Principal Landau Determinants, Part I

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Joint with Sebastian Mizera and Simon Telen

MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

MathemAmplitudes 2023: QFT at the Computational Frontier
Palazzo del Monte di Pieta' - September 25, 2023

Motivation: singularities and GKZ systems

$$A = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ m_1 & m_2 & \cdots & m_s \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \in \mathbb{Z}^{n\times s}, \quad \text{rank}(A) = n$$

$$\alpha^{m_i} = \alpha_1^{m_{1i}} \cdots \alpha_n^{m_{ni}}$$

$$f_A(\alpha; z) = z_1\,\alpha^{m_1} + z_2\,\alpha^{m_2} + \cdots + z_s\,\alpha^{m_s}$$

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

$$z_i \in \mathbb{C}$$

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Remark (actually, a theorem): Under genericity assumptions $|\chi(V_{A,z^*})| = \text{vol}(A)$ counts the number of linearly independent solutions to a GKZ system.

SST'99 - Matsubara-Heo'20   de la Cruz '19 - CGMMMMT'22 

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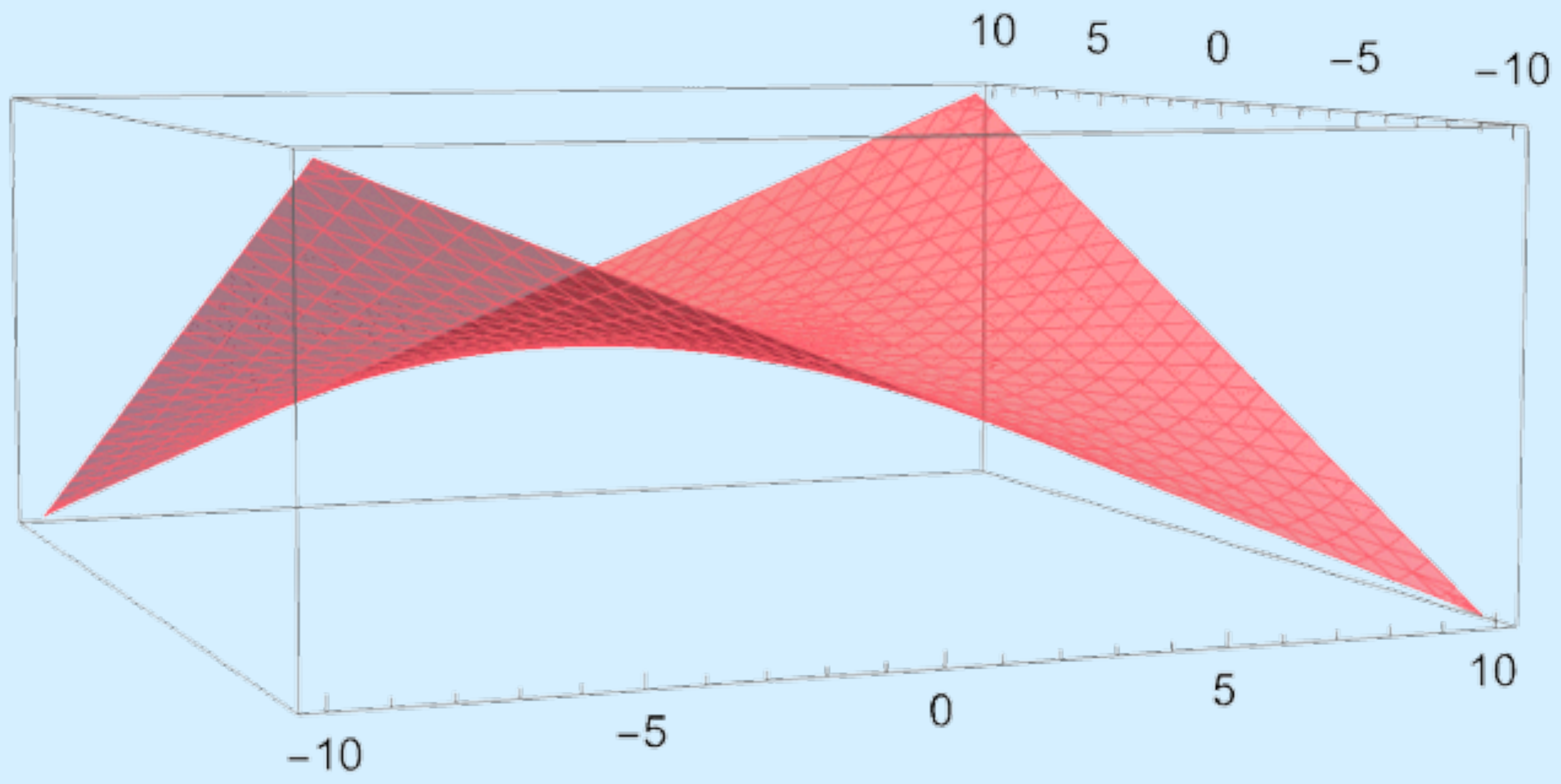
A-discriminants

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$f_A(\alpha, z) \;=\; z_1 + z_2\,\alpha_1 + z_3\,\alpha_2 + z_4\,\alpha_1\alpha_2$$



$$\Delta_A \;=\; \det\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = z_1 z_4 - z_2 z_3$$

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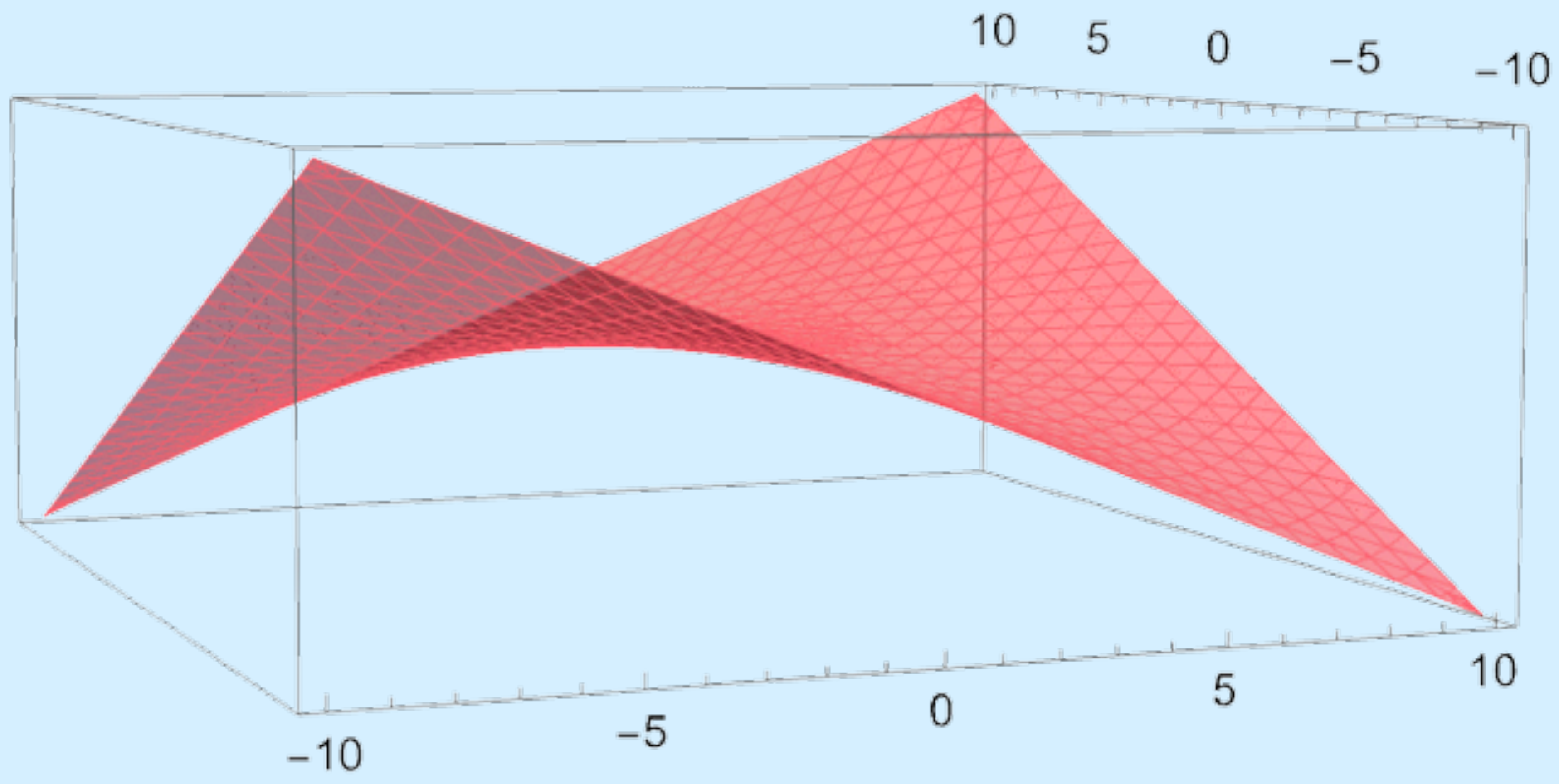

$$\nabla_A^\circ = \left\{ z \in \mathbb{C}^s \,:\, \exists \alpha \in (\mathbb{C}^*)^n \text{ s.t. } f_A(\alpha; z) = \partial_\alpha f_A(\alpha; z) = 0 \right\}$$

$$\partial_\alpha = (\partial_{\alpha_1}, \ldots, \partial_{\alpha_n})$$

Definition: The A-discriminant variety $\nabla_A = \overline{\nabla_A^\circ}$ records values of z for which $V_{A,z}$ is a singular hypersurface.

Scattering amplitudes

$G = (V, E)$ Feynman diagram

[Lee-Pomeransky, '13]

$$I_{\nu_1,\ldots,\nu_n} = \# \int_0^\infty \frac{\alpha_1^{\nu_1} \cdots \alpha_n^{\nu_n}}{\underbrace{(\mathscr{U}_G + \mathscr{F}_G)}_{\mathscr{G}_G}{}^{D/2}} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{\mathrm{d}\alpha_n}{\alpha_n}$$

Graph polynomial

$\mathscr{U}_G, \mathscr{F}_G$ homogeneous polynomials in $\alpha_1, \ldots, \alpha_n$ with coefficients in the kinematic space $\mathscr{K} \subset \mathbb{C}^m$

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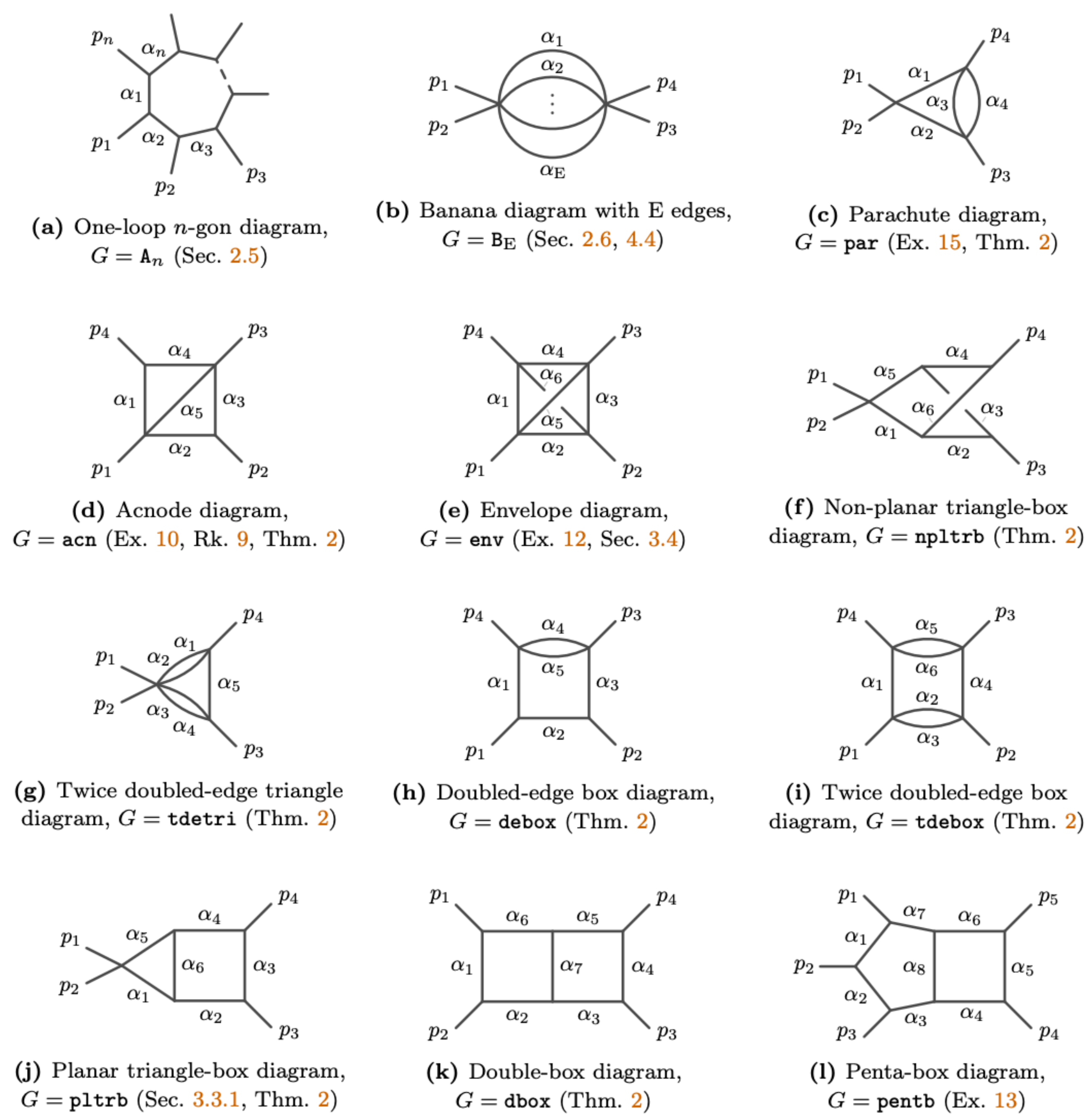
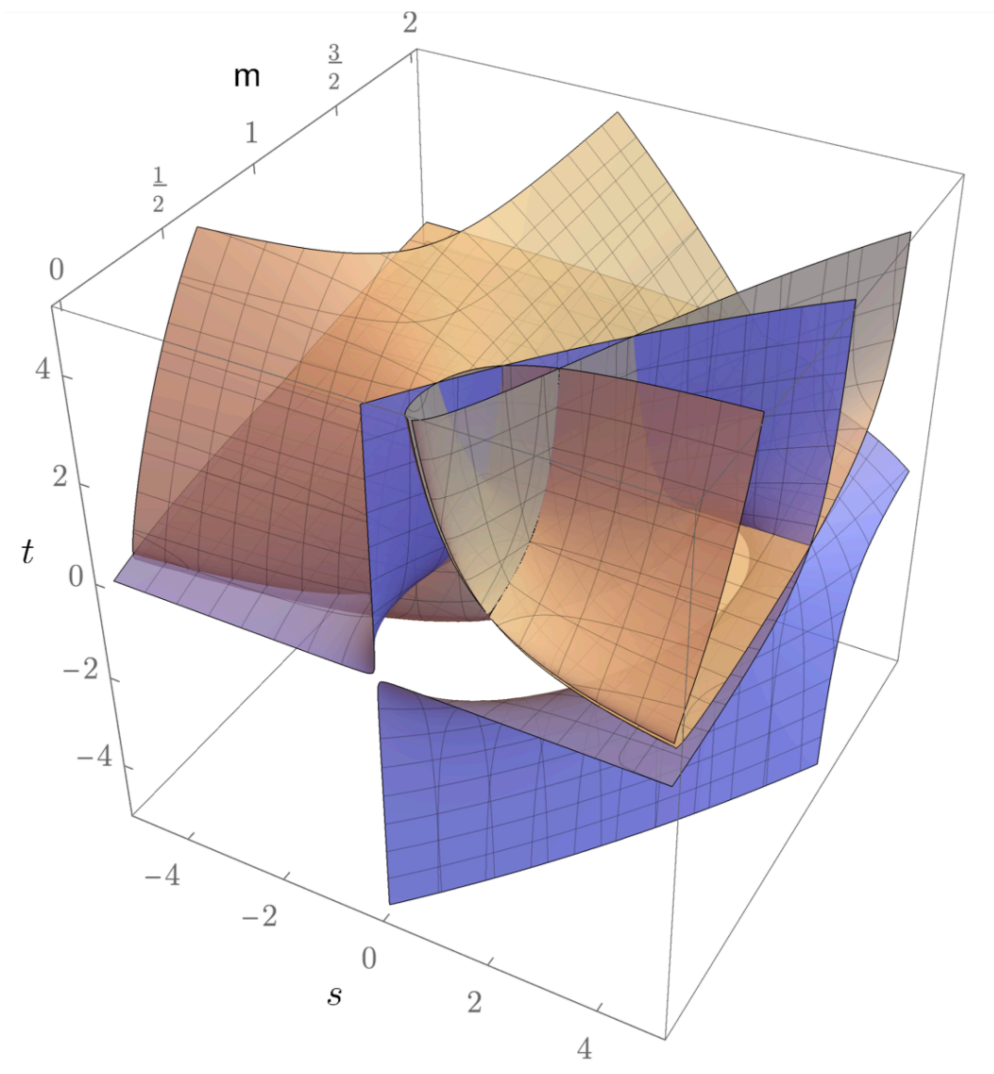
Landau discriminants

Sebastian Mizera & Simon Telen

Journal of High Energy Physics **2022**, Article number: 200 (2022)





Landau.jl

χ VS volume

Theorem (Bitoun, Bogner, Klausen, Panzer, 2018): The Euler characteristic $|\chi(V(\mathscr{G}_G))|$ counts the number of master integrals

[Submitted on 18 Aug 2022]

Vector Spaces of Generalized Euler Integrals

Daniele Agostini, Claudia Fevola, Anna-Laura Sattelberger, Simon Telen

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Theorem (Bitoun, Bogner, Klausen, Panzer, 2018): The Euler characteristic $|\chi(V(\mathcal{G}_G))|$ counts the number of master integrals

G	$\mathcal{K}$	$\mathcal{E}^{(\mathrm{M}_i,0)}$	$\mathcal{E}^{(0,\mathrm{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\mathtt{A}_4$	$(15,15)$	$(11,11)$	$(11,15)$	$(3,3)$
$\mathtt{B}_4$	$(15,35)$	$(1,1)$	$(15,35)$	$(1,1)$
par	$(19,35)$	$(4,8)$	$(13,35)$	$(1,3)$
acn	$(55,136)$	$(20,54)$	$(36,136)$	$(3,9)$
env	$(273,1496)$	$(56,262)$	$(181,1496)$	$(10,80)$
npltrb	$(116,512)$	$(28,252)$	$(77,512)$	$(5,61)$
tdetri	$(51,201)$	$(4,18)$	$(33,201)$	$(1,5)$
debox	$(43,96)$	$(11,33)$	$(31,96)$	$(3,10)$
tdebox	$(123,705)$	$(11,113)$	$(87,705)$	$(3,41)$
pltrb	$(81,417)$	$(16,201)$	$(61,417)$	$(4,80)$
dbox	$(227,1422)$	$(75,903)$	$(159,1422)$	$(12,238)$
pentb	$(543,4279)$	$(228,3148)$	$(430,4279)$	$(62,1186)$

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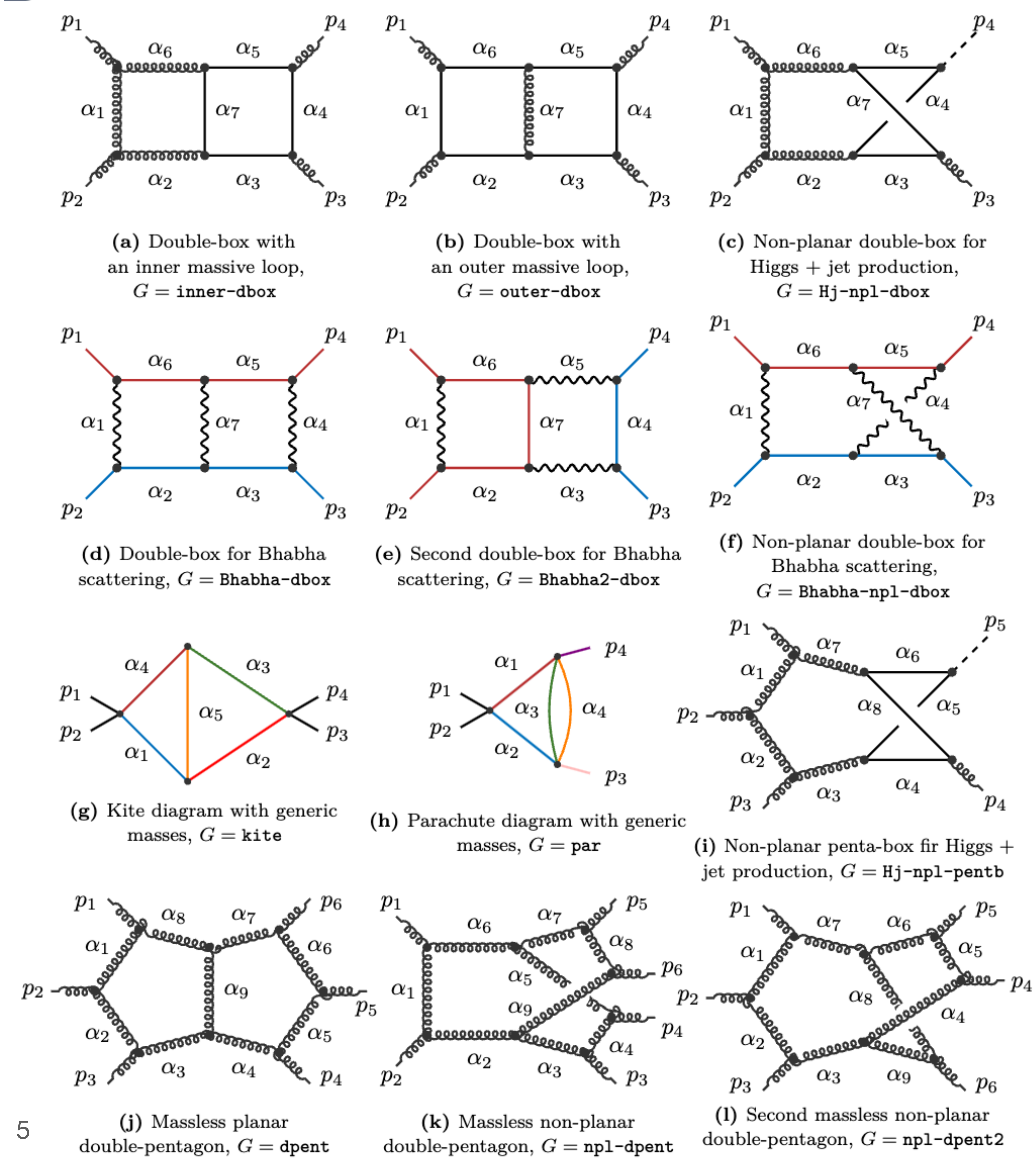
$(|\chi(V_A(\mathcal{E}))|, \mathrm{vol}(A(\mathcal{E})))$

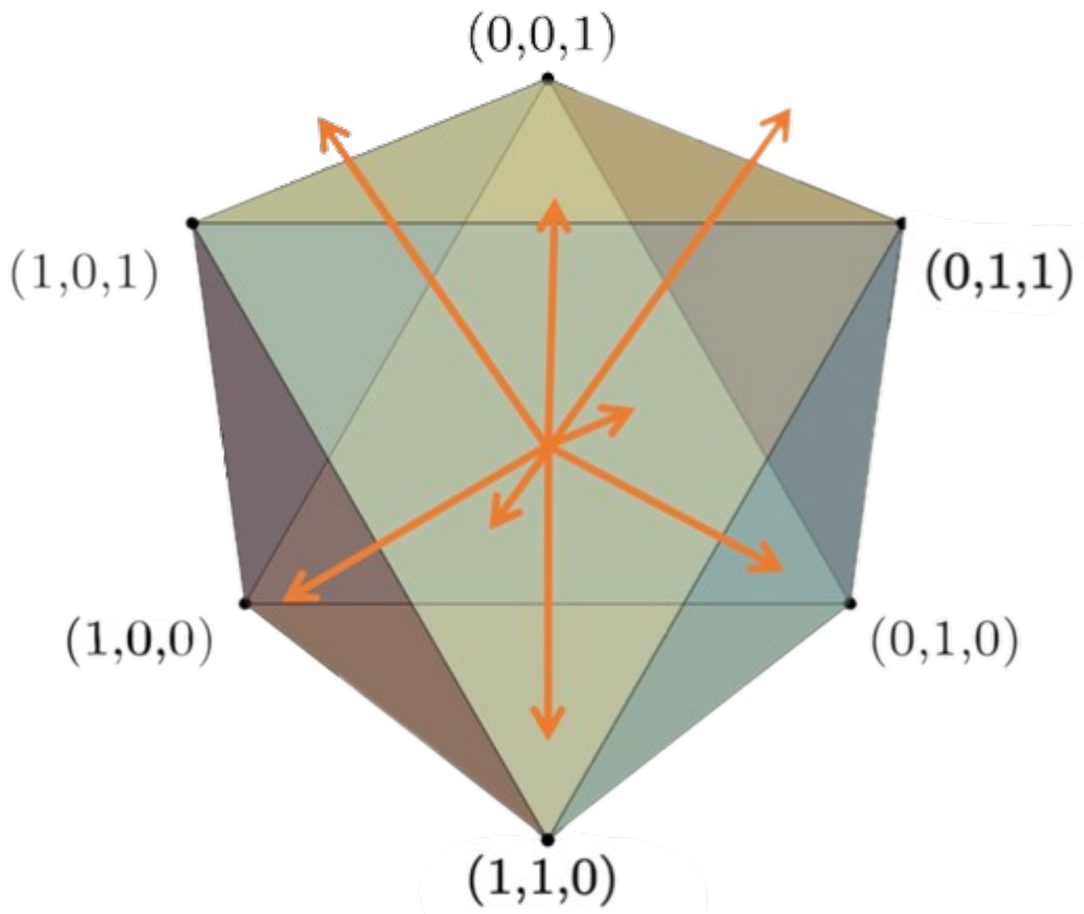
χ VS volume

G	$\mathcal{E}$
inner-dbox	$(43, 834)$
outer-dbox	$(64, 1302)$
Hj-npl-dbox	$(99, 1016)$
Bhabha-dbox	$(64, 774)$
Bhabha2-dbox	$(79, 910)$
Bhabha-npl-dbox	$(111, 936)$
kite	$(30, 136)$
par	$(19, 35)$
Hj-npl-pentb	$(330, 3144)$
dpent	$(281, 5511)$
npl-dpent	$(631, 5784)$
npl-dpent2	$(458, 5467)$



(a) Double-box with an inner massive loop, $G =$ `inner-dbox`
(b) Double-box with an outer massive loop, $G =$ `outer-dbox`
(c) Non-planar double-box for Higgs + jet production, $G =$ `Hj-npl-dbox`
(d) Double-box for Bhabha scattering, $G =$ `Bhabha-dbox`
(e) Second double-box for Bhabha scattering, $G =$ `Bhabha2-dbox`
(f) Non-planar double-box for Bhabha scattering, $G =$ `Bhabha-npl-dbox`
(g) Kite diagram with generic masses, $G =$ `kite`
(h) Parachute diagram with generic masses, $G =$ `par`
(i) Non-planar penta-box fir Higgs + jet production, $G =$ `Hj-npl-pentb`
(j) Massless planar double-pentagon, $G =$ `dpent`
(k) Massless non-planar double-pentagon, $G =$ `npl-dpent`
(l) Second massless non-planar double-pentagon, $G =$ `npl-dpent2`

Principal A-determinant [GKZ]



$P := \mathrm{conv}(A) \subset \mathbb{R}^n$

$$E_A = \prod_{Q \in F(A)} \Delta_{A\cap Q}^{e_Q}$$

$e_Q \in \mathbb{N}$

$$A \cap Q = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ m_1 & m_2 & \cdots & m_s \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

$m_i \in Q$

Set of faces of P

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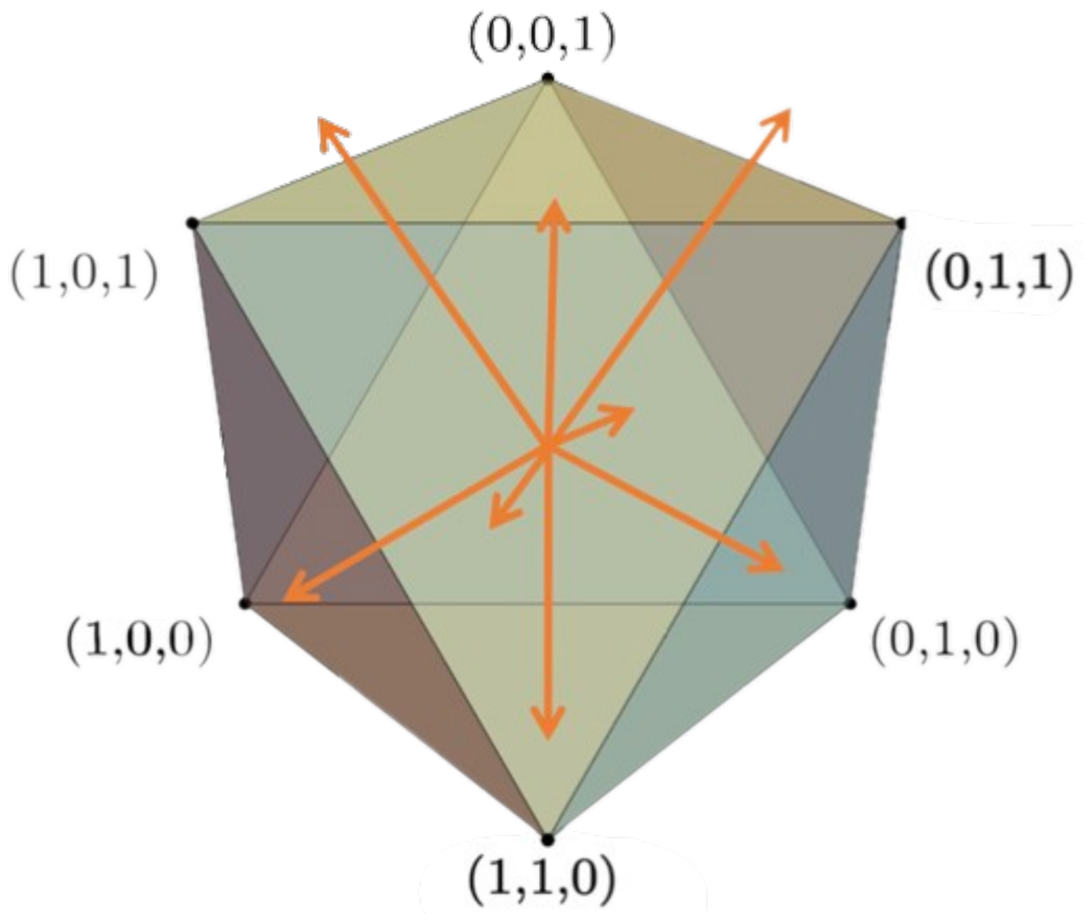

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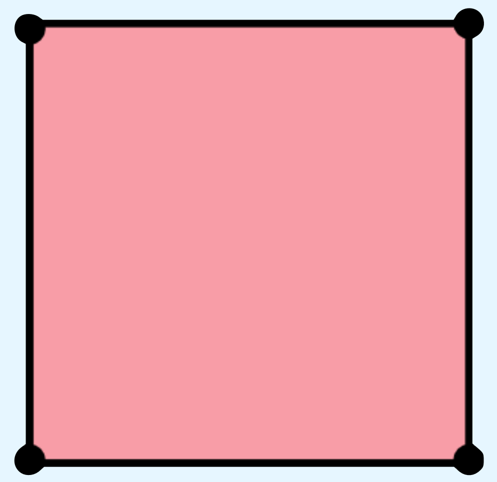
Theorem (Amendola, Bliss, Burke, Gibbons, Helmer, Hoşten, Nash, Rodriguez, Smolkin, 2012):

$$|\chi(V_{A,z})| = \mathrm{vol}(A) \quad \Longleftrightarrow \quad z \in \mathbb{C}^s \setminus \{E_A = 0\}$$

Moreover, when $E_A(z) = 0$, we have $|\chi(V_{A,z})| < \mathrm{vol}(A)$.

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$



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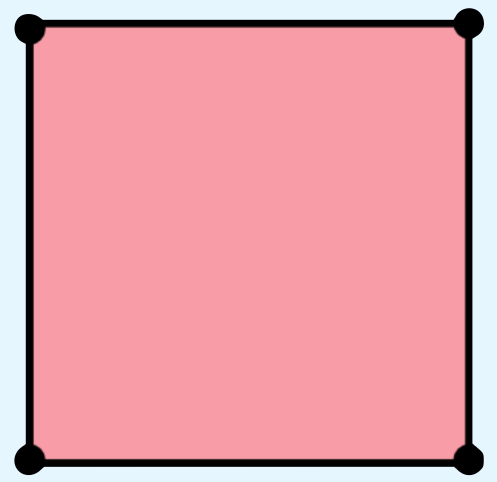

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Remark

Computing principal A-determinant → Specialising to parameter space $\neq$ Specialising to parameter space → Computing principal A-determinant

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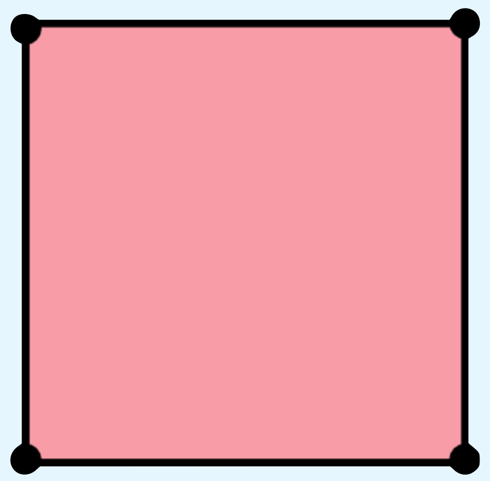

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⚠ Some factors might identically vanish

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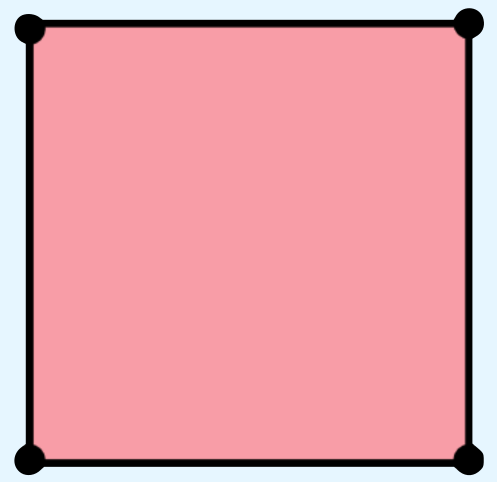

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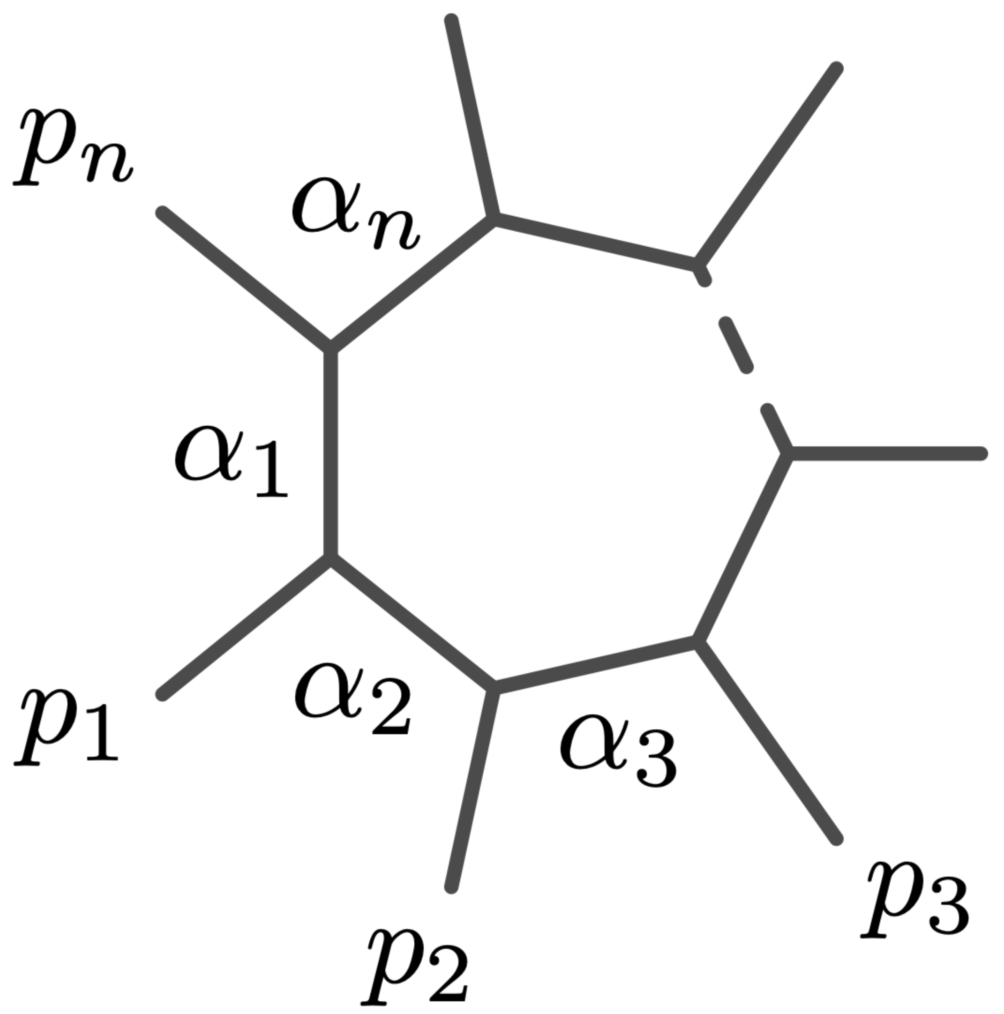
Bjorken, Landau, Nakanishi ’54

Klausen ’21 - Berghoff, Panzer ’22, - Dlapa, Helmer, Papathanasiou, Tellander ’23

Example: one-loop diagram

$\mathsf{A}_n =$ 

$$\mathscr{U}_{\mathsf{A}_n} = \alpha_1 + \ldots + \alpha_n$$

$$\mathscr{F}_{\mathsf{A}_n} = \sum_{i<n} (s_{i,i+1,\ldots,j-1} - \mathsf{m}_i - \mathsf{m}_j)\,\alpha_i\alpha_j - \sum_{i=1}^{n} \mathsf{m}_i\,\alpha_i^2$$

	$\mathcal{K}$	$\mathcal{E}^{(\mathsf{M}_i,0)}$	$\mathcal{E}^{(0,\mathsf{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\lvert\chi(V_{\mathtt{A}_n}(\mathcal{E}))\rvert$	2^n-1	2^n-1-n	2^n-1-n	$1,\ 3,\ 11, 33,\ 85,\ 199, 439,\ 933,\ \ldots$
$\mathrm{vol}(A_n(\mathcal{E}))$	2^n-1	2^n-1-n	2^n-1	$1,\ 3,\ 11, 33,\ 85,\ 199, 439,\ 933,\ \ldots$

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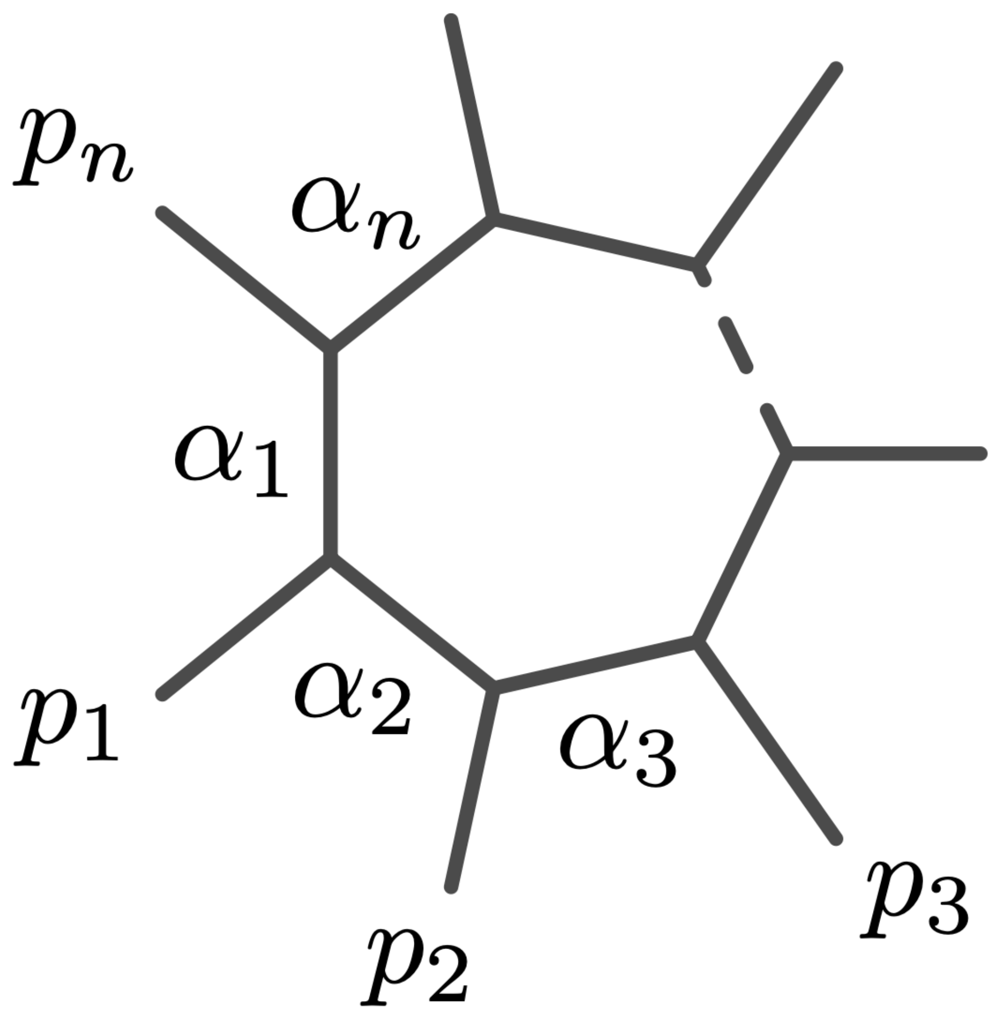
$\mathsf{A}_n =$ 

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Theorem (F,Mizera,Telen, 2023+):

Specialising to kinematic parameters in

Conjecture!

$$\{\mathscr{K}, \mathscr{E}^{(\mathsf{M}_i,0)}, \mathscr{E}^{(0,0)}\}$$

no factors of the principal A-determinant is identically zero.

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Proof idea:

1. Determine the combinatorics of conv$(A_n(\mathscr{E}))$
2. Compute the principal $A_n(\mathscr{E})$-determinant
3. Find a choice of parameters s.t. the specialisation does not vanish

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Outcome:

1. Formulae for the singular locus of A$_n$
2. Formal proof for the number of master integrals

Example: A_3

$$\mathscr{G}_{\mathsf{A}_3} = \frac{1}{2}\begin{pmatrix}1 & \alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 1\\ 1 & -2\mathsf{m}_1 & \mathsf{M}_1-\mathsf{m}_1-\mathsf{m}_2 & \mathsf{M}_3-\mathsf{m}_1-\mathsf{m}_3\\ 1 & \mathsf{M}_1-\mathsf{m}_1-\mathsf{m}_2 & -2\mathsf{m}_2 & \mathsf{M}_2-\mathsf{m}_2-\mathsf{m}_3\\ 1 & \mathsf{M}_3-\mathsf{m}_1-\mathsf{m}_3 & \mathsf{M}_2-\mathsf{m}_2-\mathsf{m}_3 & -2\mathsf{m}_3\end{pmatrix}\begin{pmatrix}1\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$$

$$\begin{aligned}E_{\mathtt{A}_3}(\mathcal{K}) = {} & \mathsf{m}_1\mathsf{m}_2\mathsf{m}_3\prod_{i=1}^{3}\lambda(\mathsf{m}_i,\mathsf{m}_{i+1},\mathsf{M}_i)\mathsf{M}_1\mathsf{M}_2\mathsf{M}_3(\mathsf{m}_1^2\mathsf{M}_2+\mathsf{m}_1\mathsf{M}_2^2\\ & +\mathsf{m}_2\mathsf{m}_1\mathsf{M}_1-\mathsf{m}_3\mathsf{m}_1\mathsf{M}_1-\mathsf{m}_2\mathsf{m}_1\mathsf{M}_2-\mathsf{m}_3\mathsf{m}_1\mathsf{M}_2-\mathsf{m}_1\mathsf{M}_1\mathsf{M}_2-\mathsf{m}_2\mathsf{m}_1\mathsf{M}_3+\mathsf{m}_3\mathsf{m}_1\mathsf{M}_3\\ & -\mathsf{m}_1\mathsf{M}_2\mathsf{M}_3+\mathsf{m}_3\mathsf{M}_1^2+\mathsf{m}_2\mathsf{M}_3^2+\mathsf{m}_3^2\mathsf{M}_1-\mathsf{m}_2\mathsf{m}_3\mathsf{M}_1+\mathsf{m}_2\mathsf{m}_3\mathsf{M}_2-\mathsf{m}_3\mathsf{M}_1\mathsf{M}_2+\mathsf{m}_2^2\mathsf{M}_3\\ & -\mathsf{m}_2\mathsf{m}_3\mathsf{M}_3-\mathsf{m}_2\mathsf{M}_1\mathsf{M}_3-\mathsf{m}_3\mathsf{M}_1\mathsf{M}_3-\mathsf{m}_2\mathsf{M}_2\mathsf{M}_3+\mathsf{M}_1\mathsf{M}_2\mathsf{M}_3)\cdot\lambda(\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3),\end{aligned}$$

$$\deg(E_{\mathsf{A}_3}(\mathscr{K})) = 17 = 2\cdot 8+1 = (n-1)\cdot 2^n+1$$

χ-discriminants

$$X_z = \{\alpha \in (\mathbb{C}^*)^n : f_i(\alpha, z) \neq 0, i = 1,\ldots,\ell\}$$

$\mathscr{E} = \mathscr{K}$

$$Z_k(\mathscr{E}) = \{z \in \mathscr{E} : |\chi(X_z)| \leq k\} \qquad V_k(\mathscr{E}) = \{z \in \mathscr{E} : |\chi(X_z)| = k\}$$

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Definition

The χ-discriminant variety of the family X_z of very affine varieties over $\mathscr{E}$ is the closed subvariety

$$\nabla_\chi(\mathscr{E}) = Z_{\chi^*-1}(\mathscr{E}) = \mathscr{E} \backslash V_{\chi^*}(\mathscr{E}) \subset \mathbb{C}^s$$

If $\nabla_\chi(\mathscr{E})$ is defined by a single equation: $\Delta_\chi(\mathscr{E}) \in \mathbb{C}[\mathscr{E}]$ → χ-discriminant

Stay tuned for Simon's talk: PLD.jl in julia and why $\mathcal{X}$-discriminants

Thank you!