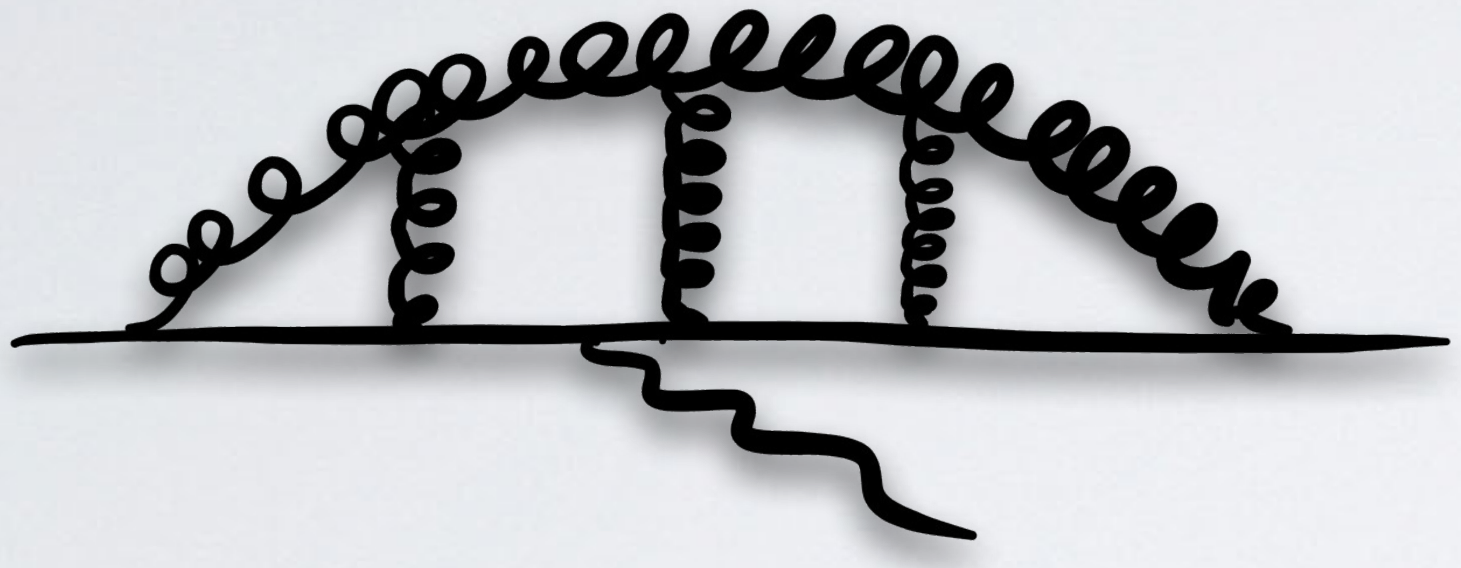
COMPUTATIONAL TOOLS FOR AMPLITUDES IN FULL-COLOR QCD



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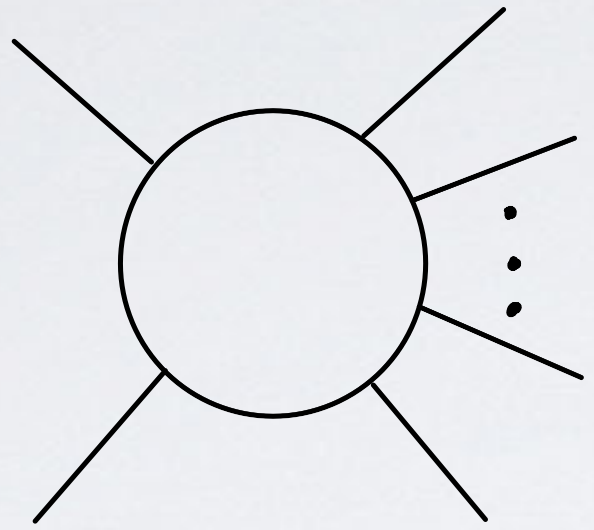
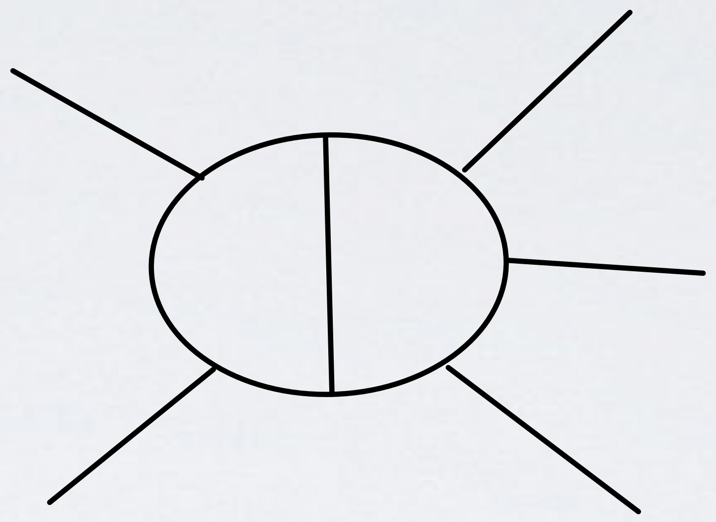
MathemAmplitudes 2023: QFT at the Computational Frontier

Padova, September 25-27, 2023

FULL-COLOR MASSLESS QCD AMPLITUDES





$q\bar{q} \to \gamma\gamma j$

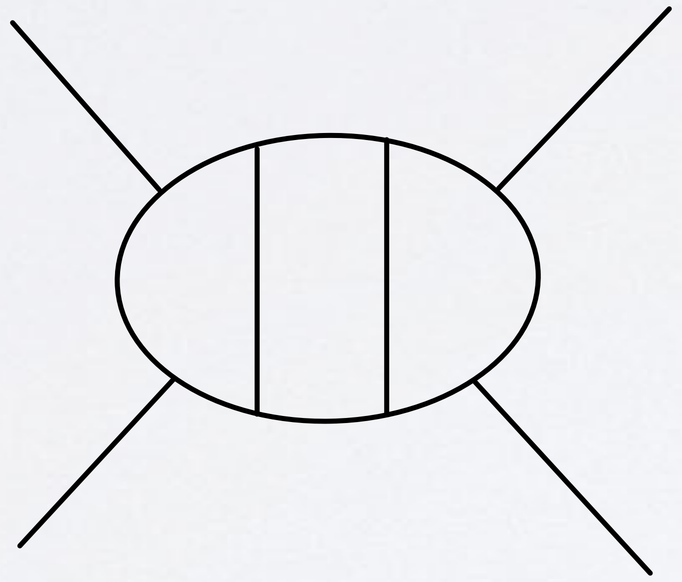
[Agarwal, Buccioni, AvM, Tancredi '21]

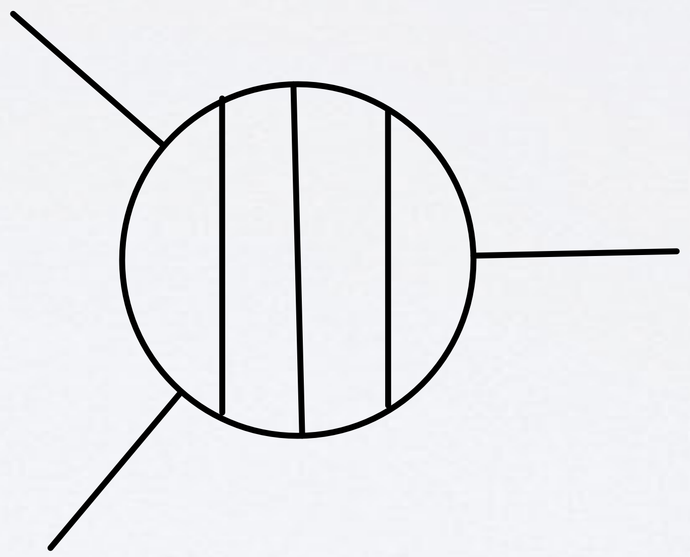
$gg \to \gamma\gamma j$

[Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]

$q\bar{q} \to \gamma\gamma\gamma$

[Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov '23]





$q\bar{q} \to \gamma\gamma$

[Caola, AvM, Tancredi '20]

$gg \to \gamma\gamma$

[Bargiela, Caola, AvM, Tancredi '21]

$q\bar{q} \to q'\bar{q}'$, $gg \to gg$, $q\bar{q} \to gg$:

[Caola, Chakraborty, Gambuti, AvM, Tancredi '21,'21,'22]

$q\bar{q} \to \gamma g$:

[Bargiela, Chakraborty, Gambuti '22]

$q\bar{q} \to \gamma^*$, $gg \to H$

[Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

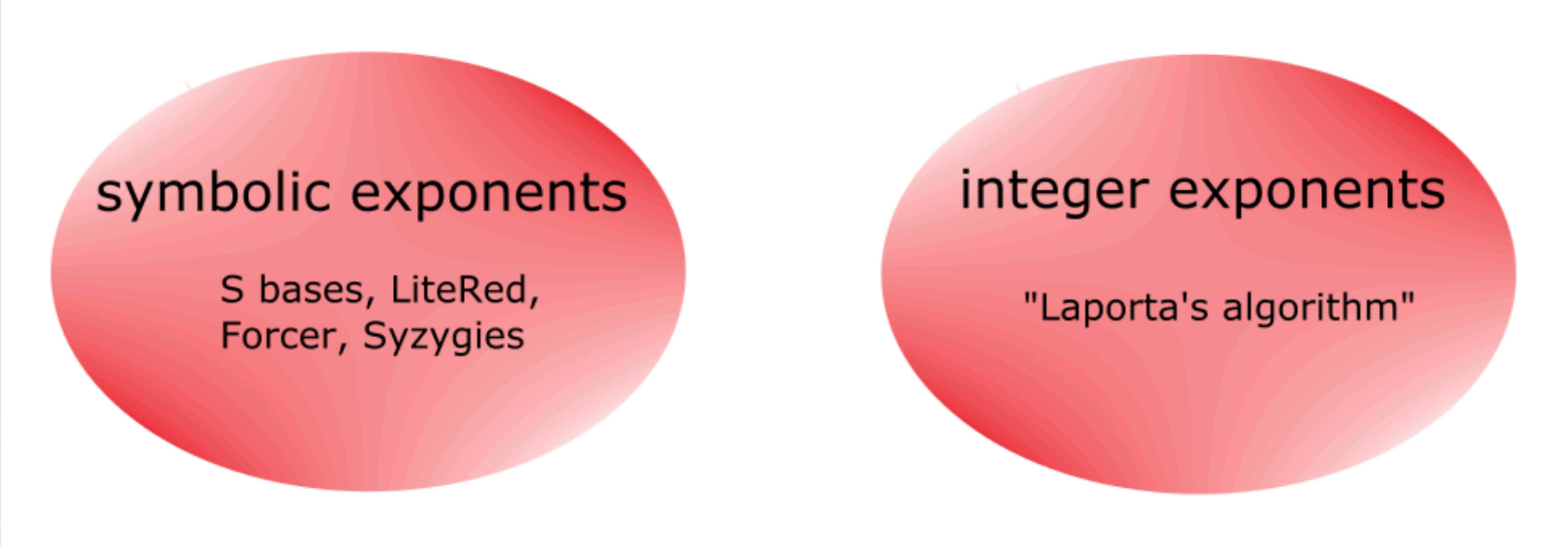
$b\bar{b} \to H$

[Chakraborty, Huber, Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

INTEGRAL REDUCTIONS

INTEGRATION-BY-PART (IBP) IDENTITIES

- IBP identities in dimensional regularization since integrals over total derivatives vanish:
$$\int d^dk_1 \cdots d^dk_L \, \frac{\partial}{\partial k_i^\mu} \left(v^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right) = 0 \,, \quad D_j = q_j^2 - m_j^2 + i\delta \,, \quad v^\mu \text{ loop or ext. mom.}$$
- Implies linear relations between loop integrals *[Chetyrkin, Tkachov ’81]*
- Integer indices: linear system of equations, allows for systematic reduction *[Laporta '00]*
- Only finite number of integrals linearly independent: basis or master integrals



- Various public reduction codes exists: Fire, Reduze, LiteRed, Kira, FiniteFlow, NeatIBP, Blade, … and many private ones
- Calculations at the symbolic level: syzygies, Gröbner bases, …
- Calculations at the linear algebra level: finite fields, …
- Often very powerful in practice: combination of both

talks: Tobias Huber, Xiao Liu, Yan-Qing Ma, Mao Zeng, Johann Usovitsch, Yang Zhang

- Alternative: intersection theory

talks: Giulio Crisanti, Gaia Fontana, Andrzej Pokraka

FINITE FIELDS AND RATIONAL RECONSTRUCTION

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

$$
\begin{array}{ccccccccc}
I_{\mathbb{Z}_p[x]} & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p[x]} & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_p,x_2} & \longrightarrow & O_{\mathbb{Z}_p,x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p,x3} & \longrightarrow & O_{\mathbb{Z}_p,x3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

[AvM, Schabinger '14; Peraro '16; …], note: parallelizable, multivariate e.g. by iteration

REDUCTIONS AND COMPLEXITY

ln (expression size)

- $\mathscr{A} = \sum a_i I_i$ (unreduced integrals I_i)
-
- $\mathscr{A} = \sum b_i M_i$ (master integrals M_i)
-
- $\mathscr{A} = \sum pf(b_i) M_i$ (partial fractioned b_i)
-
- $\mathscr{A} = \sum pf(c_i) P_i$ (transcendental functions P_i)

by-pass with finite field sampling + rat.-reconstr.

traditional symbolic algebra

manipulations

IBP reduction

partial fractioning

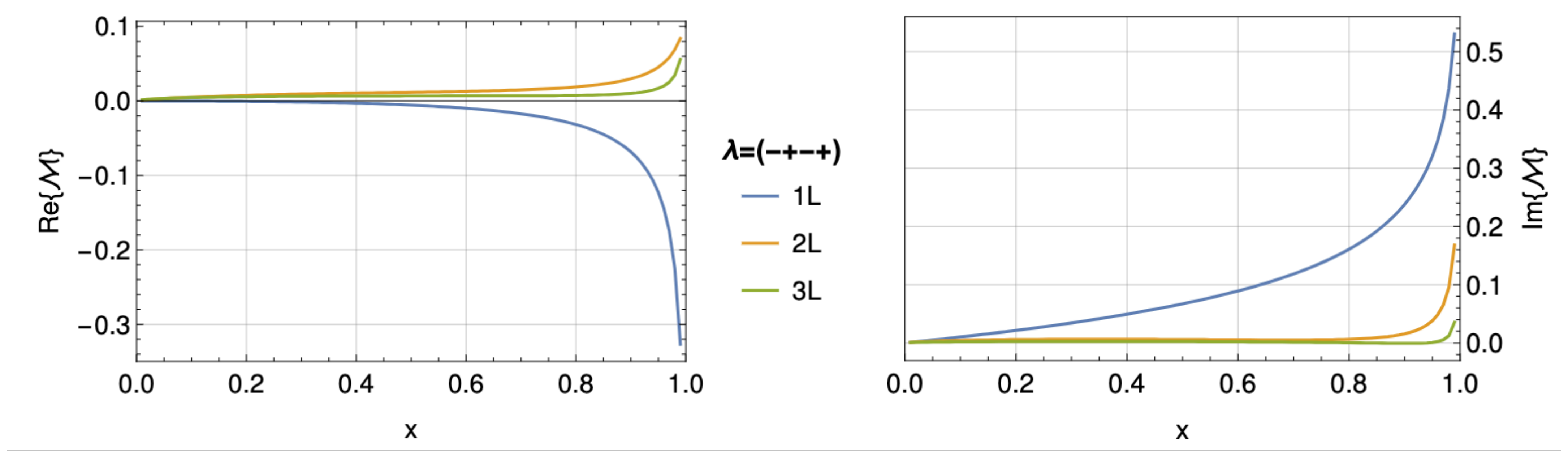
analytical integration

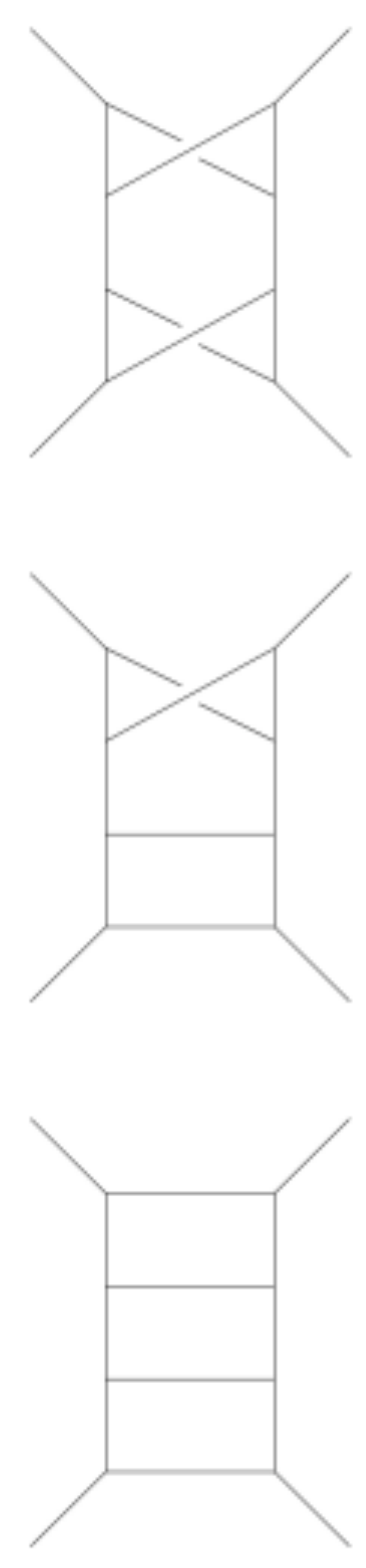
(Illustration idea by V. Sotnikov)

$gg \to \gamma\gamma$ @ 3 LOOPS

- Master integrals in terms of HPLs: *[Henn, Mistlberger, Smirnov, Wasser ’20]*
- $gg \to \gamma\gamma$ helicity amplitudes: *[Bargiela, Caola, AvM, Tancredi ’21]*
  - Symbolic intermediate expressions sizable but allow for easy crossings, simple workflow
  - Compact analytical results for amplitudes

	1L	2L	3L
Number of diagrams	6	138	3299
Number of inequivalent integral families	1	2	3
Number of integrals before IBPs and symmetries	209	20935	4370070
Number of master integrals	6	39	486
Size of the `Qgraf` result [kB]	4	90	2820
Size of the `Form` result before IBPs and symmetries [kB]	276	54364	19734644
Size of helicity amplitudes written in terms of MIs [kB]	12	562	304409
Size of helicity amplitudes written in terms of HPLs [kB]	136	380	1195





Syzygy Based IBPs Without Dots

Baikov's parametric representation of Feynman integrals:

$$I(\nu_1, \ldots, \nu_N) = \mathcal{N} \int \mathrm{d}z_1 \cdots \mathrm{d}z_m P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_N^{\nu_N}}$$

[Böhm, Georgoudis, Larsen, Schulze, Zhang '18]: useful for IBPs without dots

$$0 = \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \sum_{i=1}^{m} \frac{\partial}{\partial z_i} \left(a_i P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_N^{\nu_N}} \right)$$

$$= \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \sum_{i=1}^{N} \left(\frac{\partial a_i}{\partial z_i} + \frac{d-L-E-1}{2P} a_i \frac{\partial P}{\partial z_i} - \frac{\nu_i a_i}{z_i} \right) P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_N^{\nu_N}}$$

explicit solutions to constraint:

$$\left(\sum_{i=1}^{N} a_i \frac{\partial P}{\partial z_i} \right) + bP = 0 \qquad \text{(absence of dim. shifts)}$$

in addition, require for denominators of sector:

$$a_i = b_i z_i \qquad \text{(absence of dots)}$$

need intersection of two syzygy modules

Syzygies

- suppose that for given polynomials $f = (f_1, f_2, \ldots)$ one can find polynomials $s = (s_1, s_2, \ldots)$ such that $\sum_i f_i s_i = 0$, then s is called a syzygy
- if s is a syzygy, then $s \cdot g$ is a syzygy for any polynomial g
- the (infinite) set of syzygies for f is a syzygy module

- Reduction of numerators: "no-dot syzygies"
  [Gluza, Kajda, Kosower '11; Schabinger '11; Ita '15; Larsen, Zhang '15; Böhm, Georgoudis, Larsen Schulze, Zhang '18; …]

- Linear algebra approach: set degree restriction for monomials, use linear algebra with finite fields to determine syzygies (or intersections of syzygy modules) *[Agarwal, Jones, AvM '20]*

Syzygy Based IBPs Without Numerators

[Lee-Pomeransky '13] representation:

$$I(\nu_1,\ldots,\nu_N) = \mathcal{N}\left[\prod_{i=1}^{N}\int_0^\infty \mathrm{d}x_i x_i^{\nu_i-1}\right] G^{-d/2} \qquad \text{with } G = \mathcal{U}+\mathcal{F}$$

[Bitoun, Bogner, Klausen, Panzer '17]: define (twisted) Mellin Transform

$$\mathcal{M}\{f\}(\nu) := \left(\prod_{k=1}^{N}\int_0^\infty \frac{x_k^{\nu_k-1}\mathrm{d}x_k}{\Gamma(\nu_k)}\right) f(x_1,\ldots,x_N)$$

Feynman integrals are Mellin transforms:

$$\tilde{I}(\nu) = \mathcal{M}\left\{G^{-d/2}\right\}(\nu)$$

with $\nu = (\nu_1,\ldots,\nu_N)$ and $\tilde{I}(\nu) = \Gamma[(L+1)d/2-\nu]I(\nu)$ (remark: similar for Baikov's rep.)

Properties of Mellin transform

1. $\mathcal{M}\{\alpha f+\beta g\}(\nu) = \alpha\mathcal{M}\{f\}(\nu)+\beta\mathcal{M}\{g\}(\nu)$
2. $\mathcal{M}\{x_i f\}(\nu) = \nu_i\mathcal{M}\{f\}(\nu+e_i\}$
3. $\mathcal{M}\{-\partial_i f\}(\nu) = \mathcal{M}\{f\}(\nu-e_i\}$ (proof: partial integration + surface term is zero)

Define shift operators

$$(\hat{i}^+F)(\nu_1,\ldots,\nu_N) = \nu_i F(\nu_1,\ldots,\nu_i+1,\ldots,\nu_N)$$
$$(\hat{i}^-F)(\nu_1,\ldots,\nu_N) = F(\nu_1,\ldots,\nu_i-1,\ldots,\nu_N)$$

which form Weyl algebra, $[\hat{i}^+,\hat{j}^-] = \delta_{ij}$

Shift relations from annihilators

[Lee '14; Bitoun, Bogner, Klausen, Panzer '17]: a differential operator P which annihilates $G^{-d/2}$

$$P\,G^{-d/2}$$

generates via the substitutions $x_i \to \hat{i}^+$, $\partial_i \to -\hat{i}^-$ a shift relation according to

$$\mathcal{M}\{P\,G^{-d/2}\} = 0$$

In fact, *every* shift relation is related in this way.

consider annihilators beyond linear order:

$$\left[c_0 + \sum_{i=1}^{N} c_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{N} c_{ij} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \dots\right] G^{-d/2} = 0$$

determine $c_0(x_1,\dots,x_N),\dots$ via syzygy equations:

$$c_0\left[-\frac{2}{d}G^2\right] + \sum_{i=1}^{N} c_i \left[G\frac{\partial G}{\partial x_i}\right] + \sum_{i,j=1}^{N} c_{ij}\left[G\frac{\partial^2 G}{\partial x_i \partial x_j} + \left(-\frac{d}{2}-1\right)\frac{\partial G}{\partial x_i}\frac{\partial G}{\partial x_j}\right] + \dots = 0$$

Syzygies generate linear relations for Feynman integrals:

$$\left(\left[c_0(\hat{1}^+,\dots,\hat{N}^+) - \sum_{i=1}^{N} c_i(\hat{1}^+,\dots,\hat{N}^+)\hat{i}^- + \sum_{i,j=1}^{N} c_{ij}(\hat{1}^+,\dots,\hat{N}^+)\hat{i}^-\hat{j}^- + \dots\right]\tilde{I}\right)(\nu_1,\dots,\nu_N) = 0$$

SOLVING THE MASTER INTEGRALS

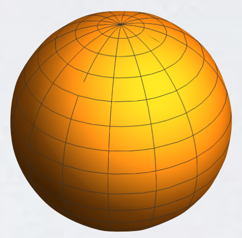
SOLVE INTEGRALS: DIFFERENTIAL EQUATIONS

- Integration of differential equations *[Kotikov '91, Remiddi '97]*:

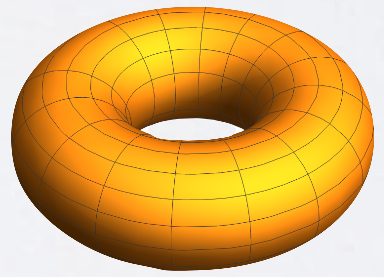
$$\partial_x \vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)$$

where $\epsilon = (4-d)/2$ (analytical or through series expansions)

- Homogeneous solutions for $\epsilon = 0$ (leading singularities):
    - ***Rational number***, e.g. $1/2$
    - ***Rational functions***, e.g. $1/x$
    - ***Algebraic functions***, e.g. $\sqrt{x(x-4)}$
    - ***Elliptic integrals***, e.g. $K(x) = \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-xz^2)}}, \ldots$



talks: William Torres Bobadilla, Jacob Bourjaily, Ekta Chaubey, Seva Chestnov, Christoph Dlapa, Martijn Hidding, Simone Zoia

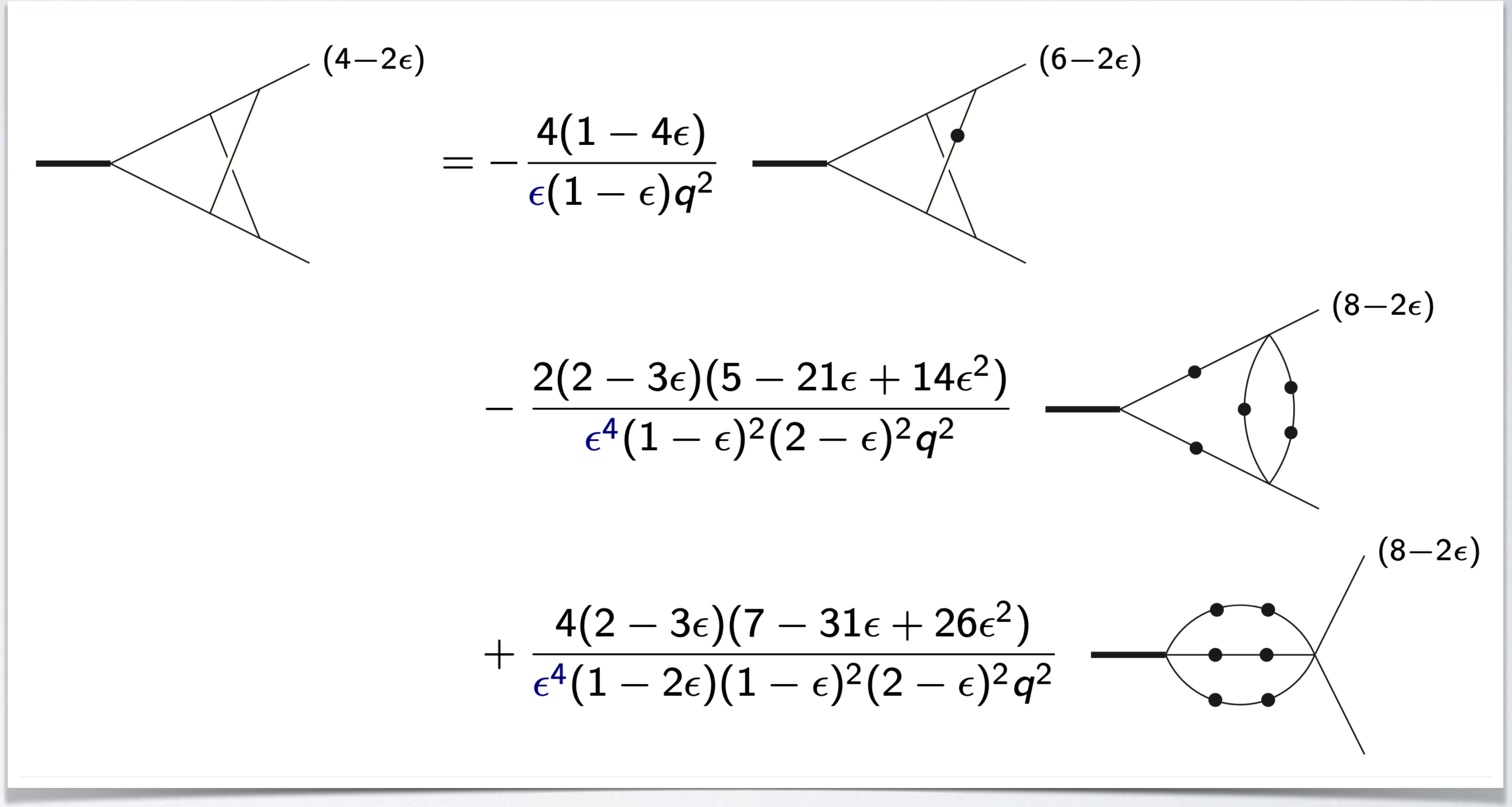


- Basis change involving homogenous solutions may allow to find ϵ-***form***:

$$d\vec{m} = \epsilon\, \mathrm{d}\ln(l_a(x))\, A^{(a)}(x)\, \vec{m}$$

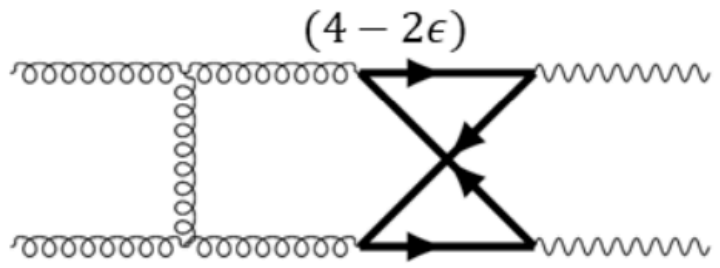
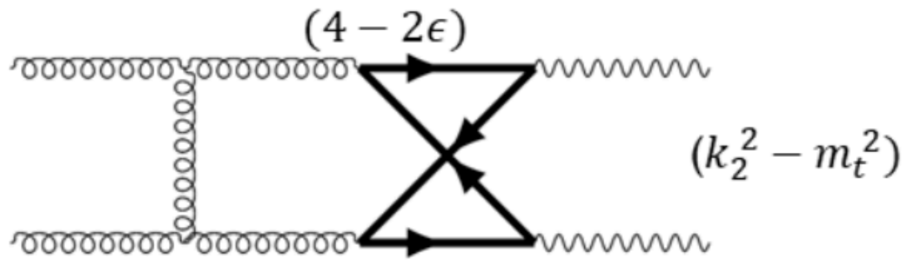
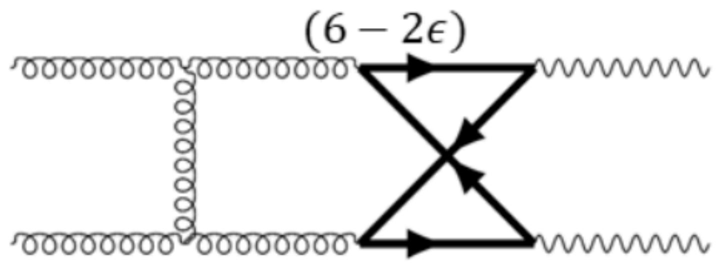
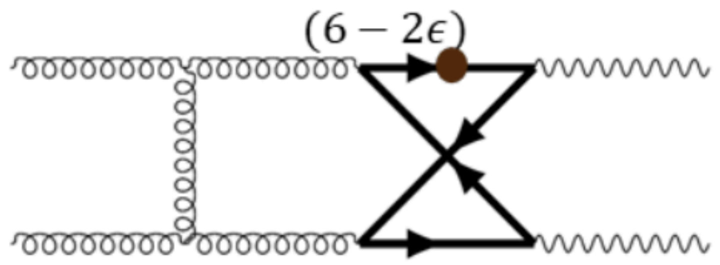
[Henn '13]

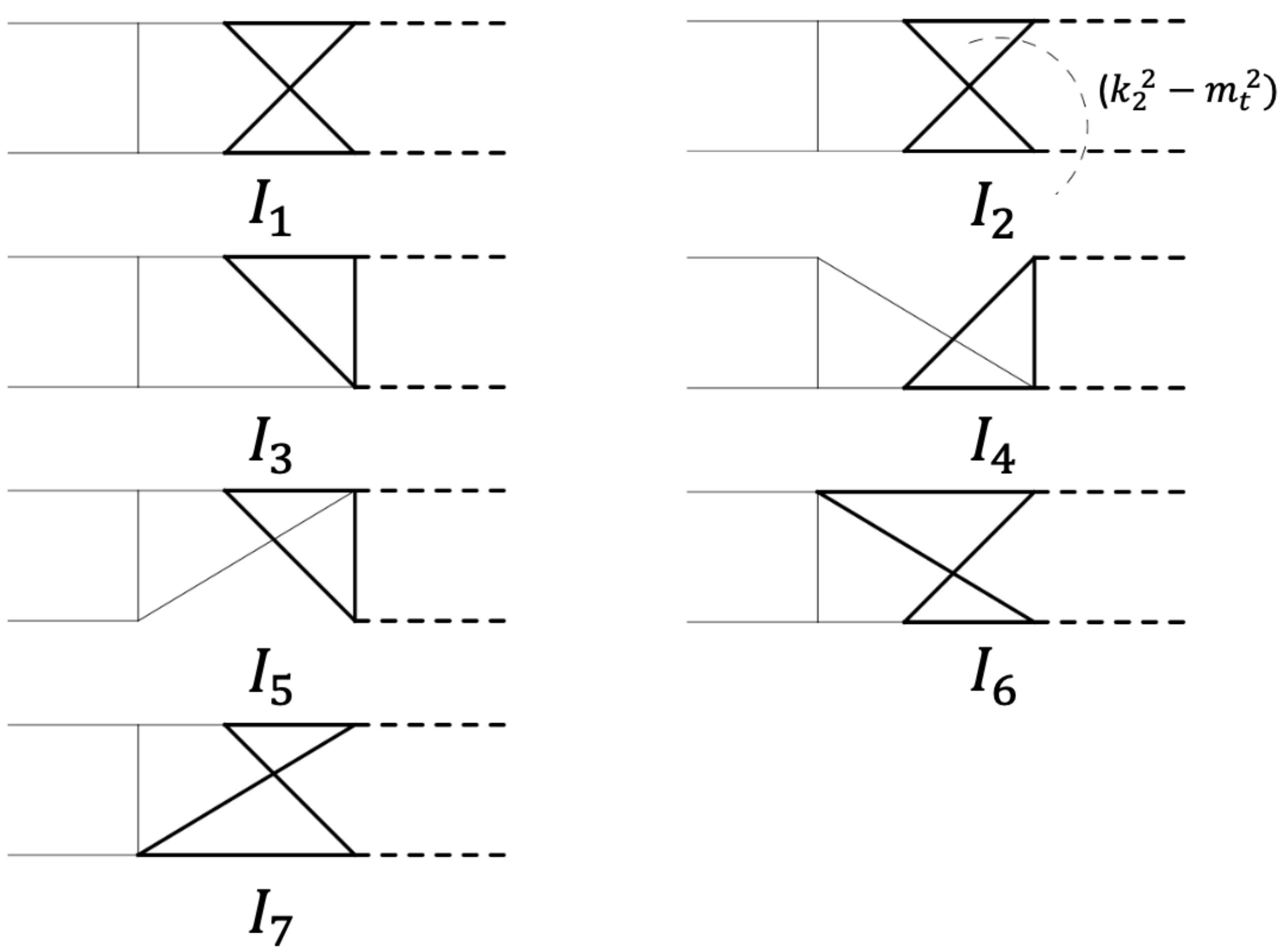
SOLVE INTEGRALS: METHOD OF FINITE INTEGRALS



- General observation *[Panzer 2014; AvM, Panzer, Schabinger 2014]*:
    - any ***divergent*** loop integral can be expressed via **finite** basis integrals
- Expand integrands of ***finite*** integrals around $\epsilon = (4-d)/2 \approx 0$
    - If linearly reducible: integrate ***analytically*** with HyperInt *[Panzer 2014]*
    - Improved ***numerical*** evaluations, used for HH *[Borowka, Greiner, Heinrich, Jones, Kerner '16]*, Hj *[Jones, Kerner, Lusioni '18]*, ZH *[Chen, Davies, Heinrich, Jones, Kerner, Mishima, Schlenk, Steinhauser '22]* …

GENERALIZED FINITE INTEGRALS

Integral	Rel.Err.	Timing(s)
	~2*10^-3	45
	~4*10^-2	63
	~8*10^-6	55
	~8*10^-4	60
Linear combination	~1*10^-4	18



$$I = (m_Z^2 - s - t)(sI_1 - I_6) + s(I_2 + I_3 - I_4 - I_5) - (m_Z^2 - t)I_7$$

$$I(\nu_1,\dots,\nu_N) = (-1)^{r+\Delta t}\,\Gamma(\nu - L\,d/2)\int\left(\prod_{j\in\mathcal{N}_T}\mathrm{d}x_j\right)\left(\prod_{j\in\mathcal{N}_t}\frac{x^{\nu_j-1}}{\Gamma(\nu_j)}\right)\delta\left(1-\sum_{j\in\mathcal{N}_T}x_j\right)$$
$$\left[\left(\prod_{j\in\mathcal{N}_{\setminus T}}\frac{\partial^{|\nu_j|}}{\partial x_j^{|\nu_j|}}\right)\left(\prod_{j\in\mathcal{N}_{\Delta t}}\frac{\partial^{|\nu_j|+1}}{\partial x_j^{|\nu_j|+1}}\right)\frac{\mathcal{U}^{\nu-(L+1)d/2}}{\mathcal{F}^{\nu-L\,d/2}}\right]_{x_j=0\ \forall\ j\in\mathcal{N}_{\setminus T}} \qquad (\nu_j\in\mathbb{Z}).$$

[Agarwal, AvM, Jones 2020]

Numerical integration used pySecDec
[Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke 2017]

talk: Gudrun Heinrich

MULTIVARIATE PARTIAL FRACTIONS

- Univariate partial fraction decomposition separates singularities:

$$\frac{x}{(x-1)(x+1)^2} = -\frac{1}{4(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{4(x-1)}$$

- Iterated partial fractioning introduces spurious poles in multivariate case:

$$\frac{1}{(x-f(y))(x-g(y))} = \frac{1}{(f(y)-g(y))}\frac{1}{(x-f(y))} - \frac{1}{(f(y)-g(y))}\frac{1}{(x-g(y))},$$

  for example:

$$\frac{1}{(x+y)(x-y)} = \frac{1}{2y}\frac{1}{(x-y)} - \frac{1}{2y}\frac{1}{(x+y)}$$

- Our approach to be discussed in the following: MultivariateApart [*Heller, AvM '21]*

- Related work:
  [Pak '11, Abreu ea '19, Boehm ea '20, Bendle ea '21, De Laurentis ea '22]]

talks: Giuseppe De Laurentis, Yang Zhang

- Motivation: (non-planar) amplitudes sometimes reduced by factor $O(100)$ in size with respect to common denominator representation

- ***Leinartas' decomposition***

$$r(x_1, \ldots) = \sum_{\mathcal{S}} \frac{n_{\mathcal{S}}(x_1, \ldots)}{\prod_{i \in \mathcal{S}} d_i^{\alpha_i}(x_1, \ldots)}$$

where denominator factors of each term

i. have common zeros

(in algebraic closure of coefficient field)

ii. are algebraically independent

(no polynomial $g(y_1, \ldots)$ such that $g(d_1(x_1, \ldots), \ldots) = 0$)

- Description (and existing algorithms) ***not unique***:

$$r(x, y) = \frac{2x - y}{x(x+y)(x-y)}$$

has e.g. these Leinartas decompositions

$$r(x, y) = \frac{1}{x(x+y)} + \frac{1}{(x-y)(x+y)}$$

and

$$r(x, y) = \frac{3}{2x(x+y)} + \frac{1}{2x(x-y)}$$

- Our ***wish-list*** for a good partial fractioning algorithm:
  i. It should give a unique answer, independent of input form.
  ii. It should not introduce spurious denominators.
  iii. It should commute with summation.
  iv. It should eliminate spurious denominators if present in input.
- Will solve (i),(ii),(iii). Also (iv) with auxiliary step.

PARTIAL FRACTIONS VIA POLYNOMIAL REDUCTIONS

- Algorithm: write inverse denominators as $q_i = 1/d_i$ and reduce polynomial in $q_1, \ldots, x_1, \ldots$ with respect to ideal

$$I = \langle q_1 d_1(x_1, \ldots) - 1, \ldots, q_m d_m(x_1, \ldots) - 1 \rangle$$

- Here, **polynomial reduction** means

$$p' = p - u \cdot g$$

such that p' "smaller" than p for some monomial ordering, u is an arbitrary polynomial and $g \in I$

- Depending on monomial ordering we can ensure specific features of output form:
    - **Theorem I:** Result is always unique if we consider all g from a Gröbner basis
    - **Theorem II:** Sorting $q_1, \ldots$ before $x_1, \ldots$ guarantees Leinartas (i)
    (useful since it separates singular behavior)
    - **Theorem III:** A lexicographic ordering of the $q_1, \ldots$ and $x_1, \ldots$ (separately) guarantees also Leinartas (ii)
    (a possible choice, but not necessarily needed)

PROOF OF THEOREM II

- Leinartas' (i): separation of zeros
- Hilbert's Nullstellensatz: polynomials d_i have common zeros if and only if there is

$$1 = \sum_i h_i(x_1, \ldots)\, d_i^{\alpha_i}(x_1, \ldots)$$

- We can then write

$$\frac{1}{d_1^{\alpha_1} \cdots d_m^{\alpha_m}} = \sum_i \frac{h_i(x_1, \ldots)}{d_1^{\alpha_1} \cdots \hat{d}_i^{\,\alpha_i} \cdots d_m^{\alpha_m}}$$

  or, using inverse denominators $q_i = 1/d_i$

$$q_1^{\alpha_1} \cdots q_m^{\alpha_m} - \sum_i h_i(x_1, \ldots)\, q_1^{\alpha_1} \cdots \hat{q}_i^{\,\alpha_i} \cdots q_m^{\alpha_m} = 0$$

- Assuming the monomial ordering sorts first for the q_i and then for the x_i, the last equation is a reduction step
- A fully reduced polynomial will therefore separate denominators with common zeros, which proves Theorem II.

PROOF OF THEOREM III

- Leinartas' (ii): separation of algebraically dependent polynomials
- If a set of denominators is algebraically dependent, there is a polynomial p with

  $$p(d_1^{\alpha_1}, \ldots, d_m^{\alpha_m}) = 0$$

- We can solve this equation for a term with lowest degree and get

  $$c_\beta (d^\alpha)^\beta = -\sum_{\gamma \in S} c_\gamma (d^\alpha)^\gamma$$

  where $\sum_i \beta_i \leq \sum_i \gamma_i$. Division by $c_\beta d^{\beta+1}$ gives then

  $$\frac{1}{d_1^{\alpha_1} \cdots d_m^{\alpha_m}} = -\sum_{\gamma \in S} \frac{c_\gamma}{c_\beta} \prod_{i=1}^m \frac{d_i^{\alpha_i \gamma_i}}{d_i^{\alpha_i(\beta_i+1)}}.$$

  which may or may correspond to a polynomial reduction in general.

- For a lexicographic ordering first for the q_i we pick the unique β' such that $(q^\alpha)^{\beta'}$ minimal, cancel $q_i d_i = 1$, and write

  $$q_1^{\alpha_1} \cdots q_m^{\alpha_m} + \sum_{\gamma \in S} \frac{c_\gamma}{c_{\beta'}} \prod_{i=1}^m d_i^{\max(\alpha_i(\gamma_i - \beta_i' - 1), 0)} q_i^{\max(\alpha_i(\beta_i' + 1 - \gamma_i), 0)} = 0.$$

  which gives a reduction. This proves Theorem III.

PERFORMANCE ORIENTED MONOMIAL ORDERING

- Issue I: lexicographic ordering may lead to high degrees in the $q_1, \ldots$
- Issue 2: lexicographic and total degree Gröbner basis often expensive to compute
- **MultivariateApart default ordering**: collect q_i which share the same variables into blocks, sort blocks lexicographically, sort by degree within block
- Sort first by spurious denominators, guarantees their elimination
- Good performance in practice, e.g. 5-pt 2-loop example:

Algorithm	Runtime	File size	Max. mon. deg.	Max. term length
Ref. [12]		25.1 MB	20	3564
Global GB	863 min	22.6 MB	12	3109
Local GB	356 min	21.3 MB	12	3048

solved also multivariate problems with up to degree 4 polynomials

EXAMPLE

- Decompose: $r(x, y) = \dfrac{x - 2y}{(x - y)y(x + y)}$
- Ideal: $I = \langle q_1(x - y) - 1,\ \ q_2 y - 1,\ \ q_3(x + y) - 1 \rangle$
- Monomal ordering: $\{\{q_3, q_1\}, \{q_2\}, \{x, y\}\}$
- Gröbner basis:

  $\{-1 + q_2 y, -1 + q_1 x - q_1 y, -1 + q_3 x + q_3 y, -q_1 q_2 + 2q_1 q_3 + q_2 q_3\}$
- Reducing polynomial $r = (x - 2y)q_1 q_2 q_3$ gives

  $r = -\dfrac{1}{2}q_1 q_2 + \dfrac{3}{2}q_2 q_3 = \dfrac{1}{2(x - y)y} + \dfrac{3}{2y(x + y)}$

PARTIAL FRACTIONS FOR AMPLITUDES

multivariateapart.nb

Univariate partial fractions separate terms with different poles:

In[1]:= $\text{Apart}\left[\frac{1}{x\,(1+x)},\, x\right]$

Out[1]= $\frac{1}{x} - \frac{1}{1+x}$

Let's consider a multivariate example:

In[2]:= $\text{multi} = \frac{2\,y - x}{y\,(x+y)\,(y-x)};$

Naive iteration introduces spurious poles (here 1/x) for multivariate case:

In[3]:= Apart[multi, y]

Out[3]= $\frac{1}{x\,y} + \frac{1}{2\,x\,(-x+y)} - \frac{3}{2\,x\,(x+y)}$

Solution: multivariate partial fractions using methods from polynomial ideal theory:

In[4]:= << MultivariateApart`

```
MultivariateApart -- Multivariate partial fractions. By Matthias Heller and Andreas von Manteuffel.
```

In[5]:= MultivariateApart[multi]

Out[5]= $-\frac{1}{2\,(x-y)\,y} + \frac{3}{2\,y\,(x+y)}$

Note: minimize denominator degrees (≠ Leinartas)

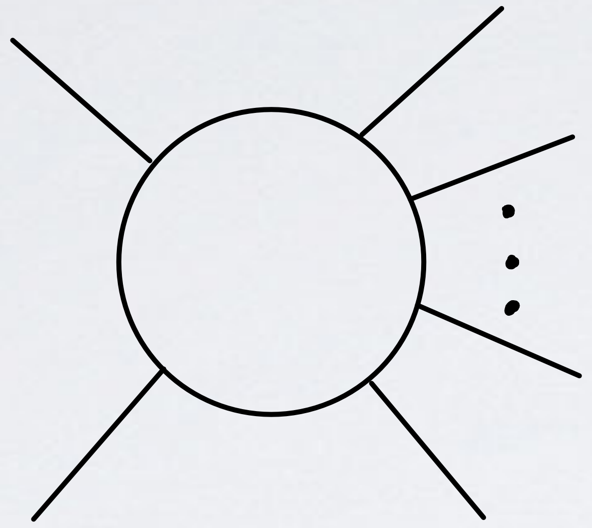
- PFD: significant ***reduction in size***
- Easy to identify ***linear relations*** between coefficients
- Easy to ***generate fast code*** even for complicated amplitudes
- Representation can be tuned for ***numerical stability*** !

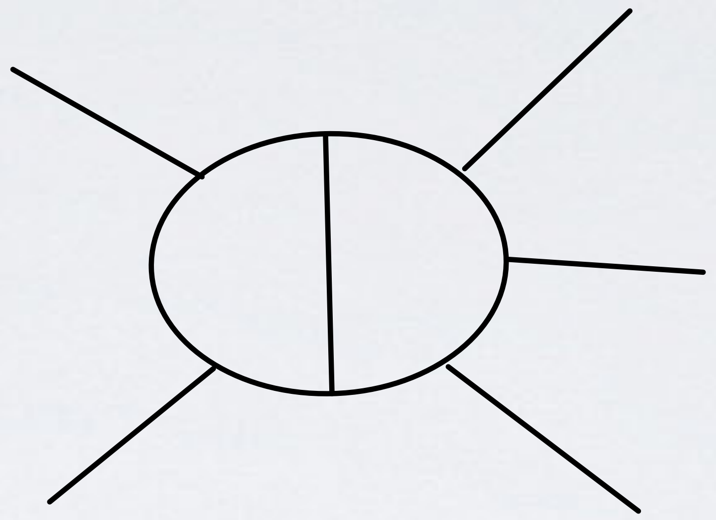
  see $q\bar{q} \to \gamma\gamma j$ @ 2-loops

  [Agarwal, Buccioni, AvM, Tancredi '21]

AMPLITUDES IN FULL-COLOR QCD

FULL-COLOR MASSLESS QCD AMPLITUDES





$q\bar{q} \to \gamma\gamma j$

[Agarwal, Buccioni, AvM, Tancredi '21]

$gg \to \gamma\gamma j$

[Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]

$q\bar{q} \to \gamma\gamma\gamma$

[Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov '23]

$q\bar{q} \to \gamma\gamma$

[Caola, AvM, Tancredi '20]

$gg \to \gamma\gamma$

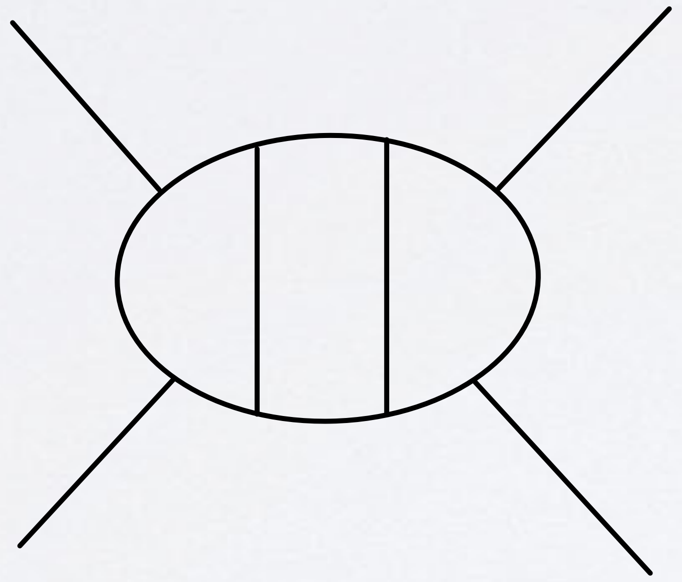
[Bargiela, Caola, AvM, Tancredi '21]

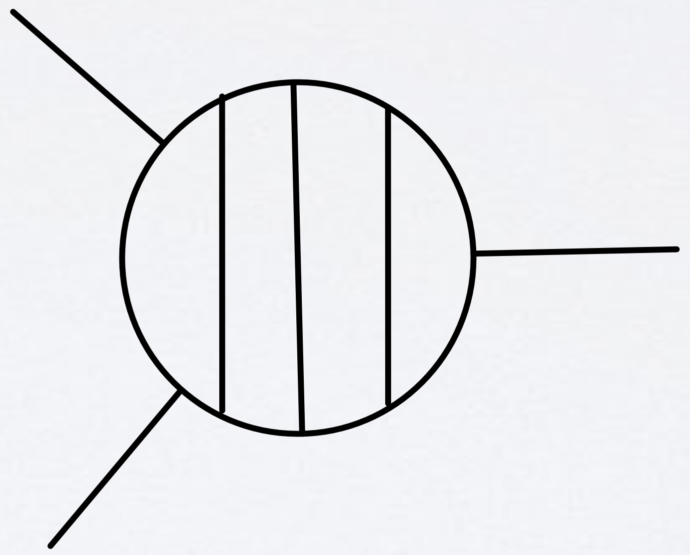
$q\bar{q} \to q'\bar{q}'$, $gg \to gg$, $q\bar{q} \to gg$:

[Caola, Chakraborty, Gambuti, AvM, Tancredi '21,'21,'22]

$q\bar{q} \to \gamma g$:

[Bargiela, Chakraborty, Gambuti '22]





$q\bar{q} \to \gamma^*$, $gg \to H$

[Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

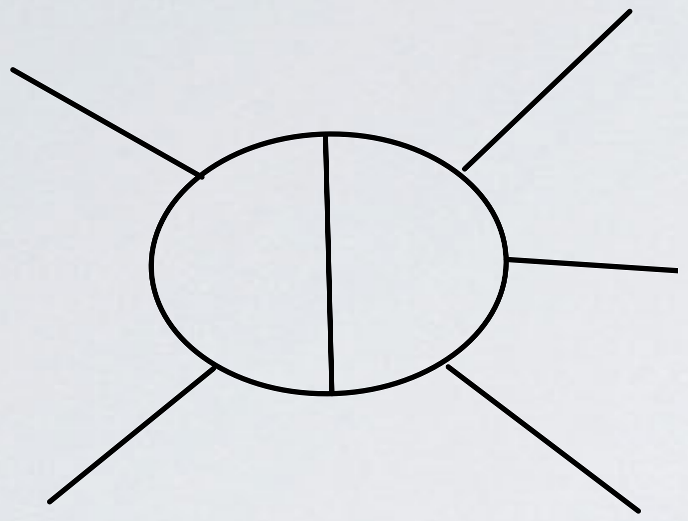
$b\bar{b} \to H$

[Chakraborty, Huber, Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

ACCURACY OF LEADING COLOR APPROXIMATIONS

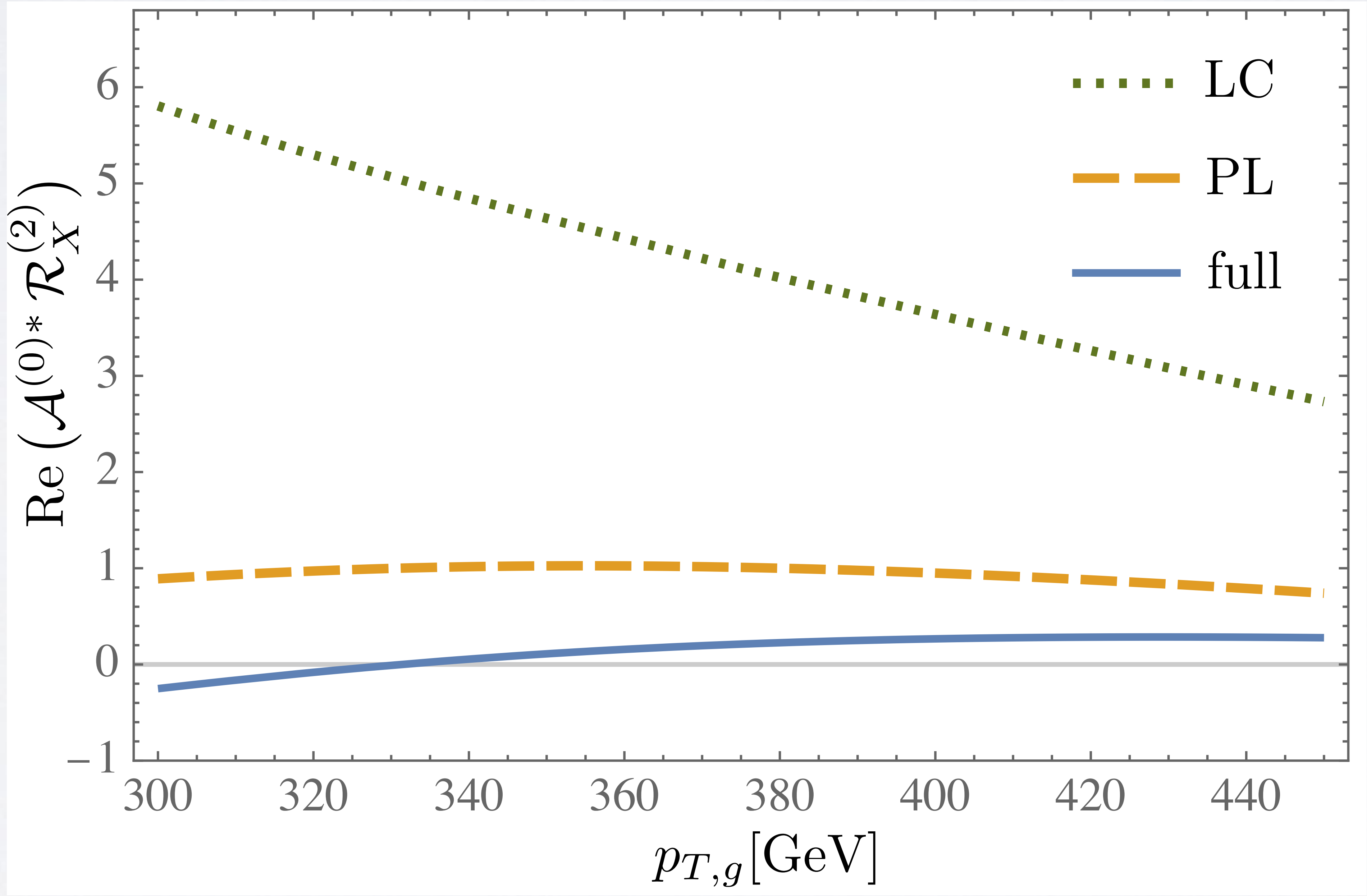
Ex.: $\gamma\gamma j$ @ NNLO: result w/ leading color virtual: *[Chawdhry, Czakon, Mitov, Poncelet 2021]*

Public library for master integrals: PentagonFunctions *[Chicherin, Sotnikov '20]*

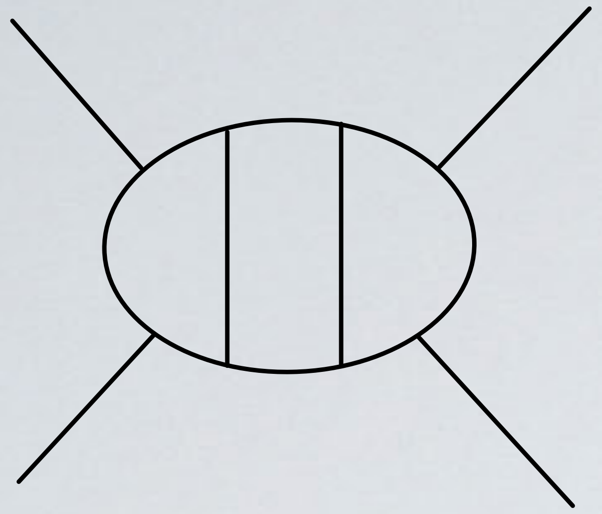


Leading color not always a good approximation:

e.g. 2-loop finite remainder for $u\bar{u} \to g\gamma\gamma$ in Catani's scheme:



[Agarwal, Buccioni, AvM, Tancredi PRL '21]



IR BEYOND DIPOLES

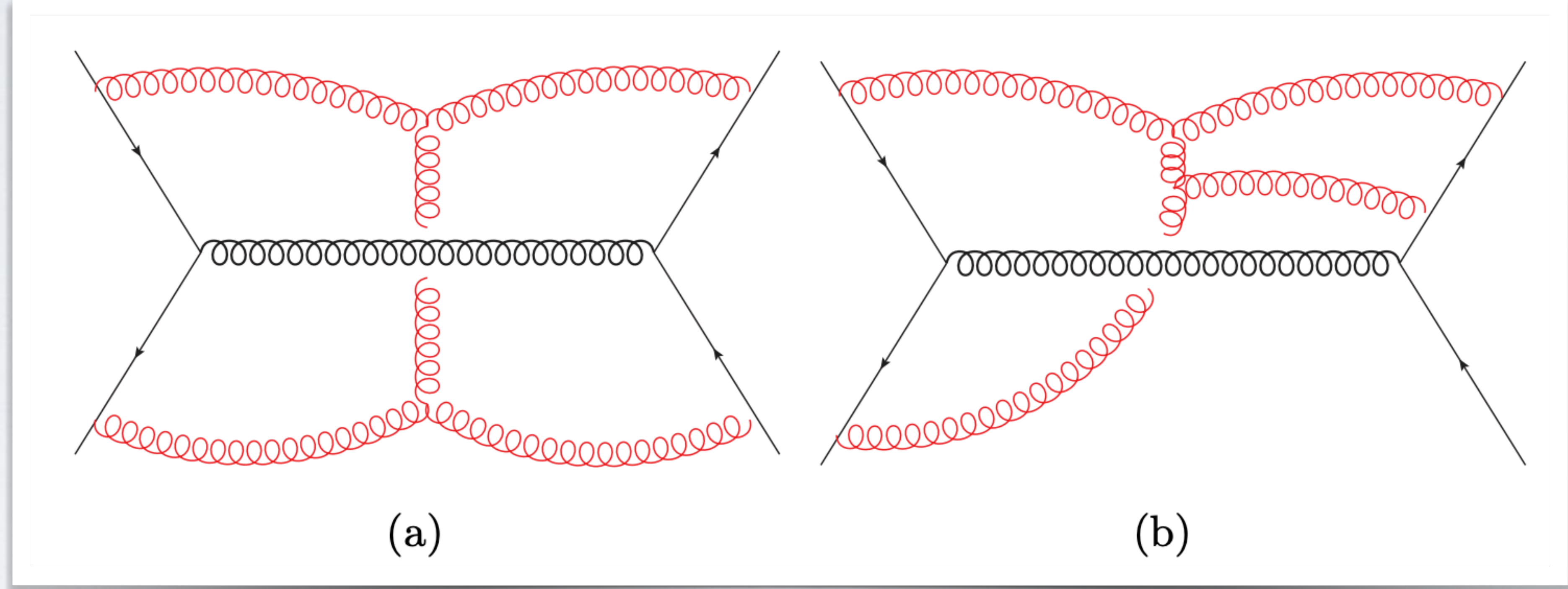
soft anomalous dimension matrix @ 3 loops
[Almelid, Duhr, Gardi '15]

$$\mathbf{\Gamma}(\{p\},\mu) = \mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) + \mathbf{\Delta}_4(\{p\})$$

$$\mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) = \sum_{1\le i<j\le 4} \frac{\mathbf{T}_i^a\,\mathbf{T}_j^a}{2}\,\gamma^{\text{cusp}}(\alpha_s)\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_s)$$

$$\mathbf{\Delta}_4^{(3)} = 128\, f_{abe}\, f_{cde}\left[\mathbf{T}_1^a\,\mathbf{T}_2^c\,\mathbf{T}_3^b\,\mathbf{T}_4^d\,D_1(x) - \mathbf{T}_4^a\,\mathbf{T}_1^b\,\mathbf{T}_2^c\,\mathbf{T}_3^d\,D_2(x)\right] - 16\,C\sum_{i=1}^{4}\sum_{\substack{1\le j<k\le 4\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\,\mathbf{T}_k^c\,,$$

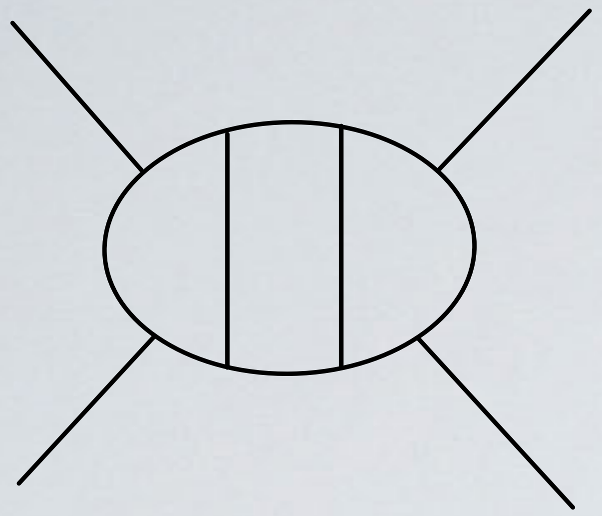
confirmed for N=4 four-point amplitude
[Henn, Mistlberger '16]



Our calculations confirm the predicted quadrupole structure for QCD in all partonic channels

$q\bar{q} \to q'\bar{q}',\ gg \to gg,\ q\bar{q} \to gg$

[Caola, Chakraborty, Gambuti, AvM, Tancredi '21,'21,'22]



HIGH ENERGY LIMIT

- Interesting to study high-energy (Regge) limit of amplitudes beyond fixed order

talk: Einan Gardi

- Regge-cut description to define Regge trajectory beyond 3-loops *[Falcioni, Gardi, Maher, Milloy, Vernazza; Nov '21]*

$$\mathcal{H}_{\mathrm{ren},\pm} = Z_g^2\, e^{L\mathbf{T}_t^2\tau_g} \sum_{n=0}^{3} \bar{\alpha}_s^n \sum_{k=0}^{n} L^k \mathcal{O}_k^{\pm,(n)} \mathcal{H}_{\mathrm{ren}}^{(0)},$$

- Our $q\bar{q} \to q'\bar{q}'$, $gg \to gg$, $q\bar{q} \to gg$ calculations *[Caola, Chakraborty, Gambuti, AvM, Tancredi '21,'21,'22]* allowed us to validate the framework and determine missing parameters:
- We extracted 3-loop gluon Regge trajectory, last building block for single-Reggeon exchanges at NNLL

$$\tau_3 = K_3 + N_c^2\left(16\zeta_5 + \frac{40\zeta_2\zeta_3}{3} - \frac{77\zeta_4}{3} - \frac{6664\zeta_3}{27} - \frac{3196\zeta_2}{81} + \frac{297029}{1458}\right) + \frac{n_f}{N_c}\left(-4\zeta_4 - \frac{76\zeta_3}{9} + \frac{1711}{108}\right)$$
$$+ N_c n_f\left(\frac{412\zeta_2}{81} + \frac{2\zeta_4}{3} + \frac{632\zeta_3}{9} - \frac{171449}{2916}\right) + n_f^2\left(\frac{928}{729} - \frac{128\zeta_3}{27}\right) + \mathcal{O}(\epsilon),$$

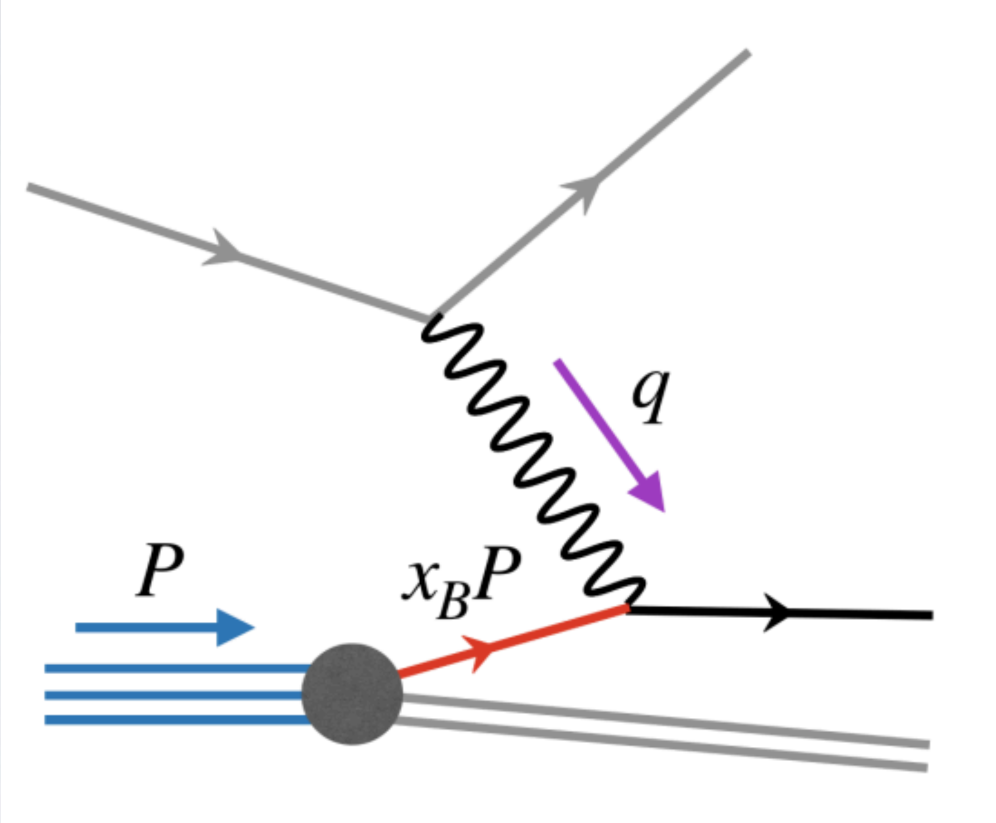
:

indep. extraction: *[Falcioni, Gardi, Maher, Milloy, Vernazza; Dec '21]*

- Gluon Regge trajectory and gluon and quark impact factors extracted from different partonic 3-loop amplitudes agree

TOWARDS ALL-N, FOUR-LOOP DGLAP EVOLUTION

SPLITTING FUNCTIONS



[Image credit: Tong-Zhi Yang]

- Factorization of hadronic cross section:

$$\sigma \sim \sum_k f_{k|N}(x) \otimes \sigma_k(x) \quad \text{with } x = -\frac{q^2}{2P \cdot q}$$

- Splitting functions P_{ik} govern DGLAP evaluations of PDFs:

$$\frac{df_{i|N}}{d \ln \mu} = 2 \sum_k P_{ik} \otimes f_{k|N}$$

- Consistent N3LO cross section requires 4-loop splitting functions, only partially known:
  - Large n_f limit *[Gracey '94,'96; Davies, Vogt, Ruijl, Ueda, Vermageren '16]*
  - Non-singlet $n \leq 16$ from off-shell OMEs *[Moch, Ruijl, Ueda, Vermaseren, Vogt '17]*
  - Singlet $n \leq 8$ from DIS *[Moch, Ruijl, Ueda, Vermaseren, Vogt '21]*
  - Pure-singlet, gluon-quark $n \leq 20$ from off-shell OMEs *[Falcioni, Herzog, Moch, Vogt '23, '23]*
  - Approximate N3LO PDF fits *[McGowan, Cridge, Harland-Lang, Thorne '22; Hekhorn, Magni '23]*
  - This talk: all-n results for pure-singlet n_f^2 splitting functions

SPLITTING FUNCTIONS FROM OPERATORS

- With Mellin transform $f_q(n) = -\int_0^1 dx\, x^{n-1} f_q(x)\ ,\quad \gamma_{ij}(n) = -\int_0^1 dx\, x^{n-1} P_{ij}(x)$

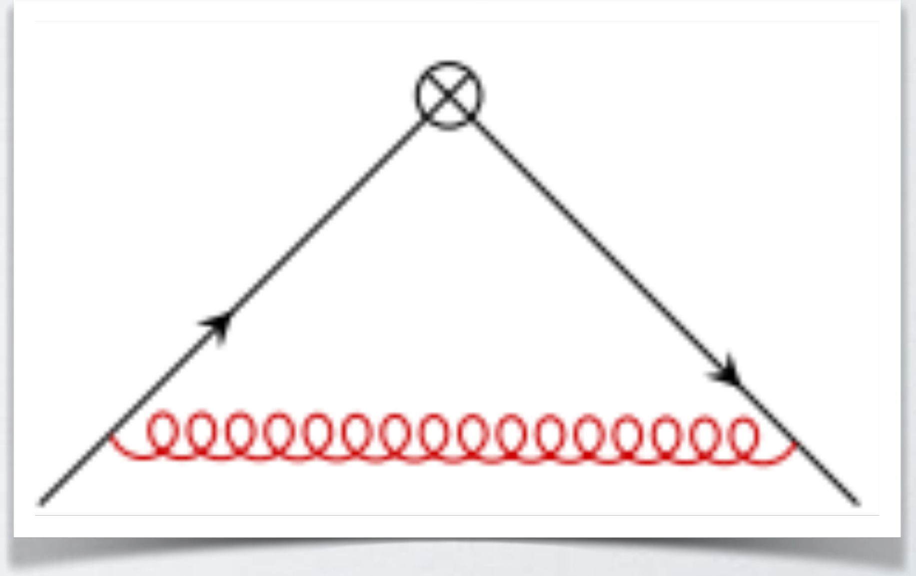
  DGLAP becomes $\dfrac{df_i(n,\mu)}{d\ln\mu} = -2\sum_j \gamma_{ij}(n) f_j(n,\mu)$

- The $\gamma_{ij}(n)$ appear as **anomalous dimensions** of **twist-two operators**,

  e.g. flavor non-singlet: $O_{q,k} = \dfrac{i^{n-1}}{2}\left[\bar{\psi}\Delta_\mu\gamma^\mu(\Delta\cdot D)^{n-1}\dfrac{\lambda_k}{2}\psi\right]$

  with multiplicative renormalization $O^R_{q,k} = Z^{ns} O^B_{q,k}$ where $\dfrac{dZ^{ns}}{d\ln\mu} = -2\gamma^{ns} Z^{ns}$

- Poles of (off-shell) operator matrix elements: **efficient** way to find $f_q(n)$



SINGLET CASE AND OPERATOR MIXING

- Singlet twist-two operators:

$$O_q = \frac{i^{n-1}}{2}\left[\overline{\psi}\Delta_\mu \gamma^\mu (\Delta \cdot D)^{n-1}\psi\right]$$

$$O_g = -\frac{i^{n-2}}{2}\left[\Delta_\mu G^{a\,\mu}{}_{\nu} (\Delta \cdot D)^{n-2}_{ab} \Delta_\kappa G_b{}^{\kappa\nu}\right]$$

- Singlet operators mix under renormalization
- For off-shell OME, also new, unknown gauge-variant operators contribute
- Gauge-variant operators caused confusion in early literature
- Construction of operators for fixed Mellin moment n from generalized BRST: *[Falcioni, Herzog '22]*
- Our goal: **all-n results**
- Our method: directly compute **counter term Feynman rules** from multi-leg off-shell OMEs *[Gehrmann, AvM, Yang '23]*

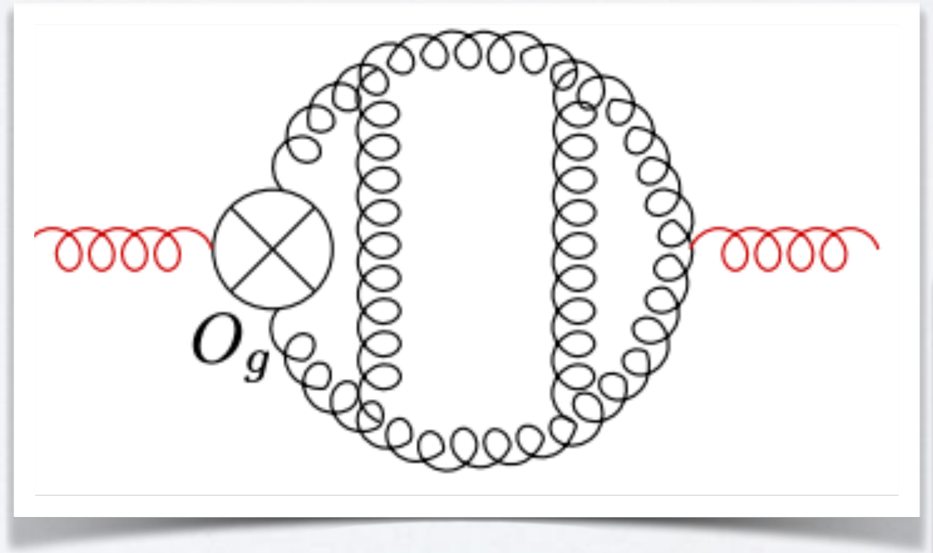
COUNTER TERMS FROM MULTI-LEG OMES

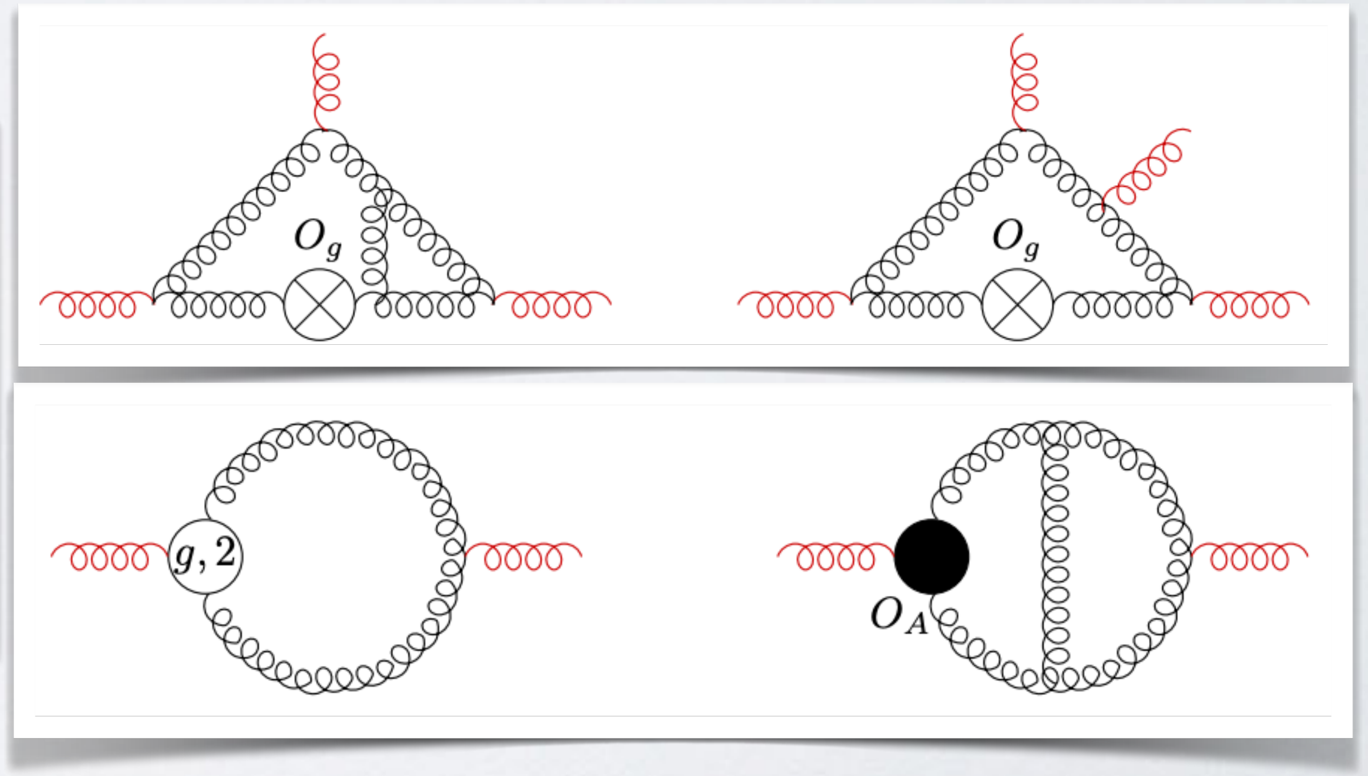
- Renormalization: $\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^R = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ Z_{Aq} & Z_{Ag} & Z_{AA} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^B + \begin{pmatrix} [ZO]_q^{GV} \\ [ZO]_g^{GV} \\ [ZO]_A^{GV} \end{pmatrix}^B$

- Take OMEs according to $\langle j \,|\, O \,|\, j + mg \rangle$ with $j = q, g, c$ and m additional gluons

- Expand $[ZO]^{GV} = \sum_l [ZO]^{GV,(l)} \alpha_s^l$, determine counter terms from OMEs with extra legs, e.g.:

Loops \ Legs	2	3	4	5
0		$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_q, O_g
1	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g	
2	O_{ABC}	O_g		
3	O_q, O_g			





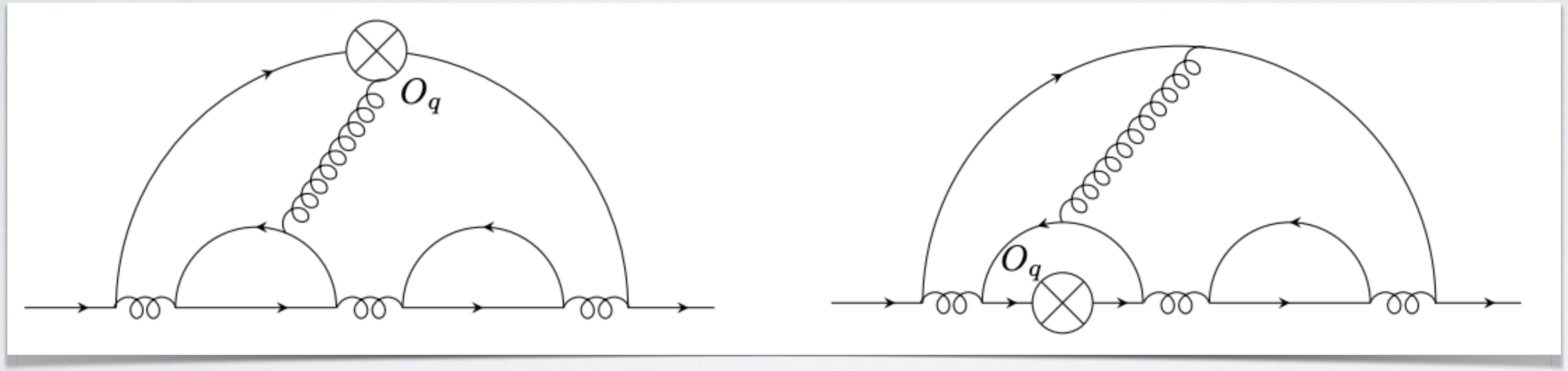
[Gehrmann, AvM, Yang '23]

THREE-LOOP SPLITTING FUNCTIONS

- Operator insertions introduce n dependent powers of scalar products
- Use **tracing parameter** t to map to standard linear propagators
  [Ablinger, Blümlein, Hasselhuhn, Schneider, Wissbrock '12]
  $$(\Delta \cdot p)^{n-1} \to \sum_{n=1}^{\infty} t^n (\Delta \cdot p)^{n-1} = \frac{t}{1 - t\,\Delta \cdot p}$$
  allows to use standard IBP technology
- We applied our method to **3-loop splitting functions**, computation in general R_ξ gauge
- Differential equations in t, find ϵ factorized form using Canonica and Libra, boundary val. known
- Complicated counter terms, involve generalized harmonic sums
- Gauge parameter ξ drops out, full agreement with *[Moch, Vermaseren, Vogt '04, '04]*

FOUR-LOOP PURE SINGLET: N_f^2, ALL-N

- Four-loop contributions for quark, with two or three closed fermion loops
  [Gehrmann, AvM, Sotnikov, Yang ’23]

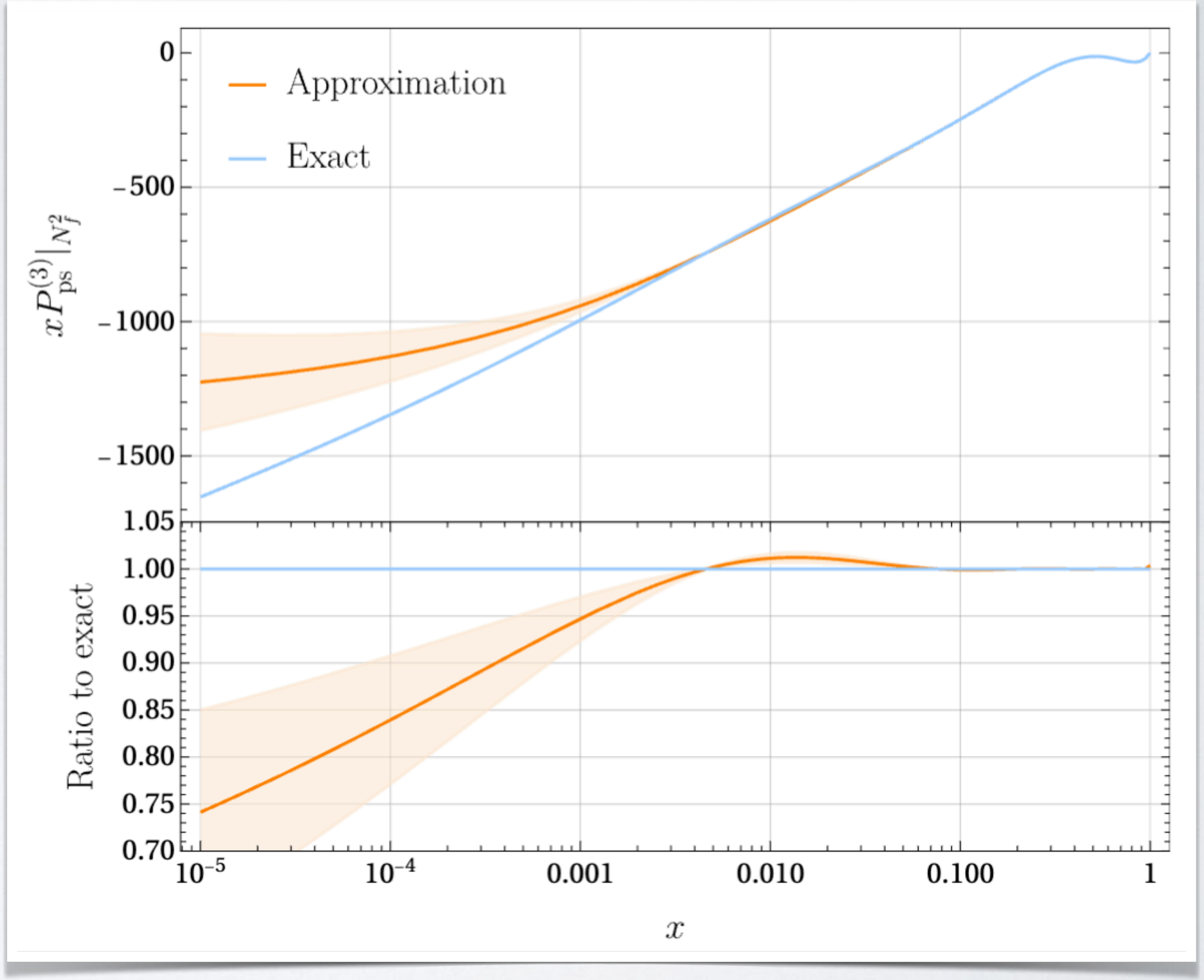


(singlet and non-singlet, also non-planar)

- Use syzygies, compute with linear algebra
- Finred with finite field sampling to derive differential equations, reduction of amplitude
- Simple analytical result for splitting functions in terms of HPLs and powers of x

ALL-N RESULT IMPROVES SMALL X KNOWLEDGE

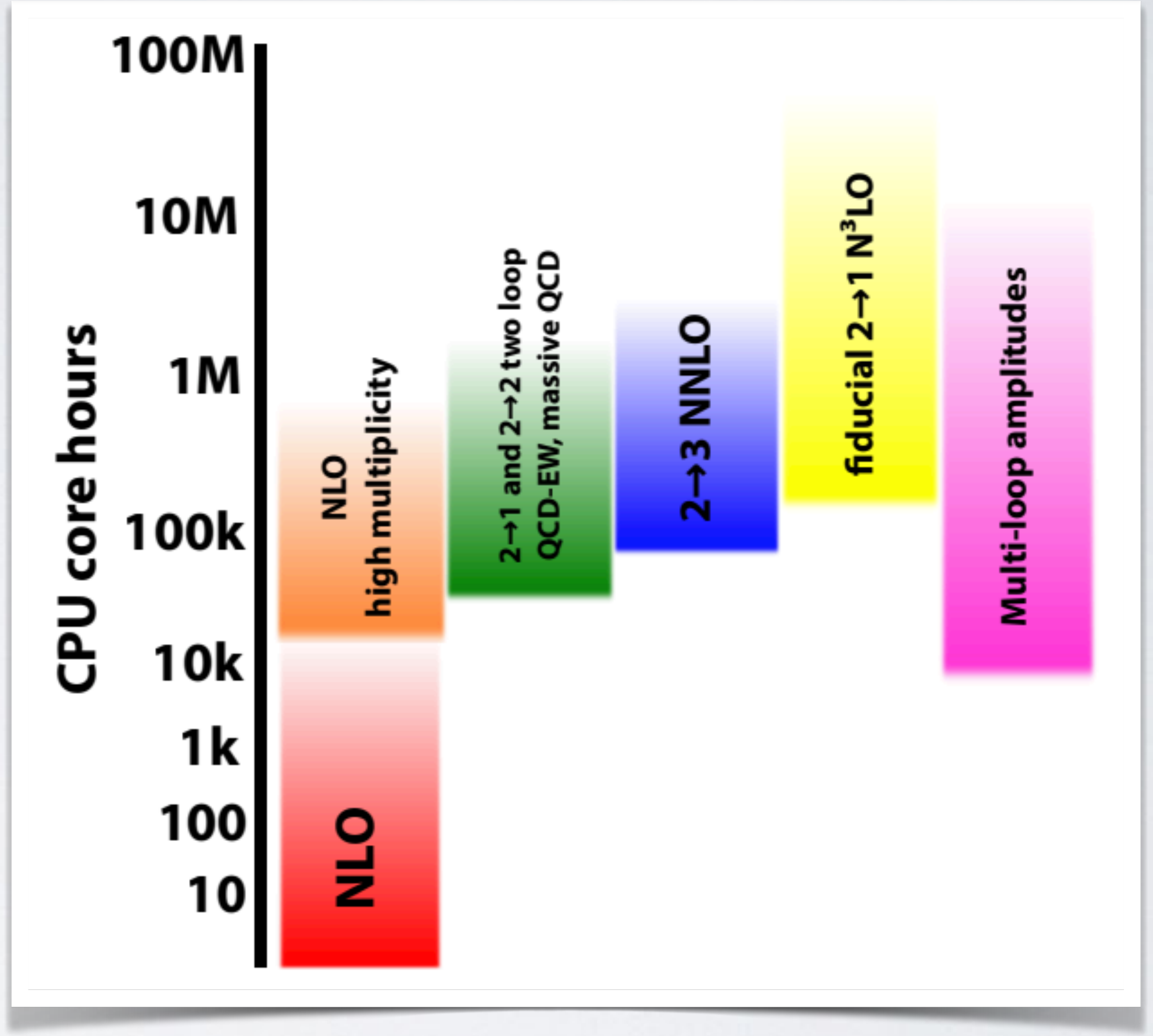
- $n \leq 20$ by *[Falcioni, Herzog, Moch, Vogt ’23]*
- partial information for $x \to 0$: *[Catani, Hautmann ’94; Davies, Kom, Moch, Vogt ’22]*
- leading terms for $x \to 1$: *[Soar, Moch, Vermaseren, Vogt ’09]*
- Generate fit similar to *[Falcioni, Herzog, Moch, Vogt ’23]*, compare to all-n result:



[Gehrmann, AvM, Sotnikov, Yang ’23]

CONCLUSIONS

- First complete calculations in full-color, massless QCD:
  - 5 points @ 2 loops
  - 4 points @ 3 loops
  - 3 points @ 4 loops
  - First steps towards exact 4-loop splitting functions
- This was possible due to progress with
  - IBP reductions
  - Solutions of master integrals
  - Treatment of rational functions
- Room for improvement for all three parts



From: *Snowmass survey of 53 recent perturbative calculations* [Febres-Cordero, AvM, Neumann '22]