Differential Equations and Hypergeometric Systems

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Theory and Phenomenology
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Based on joined work with:
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[2204.12983] [2305.01585]

`MathemAmplitudes @ Padova`
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Introduction

Today's menu

arXiv:2204.12983v2 [hep-th] 21 May 2023

Macaulay Matrix for Feynman Integrals: Linear Relations and Intersection Numbers

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Abstract: We elaborate on the connection between Gel'fand-Kapranov-Zelevinsky systems, de Rham theory for twisted cohomology groups, and Pfaffian equations for Feynman Integrals. We propose a novel, more efficient algorithm to compute Macaulay matrices, which are used to derive Pfaffian systems of differential equations. The Pfaffian matrices are then employed to obtain linear relations for $\mathcal{A}$-hypergeometric (Euler) integrals and Feynman integrals, through recurrence relations and through projections by intersection numbers.

arXiv:2305.01585v2 [hep-th] 16 May 2023

Restrictions of Pfaffian Systems for Feynman Integrals

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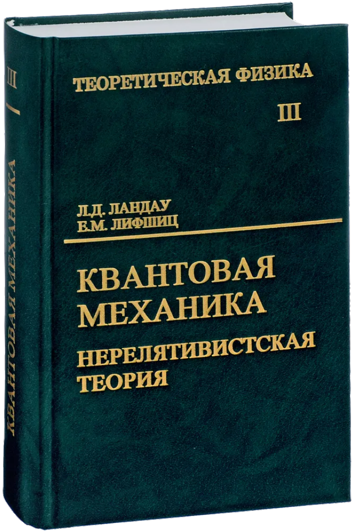
In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.

[Weyl "Invariants", 1939]

Differential Equations around us



§36. **Motion in a Coulomb field (spherical polar coordinates)**

A very important case of motion in a centrally symmetric field is that of motion in a *Coulomb field*

$$U = \pm\alpha/r$$

(where α is a positive constant). We shall first consider a Coulomb attraction, and shall therefore write $U = -\alpha/r$. It is evident from general considerations that the spectrum of negative eigenvalues of the energy will be discrete (with an infinite number of levels), while that of the positive eigenvalues will be continuous.

Equation (32.8) for the radial functions has the form

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{l(l+1)}{r^2}R+\frac{2m}{\hbar^2}\left(E+\frac{\alpha}{r}\right)R = 0. \quad (36.1)$$

If we are concerned with the relative motion of two attracting particles, m must be taken as the reduced mass.

[Landau, Lifshitz vol. 3
Quantum mechanics]

The Schrödinger equation

- The radial part R of the particle's wave function:

$$\left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2} + \frac{2m}{\hbar}\left(E + \frac{\alpha}{r}\right)\right] \bullet R(r) = 0$$

- Key idea: study derivatives $\partial_r^n \bullet R$ *modulo* this equation
  See talks tomorrow for a similar game with polynomials [Giulio] [Gaia]

The Schrödinger equation

- The radial part R of the particle's wave function:

$$\left[\partial_r^2 - c_1(r) \times \partial_r - c_0(r) \times 1\right] \bullet R(r) = 0$$

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- How many derivatives are independent?

$$\partial_r^2 \to c_1(r) \times \partial_r + c_0(r) \times 1$$

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$$\partial_r^3 \to \partial_r \left(c_1(r) \times \partial_r + c_0(r) \times 1\right)$$

The Schrödinger equation

- The radial part R of the particle's wave function:

$$\left[\partial_r^2 - c_1(r) \times \partial_r - c_0(r) \times 1\right] \bullet R(r) = 0$$

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- How many derivatives are independent?

$$\partial_r^3 \to \left(c_1'(r) + c_1^2(r) + c_0(r)\right) \times \partial_r + \left(c_0'(r) + c_1(r)c_0(r)\right) \times 1$$

The Schrödinger equation

- The radial part R of the particle's wave function:

$$\left[\partial_r^2 - c_1(r) \times \partial_r - c_0(r) \times 1\right] \bullet R(r) = 0$$

- Key idea: study derivatives $\partial_r^n \bullet R$ *modulo* this equation
  See talks tomorrow for a similar game with polynomials [Giulio] [Gaia]
- So $\partial_r R$ and R are *irreducible* $\Rightarrow$ use as a basis for a 1st-order system:

$$\partial_r \bullet \begin{bmatrix} R \\ \partial_r R \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c_0 & c_1 \end{bmatrix} \cdot \begin{bmatrix} R \\ \partial_r R \end{bmatrix}$$

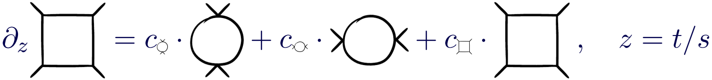
Today: how to generalize this to more variables?

Differential Equations for Feynman integrals

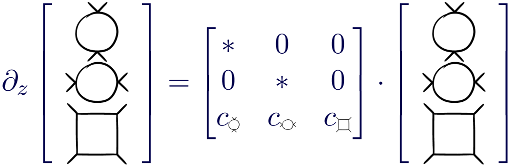
- Integration by Parts: linear relations among Feynman integrals

[Tkachov '81] [Chetyrkin Tkachov '81] [Kotikov '91] [Remiddi '97] [Laporta '00] [Gehrmann Remiddi '00] … [Henn '13] …

- Decompose any integral in terms of irreducible *master integrals*



- System of 1st-order DEs for master integrals



- In practice: many scales $\Rightarrow$ systems of 1st-order PDEs

A glimpse of twisted cohomology

- Finite dimensional vector space of integrals with *intersection number* as metric
  [Mastrolia Mizera '18] [FGMMMM '19] [FGLMMMM '20] [Agostini, Fevola '22 Sattelberger, Telen] & talks on Tuesday: [Giulio] [Gaia] [Andrzej] [Federico]

- Many ways to count master integrals r:
  1. Laporta algorithm [Laporta '01]
  2. Number of critical points $\mathrm{d}\log(\mathcal{F}+\mathcal{U})=0$ [Baikov '05] [Lee '13 Pomeransky]
  3. Number of independent integration contours [Bosma, Sogaard Zhang '17] [Primo '17 Tancredi] [Frellesvig '21]
  4. Number of independent integrands [Mastrolia Mizera '18]
  5. Holonomic rank of GKZ system (volume of Δ_A polytope) [de la Cruz '19] [Klausen '21] [2204.12983]

Also see nice reviews: [MathemAmplitudes'19] [Cacciatori '21 Conti, Trevisan]

Next

1. Derive DEs for generalized Feynman integrals using algebra of derivatives
   - Integral reduction with matrix multiplication
2. Restrict DEs back to genuine Feynman integrals
   - Asymptotic expansion of solutions revisited

Differential Equations from GKZ systems

Gelfand Kapranov Zelevinsky (GKZ) Hypergeometric system

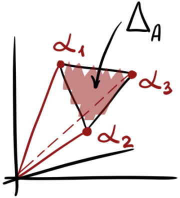
- A-hypergeometric function: [GKZ '89]

$$I_\beta(z) = \int_0^\infty f(x,z)^{\beta_0}\, x_1^{-\beta_1} \cdots x_n^{-\beta_n}\, \frac{\mathrm{d}x}{x}\,, \quad f(x,z) = \sum_{i=1}^{N} z_i\, x^{\alpha_i}\,, \quad \alpha_i \in \mathbb{Z}^n\,, \quad \beta_i \in \mathbb{C}$$

- The central object: A-matrix contains all the answers! E.g. rank $r = \mathrm{vol}(\Delta_A)$

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_N \end{bmatrix}$$

$$\begin{aligned} (A \cdot z\partial_z - \vec{\beta}) \bullet I_\beta(z) &= 0 \\ \partial_z^{\mathtt{Ker}(A)} \bullet I_\beta(z) &= 0 \end{aligned}$$

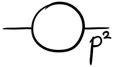


- **Feynman integrals** are A-hypergeometric for *restricted* values of z's

… [Nasrollahpoursamami '16] [de la Cruz '19] [Vanhove '18] [Klemm, Nega Safari '19] [Feng, Chang Chen, Zhang '19] [Klausen '19]
[Klausen '21] [Tellander Helmer '21] [2204.12983] [Walther '22] [Agostini, Fevola Sattelberger, Telen '22] [Feng, Zhang Chang '22] …

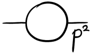
See also `FeynGKZ` package with many useful features! [Ananthanarayan, Banik Bera, Datta '22] & after lunch [Souvik]

Example: 1-loop bubble diagram

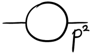


$$= \int_0^\infty (z_1 x_1^1 x_2^0 + z_2 x_1^0 x_2^1 + z_3 x_1^1 x_2^1)^{\beta_0}\, x_1^{-\beta_1} x_2^{-\beta_2}\, \frac{\mathrm{d}x}{x}\,, \qquad A = \begin{matrix} & \begin{matrix} z_1 & z_2 & z_3 \end{matrix} \\ \begin{matrix} \\ x_1 \\ x_2 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

$$(A \cdot z\partial_z - \vec{\beta}) \bullet$$



$$= \begin{bmatrix} z_1\partial_1 + z_2\partial_2 + z_3\partial_3 - \beta_0 \\ z_1\partial_1 + z_3\partial_3 - \beta_1 \\ z_2\partial_2 + z_3\partial_3 - \beta_2 \end{bmatrix} \bullet$$



$$= 0$$

A-hypergeometric function: z_1 , z_2 , z_3 are all generic

Feynman integral: keep $z_3 = -p^2$ generic, *restrict* $z_1 = z_2 = 1$

Pfaffian system of Differential Equations

- So, a generalized Feynman integral I_β satisfies a GKZ system of higher-order PDEs

$$\begin{aligned} \left(A \cdot z\partial_z - \vec{\beta}\right) \bullet I_\beta(z) &= 0 \\ \partial_z^{\mathtt{Ker}(A)} \bullet I_\beta(z) &= 0 \end{aligned}$$

- Turn it into a 1st-order **Pfaffian system** for an r-dimensional basis $\vec{I}(z)$

$$\partial_i \bullet \vec{I}(z) \;=\; P_i(z) \cdot \vec{I}(z)$$

- Pfaffian rational matrices $P_i \in \mathbb{Q}^{r\times r}(z)$ satisfy integrability condition

$$\partial_i P_j - P_i \cdot P_j = \partial_j P_i - P_j \cdot P_i$$

Singularities from A-discriminants [Klausen '21] [Mizera Telen '21] & later today [Claudia] [Simon] [Christoph]

How to compute Pfaffian matrices

- Weyl algebra of derivaitves $\mathcal{D}_N = \mathbb{C}[z_1, \ldots, z_N]\langle\partial_1, \ldots, \partial_N\rangle$, $[\partial_i, z_j] = \delta_{i\,j}$
- Ideal generated by the GKZ operators

$$H_A(\beta) \; := \; \mathcal{D}_N \left(A \cdot z\partial_z - \vec{\beta}\,\right) \; + \; \mathcal{D}_N \, \partial^{\,\mathtt{Ker}(A)}$$

- Irreducible derivatives modulo $H_A(\beta)$ are **standard monomials**: $\mathtt{Std} = \left\{\partial^{\vec{k}} \,\middle|\, \vec{k} \in \mathbb{N}^N\right\}$
- $\exists$ a fast algorithm to find $\mathtt{Std}$ via commutative Gröbner basis [Hibi, Nishiyama Takayama '17]
  $\Rightarrow$ know the master integrals $\vec{I} := \mathtt{Std} \bullet I_\beta$
- We proposed to use Laporta-like system (Macaulay matrix) to find P_i [2204.12983]

$$\partial\vec{I} = \partial\,\mathtt{Std} \bullet I_\beta = P_i \cdot \vec{I} \mod H_A(\beta)$$

instead of non-commutative Gröbner basis in CAS Asir [Asir on github]

Another example of $\mathcal{D}_N$-module techniques: [Henn, Pratt '23 Sattelberger, Zoia]

Application: IBPs with Pfaffian matrices

- Derivatives ∂_z commute with $\texttt{Std}$

$$\partial \circ \texttt{Std} = \texttt{Std} \circ \partial$$

- Acting on $I_\beta(z)$ with derivatives shifts parameters by a column of the A-matrix α_i

$$\partial_i \bullet I_\beta(z) \;=\; I_{\beta-\alpha_i}(z)$$

- Therefore, P_i yields a difference equation for $\vec{I}(\beta) := \texttt{Std} \bullet I_\beta(z)$

$$\partial_i \bullet \vec{I}(\beta) \;=\; P_i \cdot \vec{I}(\beta) \;=\; \vec{I}(\beta - \alpha_i)$$

- If β is non-resonant, can then define $Q_i(\beta) := P_i(\beta + \alpha_i)^{-1} \rightsquigarrow$ anti-differentiation

$$\partial_i^{-1} \bullet \vec{I}_\beta \;=\; Q_i \cdot \vec{I}(\beta) \;=\; \vec{I}(\beta + \alpha_i)$$

- $\Rightarrow$ Integral reduction with matrix multiplication by P_i and Q_i [2204.12983]

Restriction of Differential Equations

Restricting Differential Equations

- Consider $z_1 \to 0$ limit of a rank r Pfaffian system

$$\partial_i \vec{I}(z) = P_i(z) \cdot \vec{I}(z)\,, \quad i = 1, 2$$

  Suppose $P_1(z)$ has a pole at $z_1 = 0$.

- Observation: the rank r “decreases” at $z_1 \to 0 \Rightarrow$ new relations among $\vec{I}$ at $z_1 \to 0$

- Restriction answers the following questions: [Haraoka '20] [Bytev, Kniehl Veretin '22] [2305.01585]
  - What is the new rank?
  - How to systematically find new linear relations in the limit?
  - Given some basis $\vec{I}(z_1, z_2)$, how to find a basis $\vec{J}(z_2)$ at $z_1 \to 0$?
  - What are the restricted Pfaffians?

Rank drop

- Holomorphic solutions $\vec{I}(z)$ to

$$\partial_i \vec{I}(z) = P_i(z) \cdot \vec{I}(z)\,, \quad i = 1, 2$$

expanded in Taylor series look like

$$\vec{I}(z) = \sum_{n=0}^{\infty} \vec{I}^{(n)}(z_2)\, z_1^n$$

- *Integrable* Pfaffian system can be brought to normal form:

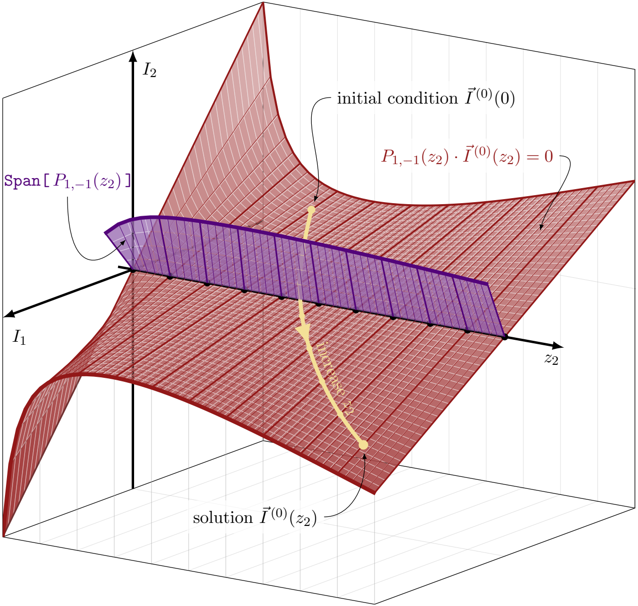
$$P_1(z) = \sum_{n=-1}^{\infty} P_{1,n}(z_2)\, z_1^n\,, \quad P_2(z) = \sum_{n=0}^{\infty} P_{2,n}(z_2)\, z_1^n$$

- In the $z_1 \to 0$ limit

PDE: $\partial_2 \vec{I}^{(0)}(z_2) = P_{2,0}(z_2) \cdot \vec{I}^{(0)}(z_2)$

Rank drop: $P_{1,-1}(z_2) \cdot \vec{I}^{(0)}(z_2) = 0$

Solution at $z_1 \to 0$ remains in the kernel of $P_{-1,1}$



Some details

- To find the restricted Pfaffian take

$$R := \texttt{RowReduce}[P_{-1,1}]$$

and pick any basis $B(z_2)$ such that the square matrix is of full rank

$$M := \left[\begin{array}{c} B \\ \hline R \end{array}\right]$$

- Gauge transformation M produces the restricted basis $\vec{J}$ and Pfaffian Q_2

$$M \cdot \vec{I}^{(0)} = \left[\begin{array}{c|cccc} \vec{J} & 0 & \dots & 0 \end{array}\right]^{\mathsf{T}} \quad \text{and} \quad (\partial_2 M + M \cdot P_{2,0}) \cdot M^{-1} = \left[\begin{array}{c|c} Q_2 & * \\ \hline 0 & * \end{array}\right]$$

- For **Feynman integrals** can find the IBPs as well as the symmetry relations in the limit

Application: asymptotic expansion of Feynman integrals

- Given z_1 (small mass, threshold, collinear ...)

$$\vec{I}(z_1, z_2, \ldots) = \sum_{\lambda,n,m} \vec{I}^{(\lambda,n,m)}(z_2,\ldots)\; z_1^{\lambda+n}\, \log^m z_1$$

- To get the power-and-log expansion at $z_1 \to 0$:

1. Solve simpler Pfaffian system for $\vec{I}^{(\lambda,0,0)}(z_2,\ldots)$

2. Get $\vec{I}^{(\lambda,n,m)}(z_2,\ldots)$ from matrix multiplication

- Doesn't require `JordanDecomposition` and `MatrixExp`

Conclusions

- GKZ hypergeometric systems offer new tools to study Feynman integrals
- Novel algorithm for computing Pfaffian systems via Macaulay matrices
- Knowing the Pfaffians provides IBPs just by matrix multiplication
- Restriction of Pfaffian systems brings GKZ to Feynman integrals
- Restriction is also suited for power-and-log expansion of generic Feynman integrals