Computational Tools For Colliders and GW Physics

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University of Padova and INFN Padova

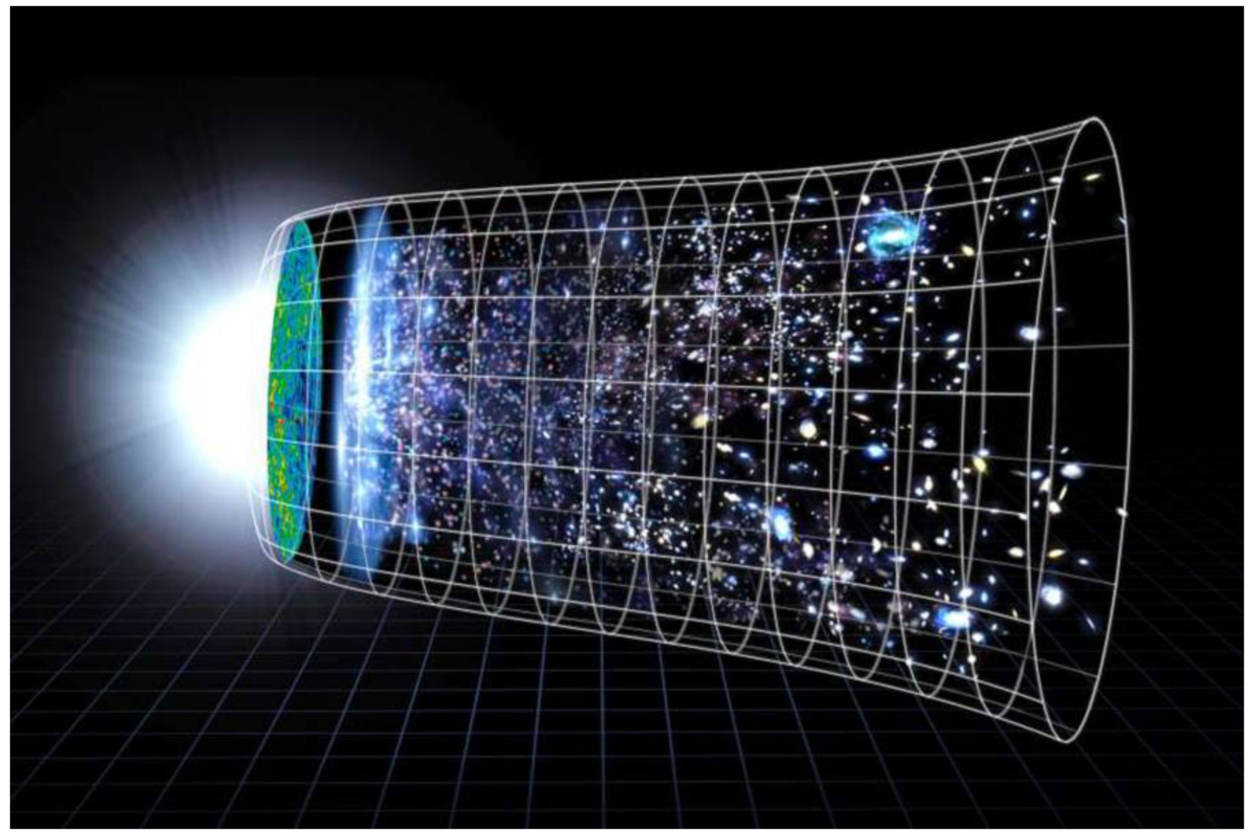
Advanced Calculus for Fundamental Interactions

10th July, 2023

Scattering Amplitudes

Collider Phenomenology

Gravitational Waves



Cosmology

Precise Observables <—> Scattering Amplitudes

Scattering Amplitude: Connecting Theory and Experiment

Perturbative Expansion of Cross-Section

$$\sigma = \sigma^{(0)} + \alpha_s \sigma^{(1)} + \alpha_s^2 \sigma^{(2)} + \cdots$$

LO NLO NNLO

Cross-section Measured in Experiment

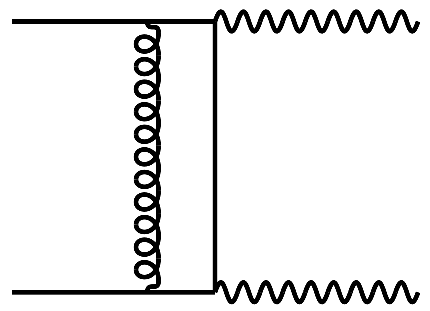
$$\sigma^0 \approx \int |\mathcal{M}_N^{(0)}|^2 d\Phi_N$$

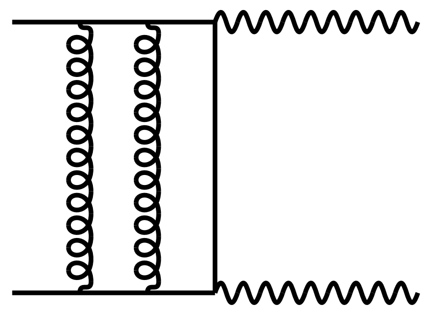
Sum of Feynman Diagrams

Theory

Scattering Amplitudes

Computation of the Loop Amplitude





Generation of the Diagrams via QGRAF

↓

Dirac algebra, Color sum, Trace in the numerators

↓

Reduction to scalar integrals

$$\mathcal{M} = \sum_i a_i I_i \qquad i = \mathcal{O}(10^5)$$

Integration-By-Parts Identity

$$\int \prod_{\alpha=1}^{l} d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = 0$$

Chetyrkin, Tkachov

Loop momenta

Loop and external momenta

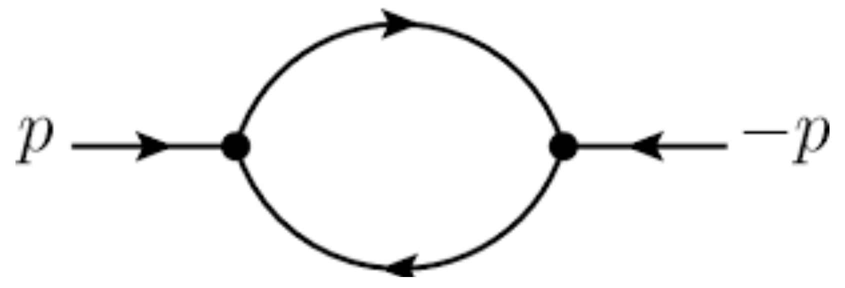
$$\int_{\alpha=1}^{l} \prod d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = \int_{\alpha=1}^{l} \prod d^d k_\alpha \left[\frac{\partial v^\mu}{\partial k_{j,\mu}} \left(\frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right) - \sum_{j=1}^{N} \frac{a_j}{D_j} \frac{\partial D_j}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) \right]$$

$$C_1\, I(a_1, \cdots a_N - 1) + \cdots + C_r\, I(a_1 + 1, \cdots a_N) = 0$$

- Gives relations between different scalar integrals with different exponents
- l(l+E) number of equations
- Solve the system symbolically : Recursion relations — LiteRed
- Solve for specific integer value of the exponents : Laporta Algorithm — Fire, Reduze, Kira,..

Integration-By-Parts Identity (Example)

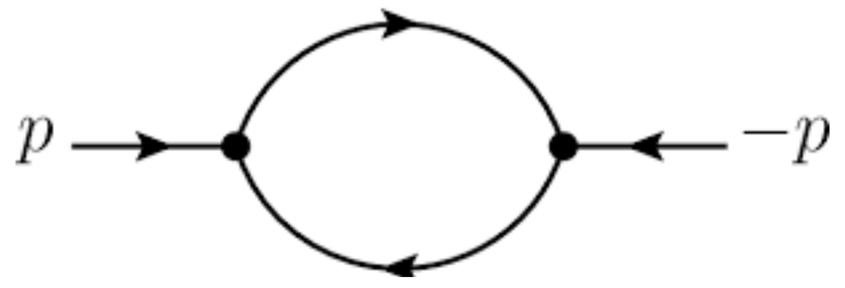
One Loop Massless Bubble



$$I(a_1, a_2) = \int \frac{d^d k_1}{(k_1^2)^{a_1} ((k_1 + p)^2)^{a_2}}$$

Integration-By-Parts Identity (Example)

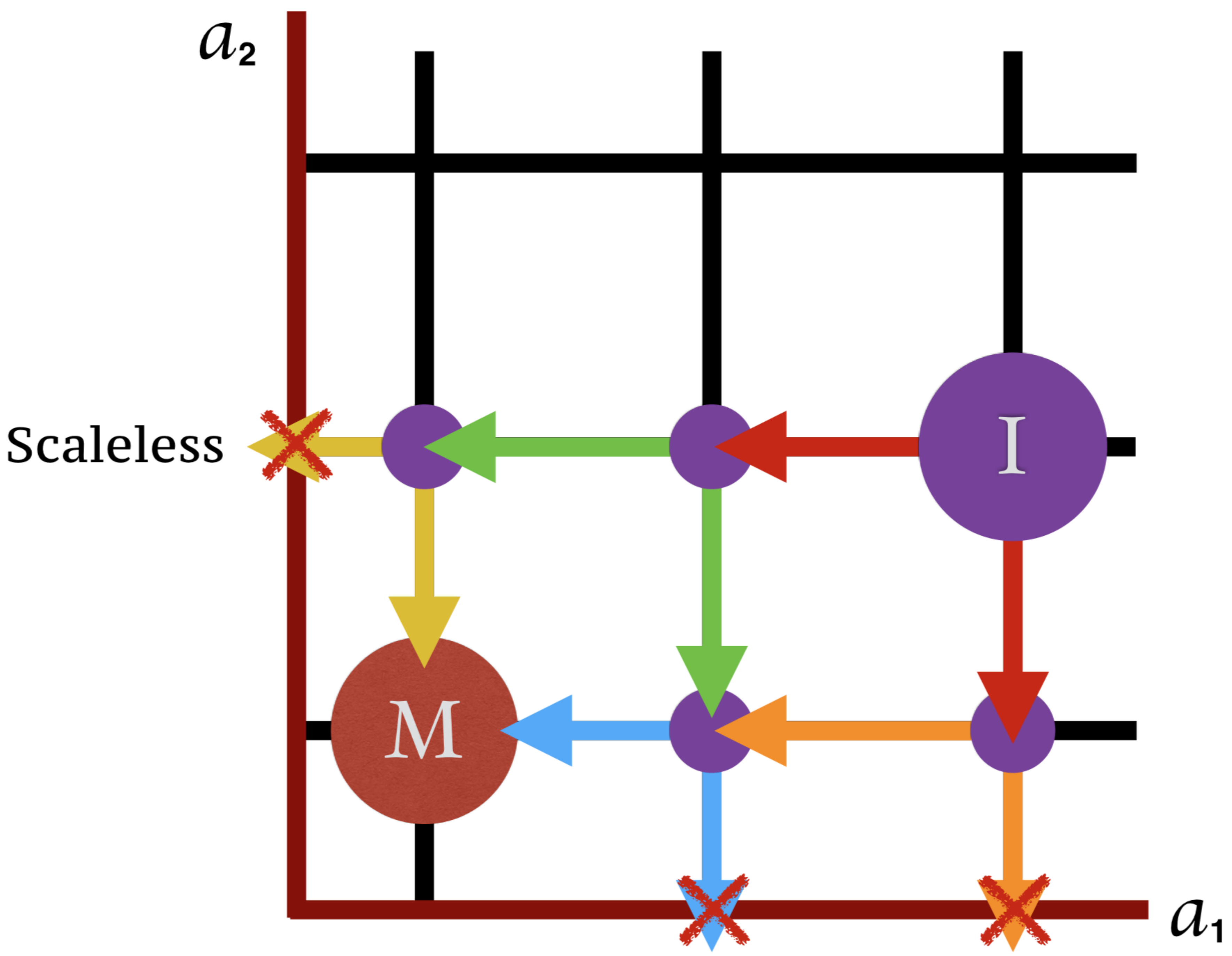
One Loop Massless Bubble



$$I(a_1, a_2) = \int \frac{d^d k_1}{(k_1^2)^{a_1} ((k_1 + p)^2)^{a_2}}$$

IBP Identity

$$I(a_1, a_2) = \frac{a_1 + a_2 - d - 1}{p^2 (a_2 - 1)} I(a_1, a_2 - 1) + \frac{1}{p^2} I(a_1 - 1, a_2)$$



Loop Amplitude

Reduction of scalar integrals to Master integrals

↓

Compute the Master Integrals

Number of Master Integrals

$$\mathcal{M} = \sum_i c_i J_i \qquad i = \mathcal{O}(10^2)$$

Computation of the Loop Amplitude

Mathematica Based Package AIDA [Mastrolia, Peraro, Primo, Ronca, Torres Bobadilla (To be Published)]

Generation of Diagram by FeynArts

↓

Spin sums, Dirac Algebra, Trace by FeynCalc

↓

Adaptive Integrand Decomposition

↓

IBP Reduction via Reduze and KIRA

↓

Master Integral evaluation
Ginac, handyG, FastGPL

$$\mathcal{M}_{\rm b}^{(n)} = (S_\epsilon)^n \int \prod_{i=1}^{n} \frac{d^d k_i}{(2\pi)^d} \sum_G \frac{N_G}{\prod_{\sigma \in G} D_\sigma}$$

$$\mathcal{M}_{\rm b}^{(n)} = \mathbb{C}^{(n)} \cdot \mathbf{I}^{(n)}$$

Master Integrals

Evaluation of the Amplitude

Amplitude

Analytic computation of MIs

Feynman parameters

Mellin Barnes Representation

Asymptotic Expansion

Differential Equation

Kotikov; Gehrmann, Remiddi; Henn; Argeri, Mastrolia; Laporta, Remiddi; Argeri et al; Moriello; Czakon; …

Numerical Evaluation of the MIs

PySecDec [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke]

DiffExp [Hidding]

SeaSyde [Armadillo, Bonciani, Devoto, Rana, Vicini]

AMFlow [Liu, Ma]

FeynTrop [Borinsky, Munch, Tellander]

Numerical Solution [MKM, Zhao]

Recent Applications

PHYSICAL REVIEW LETTERS **128,** 022002 (2022)

Two-Loop Four-Fermion Scattering Amplitude in QED

R. Bonciani,[1,*] A. Broggio,[2,†] S. Di Vita,[3,4] A. Ferroglia,[5,6,‡] M. K. Mandal,[7,8,§] P. Mastrolia,[8,7,∥] L. Mattiazzi,[7,8,¶] A. Primo,[9,**] J. Ronca,[10,††] U. Schubert,[11,‡‡] W. J. Torres Bobadilla,[12,§§] and F. Tramontano[10,∥∥]

JHEP

PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: *April 20, 2022*
REVISED: *August 10, 2022*
ACCEPTED: *August 19, 2022*
PUBLISHED: *September 16, 2022*

Two-loop scattering amplitude for heavy-quark pair production through light-quark annihilation in QCD

Manoj K. Mandal,[a] Pierpaolo Mastrolia,[a,b] Jonathan Ronca[c] and William J. Torres Bobadilla[d]

JHEP

PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: *December 23, 2022*
ACCEPTED: *January 3, 2023*
PUBLISHED: *January 20, 2023*

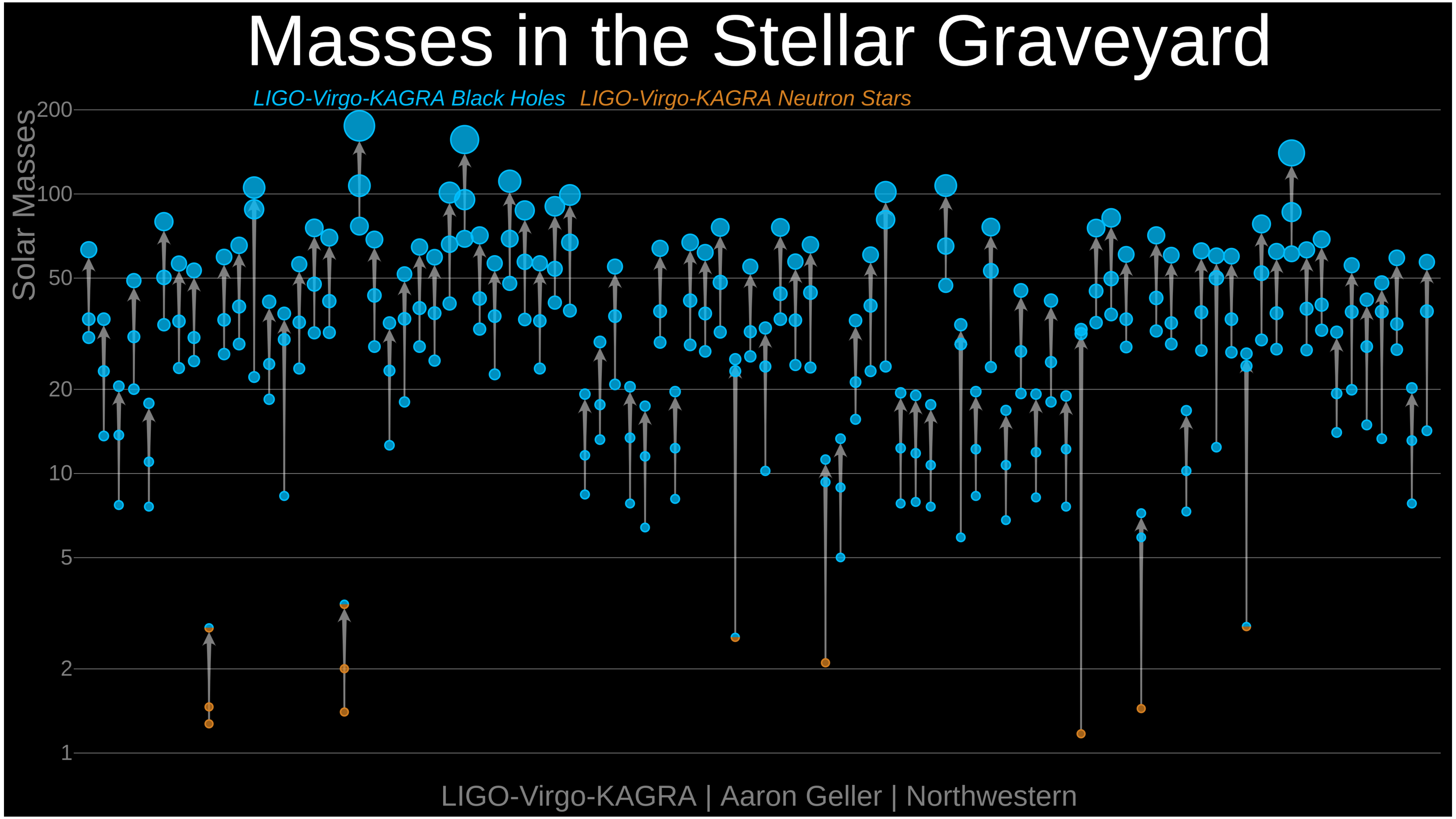
Muon-electron scattering at NNLO

A. Broggio,[a] T. Engel,[b,c,d] A. Ferroglia,[e,f] M.K. Mandal,[g,h] P. Mastrolia,[i,g] M. Rocco,[b] J. Ronca,[j] A. Signer,[b,c] W.J. Torres Bobadilla,[k] Y. Ulrich[l] and M. Zoller[b]

Gravitational Wave Observables

MKM, Mastrolia, Patil, Steinhoff (2022)
MKM, Mastrolia, Patil, Steinhoff (2022)
MKM, Mastrolia, O Silva, Patil, Steinhoff (2023)

GW observations

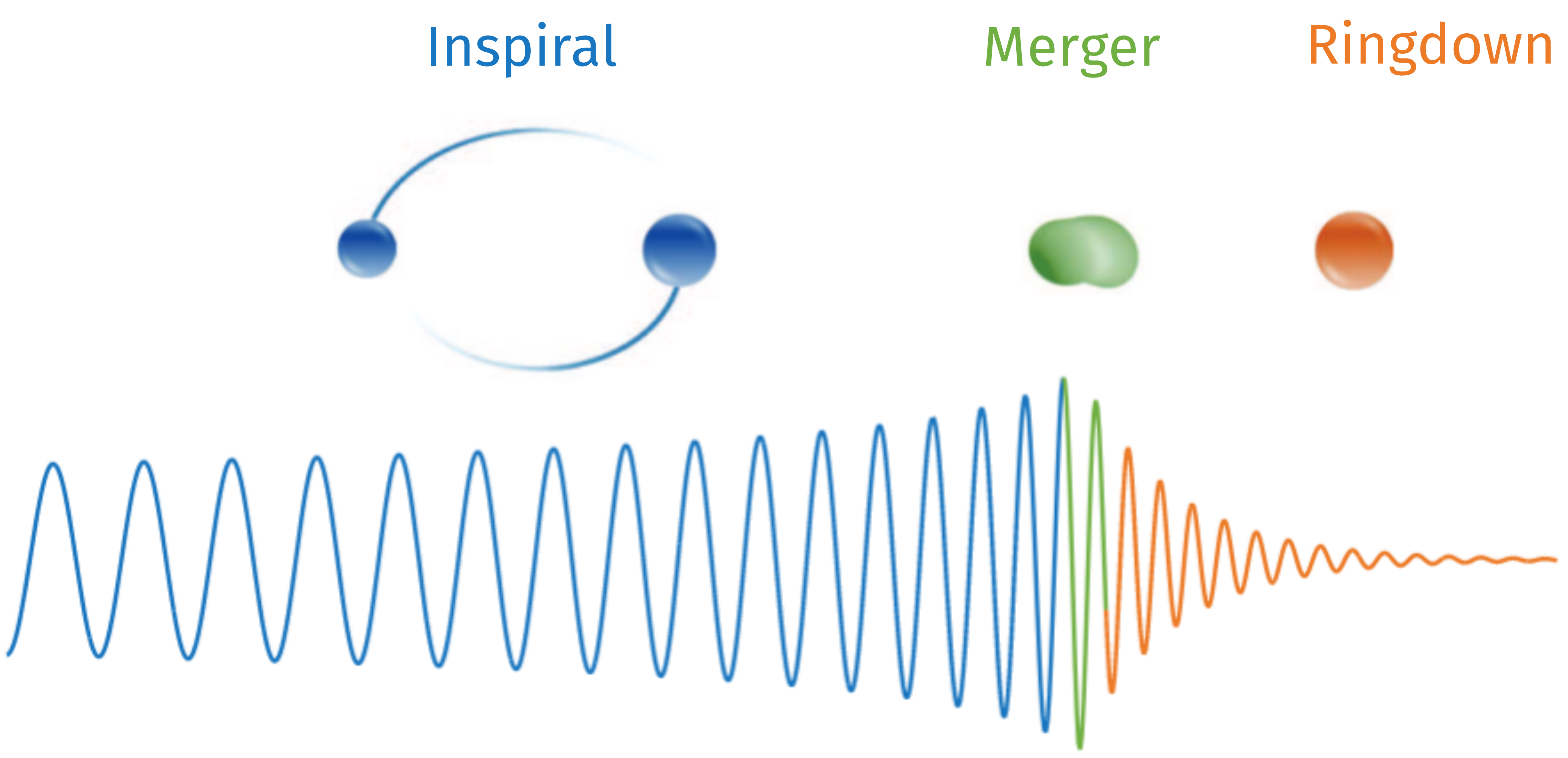


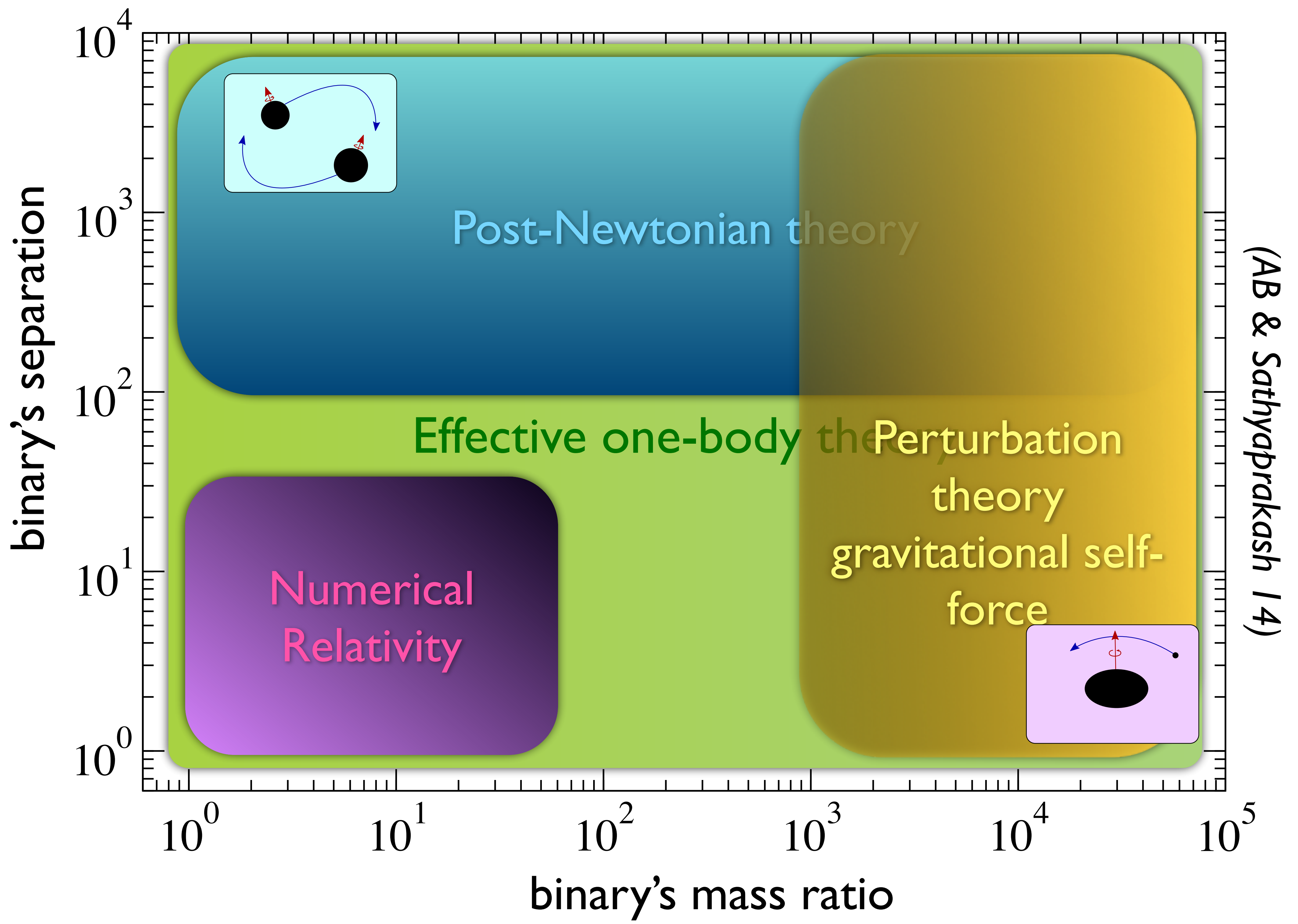
Tasks

- Supplement conventional Analysis
- Increase Theoretical Precision
- Perform Gravity phenomenology

Solving two-body problem in GR

Antelis, moreno (2016)





Post-Newtonian (PN)

Post-Minkowskian (PM)

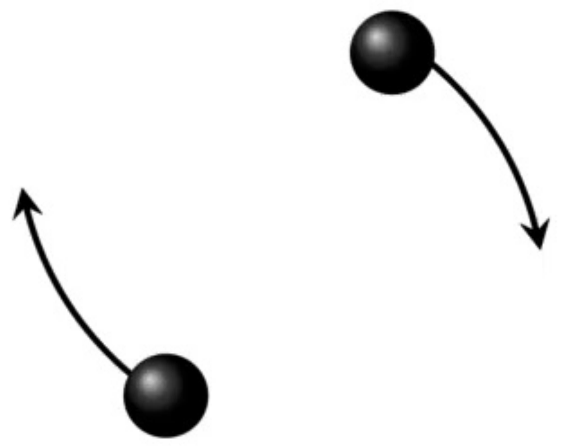
Numerical Relativity

Perturbation Theory

Analytical Approximation Methods

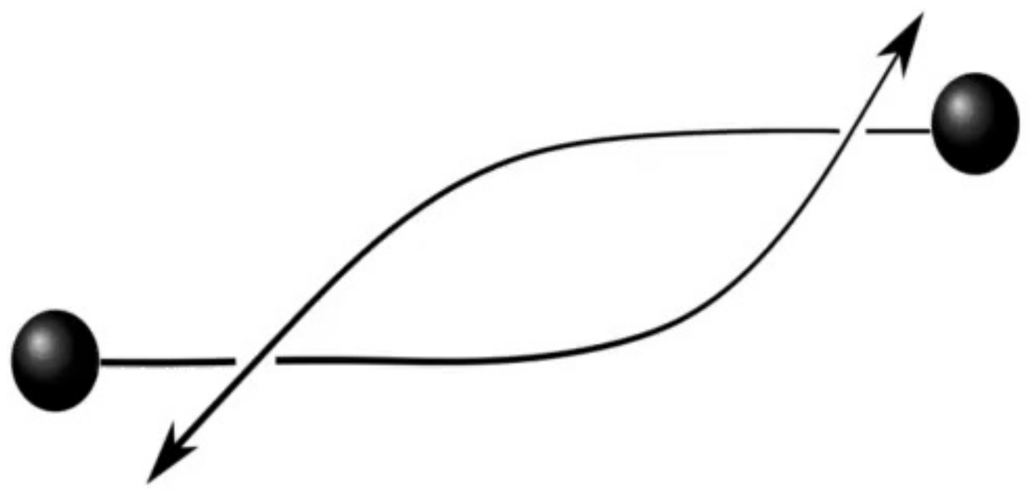
Post-Newtonian (PN)

$$\frac{v^2}{c^2} \sim \frac{GM}{rc^2} \ll 1$$



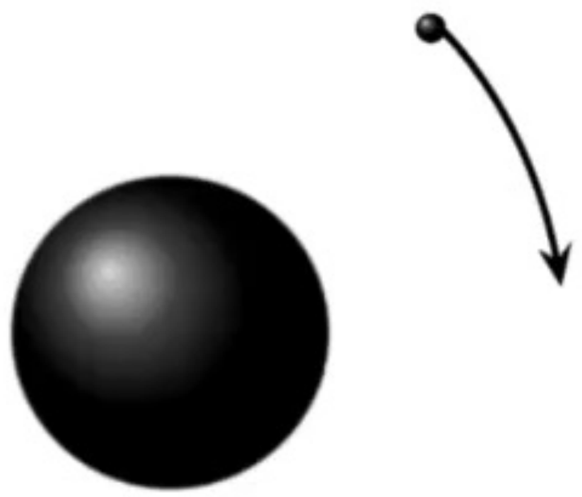
Post-Minkowskian (PM)

$$\frac{GM}{rc^2} \ll 1$$



Self-Force (SF)

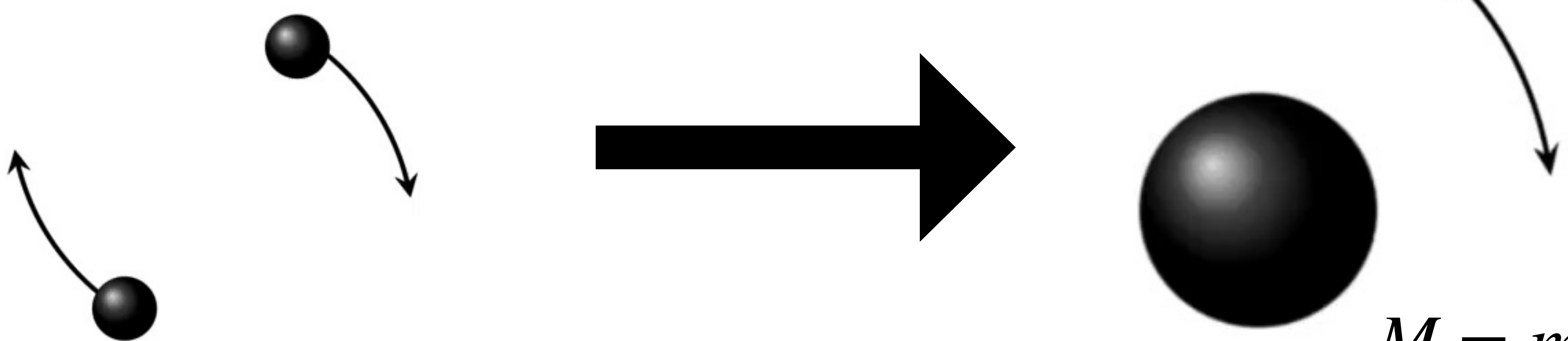
$$\frac{m_1}{m_2} \ll 1$$



Effective One-Body (EOB)

$$\mu = \frac{m_1 m_2}{M}$$

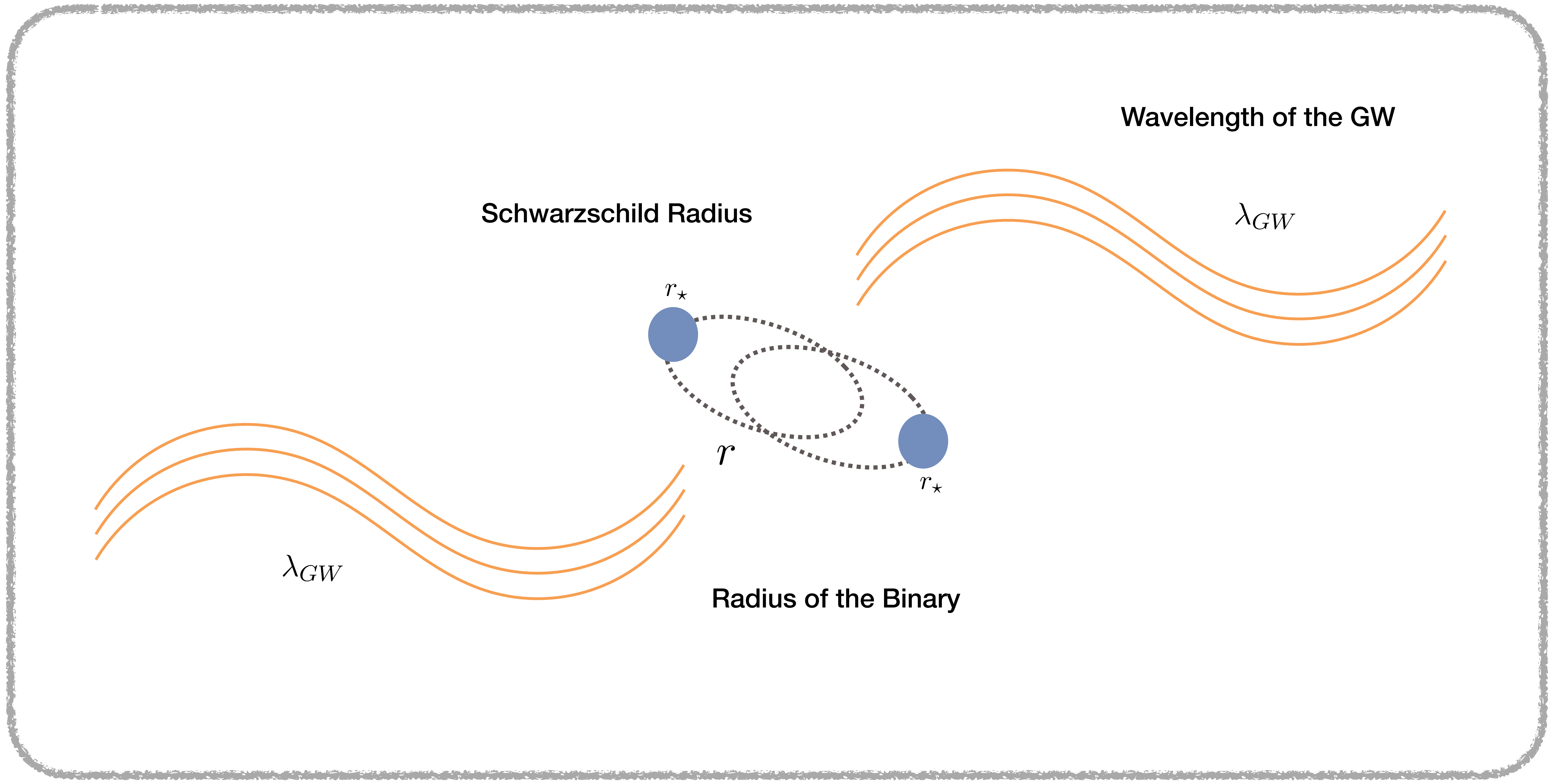
m_2, a_2



m_1, a_1

$$M = m_1 + m_2$$
$$a = a_1 + a_2$$

Post-Newtonian Expansion EFT set up



Equations of Motion

$$\dot{r} = \frac{d\mathcal{H}}{dp_r} \qquad \dot{p}_r = -\frac{d\mathcal{H}}{dr} + \mathcal{F}_r$$

$$\dot{\phi} = \frac{d\mathcal{H}}{dp_\phi} \qquad \dot{p}_\phi = -\frac{d\mathcal{H}}{d\phi} + \mathcal{F}_\phi$$

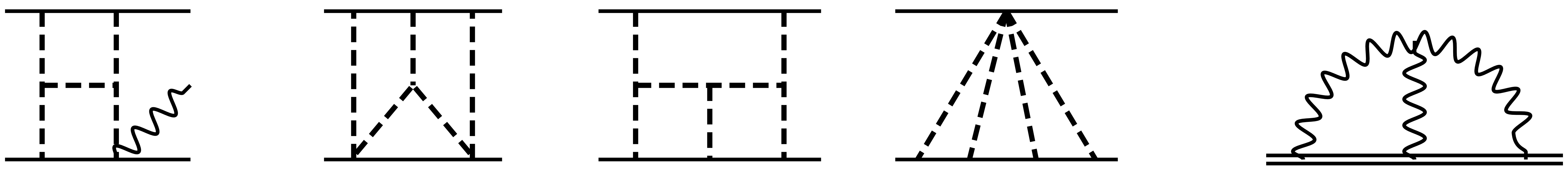
Need:

Hamiltonian $\mathcal{H}$

Radiation Reaction $\mathcal{F}$

Advantage of QFT techniques

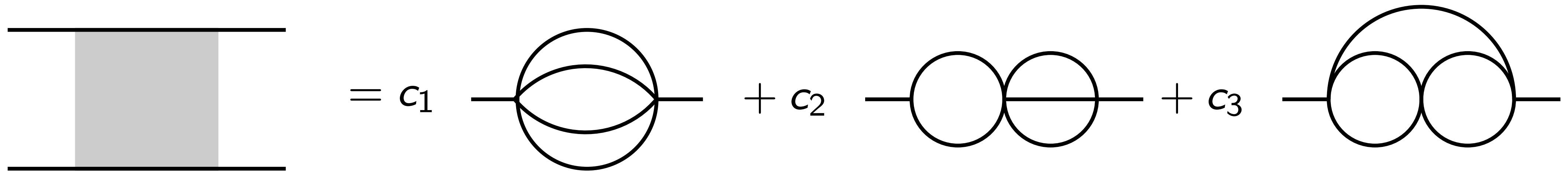
- Use of Feynman diagrams



- Dimensional Regularization

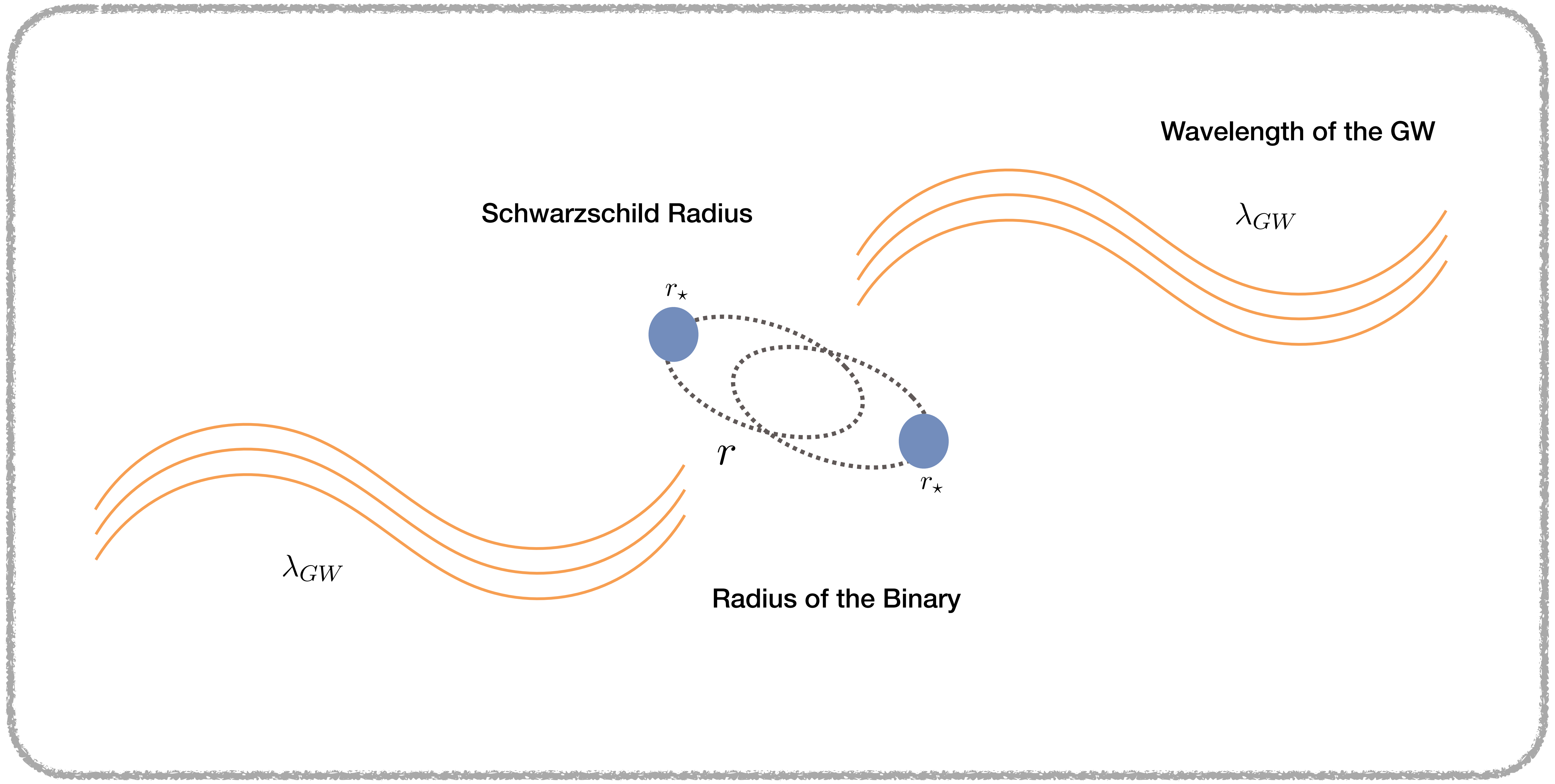
Better to handle spurious divergences

- Multi-loop Techniques



- ☑ IBP relations
- ☑ Differential Equations

Post-Newtonian Expansion EFT set up



Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

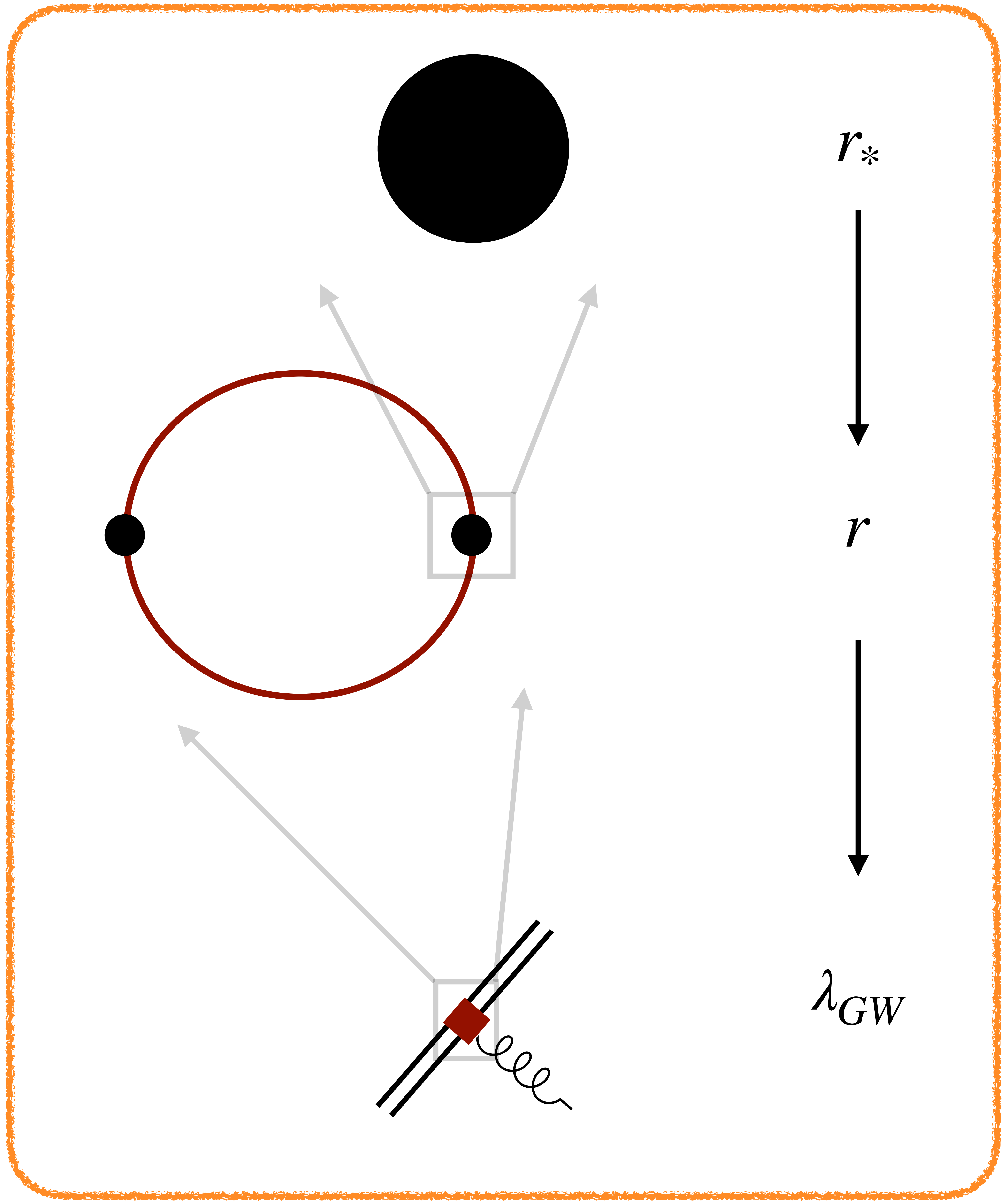
Post-Newtonian Expansion EFT set up

Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

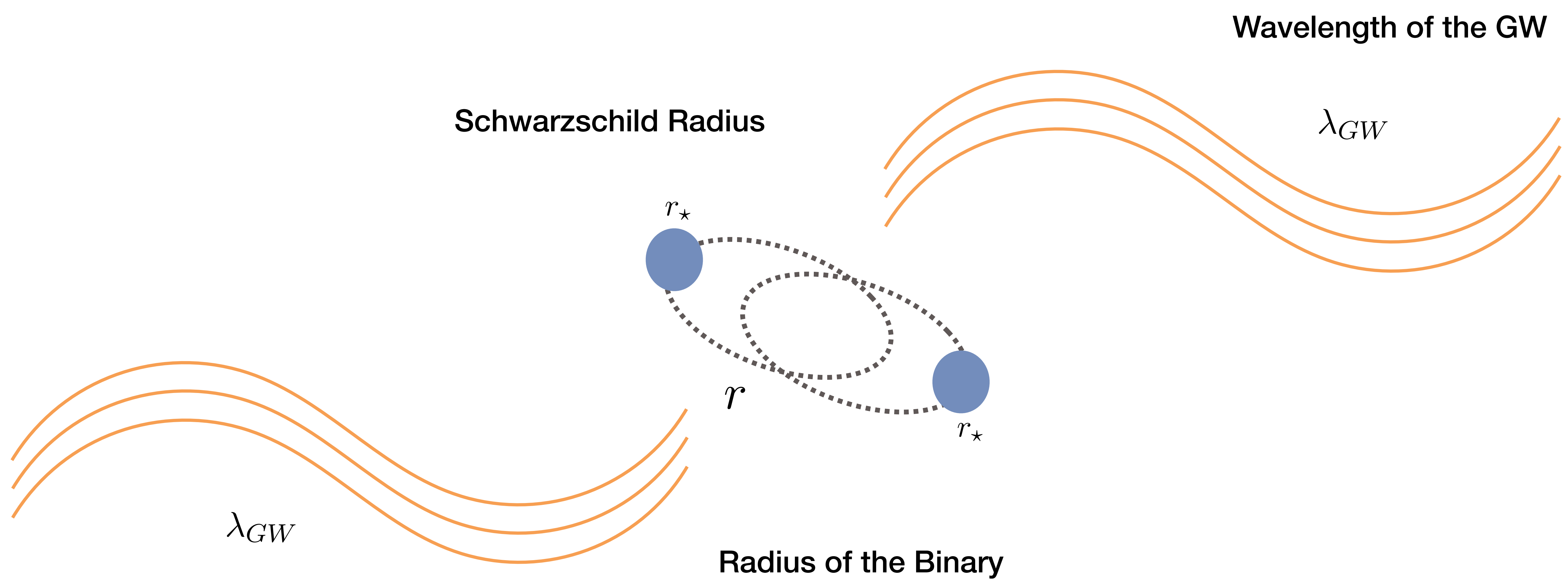
Tower of EFTs Goldberger, Rothstein

1. One-Particle EFT for Compact Object
2. EFT of Composite Particle for Binary
3. Effective Theory of Dynamical Multipoles



Post-Newtonian Expansion EFT set up

Goldberger, Rothstein



Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

Tower of EFTs

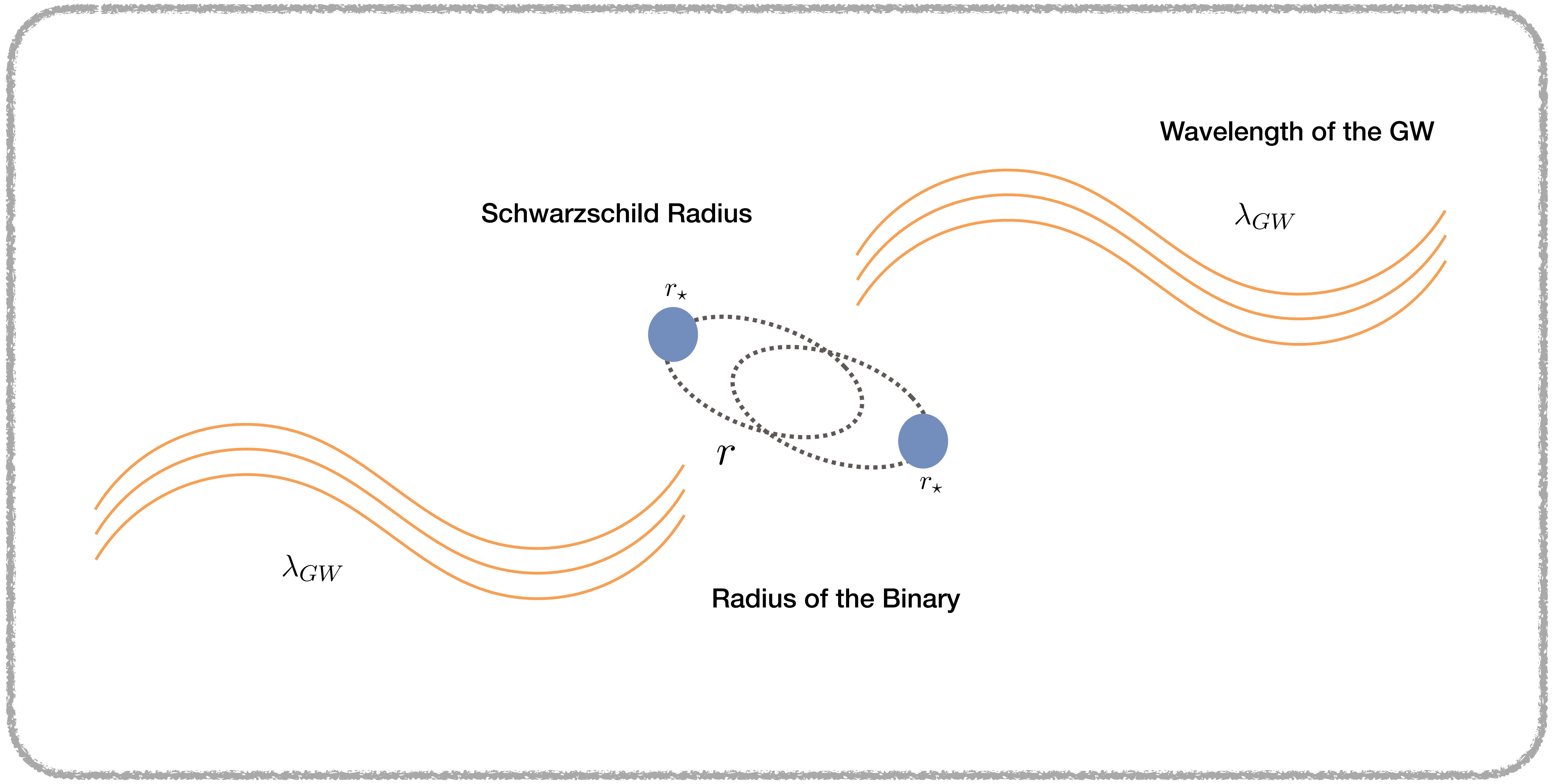
1. One-Particle EFT for Compact Object

$$S[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R$$

$$S_{pp}[g_{\mu\nu}] = -m\int d\sigma\sqrt{u^2}$$

Post-Newtonian Expansion EFT set up

Goldberger, Rothstein



Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

Tower of EFTs

2. EFT of Composite Particle for Binary

$$S[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R$$

$$g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + h_{\mu\nu}$$

$$S_{pp}[g_{\mu\nu}, x_K] = \sum_{K=1}^{2} - m_K \int d\sigma\sqrt{u_K^2}$$

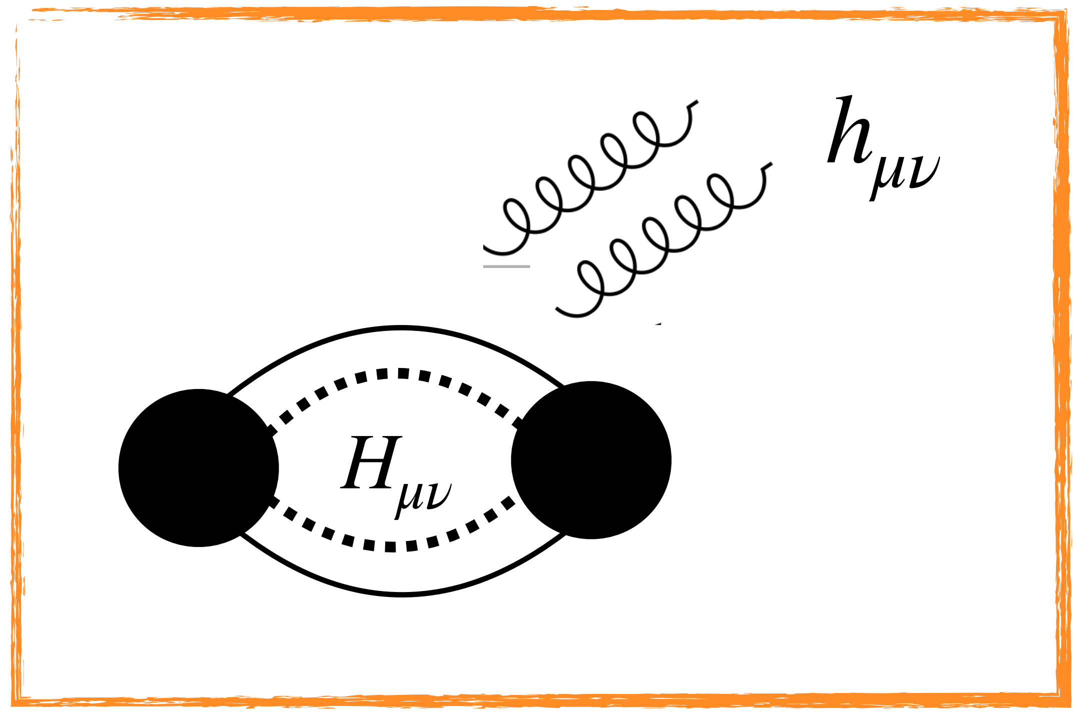
Method of Regions

potential gravitons $H_{\mu\nu}$ with scaling $(k_0, \mathbf{k}) \sim (v/r, 1/r)$

radiation gravitons $\bar{h}_{\mu\nu}$ with scaling $(k_0, \mathbf{k}) \sim (v/r, v/r)$

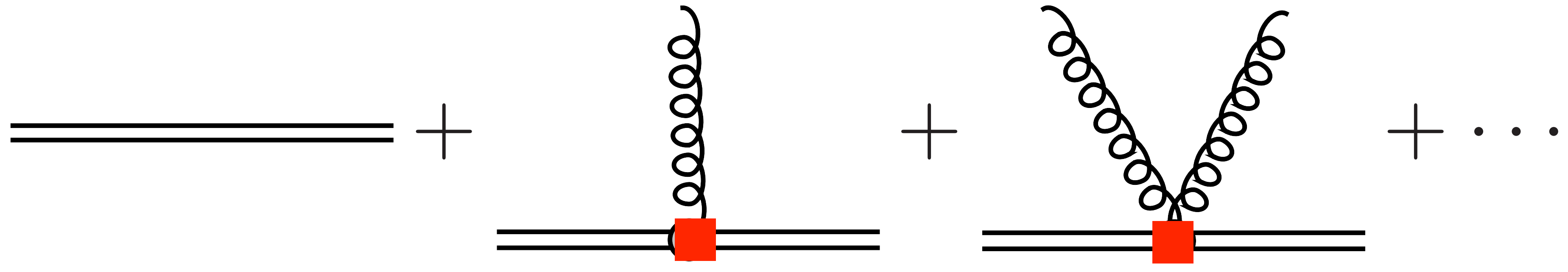
EFT at the orbital scale: Conservative Dynamics

$$e^{i S_{eff}[x_K]} = \int \mathcal{D}\bar{h}_{\mu\nu} \int \mathcal{D}H_{\mu\nu} \, \exp\left\{ iS[\eta + \bar{h} + H] + i \sum_{K=1}^{2} S_{pp}[x_K(t), \eta + \bar{h} + H] \right\}$$



Effective Action for Dynamical Multipoles

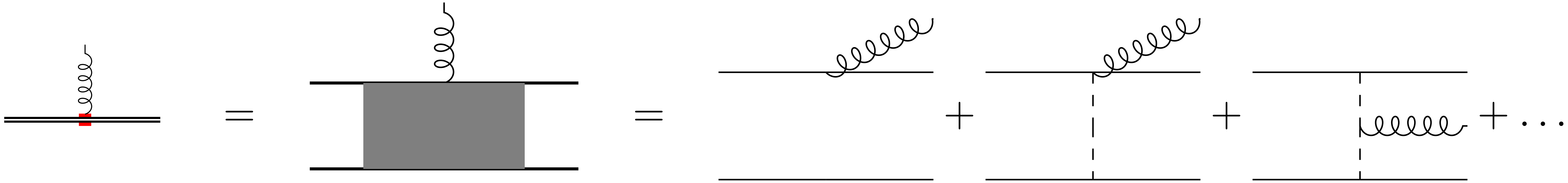
$$e^{i S_{eff}[x_K]} = \int \mathcal{D}\bar{h}_{\mu\nu} \, \exp\left\{ iS[\eta + \bar{h}] + \right.$$



$$\left. + \cdots \right\}$$

Conservative Dynamics

$$\int \mathcal{D}H \, \exp\left\{ i\, S\,[\eta + H, h = 0] + i\, S_{pp}\,[x_K, \eta + H, h = 0] \right\} = e^{i\, S_{eff}[h=0, x_K]} = e^{i \int dt\, \mathcal{L}_{eff}}$$

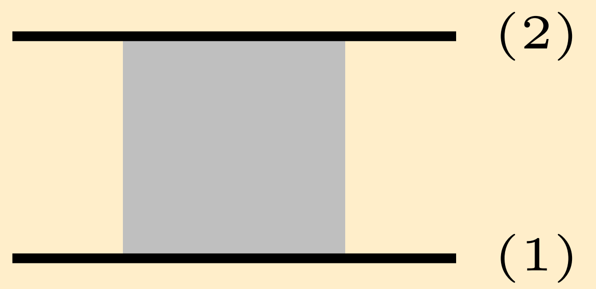


Potential for the 2-body system

Goldberger, Rothstein, Porto, Levi, …

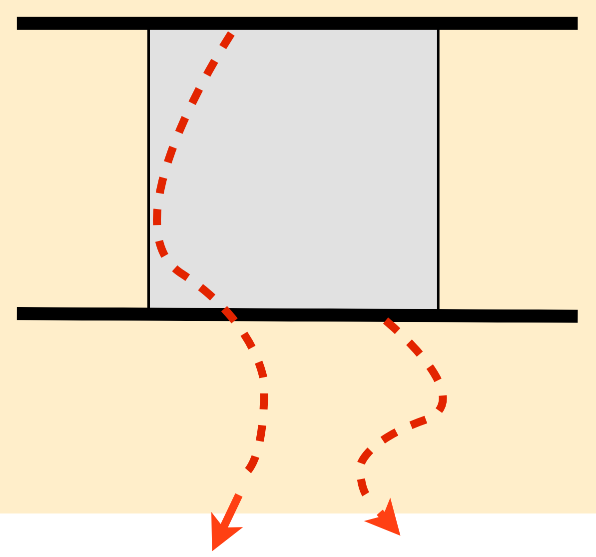
Foffa, Sturani, Sturm, Mastrolia (2016)

$$\mathcal{V}_{\rm eff} = \mathbf{i} \lim_{d \to 3} \int_{\mathbf{p}} e^{\mathbf{ip}\cdot(\mathbf{x}_{(1)} - \mathbf{x}_{(2)})}$$



Key Observation

Diagrams in GR



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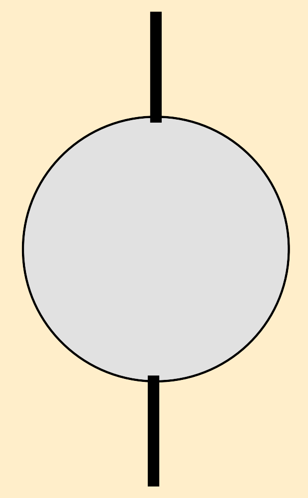
Diagrams in QFT



$p^2 \equiv s \neq 0$

Static Source :: Not Propagating

Status of PN Results

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	need up to
no spin	N	1PN		2PN		3PN		4PN		5PN		6PN		1PM
spin-orbit			LO SO		NLO SO		N2LO SO		N3LO SO		N4LO SO			2PM
spin^2				LO S2		NLO S2		N2LO S2		N3LO S2				3PM
spin^3							LO S3		NLO S3		N2LO S3			4PM
spin^4								LO S4		NLO S4				5PM
spin^5											LO S5		NLO S5	6PM
spin^6												LO S6		7PM

credit: Justin Vines

Levi, McLeod, Steinhoff, Teng, Von Hippel,..

Kim, Levi, Yin (2021)

Kim, Levi, Yin (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

Levi, Yin (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

1PN [Einstein, Infeld, Hoffman '38].

2PN [Ohta *et al.*, '73].

3PN [Jaranowski, Schaefer, '97; Damour, Jaranowski, Schaefer, '97; Blanchet, Faye, '00; Damour, Jaranowski, Schaefer, '01]

4PN [Damour, Jaranowski, Schäfer, Bernard, Blanchet, Bohe, Faye, Marsat, Marchand, Foffa, Sturani, Mastrolia, Sturm, Porto, Rothstein...]

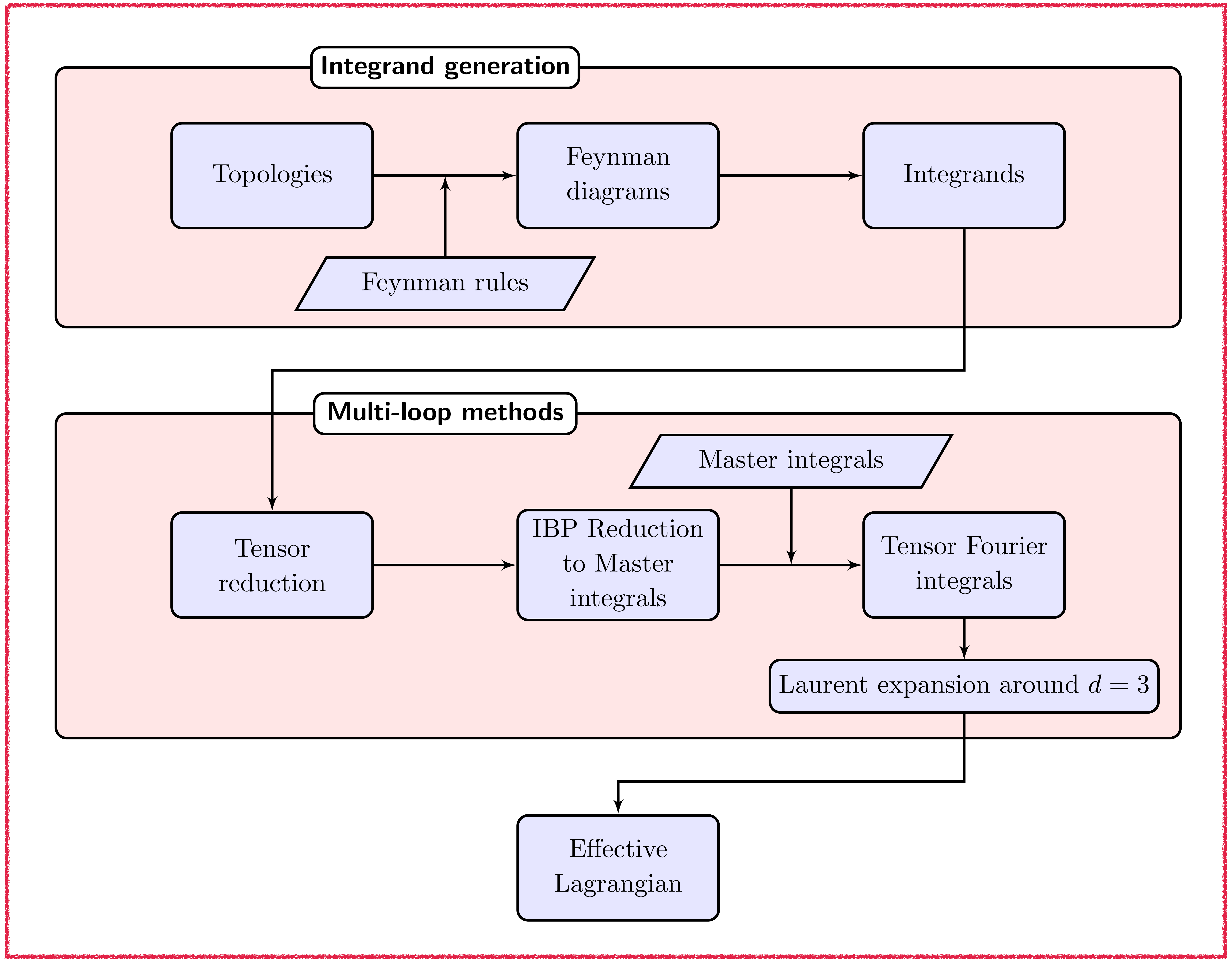
5PN [Foffa, Mastrolia, Sturani, Sturm, Bodabilla, '19; Blümlein, Maier, Marquard, '19; Bini, Damour, Geralico, '19; Blümlein, Maier, Marquard, '19; Almeida, Foffa, Sturani, '22;]

Computational Algorithm : Towards Automation

MKM, Mastrolia, Patil, Steinhoff (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

MKM, Mastrolia, O Silva, Patil, Steinhoff (2023)



☑ Automated in-house codes

Aim to publish the code in future

Diagrams for Spinning Binaries

MKM, Mastrolia, Patil, Steinhoff (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

S^0

Order	Diagrams	Loops	Diagrams
0PN	1	0	1
1PN	4	1	1
		0	3
2PN	21	2	5
		1	10
		0	6
3PN	130	3	8
		2	75
		1	38
		0	9

(a) Non-spinning sector

S^1

Order	Diagrams	Loops	Diagrams
LO	2	0	2
NLO	13	1	8
		0	5
N^2LO	100	2	56
		1	36
		0	8
N^3LO	894	3	288
		2	495
		1	100
		0	11

(b) Spin-orbit sector

S^2

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	7	1	3
		0	4
N^2LO	58	2	27
		1	24
		0	7
N^3LO	553	3	125
		2	342
		1	76
		0	10

(a) Spin1-Spin2 and Spin1^2 (Spin2^2) sector

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	4	1	1
		0	3
N^2LO	25	2	7
		1	12
		0	6
N^3LO	168	3	15
		2	101
		1	43
		0	9

(b) ES^2 sector

Order	Loops	Diagrams
LO	1	1

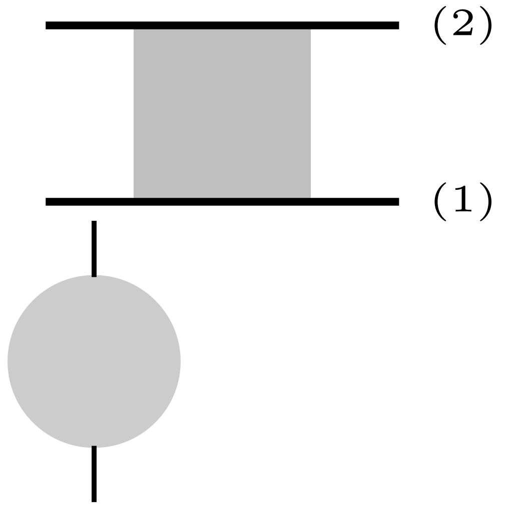
(c) E^2 sector

Order	Loops	Diagrams
LO	1	1

(d) E^2S^2 sector

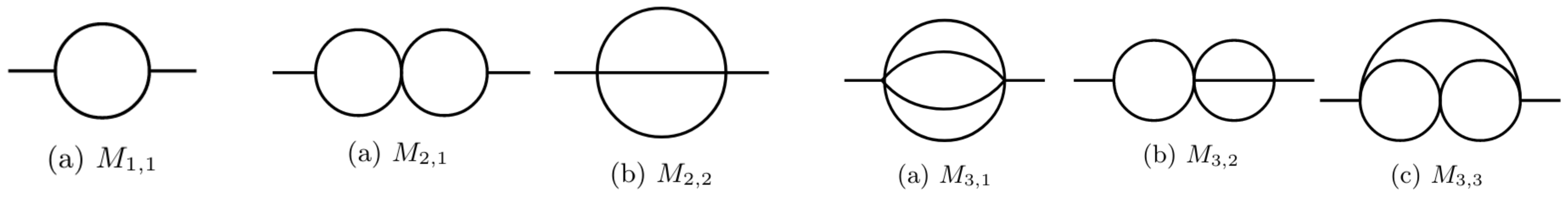
$$\mathcal{L}(x_a, \dot{x}_a, \ddot{x}_a, \ldots S_a, \dot{S}_a, \ddot{S}_a, \ldots) = \text{- i} \lim_{d\to 3} \int_{\mathbf{p}} e^{\mathbf{ip}\cdot(\mathbf{x}_{(1)} - \mathbf{x}_{(2)})}$$

$$= \text{- i} \lim_{d\to 3} \int_{\mathbf{p}} e^{\mathbf{ip}\cdot(\mathbf{x}_{(1)} - \mathbf{x}_{(2)})}$$



- Dimensional Regularization d= 3+ϵ
- IBP Decomposition

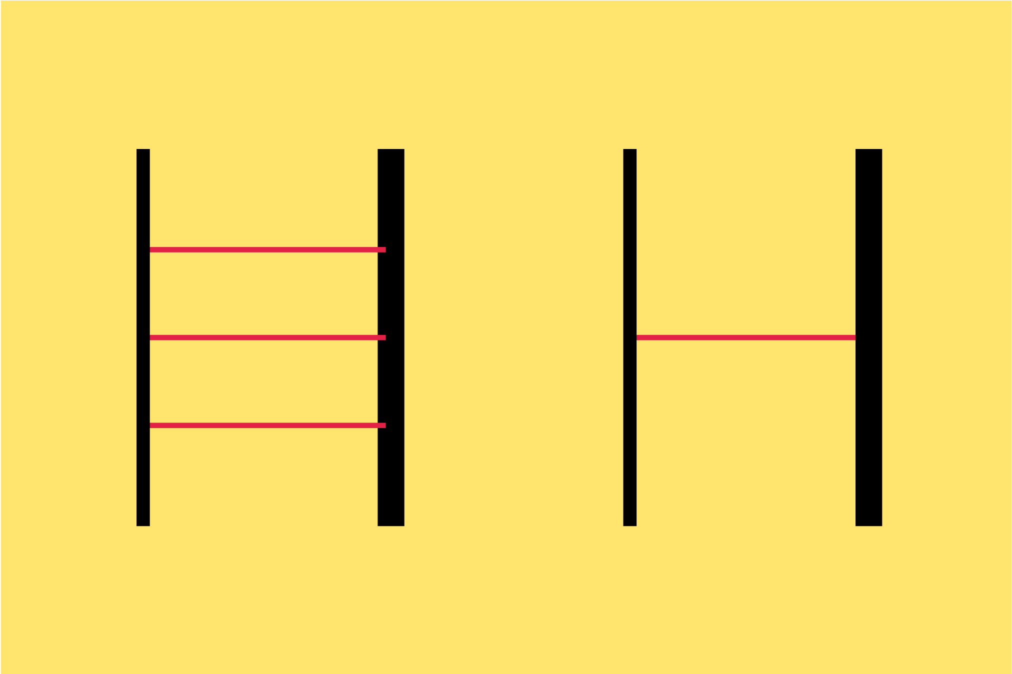
Three Loop MIs



Outlook

Collider Applications

Electron-Muon Scattering at MuonE



Mixed QCD-Electroweak Corrections

Mass effects for 4 / 5 point amplitudes at LHC

Radiation Reaction Potentials

Potentials

Power Loss / Flux

Gravitational Wave Physics

Tidal Effects

Waveforms

Spin effects

Conclusion

☑ Applications to Collider Physics

- ☑ muon-electron scattering at NNLO has been obtained
- ☑ top-pair production from quark annihilation has been computed analytically

☑ Applications to GW phenomenology

- ☑ progress in understanding spin effects / tidal effects for the compact binaries
- ☑ A number of observables e.g binding energy, scattering angle has been computed to high precision

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MathemAmpltudes 2023: QFT at the Computational Frontier

25–27 Sept 2023
Centro Universitario Padovano
Europe/Rome timezone

The workshop aims to explore the latest developments in the evaluation of Feynman integrals and Scattering Amplitude by applying advanced computer algebra and mathematical methods.

Recent studies connect Feynman Integrals and Scattering Amplitudes to concepts ranging from Differential and Algebraic Geometry, Number Theory, Combinatorics and Statistics. Pfaffian Equations, D-module theory, Stoke's theorem, Morse theory, Global Residue Theorem, Systems of linear equations, Finite Fields, Groebner bases, Tropical Geometry, Intersection Theory, to name a few, inspired the development of novel algorithms and software that pushed forward the computational frontier of scattering amplitudes, Feynman integrals, along with Euler-Mellin integrals and GKZ systems.

This initiative aims at bringing together mathematicians and theoretical physicists, interested in computational aspects of algebraic geometry and quantum field theory with the goal of proposing new, interdisciplinary research directions.

Confirmed Speakers

- Souvik Bera
- Zvi Bern
- Michael Borinski
- Seva Chestnov
- Giulio Crisanti
- Giuseppe De Laurentis
- Lance Dixon
- Ekta Ekta
- Claudia Fevola
- Gaia Fontana
- Federico Gasparotto
- Gudrun Heinrich
- Martijn Hidding
- Tobias Huber
- Harald Ita
- David Kosower
- Xiao Liu
- Yan-Qing Ma
- Andrzej Pokraka
- Simon Telen
- William Torres Bobadilla
- Johann Usovitsch
- Andreas Von Manteuffel
- Mao Zeng
- Yang Zhang
- Simone Zoia

Organizers

Hjalte Frellesvig

Ramona Groeber

Daniel Maitre

Manoj K. Mandal

Pierpaolo Mastrolia

Tiziano Peraro

Thank You