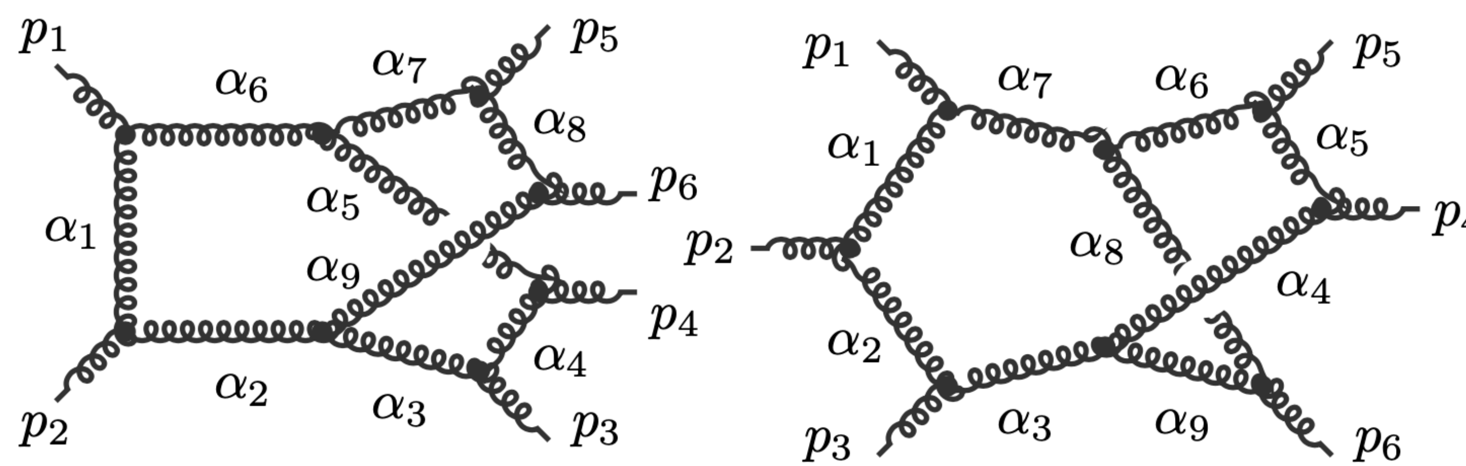
Max-Planck-Institut für Mathematik in den Naturwissenschaften

Principal Landau Determinants

Part II

Simon Telen





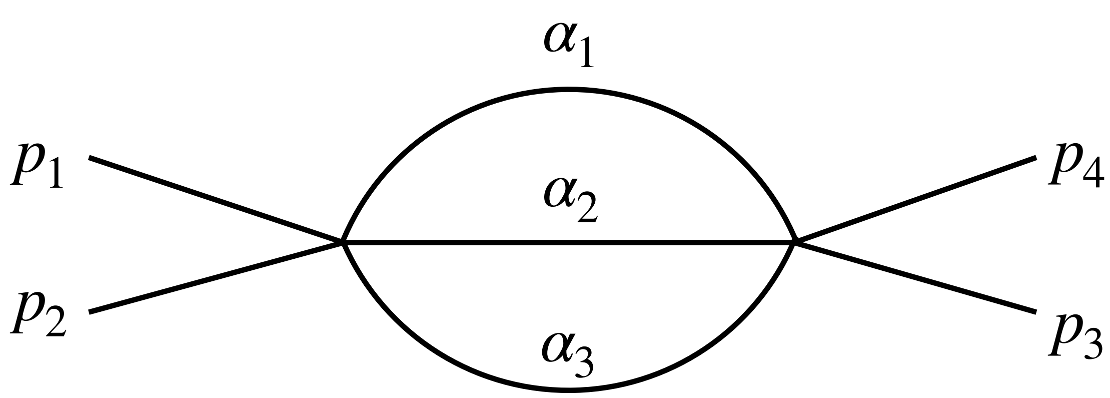
MathemAmplitudes 2023

September 25

Joint with C. Fevola and S. Mizera

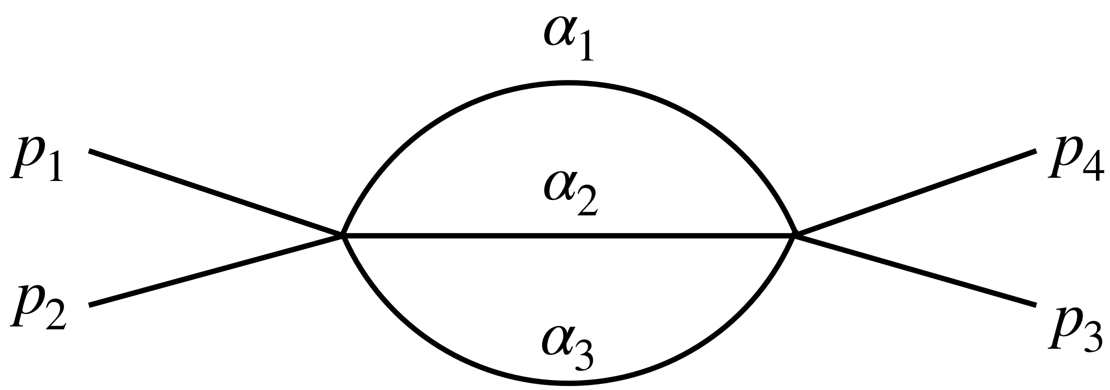
Sunrise problem

Sunrise problem



$$\mathcal{I} = \int_\Gamma [(1 - \textstyle\sum_{i=1}^{3} \mathsf{m}_i \alpha_i)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + s\alpha_1\alpha_2\alpha_3]^\mu \alpha_1^{\nu_1}\alpha_2^{\nu_2}\alpha_3^{\nu_3} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \frac{\mathrm{d}\alpha_2}{\alpha_2} \wedge \frac{\mathrm{d}\alpha_3}{\alpha_3}$$

Sunrise problem

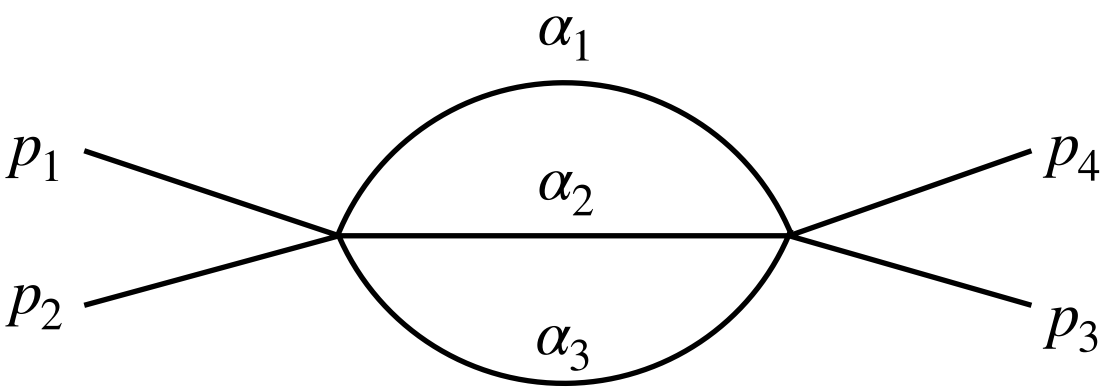


$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathcal{I} = \int_\Gamma [(1 - \textstyle\sum_{i=1}^{3} \mathsf{m}_i \alpha_i)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + s\alpha_1\alpha_2\alpha_3]^\mu \alpha_1^{\nu_1}\alpha_2^{\nu_2}\alpha_3^{\nu_3} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \frac{\mathrm{d}\alpha_2}{\alpha_2} \wedge \frac{\mathrm{d}\alpha_3}{\alpha_3}$$

$$z_1\,\alpha_1\alpha_2 + z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_4\,\alpha_1^2\alpha_2 + z_5\,\alpha_1^2\alpha_3 + z_6\,\alpha_2^2\alpha_3 + z_7\,\alpha_1\alpha_2^2 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 + z_{10}\,\alpha_1\alpha_2\alpha_3$$

Sunrise problem



$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

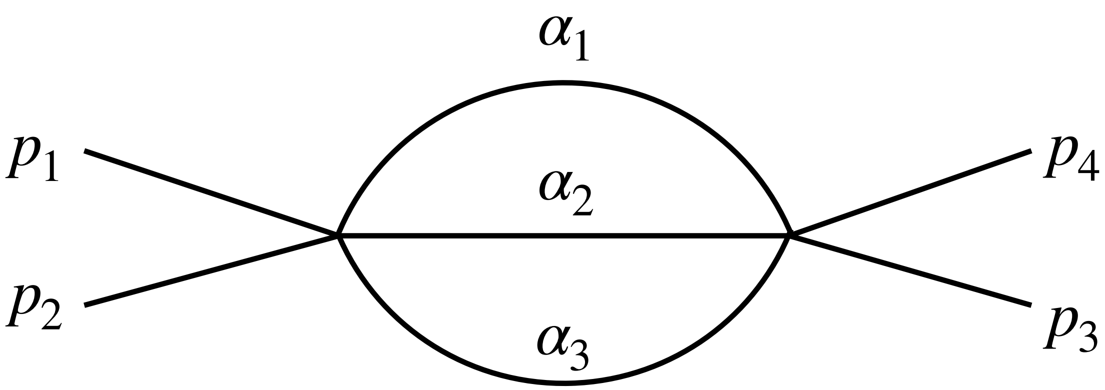
$$\mathcal{I} = \int_\Gamma [(1 - \textstyle\sum_{i=1}^{3} \mathsf{m}_i \alpha_i)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + s\alpha_1\alpha_2\alpha_3]^\mu \alpha_1^{\nu_1}\alpha_2^{\nu_2}\alpha_3^{\nu_3} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \frac{\mathrm{d}\alpha_2}{\alpha_2} \wedge \frac{\mathrm{d}\alpha_3}{\alpha_3}$$

restrict to $\mathscr{E} = \mathscr{K}$

$$(z_1, \ldots, z_{10}) = (1, 1, 1, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, -\mathsf{m}_3, -\mathsf{m}_3, s - \mathsf{m}_1 - \mathsf{m}_2 - \mathsf{m}_3)$$

$$z_1\,\alpha_1\alpha_2 + z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_4\,\alpha_1^2\alpha_2 + z_5\,\alpha_1^2\alpha_3 + z_6\,\alpha_2^2\alpha_3 + z_7\,\alpha_1\alpha_2^2 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 + z_{10}\,\alpha_1\alpha_2\alpha_3$$

Sunrise problem



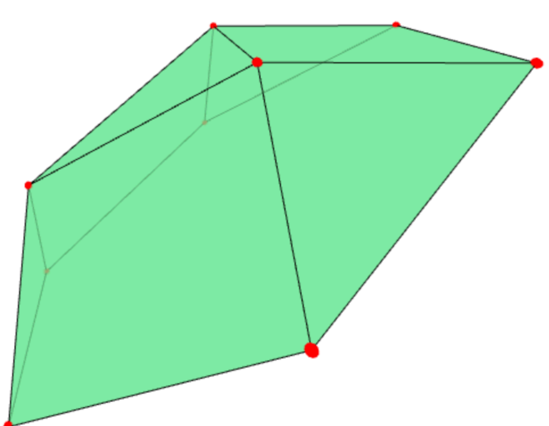
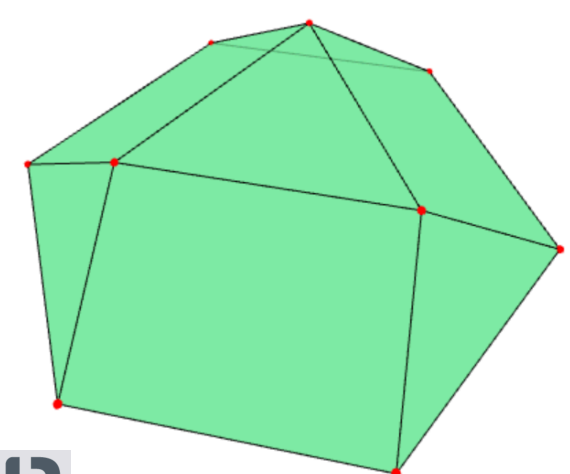
$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathcal{I} = \int_\Gamma [(1 - \textstyle\sum_{i=1}^3 \mathsf{m}_i \alpha_i)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + s\alpha_1\alpha_2\alpha_3]^\mu \alpha_1^{\nu_1}\alpha_2^{\nu_2}\alpha_3^{\nu_3} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \frac{\mathrm{d}\alpha_2}{\alpha_2} \wedge \frac{\mathrm{d}\alpha_3}{\alpha_3}$$

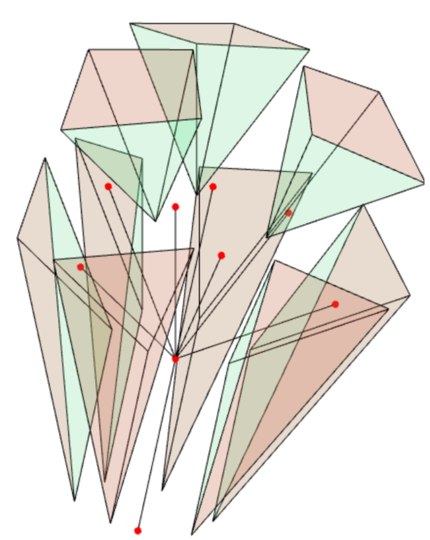
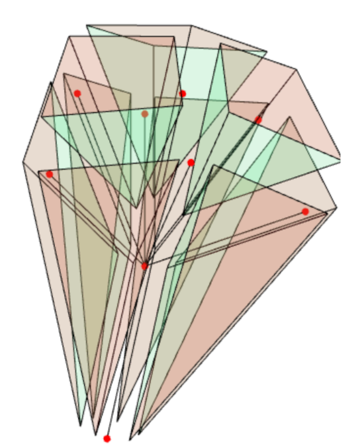
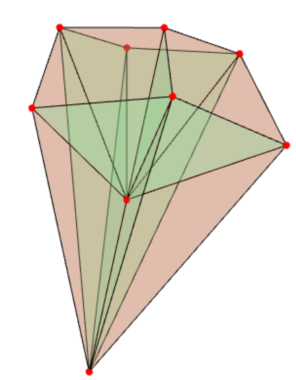
restrict to $\mathscr{E} = \mathscr{K}$

$$(z_1, \ldots, z_{10}) = (1, 1, 1, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, -\mathsf{m}_3, -\mathsf{m}_3, s - \mathsf{m}_1 - \mathsf{m}_2 - \mathsf{m}_3)$$

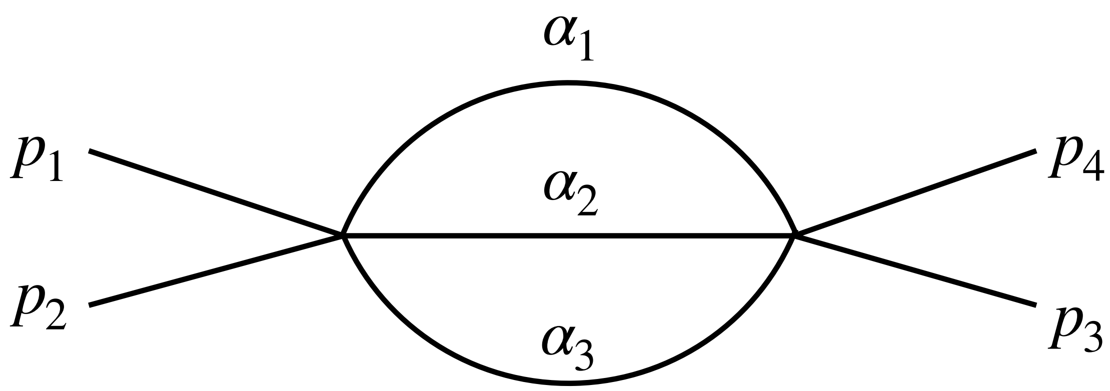
$$z_1\,\alpha_1\alpha_2 + z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_4\,\alpha_1^2\alpha_2 + z_5\,\alpha_1^2\alpha_3 + z_6\,\alpha_2^2\alpha_3 + z_7\,\alpha_1\alpha_2^2 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 + z_{10}\,\alpha_1\alpha_2\alpha_3$$

OSCAR SYMBOLIC TOOLS

Sunrise problem



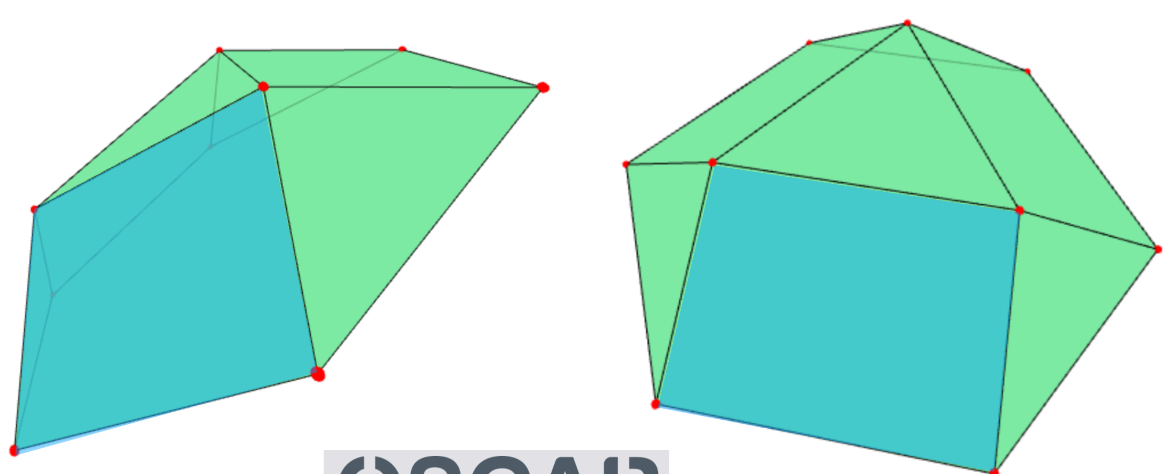
$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathcal{I} = \int_\Gamma [(1 - \textstyle\sum_{i=1}^3 \mathsf{m}_i \alpha_i)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + s\alpha_1\alpha_2\alpha_3]^\mu \alpha_1^{\nu_1}\alpha_2^{\nu_2}\alpha_3^{\nu_3} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \frac{\mathrm{d}\alpha_2}{\alpha_2} \wedge \frac{\mathrm{d}\alpha_3}{\alpha_3}$$

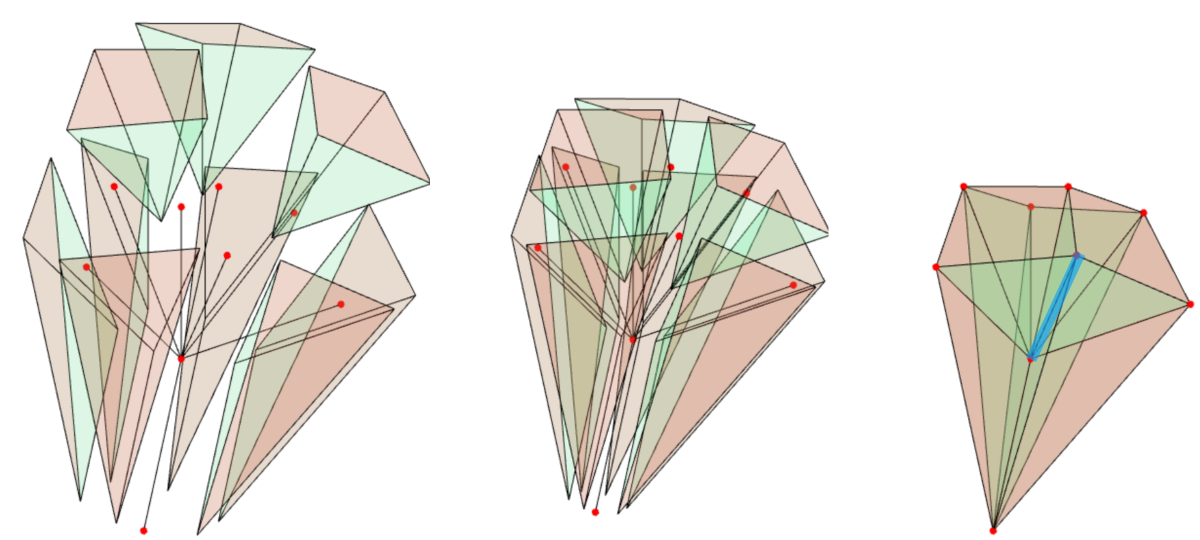
restrict to $\mathscr{E} = \mathscr{K}$

$$(z_1, \ldots, z_{10}) = (1, 1, 1, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, -\mathsf{m}_3, -\mathsf{m}_3, s - \mathsf{m}_1 - \mathsf{m}_2 - \mathsf{m}_3)$$

$$z_1\,\alpha_1\alpha_2 + \underline{z_2\,\alpha_1\alpha_3} + \underline{z_3\,\alpha_2\alpha_3} + z_4\,\alpha_1^2\alpha_2 + z_5\,\alpha_1^2\alpha_3 + z_6\,\alpha_2^2\alpha_3 + z_7\,\alpha_1\alpha_2^2 + \underline{z_8\,\alpha_1\alpha_3^2} + \underline{z_9\,\alpha_2\alpha_3^2} + z_{10}\,\alpha_1\alpha_2\alpha_3$$



OSCAR SYMBOLIC TOOLS



Sunrise problem

Sunrise problem

$$
\begin{aligned}
z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 &= 0 \\
z_2\,\alpha_3 + z_8\,\alpha_3^2 &= 0 \\
z_3\,\alpha_3 + z_9\,\alpha_3^2 &= 0 \\
z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 &= 0 \\
\alpha_1\alpha_2\alpha_3 y - 1 &= 0
\end{aligned}
\quad \Longrightarrow \quad \Delta_{A\cap Q} = z_2 z_9 - z_3 z_8 = 0
$$

Sunrise problem

$$z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 \;=\; 0$$
$$z_2\,\alpha_3 + z_8\,\alpha_3^2 \;=\; 0$$
$$z_3\,\alpha_3 + z_9\,\alpha_3^2 \;=\; 0$$
$$z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 \;=\; 0$$
$$\alpha_1\alpha_2\alpha_3 y - 1 \;=\; 0$$

$$\Longrightarrow \quad \Delta_{A\cap Q} \;=\; z_2 z_9 - z_3 z_8 \;=\; 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1, \ldots, z_{10}) \;=\; (1, \underline{1}, \underline{1}, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, \underline{-\mathsf{m}_3}, \underline{-\mathsf{m}_3}, s - \mathsf{m}_1 - \mathsf{m}_2 - \mathsf{m}_3)$$

Sunrise problem

$$
\begin{aligned}
z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 &= 0\\
z_2\,\alpha_3 + z_8\,\alpha_3^2 &= 0\\
z_3\,\alpha_3 + z_9\,\alpha_3^2 &= 0\\
z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 &= 0\\
\alpha_1\alpha_2\alpha_3 y - 1 &= 0
\end{aligned}
$$

$$\Longrightarrow \quad \Delta_{A\cap Q} = z_2 z_9 - z_3 z_8 = 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1, \ldots, z_{10}) = (1, \underline{1}, \underline{1}, -\mathrm{m}_1, -\mathrm{m}_1, -\mathrm{m}_2, -\mathrm{m}_2, \underline{-\mathrm{m}_3}, \underline{-\mathrm{m}_3}, s - \mathrm{m}_1 - \mathrm{m}_2 - \mathrm{m}_3)$$

all parameters in $\mathscr{E}$ lie inside the principal A-determinant

$\Updownarrow$

the generic Euler characteristic on $\mathscr{E}$ is strictly smaller than $\mathrm{vol}(A)$

Sunrise problem

$$z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 \;=\; 0$$
$$z_2\,\alpha_3 + z_8\,\alpha_3^2 \;=\; 0$$
$$z_3\,\alpha_3 + z_9\,\alpha_3^2 \;=\; 0$$
$$z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 \;=\; 0$$
$$\alpha_1\alpha_2\alpha_3 y - 1 \;=\; 0$$

$$\Longrightarrow \quad \Delta_{A\cap Q} \;=\; z_2 z_9 - z_3 z_8 \;=\; 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1,\ldots,z_{10}) \;=\; (1,\underline{1},\underline{1},-\mathrm{m}_1,-\mathrm{m}_1,-\mathrm{m}_2,-\mathrm{m}_2,\underline{-\mathrm{m}_3},\underline{-\mathrm{m}_3},s-\mathrm{m}_1-\mathrm{m}_2-\mathrm{m}_3)$$

all parameters in $\mathscr{E}$ lie inside the principal A-determinant

$\Updownarrow$

the generic Euler characteristic on $\mathscr{E}$ is strictly smaller than $\mathrm{vol}(A)$

G	$\mathcal{K}$	$\mathcal{E}^{(\mathsf{M}_i,0)}$	$\mathcal{E}^{(0,\mathsf{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\mathtt{A}_4$	(15, 15)	(11, 11)	(11, 15)	(3, 3)
$\mathtt{B}_4$	(15, 35)	(1,1)	(15,35)	(1,1)
par	(19, 35)	(4, 8)	(13, 35)	(1, 3)
acn	(55, 136)	(20, 54)	(36, 136)	(3, 9)
env	(273, 1496)	(56, 262)	(181, 1496)	(10, 80)
npltrb	(116, 512)	(28, 252)	(77, 512)	(5, 61)
tdetri	(51, 201)	(4, 18)	(33, 201)	(1, 5)
debox	(43, 96)	(11, 33)	(31, 96)	(3, 10)
tdebox	(123, 705)	(11, 113)	(87, 705)	(3, 41)
pltrb	(81, 417)	(16, 201)	(61, 417)	(4, 80)
dbox	(227, 1422)	(75, 903)	(159, 1422)	(12, 238)
pentb	(543, 4279)	(228, 3148)	(430, 4279)	(62, 1186)

G	$\mathcal{E}$
inner-dbox	(43, 834)
outer-dbox	(64, 1302)
Hj-npl-dbox	(99, 1016)
Bhabha-dbox	(64, 774)
Bhabha2-dbox	(79, 910)
Bhabha-npl-dbox	(111, 936)
kite	(30, 136)
par	(19, 35)
Hj-npl-pentb	(330, 3144)
dpent	(281, 5511)
npl-dpent	(631, 5784)
npl-dpent2	(458, 5467)

Sunrise problem

$$z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 = 0$$
$$z_2\,\alpha_3 + z_8\,\alpha_3^2 = 0$$
$$z_3\,\alpha_3 + z_9\,\alpha_3^2 = 0$$
$$z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 = 0$$
$$\alpha_1\alpha_2\alpha_3 y - 1 = 0$$

$$\Longrightarrow \quad \Delta_{A\cap Q} = z_2 z_9 - z_3 z_8 = 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1,\ldots,z_{10}) = (1, \underline{1}, \underline{1}, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, \underline{-\mathsf{m}_3}, \underline{-\mathsf{m}_3}, s-\mathsf{m}_1-\mathsf{m}_2-\mathsf{m}_3)$$

all parameters in $\mathscr{E}$ lie inside the principal A-determinant

$\Updownarrow$

the generic Euler characteristic on $\mathscr{E}$ is strictly smaller than $\mathrm{vol}(A)$

G	$\mathcal{K}$	$\mathcal{E}^{(\mathsf{M}_i,0)}$	$\mathcal{E}^{(0,\mathsf{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\mathtt{A}_4$	$(15,15)$	$(11,11)$	$(11,15)$	$(3,3)$
$\mathtt{B}_4$	$(15,35)$	$(1,1)$	$(15,35)$	$(1,1)$
par	$(19,35)$	$(4,8)$	$(13,35)$	$(1,3)$
acn	$(55,136)$	$(20,54)$	$(36,136)$	$(3,9)$
env	$(273,1496)$	$(56,262)$	$(181,1496)$	$(10,80)$
npltrb	$(116,512)$	$(28,252)$	$(77,512)$	$(5,61)$
tdetri	$(51,201)$	$(4,18)$	$(33,201)$	$(1,5)$
debox	$(43,96)$	$(11,33)$	$(31,96)$	$(3,10)$
tdebox	$(123,705)$	$(11,113)$	$(87,705)$	$(3,41)$
pltrb	$(81,417)$	$(16,201)$	$(61,417)$	$(4,80)$
dbox	$(227,1422)$	$(75,903)$	$(159,1422)$	$(12,238)$
pentb	$(543,4279)$	$(228,3148)$	$(430,4279)$	$(62,1186)$

G	$\mathcal{E}$
inner-dbox	$(43,834)$
outer-dbox	$(64,1302)$
Hj-npl-dbox	$(99,1016)$
Bhabha-dbox	$(64,774)$
Bhabha2-dbox	$(79,910)$
Bhabha-npl-dbox	$(111,936)$
kite	$(30,136)$
par	$(19,35)$
Hj-npl-pentb	$(330,3144)$
dpent	$(281,5511)$
npl-dpent	$(631,5784)$
npl-dpent2	$(458,5467)$

The $\mathcal{X}$-discriminant can usually **not** be obtained by restricting the principal A-determinant

Sunrise problem

$$
\begin{aligned}
z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 &= 0\\
z_2\,\alpha_3 + z_8\,\alpha_3^2 &= 0\\
z_3\,\alpha_3 + z_9\,\alpha_3^2 &= 0\\
z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 &= 0\\
\alpha_1\alpha_2\alpha_3 y - 1 &= 0
\end{aligned}
$$

$$\Longrightarrow \quad \Delta_{A\cap Q} = z_2 z_9 - z_3 z_8 = 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1, \ldots, z_{10}) = (1, \underline{1}, \underline{1}, -\mathrm{m}_1, -\mathrm{m}_1, -\mathrm{m}_2, -\mathrm{m}_2, \underline{-\mathrm{m}_3}, \underline{-\mathrm{m}_3}, s - \mathrm{m}_1 - \mathrm{m}_2 - \mathrm{m}_3)$$

all parameters in $\mathscr{E}$ lie inside the principal A-determinant

$\Updownarrow$

the generic Euler characteristic on $\mathscr{E}$ is strictly smaller than $\mathrm{vol}(A)$

G	$\mathcal{K}$	$\mathcal{E}^{(\mathsf{M}_i,0)}$	$\mathcal{E}^{(0,\mathsf{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\mathtt{A}_4$	$(15,15)$	$(11,11)$	$(11,15)$	$(3,3)$
$\mathtt{B}_4$	$(15,35)$	$(1,1)$	$(15,35)$	$(1,1)$
par	$(19,35)$	$(4,8)$	$(13,35)$	$(1,3)$
acn	$(55,136)$	$(20,54)$	$(36,136)$	$(3,9)$
env	$(273,1496)$	$(56,262)$	$(181,1496)$	$(10,80)$
npltrb	$(116,512)$	$(28,252)$	$(77,512)$	$(5,61)$
tdetri	$(51,201)$	$(4,18)$	$(33,201)$	$(1,5)$
debox	$(43,96)$	$(11,33)$	$(31,96)$	$(3,10)$
tdebox	$(123,705)$	$(11,113)$	$(87,705)$	$(3,41)$
pltrb	$(81,417)$	$(16,201)$	$(61,417)$	$(4,80)$
dbox	$(227,1422)$	$(75,903)$	$(159,1422)$	$(12,238)$
pentb	$(543,4279)$	$(228,3148)$	$(430,4279)$	$(62,1186)$

G	$\mathcal{E}$
inner-dbox	$(43,834)$
outer-dbox	$(64,1302)$
Hj-npl-dbox	$(99,1016)$
Bhabha-dbox	$(64,774)$
Bhabha2-dbox	$(79,910)$
Bhabha-npl-dbox	$(111,936)$
kite	$(30,136)$
par	$(19,35)$
Hj-npl-pentb	$(330,3144)$
dpent	$(281,5511)$
npl-dpent	$(631,5784)$
npl-dpent2	$(458,5467)$

The $\mathcal{X}$-discriminant can usually **not** be obtained by restricting the principal A-determinant

The **principal Landau determinant** is a computable subset of the χ-discriminant, whose definition is inspired by GKZ

A first guess

Why not $\displaystyle\prod_{(\Delta_{A\cap Q})_{|\mathscr{E}}\neq 0} (\Delta_{A\cap Q})_{|\mathscr{E}}$?

A first guess

Why not $\prod_{(\Delta_{A\cap Q})_{|\mathscr{E}} \neq 0} (\Delta_{A\cap Q})_{|\mathscr{E}}$?

$$(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$E_A = z_1z_3z_4z_6(z_2^2-4z_1z_4)(z_5^2-4z_3z_6)$$
$$(z_3^2z_4^2 - z_2z_3z_4z_5 + z_1z_4z_5^2 + z_2^2z_3z_6 - 2z_1z_3z_4z_6 - z_1z_2z_5z_6 + z_1^2z_6^2)$$

A first guess

Why not $\prod_{(\Delta_{A\cap Q})_{|\mathscr{E}}\neq 0} (\Delta_{A\cap Q})_{|\mathscr{E}}$?

$$(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} E_A = {} & z_1z_3z_4z_6(z_2^2-4z_1z_4)(z_5^2-4z_3z_6) \\ & (z_3^2z_4^2 - z_2z_3z_4z_5 + z_1z_4z_5^2 + z_2^2z_3z_6 - 2z_1z_3z_4z_6 - z_1z_2z_5z_6 + z_1^2z_6^2) \end{aligned}$$

$$(z_1,z_2,z_3,z_4,z_5,z_6) = (a, a+b, c, b, c+d, d) \quad \Rightarrow \quad abcd(a-b)(c-d)$$

A first guess

Why not $\prod_{(\Delta_{A\cap Q})_{|\mathscr{E}}\neq 0} (\Delta_{A\cap Q})_{|\mathscr{E}}$? $(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} E_A = {} & z_1 z_3 z_4 z_6 (z_2^2 - 4 z_1 z_4)(z_5^2 - 4 z_3 z_6) \\ & (z_3^2 z_4^2 - z_2 z_3 z_4 z_5 + z_1 z_4 z_5^2 + z_2^2 z_3 z_6 - 2 z_1 z_3 z_4 z_6 - z_1 z_2 z_5 z_6 + z_1^2 z_6^2) \end{aligned}$$

$$(z_1, z_2, z_3, z_4, z_5, z_6) = (a, a+b, c, b, c+d, d) \quad \Rightarrow \quad abcd(a-b)(c-d)$$

$$\begin{aligned} & \int_{\mathbb{R}_+^2} \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{[(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)]^2} \\ & \qquad = \frac{1}{(a-b)(c-d)} - \frac{1}{bc-ad}\left[\frac{b^2\log(a/b)}{(a-b)^2} + \frac{d^2\log(d/c)}{(c-d)^2}\right] \end{aligned}$$

A first guess

Why not $\displaystyle\prod_{(\Delta_{A\cap Q})_{|\mathscr{E}}\neq 0} (\Delta_{A\cap Q})_{|\mathscr{E}}$?

$$(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} E_A = {} & z_1z_3z_4z_6(z_2^2-4z_1z_4)(z_5^2-4z_3z_6) \\ & (z_3^2z_4^2 - z_2z_3z_4z_5 + z_1z_4z_5^2 + z_2^2z_3z_6 - 2z_1z_3z_4z_6 - z_1z_2z_5z_6 + z_1^2z_6^2) \end{aligned}$$

$$(z_1,z_2,z_3,z_4,z_5,z_6) = (a, a+b, c, b, c+d, d) \quad \Rightarrow \quad abcd(a-b)(c-d)$$

$$\int_{\mathbb{R}_+^2} \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{[(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)]^2} = \frac{1}{(a-b)(c-d)} - \frac{1}{bc-ad}\left[\frac{b^2\log(a/b)}{(a-b)^2} + \frac{d^2\log(d/c)}{(c-d)^2}\right]$$

R.P. Klausen, *Kinematic singularities of Feynman integrals and principal A-determinants*, *JHEP* **02** (2022) 004 [2109.07584].

C. Dlapa, M. Helmer, G. Papathanasiou and F. Tellander, *Symbol Alphabets from the Landau Singular Locus*, 2304.02629.

A first guess

Why not $\prod_{(\Delta_{A\cap Q})_{|\mathscr{E}}\neq 0} (\Delta_{A\cap Q})_{|\mathscr{E}}$?

$$(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

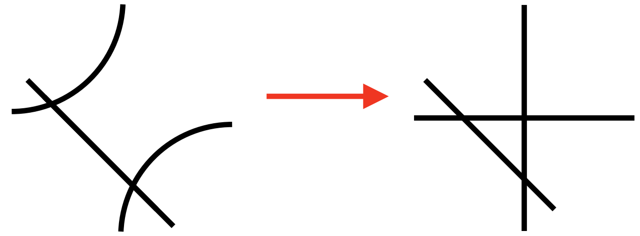
$$E_A = z_1 z_3 z_4 z_6 (z_2^2 - 4 z_1 z_4)(z_5^2 - 4 z_3 z_6) \\ (z_3^2 z_4^2 - z_2 z_3 z_4 z_5 + z_1 z_4 z_5^2 + z_2^2 z_3 z_6 - 2 z_1 z_3 z_4 z_6 - z_1 z_2 z_5 z_6 + z_1^2 z_6^2)$$

$$(z_1, z_2, z_3, z_4, z_5, z_6) = (a, a+b, c, b, c+d, d) \quad \Rightarrow \quad abcd(a-b)(c-d)$$

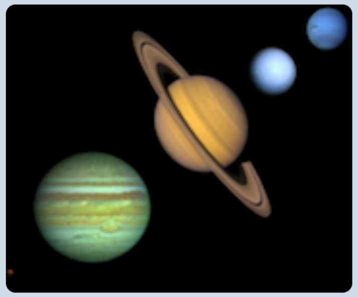
$$\int_{\mathbb{R}_+^2} \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{[(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)]^2} \\ = \frac{1}{(a-b)(c-d)} - \frac{1}{bc-ad}\left[\frac{b^2\log(a/b)}{(a-b)^2} + \frac{d^2\log(d/c)}{(c-d)^2}\right]$$

R.P. Klausen, *Kinematic singularities of Feynman integrals and principal A-determinants*, *JHEP* **02** (2022) 004 [2109.07584].

C. Dlapa, M. Helmer, G. Papathanasiou and F. Tellander, *Symbol Alphabets from the Landau Singular Locus*, 2304.02629.



Incidence variety



Welcome to the Macaulay2Web interface

```
i1 : R = QQ[x1,x2,a,b,c,d,y];

i2 : f = (1+x1)*(a + b*x1 + c*x2 + d*x1*x2);

i3 : I = ideal(f,diff(x1,f),diff(x2,f),x1*x2*y-1);

o3 : Ideal of R

i4 : PD = primaryDecomposition I;

i5 : netList PD

     +------------------------------------------------------------------------
o5 = |ideal (x1 + 1, a*y - b*y - c + d, x2*y + 1, x2*c - x2*d + a - b)
     +------------------------------------------------------------------------
     |ideal (a*y - d, x2*d + b, x1*d + c, b*c - a*d, x2*c + a, x1*b + a, x1*x2*y - 1)
     +------------------------------------------------------------------------
     |                           2            2                         2
     |ideal (a - b, x2*y + x1 + 2, c  - 2c*d + d , x1*c - x1*d + c - d, x1  + 2x1 + 1, 2x
     +------------------------------------------------------------------------

i6 : apply(PD,i->eliminate(i,{x1,x2,y}))
```

o6 = $\{\mathtt{ideal}(), \mathtt{ideal}(b\,c - a\,d), \mathtt{ideal}\left(a - b, c^2 - 2\,c\,d + d^2\right)\}$

Principal Landau determinant

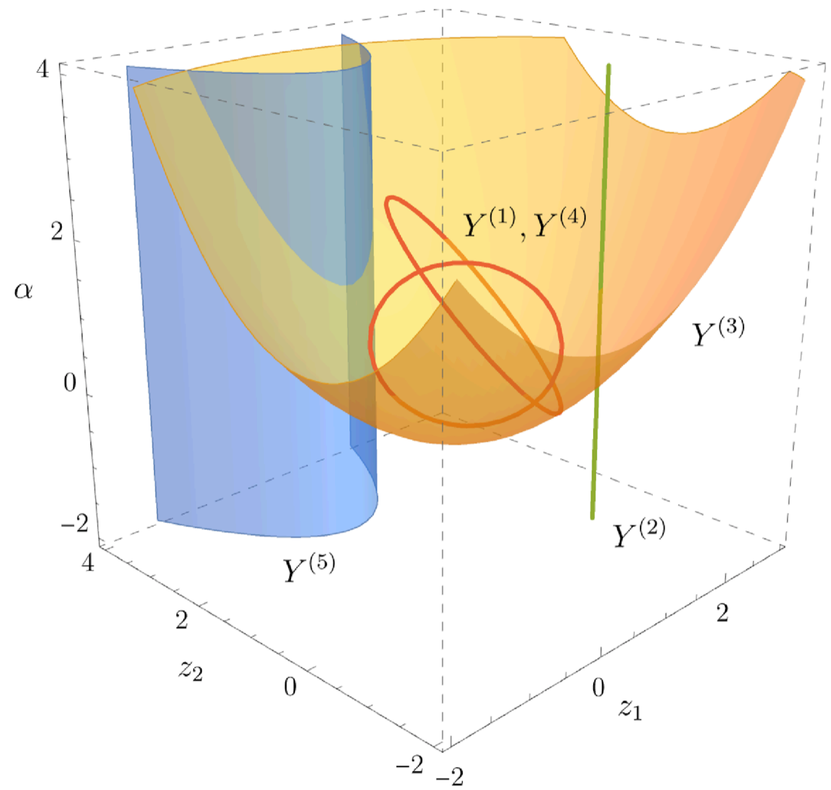
$$Q \subset \mathrm{conv}(A) \quad \text{face}, \qquad \mathcal{G}_G = \sum_{a \in A} c_a(m, M, s, t)\, \alpha^a, \qquad \mathcal{G}_{G,Q} = \sum_{a \in A \cap Q} c_a(m, M, s, t)\, \alpha^a$$

Principal Landau determinant

$$Q \subset \mathrm{conv}(A) \quad \text{face}, \quad \mathcal{G}_G = \sum_{a \in A} c_a \underbrace{(m, M, s, t)}_{z} \, \alpha^a, \quad \mathcal{G}_{G,Q} = \sum_{a \in A \cap Q} c_a \underbrace{(m, M, s, t)}_{z} \, \alpha^a$$

Principal Landau determinant

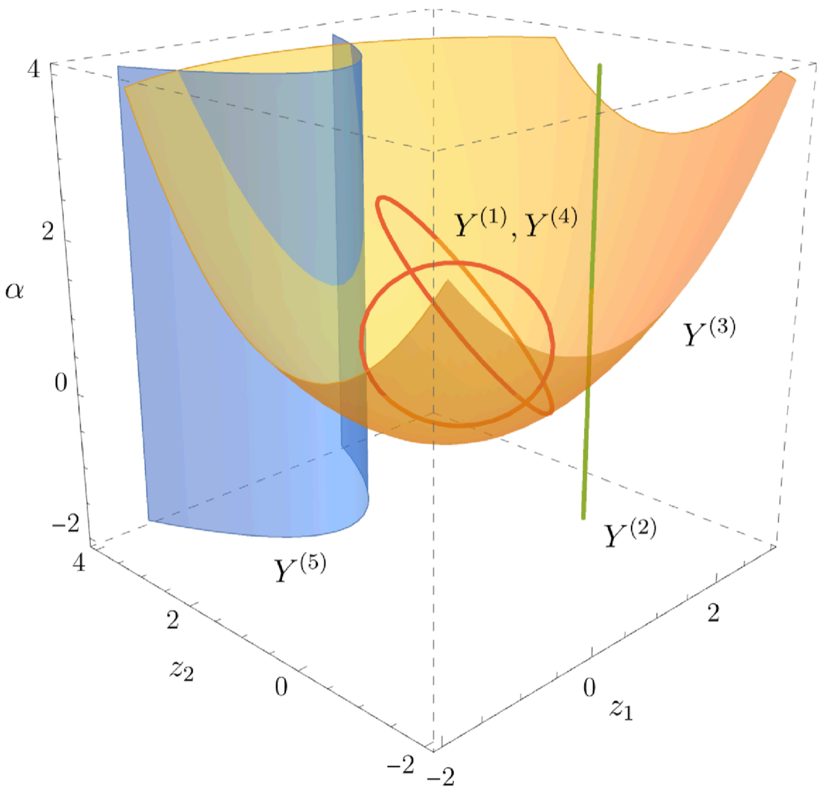
$Q \subset \mathrm{conv}(A)$ face, $\mathscr{G}_G = \sum_{a \in A} c_a(\underbrace{m, M, s, t}_{z})\, \alpha^a$, $\mathscr{G}_{G,Q} = \sum_{a \in A \cap Q} c_a(\underbrace{m, M, s, t}_{z})\, \alpha^a$



$$Y_{G,Q}(\mathscr{E}) = \{(\alpha, z) \in (\mathbb{C}^*)^E \times \mathscr{E} \, : \, \mathscr{G}_{G,Q} = 0, \partial_\alpha \mathscr{G}_{G,Q} = 0\}$$

Principal Landau determinant

$Q \subset \mathrm{conv}(A)$ face, $\quad \mathscr{G}_G = \sum_{a \in A} c_a(\underbrace{m, M, s, t}_{z})\, \alpha^a, \quad \mathscr{G}_{G,Q} = \sum_{a \in A \cap Q} c_a(\underbrace{m, M, s, t}_{z})\, \alpha^a$

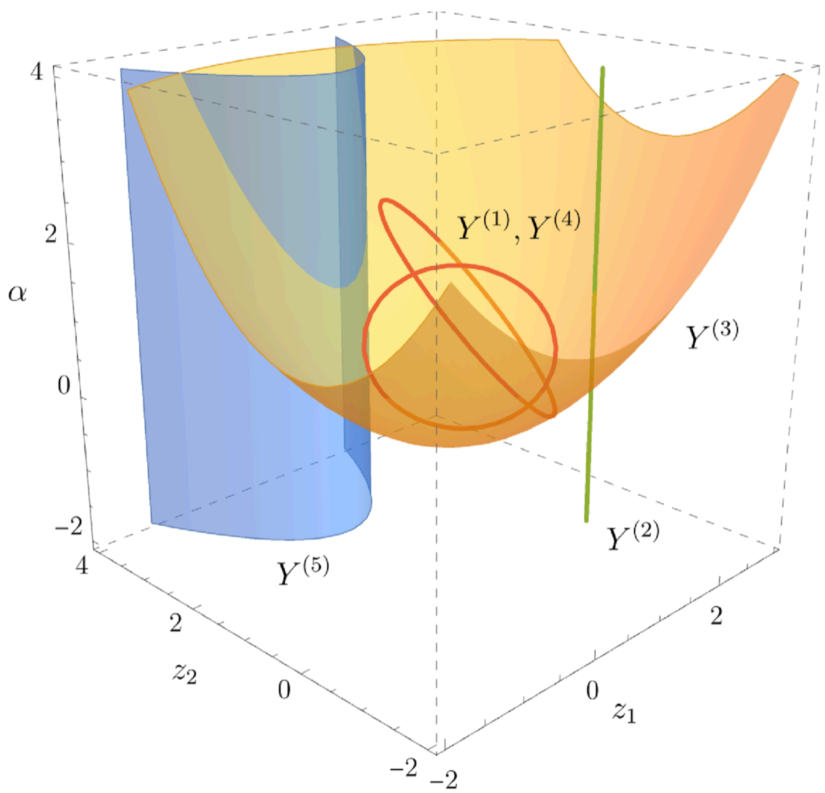


$$Y_{G,Q}(\mathscr{E}) = \{(\alpha, z) \in (\mathbb{C}^*)^E \times \mathscr{E} \,:\, \mathscr{G}_{G,Q} = 0, \partial_\alpha \mathscr{G}_{G,Q} = 0\}$$

$$Y_{G,Q}(\mathscr{E}) = \bigcup_{i \in \mathbb{I}(G,Q)} Y^{(i)}_{G,Q}(\mathscr{E})$$ irreducible decomposition

Principal Landau determinant

$$Q \subset \mathrm{conv}(A) \quad \text{face}, \quad \mathcal{G}_G = \sum_{a \in A} c_a(\underbrace{m, M, s, t}_{z})\, \alpha^a, \quad \mathcal{G}_{G,Q} = \sum_{a \in A \cap Q} c_a(\underbrace{m, M, s, t}_{z})\, \alpha^a$$



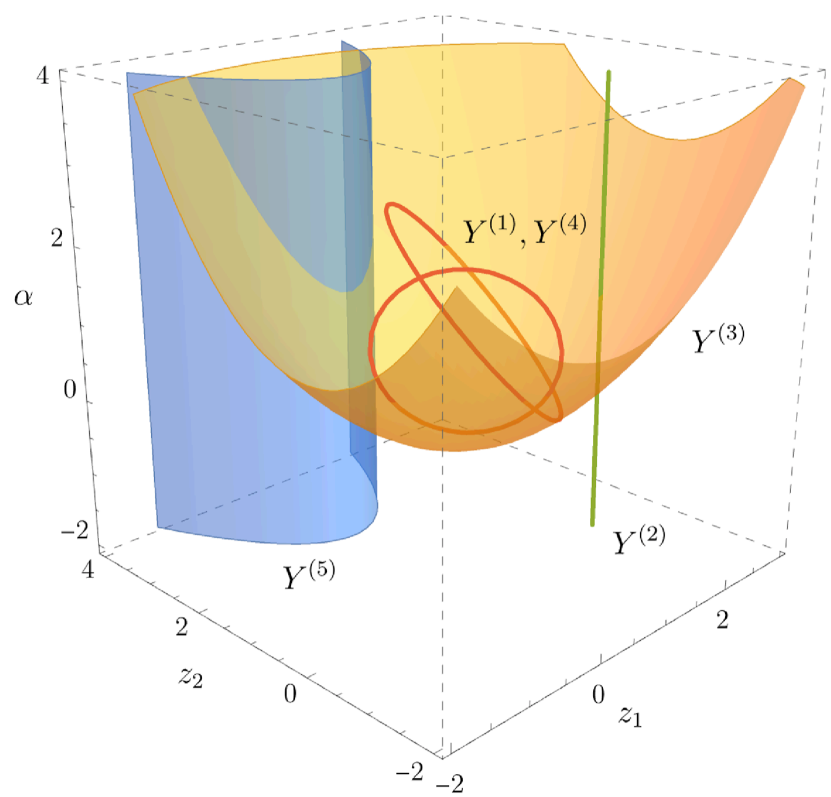
$$Y_{G,Q}(\mathcal{E}) = \{(\alpha, z) \in (\mathbb{C}^*)^E \times \mathcal{E} \,:\, \mathcal{G}_{G,Q} = 0, \partial_\alpha \mathcal{G}_{G,Q} = 0\}$$

$$Y_{G,Q}(\mathcal{E}) = \bigcup_{i \in \mathbb{I}(G,Q)} Y^{(i)}_{G,Q}(\mathcal{E})$$ irreducible decomposition

$$\nabla^{(i)}_{G,Q}(\mathcal{E}) = \overline{\pi_{\mathcal{E}}\left(Y^{(i)}_{G,Q}(\mathcal{E})\right)}$$ projection to $\mathcal{E}$

Principal Landau determinant

$Q \subset \mathrm{conv}(A)$ face, $\quad \mathcal{G}_G = \sum_{a \in A} c_a \underbrace{(m, M, s, t)}_{z} \alpha^a, \quad \mathcal{G}_{G,Q} = \sum_{a \in A \cap Q} c_a \underbrace{(m, M, s, t)}_{z} \alpha^a$



$$Y_{G,Q}(\mathcal{E}) = \{(\alpha, z) \in (\mathbb{C}^*)^E \times \mathcal{E} \,:\, \mathcal{G}_{G,Q} = 0, \partial_\alpha \mathcal{G}_{G,Q} = 0\}$$

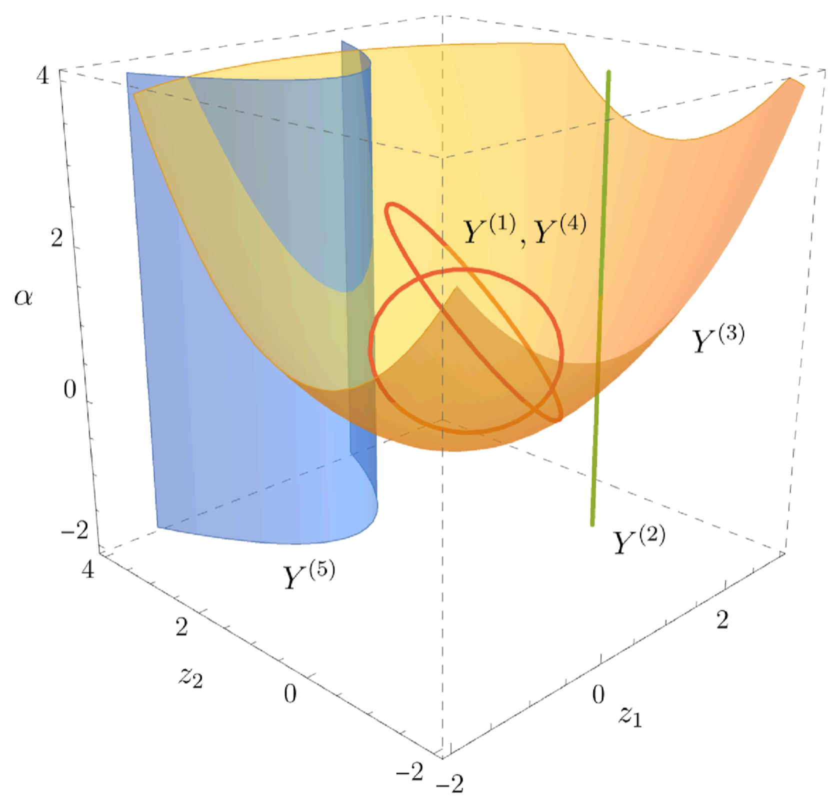
$$Y_{G,Q}(\mathcal{E}) = \bigcup_{i \in \mathbb{I}(G,Q)} Y^{(i)}_{G,Q}(\mathcal{E})$$ irreducible decomposition

$$\nabla^{(i)}_{G,Q}(\mathcal{E}) = \overline{\pi_{\mathcal{E}}\left(Y^{(i)}_{G,Q}(\mathcal{E})\right)}$$ projection to $\mathcal{E}$

$$\mathbb{I}(G,Q)_1 = \{i \in \mathbb{I}(G,Q) \,:\, \mathrm{codim}\, \nabla^{(i)}_{G,Q}(\mathcal{E}) = 1\}$$

Principal Landau determinant

$Q \subset \mathrm{conv}(A)$ face, $\mathcal{G}_G = \sum_{a \in A} c_a\underbrace{(m, M, s, t)}_{z}\, \alpha^a$, $\mathcal{G}_{G,Q} = \sum_{a \in A \cap Q} c_a\underbrace{(m, M, s, t)}_{z}\, \alpha^a$



$$Y_{G,Q}(\mathcal{E}) = \{(\alpha, z) \in (\mathbb{C}^*)^E \times \mathcal{E} \; : \; \mathcal{G}_{G,Q} = 0, \partial_\alpha \mathcal{G}_{G,Q} = 0\}$$

$$Y_{G,Q}(\mathcal{E}) = \bigcup_{i \in \mathbb{I}(G,Q)} Y^{(i)}_{G,Q}(\mathcal{E})$$ irreducible decomposition

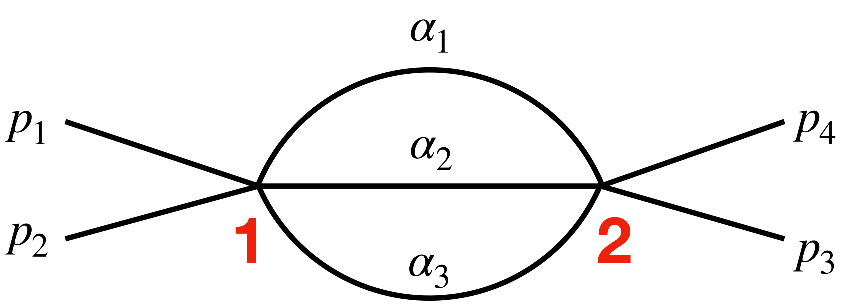
$$\nabla^{(i)}_{G,Q}(\mathcal{E}) = \overline{\pi_{\mathcal{E}}\left(Y^{(i)}_{G,Q}(\mathcal{E})\right)}$$ projection to $\mathcal{E}$

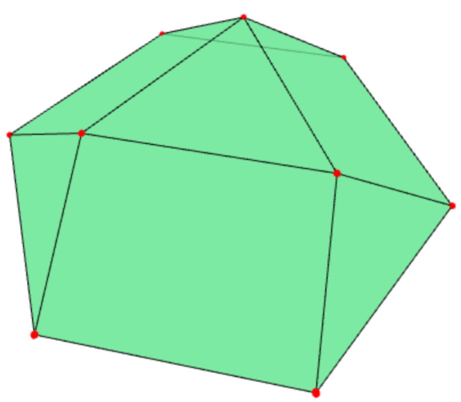
$$\mathbb{I}(G,Q)_1 = \{i \in \mathbb{I}(G,Q) \; : \; \mathrm{codim}\, \nabla^{(i)}_{G,Q}(\mathcal{E}) = 1\}$$

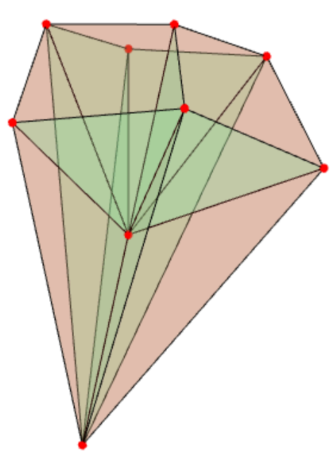
Definition. The **principal Landau determinant** of the diagram G with respect to the parameter space $\mathcal{E}$ is the defining polynomial E_G of

$$\mathrm{PLD}_G(\mathcal{E}) = \bigcup_{Q \subset \mathrm{conv}(A)} \bigcup_{i \in \mathbb{I}(G,Q)_1} \nabla^{(i)}_{G,Q}(\mathcal{E})$$

Sunrise solution: PLD.jl



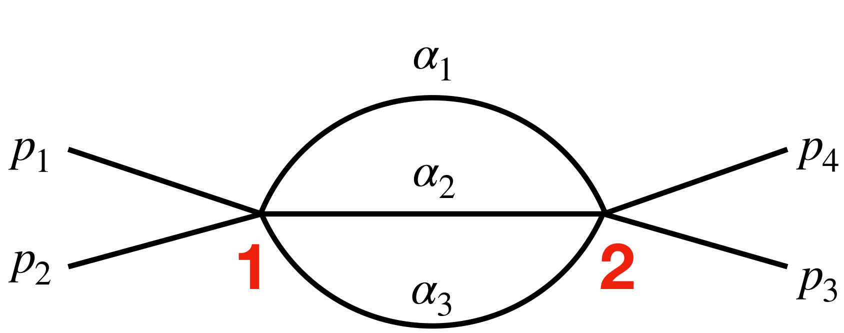


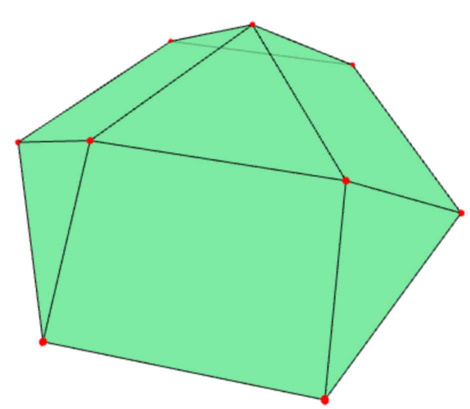


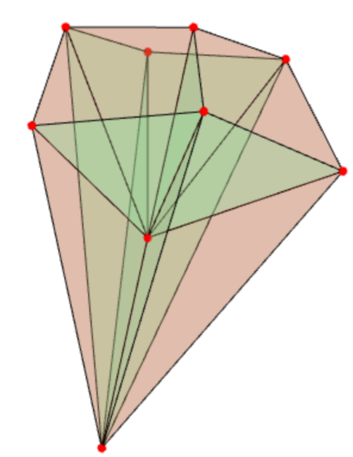
```
julia> edges = [[1,2],[1,2],[1,2]]; nodes = [1,1,2,2]; @var m[1:3] M[1:4];

julia> getPLD(edges, nodes; internal_masses = m, external_masses = M, method = :num)
-------- codim = 3, 9 faces
codim: 3, face: 1/9, weights: [-1, 0, 1], discriminant: m₁
New discriminants after codim 3, face 1/9. The list is: m₁
codim: 3, face: 2/9, weights: [-1, 1, 0], discriminant: m₁
codim: 3, face: 3/9, weights: [0, -1, 1], discriminant: m₂
New discriminants after codim 3, face 3/9. The list is: m₁, m₂
codim: 3, face: 4/9, weights: [2, 2, 4], discriminant: 1
New discriminants after codim 3, face 4/9. The list is: 1, m₁, m₂
codim: 3, face: 5/9, weights: [0, 1, -1], discriminant: m₃
New discriminants after codim 3, face 5/9. The list is: 1, m₁, m₂, m₃
codim: 3, face: 6/9, weights: [2, 4, 2], discriminant: 1
codim: 3, face: 7/9, weights: [1, -1, 0], discriminant: m₂
codim: 3, face: 8/9, weights: [1, 0, -1], discriminant: m₃
codim: 3, face: 9/9, weights: [4, 2, 2], discriminant: 1
Unique discriminants after codim 3: 1, m₁, m₂, m₃
-------- codim = 2, 15 faces
codim: 2, face: 1/15, weights: [-1, 0, 0], discriminant: m₁
codim: 2, face: 2/15, weights: [-1, -1, 0], discriminant: 1
codim: 2, face: 3/15, weights: [0, 1, 2], discriminant: 1
codim: 2, face: 4/15, weights: [-1, 0, -1], discriminant: 1
codim: 2, face: 5/15, weights: [0, 2, 1], discriminant: 1
codim: 2, face: 6/15, weights: [1, 0, 2], discriminant: 1
codim: 2, face: 7/15, weights: [0, -1, 0], discriminant: m₂
codim: 2, face: 8/15, weights: [1, 2, 2], discriminant: 1
codim: 2, face: 9/15, weights: [2, 1, 2], discriminant: 1
codim: 2, face: 10/15, weights: [1, 2, 0], discriminant: 1
codim: 2, face: 11/15, weights: [0, 0, -1], discriminant: m₃
codim: 2, face: 12/15, weights: [2, 2, 1], discriminant: 1
codim: 2, face: 13/15, weights: [0, -1, -1], discriminant: 1
codim: 2, face: 14/15, weights: [2, 0, 1], discriminant: 1
codim: 2, face: 15/15, weights: [2, 1, 0], discriminant: 1
Unique discriminants after codim 2: 1, m₁, m₂, m₃
```

Sunrise solution: PLD.jl

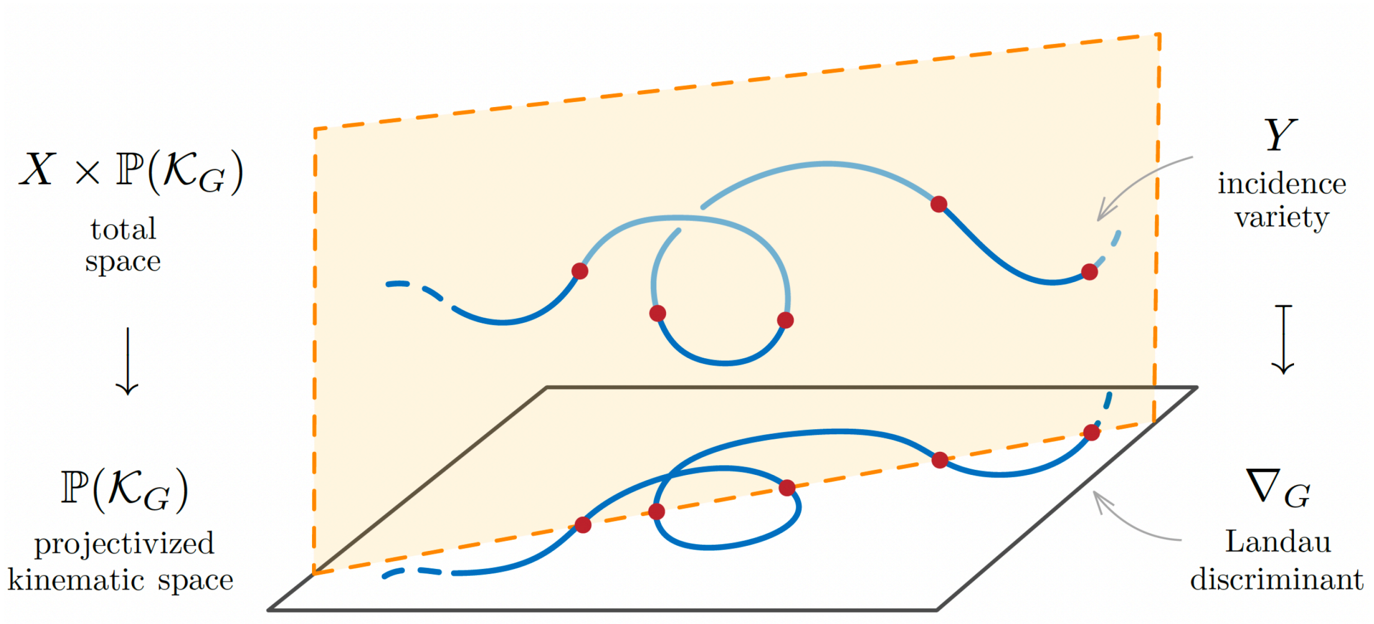
```
-------- codim = 1, 8 faces
codim: 1, face: 1/8, weights: [-1, -1, -1], discriminant: m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ +
4*m₁^2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*
m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s
^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, s
New discriminants after codim 1, face 1/8. The list is: 1, m₁, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*
m₃ + 4*m₁^2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s
 + 4*m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m
₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, m₂, m₃, s
codim: 1, face: 2/8, weights: [0, 1, 1], discriminant: 1
codim: 1, face: 3/8, weights: [0, 0, 1], discriminant: 1
codim: 1, face: 4/8, weights: [0, 1, 0], discriminant: 1
codim: 1, face: 5/8, weights: [1, 0, 1], discriminant: 1
codim: 1, face: 6/8, weights: [1, 1, 1], discriminant: 1
codim: 1, face: 7/8, weights: [1, 1, 0], discriminant: 1
codim: 1, face: 8/8, weights: [1, 0, 0], discriminant: 1
Unique discriminants after codim 1: 1, m₁, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6
*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m
₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3
+ 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, m₂, m₃, s
-------- codim = 0, 1 faces
codim: 0, face: 1/1, weights: [0, 0, 0], discriminant: s
Unique discriminants after codim 0: 1, m₁, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6
*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m
₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3
+ 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, m₂, m₃, s
```







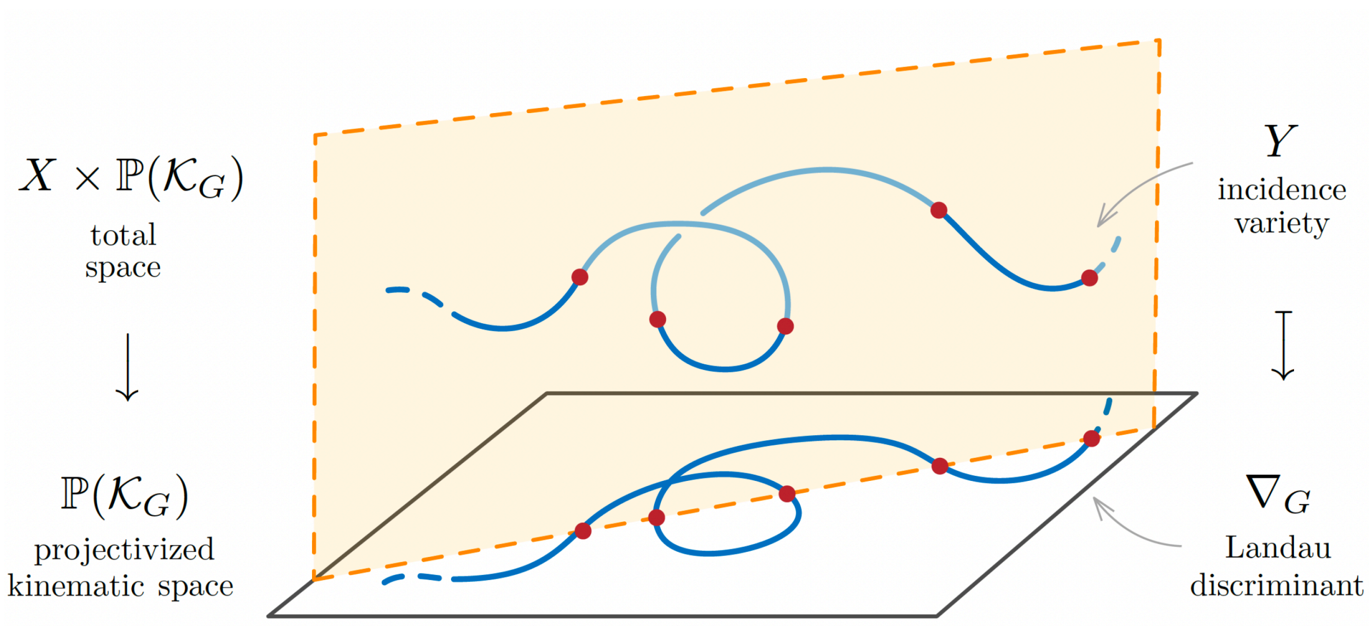
method = :num



S. Mizera and S. Telen, *Landau discriminants*, *JHEP* **08** (2022) 200 [2109.08036].

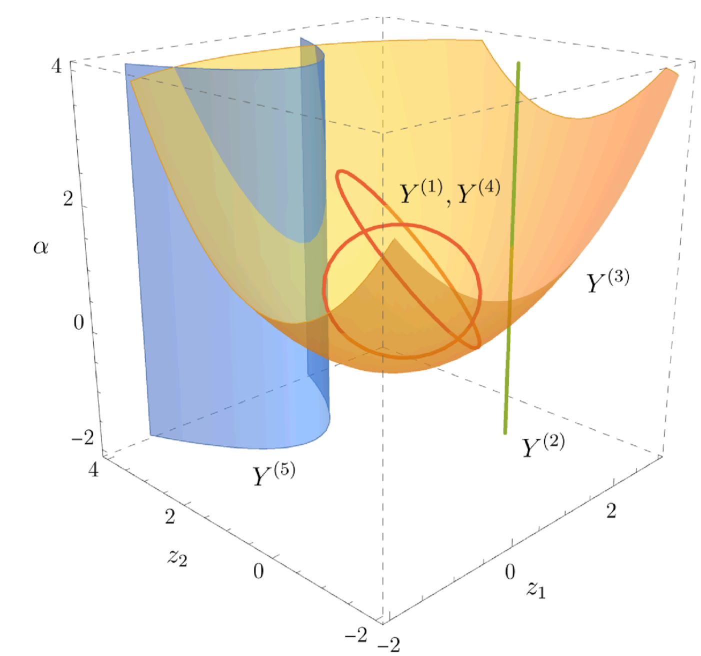
HomotopyContinuation.jl + numerical interpolation

method = :num

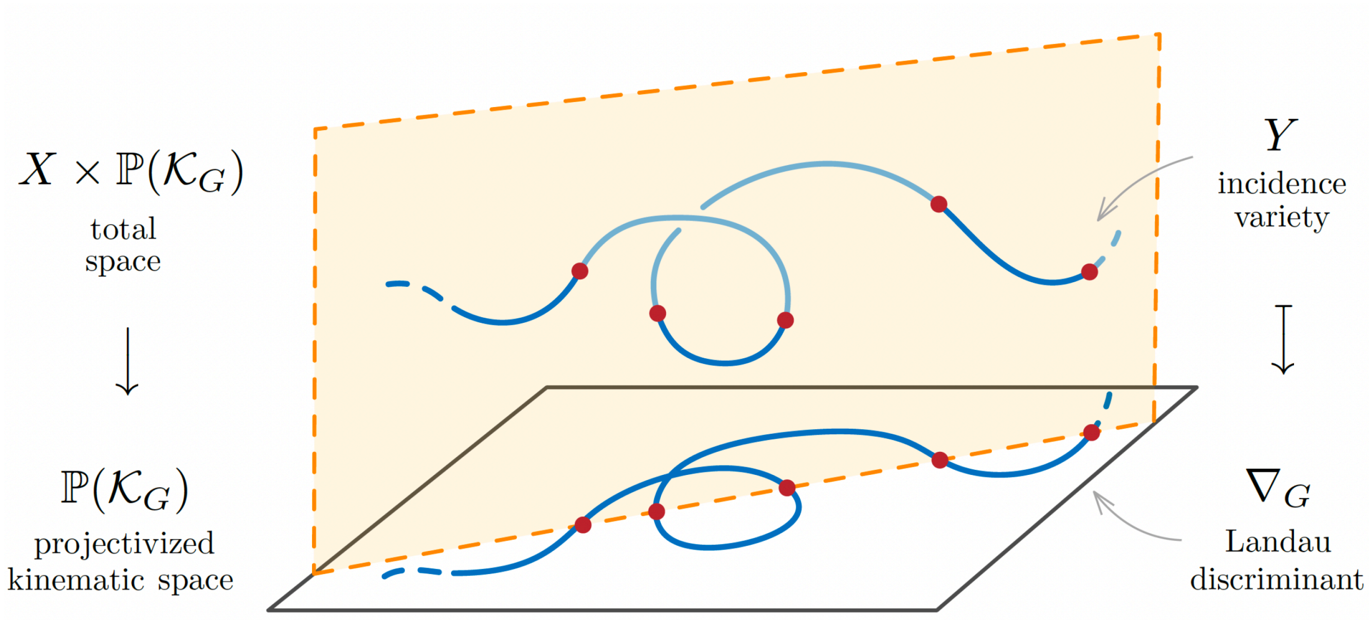


S. Mizera and S. Telen, *Landau discriminants*, *JHEP* **08** (2022) 200 [2109.08036].

Homotopy Continuation.jl **+ numerical interpolation**

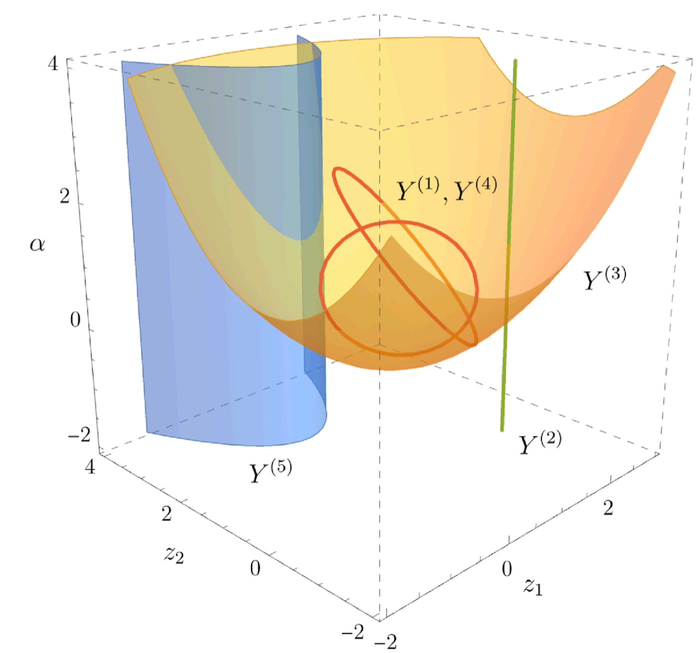


method = :num



S. Mizera and S. Telen, *Landau discriminants*, *JHEP* **08** (2022) 200 [2109.08036].

Homotopy Continuation.jl **+ numerical interpolation**



1. Compute samples on the incidence variety as regular solutions to systems of polynomial equations. We pick up points on desired components + dominant components
2. Filter out such dominant points, and continue with the remaining samples
3. Divide the samples into groups corresponding to their incidence components
4. Deduce the degree of the projected components from the number of samples
5. Collect enough samples to find a unique interpolant

Conjectures

$\mathrm{PLD}_{\mathrm{G}}(\mathscr{E}) \subset \nabla_{\chi}(\mathscr{E})$

Conjectures

$$\mathrm{PLD_G}(\mathscr{E}) \subset \nabla_\chi(\mathscr{E})$$

```
julia> χdiscriminantQ(U+F,pp,vv,unique(vcat(Δ...)))
Generic |Euler characteristic|, χ* = 7
candidates = Any[m₁, m₂, m₃, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^
2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 -
 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂
^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 -
 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, s]
Subspace m₁ has χ = 4 < χ*
Subspace m₂ has χ = 4 < χ*
Subspace m₃ has χ = 4 < χ*
Subspace m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6*m₁^2*m₃^2
 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m
₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 +
 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s
^2 - 4*m₃*s^3 + s^4 has χ = 6 < χ*
Subspace s has χ = 4 < χ*
```

Conjectures

$\mathrm{PLD}_G(\mathscr{E}) \subset \nabla_\chi(\mathscr{E})$

but we know $\mathrm{PLD}_G(\mathscr{E}) \not\supset \nabla_\chi(\mathscr{E})$

```
julia> χdiscriminantQ(U+F,pp,vv,unique(vcat(Δ...)))
Generic |Euler characteristic|, χ∗ = 7
candidates = Any[m₁, m₂, m₃, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^
2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 -
 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂
^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 -
 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, s]
Subspace m₁ has χ = 4 < χ∗
Subspace m₂ has χ = 4 < χ∗
Subspace m₃ has χ = 4 < χ∗
Subspace m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6*m₁^2*m₃^2
 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m
₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 +
 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s
^2 - 4*m₃*s^3 + s^4 has χ = 6 < χ∗
Subspace s has χ = 4 < χ∗
```

Conjectures

$\mathrm{PLD}_G(\mathscr{E}) \subset \nabla_\chi(\mathscr{E})$ but we know $\mathrm{PLD}_G(\mathscr{E}) \not\supset \nabla_\chi(\mathscr{E})$

```
julia> χdiscriminantQ(U+F,pp,vv,unique(vcat(Δ...)))
Generic |Euler characteristic|, χ∗ = 7
candidates = Any[m₁, m₂, m₃, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^
2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 -
 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂
^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 -
 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, s]
Subspace m₁ has χ = 4 < χ∗
Subspace m₂ has χ = 4 < χ∗
Subspace m₃ has χ = 4 < χ∗
Subspace m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6*m₁^2*m₃^2
 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m
₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 +
 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s
^2 - 4*m₃*s^3 + s^4 has χ = 6 < χ∗
Subspace s has χ = 4 < χ∗
```

$\mathrm{PLD}_G(\mathscr{E}) \subset \nabla_\chi(\mathscr{E}) = \text{Landau variety} \subset \text{result of } \texttt{HyperInt}$

F.C.S. Brown, *On the periods of some Feynman integrals*, 0910.0114.

E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, Comput. Phys. Commun. **188** (2015) 148 [1403.3385].

Conjectures

$\mathrm{PLD}_G(\mathscr{E}) \subset \nabla_\chi(\mathscr{E})$

but we know $\mathrm{PLD}_G(\mathscr{E}) \not\supset \nabla_\chi(\mathscr{E})$

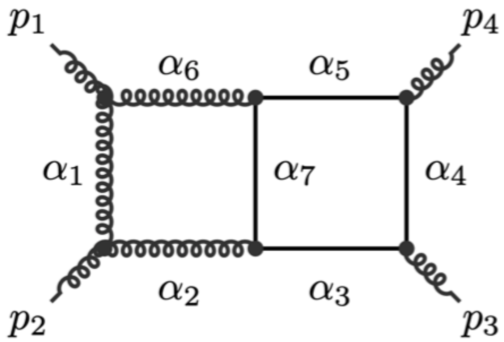
```
julia> χdiscriminantQ(U+F,pp,vv,unique(vcat(Δ...)))
Generic |Euler characteristic|, χ* = 7
candidates = Any[m₁, m₂, m₃, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, s]
Subspace m₁ has χ = 4 < χ*
Subspace m₂ has χ = 4 < χ*
Subspace m₃ has χ = 4 < χ*
Subspace m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4 has χ = 6 < χ*
Subspace s has χ = 4 < χ*
```

$\mathrm{PLD}_G(\mathscr{E}) \subset \nabla_\chi(\mathscr{E}) = \text{Landau variety} \subset \text{result of } \texttt{HyperInt}$

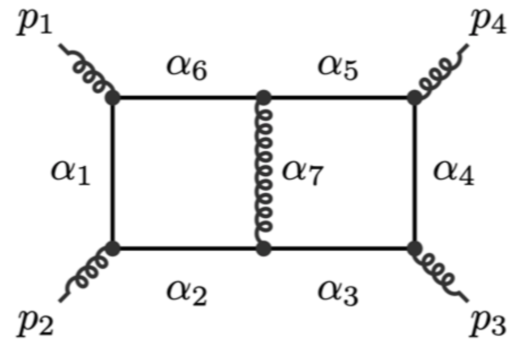
F.C.S. Brown, *On the periods of some Feynman integrals*, 0910.0114.

E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, Comput. Phys. Commun. **188** (2015) 148 [1403.3385].

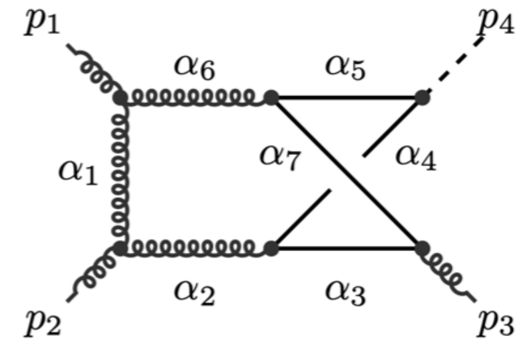
Can we compute $\nabla_\chi(\mathscr{E})$ from the primary decomposition of an ideal in the Cox ring of $X_{\mathrm{conv}(A)}$?



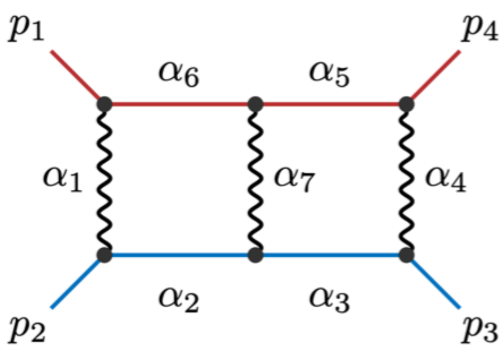
(a) Double-box with an inner massive loop, $G =$ `inner-dbox`



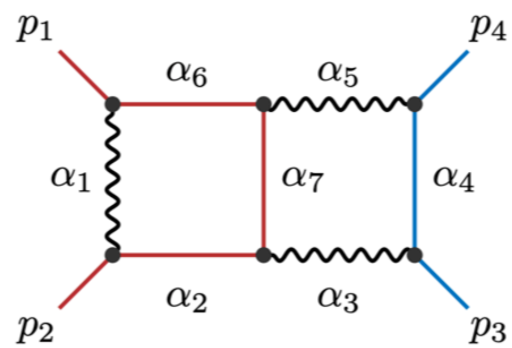
(b) Double-box with an outer massive loop, $G =$ `outer-dbox`



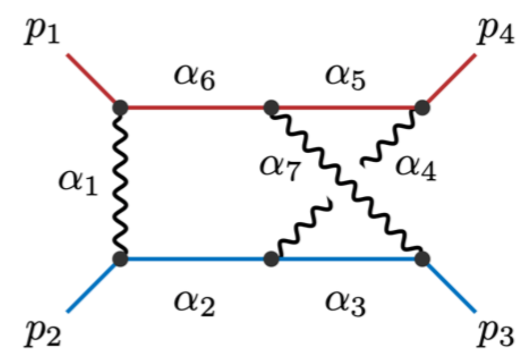
(c) Non-planar double-box for Higgs + jet production, $G =$ `Hj-npl-dbox`



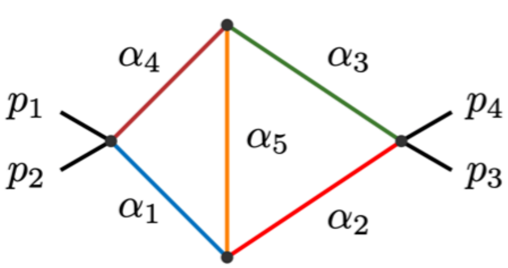
(d) Double-box for Bhabha scattering, $G =$ `Bhabha-dbox`



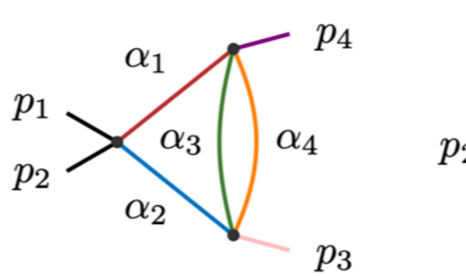
(e) Second double-box for Bhabha scattering, $G =$ `Bhabha2-dbox`



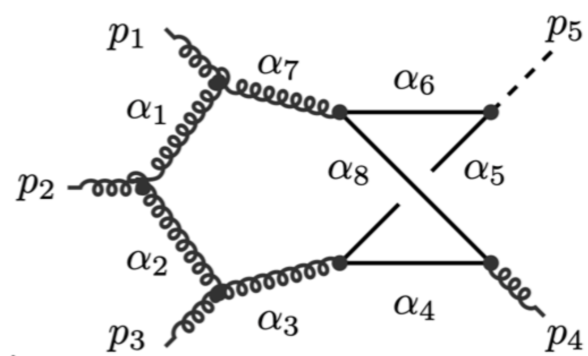
(f) Non-planar double-box for Bhabha scattering, $G =$ `Bhabha-npl-dbox`



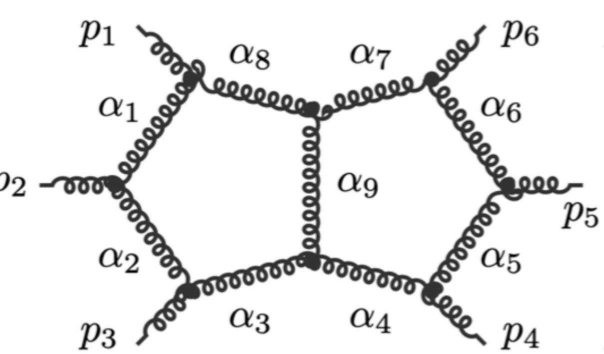
(g) Kite diagram with generic masses, $G =$ `kite`



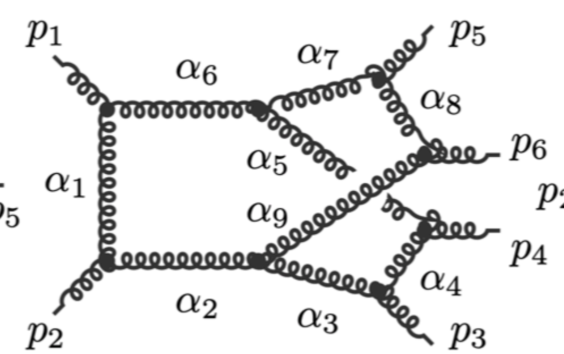
(h) Parachute diagram with generic masses, $G =$ `par`



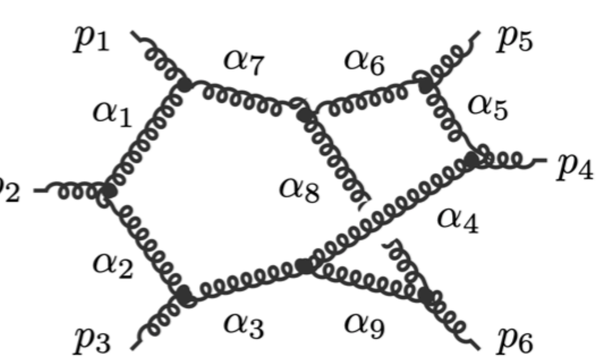
(i) Non-planar penta-box fir Higgs + jet production, $G =$ `Hj-npl-pentb`



(j) Massless planar double-pentagon, $G =$ `dpent`



(k) Massless non-planar double-pentagon, $G =$ `npl-dpent`



(l) Second massless non-planar double-pentagon, $G =$ `npl-dpent2`

Database coming soon at

MATHREPO
MATHEMATICAL RESEARCH-DATA REPOSITORY

Thank you!