Series expansion of multivariate hypergeometric series around its integer-valued parameters

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Outline

Motivation

Definitions

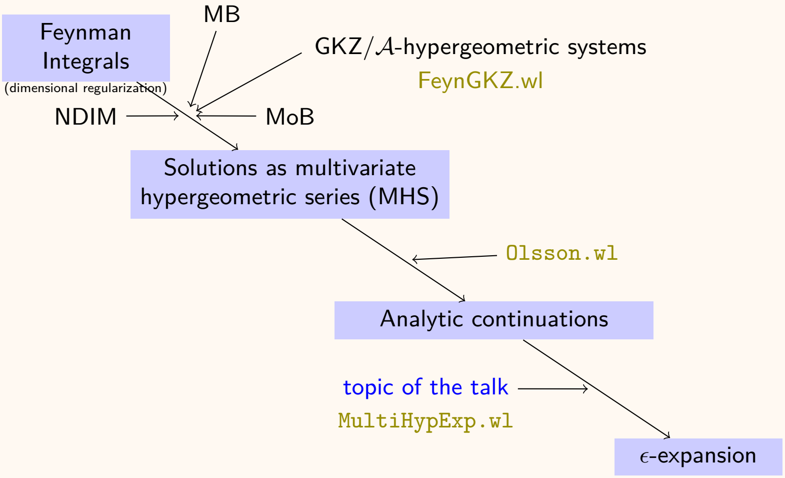
Feynman integrals and hypergeometric functions

Series expansion

Algorithm

Mathematica package : `MultiHypExp`

The Big Picture



Definitions

- **Pochhammer symbol** :

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(x)_n &= \frac{\Gamma(x+n)}{\Gamma(x)}, \\
&= x(x+1)\dots(x+n-1), \qquad x \in \mathbb{C} \setminus \mathbb{Z}_0^-, n \in \mathbb{Z}_0^+ \\
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- General hypergeometric functions

$$
{}_pF_{p-1}\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) = \sum_{n=0}^{\infty} \frac{\left(a_1\right)_n \cdots \left(a_p\right)_n}{\left(b_1\right)_n \cdots \left(b_{p-1}\right)_n} \frac{z^n}{n!}, \qquad |z| < 1
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- **Lauricella functions**: F_A, F_B, F_C and F_D

$$F_C^{(3)} = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(a_1)_{n_1+n_2+n_3}(a_2)_{n_1+n_2+n_3}}{(c_1)_{n_1}(c_2)_{n_3}(c_3)_{n_2}} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{n_1!n_2!n_3!}$$

with domain of convergence : $\sqrt{|z_1|} + \sqrt{|z_2|} + \sqrt{|z_3|} < 1$

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- ϵ-expansion of the multivariate hypergeometric series (MHS) are needed

From the literature

- Each of the representations of the MHS can be used
  - Series
  - Integral and Mellin-Barnes representation
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- Available packages:
  - **Analytical** :
    - `HypExp` , `HypExp2` (Huber et. al. [3, 4])
    - `XSummer` (Moch et. al. [5])
    - `nestedsums` (Weinzierl [6, 7])
  - **Numerical** : `NumExp` (Huang et. al. [8])

Demonstration

- Case I :

$$ {}_2F_1(1,1;\epsilon+1;x) = \sum_{m=0}^{\infty} \frac{(1)_m(1)_m}{(\epsilon+1)_m}\frac{x^m}{m!} = \sum_{m=0}^{\infty} x^m + O(\epsilon) = \frac{1}{1-x} + O(\epsilon) $$

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- MHS with *singular* parameters may have Laurent series expansion

Algorithm

- *Step 1*: Check if the Pochhammer parameters of the given function ($F(\epsilon)$) are *singular* or not

- *Step 2*: If those are non-singular, find the Taylor expansion of $F(\epsilon)$

- *Step 3*: If any of the Pochhammer parameter of $F(\epsilon)$ is *singular* then find a new function ($G(\epsilon)$) by replacing

$$\textit{singular}\text{ Pochhammer } \longrightarrow \text{non-}\textit{singular}\text{ Pochhammer}$$

- *Step 4*: Relate them using a differential operator (H)

$$F(\epsilon) = H \bullet G(\epsilon)$$
$$\left[\sum_{i=-n}^{\infty} \epsilon^i H_i\right] \bullet \left[\sum_{j=0}^{\infty} \epsilon^j G_j\right]$$

Step 1 & 3 : Checking the Pochhammers

- *Singular* Pochhammers :

  1. When one or more lower Pochhammer parameters (i.e., Pochhammer parameters in the denominator) are of the form

  $$(n + q\epsilon)_p$$

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- The Gauss ${}_2F_1$ example

$$A_1 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon-1)_m}\frac{x^m}{m!} \quad , \quad A_2 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon+1)_m}\frac{x^m}{m!}$$

singular Pochhammer in A_1 : $(-1+\epsilon)_m$
non-*singular* Pochhammer in A_2 : $(1+\epsilon)_m$

Step 2 :Taylor Expansion

Obtaining DE

- From the denition of Taylor expansion

$$F(\epsilon) = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \frac{d^i}{d\epsilon^i} F(\epsilon)\Big|_{\epsilon=0}$$

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$$\frac{A_{n+1}}{A_n} = \frac{(a+n)(b+n)}{(n+1)(c+n)} = \frac{g(n)}{h(n)}$$

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- The annihilator

$$L = \left[h(\theta) \frac{1}{x} - g(\theta) \right]$$

where $\theta = x\partial_x$

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Pfaff System

- For Gauss ${}_2F_1$: $L \bullet {}_2F_1(a, b; c; x) = 0$

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- Consider $g = ({}_2F_1, x\partial_x \bullet {}_2F_1)^T$

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- Find a transformation T to bring the system into *canonical* form (Henn [10])

$$dg' = \epsilon \Omega' g'$$

with

$$g = Tg' \qquad , \qquad \Omega' = T^{-1}\Omega T - T^{-1}dT$$

Step 2 :Taylor Expansion
Example

- For our example of $A_2 = \ {}_2F_1(\epsilon, -\epsilon; 1+\epsilon; x)$

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- **Solution :**

$$A_2 = 1 + \epsilon^2 G(0,1;x) + \epsilon^3(-G(0,0,1;x) + G(0,1,1;x)) + O\left(\epsilon^4\right)$$

Step 4 : Differential operator

Contiguous relations

There exist contiguous relations that relate

$$ {}_2F_1(a \pm 1, b; c; x) \ , \ {}_2F_1(a, b \pm 1; c; x) \ , \ {}_2F_1(a, b; c \pm 1; x) $$

These can be obtained by applying differential operators

- *Example*:
  The unit step down operator for the GHS is given by $H(c) = \frac{1}{c}(\theta + c)$, i.e.,

$$ {}_2F_1(a, b; c; x) = H(c) \bullet \ {}_2F_1(a, b; c+1; x) $$

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- Another example

$$ {}_2F_1(a+1, b; c; x) = \frac{1}{a}\,(\theta + a) \bullet \; {}_2F_1(a, b; c; x) $$

Step 4 : Differential operator

Step Down Operators

- If needed, apply the step down operator multiple times

$$ {}_2F_1(a, b; c-1; x) = H(c-1)H(c) \bullet {}_2F_1(a, b; c+1; x) $$

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$$ {}_2F_1(a, b; c-1; x) = H(c-1)H(c) \bullet \ {}_2F_1(a, b; c+1; x) $$

- Take quotient by the annihilator L

$$ L \bullet \ {}_2F_1(a, b; c+1, x) = 0 $$
$$ L = \left[-ab + (-x(a+b+1) + c + 1)\,\partial_x - (x-1)x\partial_x^2\right] $$

Step 4 : Differential operator

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- The step down operator :

$$ H = H(c-1)H(c) $$
$$ = \left(1 - \frac{abx}{(c-1)c(x-1)}\right) - \frac{x(a+b+1)-2cx+c-1}{(c-1)c(x-1)}\theta $$

Step 4 : Differential operator

Example

- $$A_1 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon-1)_m}\frac{x^m}{m!} \quad , \quad A_2 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon+1)_m}\frac{x^m}{m!}$$

Step 4 : Differential operator

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- $$A_1 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon-1)_m} \frac{x^m}{m!} \quad , \quad A_2 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon+1)_m} \frac{x^m}{m!}$$

- So $A_1 = H \bullet A_2$

$$\begin{aligned} H &= \frac{(\epsilon(2x-1)-x+1)}{(\epsilon-1)\epsilon(x-1)}\theta + \frac{\epsilon(2x-1)-x+1}{(\epsilon-1)(x-1)} \\ &= \frac{1}{\epsilon}\theta + \left(1 - \frac{x}{x-1}\theta\right) + O(\epsilon) \end{aligned}$$

Step 4 : Differential operator

Example

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$$H = \frac{(\epsilon(2x-1)-x+1)}{(\epsilon-1)\epsilon(x-1)}\theta + \frac{\epsilon(2x-1)-x+1}{(\epsilon-1)(x-1)}$$
$$= \frac{1}{\epsilon}\theta + \left(1 - \frac{x}{x-1}\theta\right) + O(\epsilon)$$
- $$A_2 = 1 + \epsilon^2 G(0,1;x) + \epsilon^3(-G(0,0,1;x) + G(0,1,1;x)) + O\left(\epsilon^4\right)$$

Step 4 : Differential operator

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- $$A_2 = 1 + \epsilon^2 G(0,1;x) + \epsilon^3(-G(0,0,1;x) + G(0,1,1;x)) + O\left(\epsilon^4\right)$$
- $$A_1 = 1 + \epsilon\left[G(1;x) - \frac{x}{x-1}\right] + \epsilon^2\left[-\frac{x}{x-1}G(1;x) + G(1,1;x) - \frac{x}{x-1}\right] + O(\epsilon^3)$$

Mathematica Package

MultiHypExp

MultiHypExp

Available at GitHub [11]

Dependencies :

- RISC`HolonomicFunctions (Koutschan [12, 13]) : To find the PDE associated with the given MHS and to form the Pfaff system
- HYPERDIRE (Bytev et.al. [14, 15, 16]) : For step up/down operations
- CANONICA (Meyer [17]) : To bring the Pfaff system into canonical form
- PolyLogTools (Duhr et. al. [18]) : To handle MPLs

Mathematica Package

MultiHypExp

The package is able to expand the following series

- **One variable** : ${}_pF_{p-1}$
- **Two variables** : Appell F_1, F_2, F_3, F_4, Horn $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_6$ and H_7 and certain KdF functions
- **Three variables** : Lauricella Saran $F_A, F_B, F_D, F_K, F_M, F_N$ and F_S
- Apart from Appell F_1, F_2, F_3 and Horn H_2, other Appell-Horn series are expanded using their relation to the former functions
- Series expansion of Appell F_4 and Horn H_1 is possible with certain restriction on the Pochhammer parameters

MultiHypExp

Commands for one variable

To obtain the series expansion ${}_2F_1(\epsilon, -\epsilon; \epsilon - 1; x)$

In[1]:=
```
SeriesExpand[{{e, -e}, {e - 1}}, {x}, e, 3]
```

Out[1]=
```
1 + (-(x/(-1+x))+G[1,x]) e + (-(x/(-1+x))-(x G[1,x])
/(-1+x)+G[1,1,x]) e^2 + O[e]^3
```

Alternatively,

In[2]:=
```
SeriesExpand[{n},(Pochhammer[e,n]Pochhammer[-e,n]x^n)
/(Pochhammer[e-1,n]n!),{x},e,3]
```

yields the same result.

MultiHypExp

Commands for bi- and tri-variate HS

In[3]:= `SeriesExpand[F2,{1,1,e,1+e,1-e},{x,y},e,3]`

Out[3]=

```
 -(1/(-1+x))+((-2 G[1,x]+G[1,y]+G[1-y,x]) e)/(-1+x)
+(1/(-1+x))(2 G[1,x] G[1,y]-2 G[1,y] G[1-y,x]
+2 G[0,1,x]+G[0,1,y]-G[0,1-y,x]-4 G[1,1,x]-2 G[1,1,y]
+2 G[1,1-y,x]+2 G[1-y,1,x]-G[1-y,1-y,x]) e^2+O[e]^3
```

yields the first four series expansion coefficients of Appell $F_2(1, 1, e; 1 + e, 1 - e; x, y)$ with respect to e in terms of MPLs.

In[4]:= `SeriesExpand[{m,n}, exp, {x, y}, e, 4]`

exp must be a series presentation of a MHF with summation indices m and n.

MultiHypExp

Commands for obtaining reduction formulae

To find reduction formulae of MHS

In[5]:=
```
ReduceFunction[F2,{3,2,1,3,2},{x,y}]
```

Out[5]=
```
1/((-1+x) x (-1+x+y))-G[1,x]/(x^2 y)+G[1-y,x]/(x^2 y)
```

In terms of logarithms

$$F_2(3,2,1;3,2;x,y) = -\frac{\log(1-x)}{x^2 y} + \frac{\log\left(1-\frac{x}{1-y}\right)}{x^2 y} + \frac{1}{(x-1)x(x+y-1)}$$

This command can find reduction formulae of Appell F_1, F_2, F_3, F_4 and Lauricella-Saran $F_D^{(3)}$ and $F_S^{(3)}$

MultiHypExp

Conclusions

- Applicable when the parameter ϵ appears linearly inside the Pochhammer symbols
- The package can find the expansion of most of the MHS around integer values of Pochhammer parameters
- It can find at most first 6 coefficients
- Takes 3-4 hours to find series expansion of a three variable HS in an ordinary personal computer

Thank You

References I

[1] C. Anastasiou, E. W. Nigel Glover, and C. Oleari. *Nucl. Phys. B* 572 (2000), pp. 307–360. arXiv: hep-ph/9907494.

[2] Frits A. Berends et al. *Z. Phys. C* 63 (1994), pp. 227–234.

[3] T. Huber and Daniel Maitre. *Comput. Phys. Commun.* 175 (2006), pp. 122–144. arXiv: hep-ph/0507094.

[4] Tobias Huber and Daniel Maitre. *Comput. Phys. Commun.* 178 (2008), pp. 755–776. arXiv: 0708.2443 [hep-ph].

[5] S. Moch and P. Uwer. *Comput. Phys. Commun.* 174 (2006), pp. 759–770. arXiv: math-ph/0508008.

[6] Stefan Weinzierl. *Comput. Phys. Commun.* 145 (2002), pp. 357–370. arXiv: math-ph/0201011.

[7] Stefan Weinzierl. *J. Math. Phys.* 45 (2004), pp. 2656–2673. arXiv: hep-ph/0402131.

[8] Zhi-Wei Huang and Jueping Liu. *Comput. Phys. Commun.* 184 (2013), pp. 1973–1980. arXiv: 1209.3971 [physics.comp-ph].

[9] Souvik Bera. *Nucl. Phys. B* 989 (2023), p. 116145. arXiv: 2208.01000 [math-ph].

References II

[10] Johannes M. Henn. *Phys. Rev. Lett.* 110 (2013), p. 251601. arXiv: 1304.1806 [hep-th].

[11] Souvik Bera. (June 2023). arXiv: 2306.11718 [hep-th].

[12] Christoph Koutschan. *ACM Commun. Comput. Algebra* 43.3/4 (June 2010), p. 119. ISSN: 1932-2240. URL: https://doi.org/10.1145/1823931.1823954.

[13] Christoph Koutschan. *Mathematics in Computer Science* 4.2 (2010), pp. 259–266.

[14] Vladimir V. Bytev, Mikhail Yu. Kalmykov, and Bernd A. Kniehl. *Comput. Phys. Commun.* 184 (2013), pp. 2332–2342. arXiv: 1105.3565 [math-ph].

[15] Vladimir V. Bytev, Mikhail Yu. Kalmykov, and Sven-Olaf Moch. *Comput. Phys. Commun.* 185 (2014), pp. 3041–3058. arXiv: 1312.5777 [math-ph].

[16] Vladimir V. Bytev and Bernd A. Kniehl. *Comput. Phys. Commun.* 189 (2015), pp. 128–154. arXiv: 1309.2806 [math-ph].

[17] Christoph Meyer. *Comput. Phys. Commun.* 222 (2018), pp. 295–312. arXiv: 1705.06252 [hep-ph].

References III

[18] Claude Duhr and Falko Dulat. *JHEP* 08 (2019), p. 135. arXiv: 1904.07279 [hep-th].