Reduction of symbolic propagator powers in Kira

(in collaboration with Fabian Lange)

MathemAmplitudes 2023: QFT at the Computational Frontier

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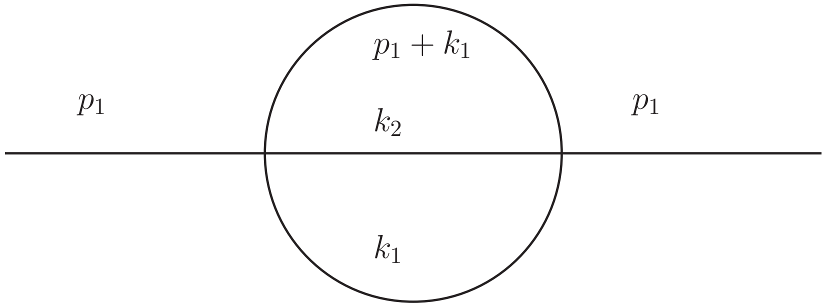
Integral family



$$I(\vec{s}, D|a_1, \ldots, a_5) = \frac{\int d^D k_1 d^D k_2}{\underbrace{[{k_1}^2 - m_1^2]^{a_1}}_{P_1^{a_1}} \underbrace{[(p_1 + k_1)^2]^{a_2}}_{P_2^{a_2}} \underbrace{[{k_2}^2]^{a_3}}_{P_3^{a_3}} \underbrace{[(p_1 + k_2)^2]^{a_4}}_{P_4^{a_4}} \underbrace{[(k_2 - k_1)^2]^{a_5}}_{P_5^{a_5}}}$$

- $q_j = k_1, \ldots, k_L, p_1, \ldots, p_E$
- $s_{ij} = q_i\, q_j, \quad i = 1, \ldots, L, \quad j = i, \ldots, L + E$
- $\vec{s} = (\{s_i\}, \{m_i^2\})$, dimensional regularization parameter $D = 4 - 2\epsilon$
- The integral family definition is complete, if all P_i are linearly independent in the s_{ij}
- $s_{11} = m_1^2 + P_1, \quad s_{12} = \frac{1}{2}(m_1^2 + P_1 + P_3 - P_5), \quad s_{22} = P_3,$
  $s_{13} = \frac{1}{2}(-m_1^2 - P_1 - p_1\, p_1 + P_2), \quad s_{23} = \frac{1}{2}(-p_1\, p_1 - P_3 + P_4)$

Integration-by-parts (IBP) identities

$$I(a_1,\ldots,a_5) = \int \frac{d^D k_1 d^D l_2}{[k_1{}^2 - m_1^2]^{a_1} [(p_1+k_1)^2]^{a_2} [k_2{}^2]^{a_3} [(p_1+k_2)^2]^{a_4} [(k_2-k_1)^2]^{a_5}}$$

$$\int d^D \boldsymbol{k_1} \ldots d^D \boldsymbol{k_L} \frac{\partial}{\partial(\boldsymbol{k_i})_\mu} \left((q_j)_\mu \frac{1}{[P_1]^{a_1} \ldots [P_N]^{a_N}} \right) \text{[Chetyrkin, Tkachov, 1981]} = 0$$

$$c_1(\{a_f\}, \vec{s}, D) I(a_1,\ldots,a_N \mathbf{-1}) + \cdots + c_m(\{a_f\}, \vec{s}, D) I(a_1 \mathbf{+1},\ldots,a_N) = 0$$

m number of terms generated by one IBP identity

Reduction: express all integrals with the same set of propagators but with different exponents a_f as a linear combination of some basis integrals (master integrals)

- Gives relations between the scalar integrals with different exponents a_f
- Number of $L(E+L)$ IBP equations, for each choice of $i = 1,\ldots,L$ and $j = 1,\ldots,E+L$
- a_f = symbols: Seek for recursion relations, `LiteRed` [Lee, 2012]
- a_f = integers: Sample a system of equations, **Laporta algorithm** [Laporta, 2000]
- Seeds: $I(a_1,\ldots,a_5) = [P_1]^{a_1} \ldots [P_N]^{a_N}$

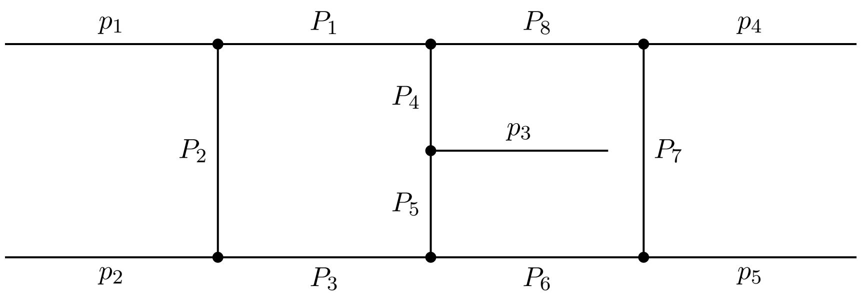
General features of Kira

- MPI support
- Finite field support
- Reduction of general linear system of equations
- Automatic generation of IBPs and symmetry finder for multiple integral toplogies

General purpose of Kira

- Reduction of $2 \to 2$ doublebox integrals (first application to single top production in t-channel)
- Reduction of $1 \to 2$ three-loop form factors (first application $H \to gg$ 3-loop form factor)
- Application of user defined systems
  - Gradient flow formalism [R. V. Harlander, F. Lange, 2022]
  - Phase-space integrals with heaviside functions [D. Baranowski, M. Delto, K. Melnikov, C.-Y. Wang, 2021]
  - Solving system of differential equations (used in Feynman integral reduction through differential equations [JU,Hidding, 2022], used in AMFlow [Xiao Liu,Yan-Qing Ma, 2022])
  - Double-pentagon topology in five-light-parton scattering (solves block triangular form: [Xin Guan, Xiao Liu, Yan-Qing Ma, 2019])

Double-pentagon topology in five-light-parton scattering



Runtime	Memory	Probes	CPU time per probe	CPU time for probes	
12 d	540 GiB	38278000	0.37 s 1 s	25 %	block triangular native `Kira`

- The reduction is a six variable problem
- We use a system of equations which is in **block-triangular form** taken from [Xin Guan, Xiao Liu, Yan-Qing Ma, 2019]
- We benchmark the reduction of all integrals including five scalar products
- `Kira` run specs: system generated with r: 8, s: 5, d: 0 in 5 min

Double-pentagon topology in five-light-parton scattering

- From [JU, 2020] one denominator coefficient in the IBP table

$$(-8+d)*(-6+d)^3*(-5+d)^3*(-4+d)^3*(-3+d)^2*(-2+d)*(-1+d)*(-11+2*d)*(-9+2*d)*(-7+2*d)*s15^2*(s15-s23)*s23^4*(1+s15-s34)^5*(s15-s23-s34)^4*(-1+s34)*s34^6*(-1+s45)^4*s45^3*(-1-s23+s45)^3*(s15-s23+s45)^4*(-1+s34+s45)^5*(s34+s45)^2*(-1+s34+s45+s34*s45)*(s15-s23+s23*s34-s15*s45+s34*s45)*(-s15+s23-s23*s34-s45+s15*s45-2*s23*s45+s45^2)*(1+s23-s34-2*s23*s34-2*s45-s23*s45+s34*s45+s45^2)*(-(s15*s34)+s23*s34-s23*s34^2+s15*s45-s23*s45-2*s34*s45-s15*s34*s45-s23*s34*s45+s34^2*s45-s15*s45^2+s34*s45^2)*(s15^2-2*s15*s23+s23^2+2*s15*s23*s34-2*s23^2*s34+s23^2*s34^2-2*s15^2*s45+2*s15*s23*s45+2*s15*s34*s45+2*s23*s34*s45+2*s15*s23*s34*s45-2*s23*s34^2*s45+s15^2*s45^2-2*s15*s34*s45^2+s34^2*s45^2)$$

- One term after the expansion: $8d^{17}[l^{59}]s15^2 s23^5 s34^{21} s45^{31}$

- The option `insert_prefactors` in `Kira` allows to cancel known parts of a coefficient

Trick to simplify a reduction

Example integral $I(1,1,1,1,1,1,0,1,0,1,1,1,-5,0,0,0,0,0,0,0)$

- For a project in collaboration with Matteo Fael it is necessary to reduce 5 scalar products
- Very difficult with public tools out of the box Kira [Klappert, Lange, Maierhöfer, Usovitsch, 1705.05610, 2008.06494], Reduze 2 [von Manteuffel, Studerus, 1201.4330], FIRE 6 [Smirnov, Chuharev, 1901.07808], FiniteFlow [Peraro, 1905.08019]+LiteRed, Blade
- But it works for sure with Kira if we integrate out one-loop self-energy analytically
- $\int d^D k \frac{k^{\alpha_1} \ldots k^{\alpha_n}}{(-k^2)^{\lambda_1}[-(q-k)^2]^{\lambda_2}} =$

  $\frac{i\pi^{D/2}}{(-q^2)^{\lambda_1+\lambda_2+\epsilon-2}} \sum_{r=0}^{[n/2]} A_{NT}(\lambda_1,\lambda_2;r,n)(\frac{q^2}{2})^r\{[g]^r[g]^{n-2r}\}^{\alpha_1\ldots\alpha_n}$ with

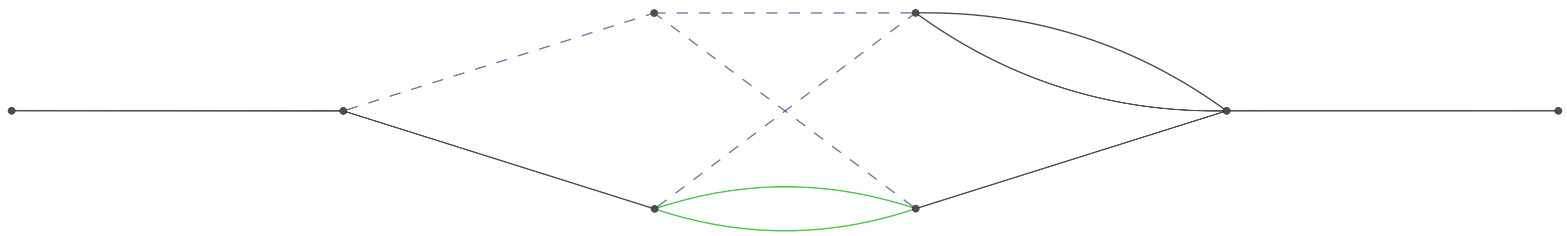
  $A_{NT}(\lambda_1,\lambda_2;r,n) = \frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2-r)\Gamma(n+2-\epsilon-\lambda_1-r)\Gamma(2-\epsilon-\lambda_2+r)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4+n-\lambda_1-\lambda_2-2\epsilon)}$

Integral family with symbolic propagator power

- $I[1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, -1, -1]$
  =(19 terms)
  $-3\frac{\Gamma(2-d/2)\Gamma(-1+d/2)\Gamma(d/2)}{\Gamma(-1+d)}\ I_x[b_1 - 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0, -1, 0]$

- $b_1 = (4 - D)/2$

- New integral family has one-loop less (20 propagator power indices reduce to 14 indices), but one propagator is raised to a symbolic propagator power



Integral family with symbolic propagator power

- Choose master integrals such that b_1 is without integer shifts
- Reintroduce $\frac{\Gamma(-2+D)}{\Gamma(1+\frac{-4+D}{2})^2\Gamma(\frac{4-D}{2})}$, when 4-loop $\to$ 5-loop conversion
- `Kira` does support symbolic reduction for many years
- But it was hardly ever used

Tricks in symbolic reduction with Kira

$$\int d^D\boldsymbol{k_1}\dots d^D\boldsymbol{k_L}\frac{\partial}{\partial(\boldsymbol{k_i})_\mu}\left((q_j)_\mu\frac{1}{[P_1]^{a_1}\dots[P_N]^{a_N}}\right) \quad = 0$$

[Chetyrkin, Tkachov, 1981]

$$c_1(\{a_f\},\vec{s},D)I(a_1,\dots,a_N\mathbf{-1})+\dots+c_m(\{a_f\},\vec{s},D)I(a_1\mathbf{+1},\dots,a_N) \quad = 0$$

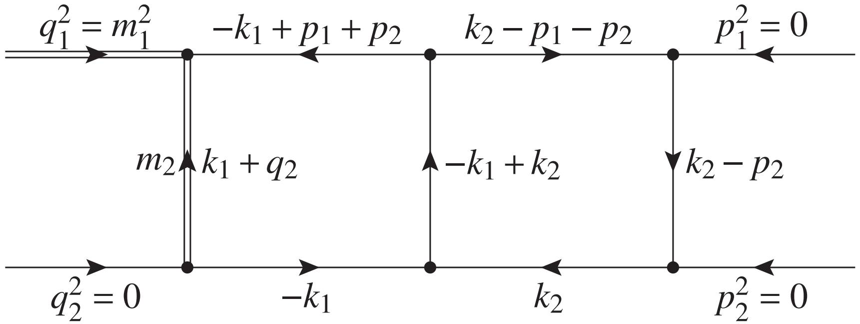
Number of IBP (identities) generators: $L(E+L)$

- The IBP generators are highly linearly dependent, especially at 5-loop
- I eliminate many of the operators
- I prefer to eliminate operators, which result in a positive shift to the symbolic power
- Allowed seeds for $I(b_1+a_1,a_2,\dots,a_{14})$ are: $a_2,\dots,a_{14}$ can take positive and negative values, but a_1 is only allowed to take negative values
- Especially the last point gives orders of magnitude better reduction results

Amplitude reduction

- Amplitude reduction for the 5-loop process finished in 2 month
- Timing: worst case is 10 days on 12 cores
- All integrals from the squared amplitude are expressed through master integrals, which have either at most 2 dots or 2 scalar products
- Unfortunately I do not have any log files anymore, to go into the rich details of reduction specific properties

More examples for symbolic reductions

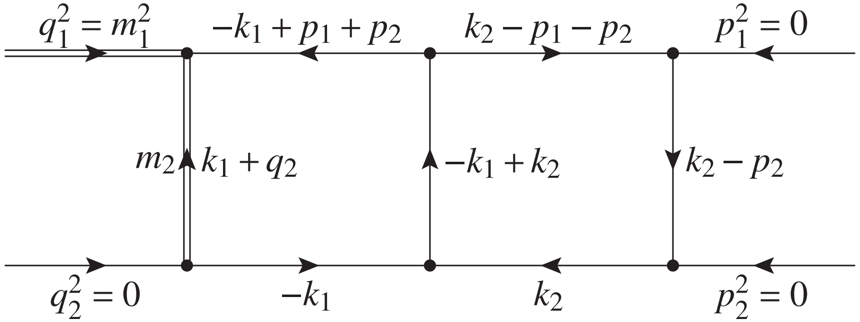


- (case: first two propagators are symbolic) generate: r: 5, s: 2, d: 1
- select: r: 5, s: 2, d: 0
- $topo7[0, 0, 1, 1, 1, 1, 1, -2, 0] =$
  $c(b_0, b_1, s_{ij}, d) topo7[-1, -1, 1, 1, 1, 1, 1, 0, 0] + ...$ in 2.5 hours

More examples for symbolic reductions

- (case: first propagator is symbolic) generate: r: 6, s: 2, d: 1
- select: r: 6, s: 2, d: 0
- $topo7[0,1,1,1,1,1,1,-2,0] =$ $c(b_0, s_{ij}, d) topo7[-1,1,1,1,1,1,1,0,0] + \ldots$ in 317 s
- (case: no symbolic propagator) generate: r: 7, s: 2, d: 1
- select: r: 7, s: 2, d: 0
- $topo7[1,1,1,1,1,1,1,-2,0] =$ $c(s_{ij}, d) topo7[1,1,1,1,1,1,1,-1,0] + \ldots$ in 18 s

NeatIBP + Kira



- Here I discuss a common collaboration with Zihao Wu, Rourou Ma, Hefeng Xu, Yang Zhang
- We develop an interface in NeatIBP [Zihao Wu, Janko Boehm, Rourou Ma, Hefeng Xu, Yang Zhang,2022], to improve IBP reductions in `Kira`
- The interface includes an automatic sorting of the system of equations to allow for the most efficient evaluation within `Kira`
- We reduce integrals with 5 scalar products
- NeatIBP run time 260 s + `Kira` run time 80 s = 340 s
- Fair comparisson with `Kira`: run time stand alone 6463 s
- Does NeatIBP work with symbolic powers?

Equation selection improvements

- Kira implements a bottom-up solver, equations are generated for the lowest sectors first
- The selector in Kira appears not to be optimal
- select, r: 7, s: 1, d: 0
- case 1: generate, r: 7, s: 2, d: 0; 4316 equations selected
- case 2: generate, r: 7, s: 1, d: 0; 1934 equations selected

- Solves equations, which comes first is solved first
- Current strategy in Kira generates each sector at a time
- But it would be smarter to generate all sectors with, e.g.: r: 7, s: 1 first
- Finish the generation of equations with r: 7, s: 2
- This strategy is under investigation

Summary and Outlook

- Introduced several features in `Kira`, e.g.: user defined systems
- For many calculations the option **d** in, r: 7, s: 2, **d: 0**, is very important
- For all calculations the option `permutation_option` is mandatory
- Improved symbolic IBP reductions with `Kira`
- `Kira` supports the reduction of arbitrary number of symbolic propagator powers
- Symbolic reductions are very useful, if an inner one-loop self-energy loop can be integrated out
- Introduced a new coming up feature in `Kira`: NeatIBP+`Kira`
- Uncovered bottlenecks in `Kira`, which are in fixing process right now