Gauge Invariance and γ_5 in Chiral Gauge Theories

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Motivations

- Electroweak corrections will become more and more relevant
- Question of principle: find 100% reliable results for chiral gauge theories
- Long term RG program has to start basically from scratch (low-hanging fruit)

Gauge Anomaly Cancellation

- Self-consistency (unitarity, physical dof, renormalizability) ➔ anomaly cancellation:

$$D^{abc} = \mathrm{tr}(T_L^a\{T_L^b, T_L^c\}) - \mathrm{tr}(T_R^a\{T_R^b, T_R^c\}) = 0\,.$$

Georgi-Glashow (1972)

- No new **relevant** anomalies emerge at non-renormalizable level. See, e.g., Gomis-Weinberg (1995)

In practical perturbative calculations, however, Gauge Invariance is <u>explicitly</u> broken:

- by **gauge fixing**

- by **regularization** (action and/or measure not invariant)

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- by **gauge fixing:** BRST symmetry ➔ self-consistent

- by **regularization:** unphysical ➔ must be removable by counterterms

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- ⁕ by **gauge fixing:** BRST symmetry ➔ self-consistent

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↕

Sometimes it is not possible to find a regulator that respects all symmetries.
In Dimensional-Regularization breaking is unavoidable if the theory is chiral (Standard Model).

$\Longrightarrow$ **we must add counterterms to amplitudes!**

Dimensional Regularization
and the BMHV scheme

't Hooft-Veltman (1972)
Breitenlohner-Maison (1977)

Dimensional Regularization

4-dimensional (d-4)-dimensional

Recipe:

— Space-time dimension continued to (complex) d. Coordinates split $\mu = \bar{\mu} \oplus \hat{\mu}$
— Kinetic terms (propagators) promoted to d-dimensions ➔ UV convergence
— Interactions (vertices) are scheme-dependent: just need to reduce to the familiar 4-dim theory
— Regularized bosonic part can respect all 4-dim symmetries (we make natural choice).
— **Regularized fermionic action cannot respect the 4-dim chiral symmetries…**

Chirality?

There is no notion of chirality in arbitrary d-dimensions

➔ Chirality-projectors are trivial

➔ The usual 4-dim relations must become inconsistent

$$\left.\begin{array}{r}\{\gamma^\mu, \gamma_5\} = 0 \\ \text{Cyclicity of the trace}\end{array}\right\} \Rightarrow$$

All traces with one γ_5 vanish: cannot have

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = 4i\epsilon^{\mu\nu\rho\sigma}$$

’t Hooft-Veltman:
— Cyclicity of trace is sacred (no anti-commuting ɣ5)
— Chirality (and hence Levi-Civita) is a purely 4-dimensional concept

$$\gamma_5 = -\frac{i}{4!}\epsilon_{\bar{\mu}\bar{\nu}\bar{\alpha}\bar{\beta}}\gamma^{\bar{\mu}}\gamma^{\bar{\nu}}\gamma^{\bar{\alpha}}\gamma^{\bar{\beta}} \quad \Longrightarrow \quad \begin{cases} \{\gamma^{\bar{\mu}}, \gamma_5\} = 0 \\ [\gamma^{\hat{\mu}}, \gamma_5] = 0 \end{cases}$$

Breitenlohner-Maison:
— Algebraically consistent <u>definitions</u>
— Allows to identify an unambiguous scheme <u>at all orders</u>

Alternatives?

No alternative prescription has ever been proven consistent

— **BMTV** based on a definition of γ_5, the other prescriptions DO NOT
— All alternatives have ambiguities
Example in naive anti-commuting approach:
Yukawa coupling at 3-loops and QCD gauge coupling at 4-loops in the SM have ambiguity

Bednyakov, A. F. Pikelner and V. N. Velizhanin (2014), F. Herren, L. Mihaila and M. Steinhauser (2017), Florian Herren and Anders Eller Thomsen (2021)

Current Status

- <u>Full 2-loop counterterms</u> (finite and divergent) **for single fermion chirality** (Algebraic method, BRST) Stockinger et al (2021)

- <u>Finite 1-loop counterterm</u> (and divergent) **for general gauge theories (SM)** (Background field method) Cornella et al (2022)

- <u>2-loop beta functions and anomalous dimensions</u> **for U(1) gauge** (Algebraic method Vs Standard Method) BeluscaMaito (2022)

A low-hanging Program…

- ❑ Reduction to scalar integrals
- ❑ Decompositions into master integrals
- ❑ 2-loop anomalous dimensions and beta functions
- ❑ ….

General Chiral Theories in Background Field Gauge with BMHV

Cornella et al (2022)

Consider the following 4-dimensional theory

$$S_{\text{Fermions}} = \int d^4x\, \overline{\Psi} i\gamma^\mu [\partial_\mu - iA_\mu^a T^a]\Psi$$

Arbitrary compact gauge theory: product of U(1)'s and simple factors.
Arbitrary fermion content: LH and RH charged under different (reducible) representations

$$T^a = T_L^a P_L + T_R^a P_R, \qquad [T^a, T^b] = if^{abc}T^c \qquad P_L = \frac{1}{2}(1-\gamma_5) \qquad P_R = \frac{1}{2}(1+\gamma_5)$$

Same for LH and RH generators

Chiral projectors

Regularized version in Dim-Reg with BMHV:

— Kinetic term must be promoted to d-dimensions.

Denominator of propagator falls off as p^2 in any d-direction

$$S = \int d^4x\, \overline{\Psi} i\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\Psi + \cdots \to S^{(d)} = \int d^dx\, \overline{\Psi} i\gamma^{\mu}\partial_{\mu}\Psi + \cdots$$

— Interaction: a lot of freedom, just needs to recover the familiar 4-dim limit

$$J^{\mu}_{L} = \overline{\Psi}\gamma^{\mu}P_L\Psi \quad \text{or} \quad \overline{\Psi}P_R\gamma^{\mu}\Psi \quad \text{or} \quad \overline{\Psi}P_R\gamma^{\mu}P_L\Psi \ ???$$

Chiral projectors defined as in d=4

$$\boxed{J^{\mu}_{L} \equiv \overline{\Psi}P_R\gamma^{\mu}P_L\Psi = \overline{\Psi}\gamma^{\bar{\mu}}P_L\Psi = [J^{\mu}_{L}]^{\dagger}}$$

Our choice:
It is hermitian (unitarity retained by regulator)
It minimizes the spurious anomaly.
Hermitian ➔ 4-dimensional!

Our regularized fermion action finally reads (= choice of scheme):

d-dimensional

$$S^{(d)}_{\rm Fermions} \equiv \int d^dx \left[\overline{\Psi} i\gamma^{\mu}\partial_{\mu}\Psi + \overline{\Psi}\gamma^{\bar{\mu}}A^a_{\bar{\mu}}T^a\Psi\right]$$

4-dimensional

$$= \int d^dx \left[\overline{\Psi} i\gamma^{\bar{\mu}}(\partial_{\bar{\mu}} - iA^a_{\bar{\mu}}T^a)\Psi + \overline{\Psi} i\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\Psi\right]$$

Conserved global symmetries:

— **SO(1,3)xSO(d-4)** → no need of Lorentz-restoring counterterms
— **CP**
— **Spurious P** (under which generators transform)
— **Vector-like rotations**
— **Chiral rotations (even non-abelian) are classically anomalous!**

Our regularized fermion action finally reads (choice of scheme):

$$S^{(d)}_{\text{Fermions}} \equiv \int d^d x \left[\overline{\Psi} i\gamma^{\mu}\partial_{\mu}\Psi + \overline{\Psi}\gamma^{\bar{\mu}} A^a_{\bar{\mu}} T^a \Psi\right]$$

$$= \int d^d x \left[\overline{\Psi} i\gamma^{\bar{\mu}}\left(\partial_{\bar{\mu}} - iA^a_{\bar{\mu}}T^a\right)\Psi + \overline{\Psi} i\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\Psi\right]$$

Local symmetries?

They must be defined in d-dimensions…
We declare they are purely 4-dimensional.
— This way vector-like symmetries are preserved.
— Axial symmetries are broken….

$$U = e^{i\alpha^a(\bar{x})T^a} \quad \rightarrow \quad \begin{cases} \Psi \to U\Psi \\ \overline{\Psi} \to \overline{\Psi}\gamma_0 U^{\dagger}\gamma_0 \\ A_{\bar{\mu}} \to U A_{\bar{\mu}} U^{\dagger} - iU\partial_{\bar{\mu}} U^{\dagger} \\ A_{\hat{\mu}} \to U A_{\hat{\mu}} U^{\dagger} \end{cases}$$

Classical anomaly!

$$\delta_\alpha S^{(d)} \equiv \int d^dx\ \alpha_a(\bar{x}) L_a S^{(d)} = -\int d^dx\ \alpha_a(\bar{x}) \overline{\Psi}(T_R^a - T_L^a)\gamma_5 \gamma^{\hat{\mu}} \partial_{\hat{\mu}} \Psi$$

Small parameter

Generator of infinitesimal gauge transformations of fields

Unavoidable: d-dim kinetic term mixes L with R → explicit breaking of chiral symmetry.
Evanescent: the anomaly must vanish as d → 4.
Minimality: our assumptions lead to minimal, irreducible anomaly (practical utility).
Generality: same anomaly found including Yukawas (result is general at ren. level).

Quantum Symmetries:
Spurious anomalies and Counterterms

Quantum Symmetries in Dim-Reg

$$e^{i\Gamma[\phi_c]} = \int_{\rm 1PI} \mathcal{D}\phi \; e^{iS[\phi+\phi_c]}$$

1) In Dim-Reg <u>the measure is invariant</u> under local transformations of fields since δ(0)=0:

$$\mathcal{D}\phi' = e^{i\delta^{(d)}(0)\int d^d x\, f(x)}\mathcal{D}\phi = \mathcal{D}\phi$$

2) The transformation of the background is given by:

$$e^{i\Gamma[\phi_c']} = \int_{\rm 1PI} \mathcal{D}\phi \; e^{iS[\phi+\phi_c']} = \int_{\rm 1PI} \mathcal{D}\phi' \; e^{iS[\phi'+\phi_c']} = \int_{\rm 1PI} \mathcal{D}\phi \; e^{iS[\phi'+\phi_c']}$$

At infinitesimal level the variation of the 1PI effective action is given by the matrix elements of the classical anomaly (Quantum Action Principle)

$$L_a^{\mathrm{bckgrd}} \Gamma[\phi_c] = \frac{\int_{\mathrm{1PI}} \mathcal{D}\phi \; e^{iS[\phi+\phi_c]} L_a S[\phi+\phi_c]}{\int_{\mathrm{1PI}} \mathcal{D}\phi \; e^{iS[\phi+\phi_c]}}$$

Infinitesimal transformation of background fields

Infinitesimal transformation of quantum+classical fields

— <u>Symmetries of the classical action hold at all orders</u> (4-dim Lorentz, vector-like, CP, P).
— What happens to anomalous symmetries?
<u>Spurious</u> (gauge, non-abelian axial) <u>or Physical</u> (abelian axial, scale invariance)

Gauge anomaly can be removed by a local counterterm, order by order.

That is, we can find S_{ct} such that $L^{\text{bckgrd}} S_{\text{ct}} = -L^{\text{bckgrd}} \Gamma$ and so

$$\begin{cases} \Gamma_{\text{inv}}^{(n)} \equiv \Gamma^{(n)} + S_{\text{ct}}^{(n)} \\ L^{\text{bckgrd}} \Gamma_{\text{inv}}^{(n)} = 0 \end{cases}$$

Theorem (anomaly of non-abelian global symmetries):

If the **renormalized** 1PI effective action is symmetric up to order (n-1) in $\hbar$

then the anomaly is the variation of a local functional ➔ it is spurious.

Proof:

Casimir
— invertible because anomaly is non-trivial
— trivial for abelian symmetries (axial, scale-inv.)

$$\left.\begin{array}{l} L_a\Gamma^{(n)} = \mathcal{A}_a^{(n)} \\ [L_a, L_b] = i f_{abc} L_c \end{array}\right\} L_a\mathcal{A}_b^{(n)} - L_b\mathcal{A}_a^{(n)} = i f_{abc}\mathcal{A}_c^{(n)} \quad \Rrightarrow \quad L^2\mathcal{A}_a^{(n)} = L_a(L_b\mathcal{A}_b^{(n)})$$

$$\Rrightarrow \quad \mathcal{A}_a^{(n)} = L_a\Big[\underbrace{L^{-2}L_b\mathcal{A}_b^{(n)}}_{-S_{\rm ct}^{(n)}}\Big]$$

It is local at each order!

Result:
At order n the spurious anomaly can be removed by an appropriate counterterm.

$$\Gamma_{\rm inv}^{(n)} \equiv \Gamma^{(n)} + S_{\rm ct}^{(n)}$$

Gauge theories are self-consistent as long as

$$D^{abc} = \mathrm{tr}(T_L^a\{T_L^b, T_L^c\}) - \mathrm{tr}(T_R^a\{T_R^b, T_R^c\}) = 0\,.$$

Georgi-Glashow (1972)

No new anomalies emerge in perturbation theory (even beyond renormalizable). See, e.g., Gomis-Weinberg (1995)

Luscher (1999)

Breaking due to Dim-Reg is artificial $\nRightarrow$ the anomaly can be removed via counterterms.

Tonin et al. (1977)

Explicit form of the Counterterm?
Background Field Method:
1-loop results

Gauge theories: Background Field Method

See, e.g. Abbott

$$e^{i\Gamma[\phi_c]} = \int_{1\mathrm{PI}} \mathcal{D}\phi \; e^{iS[\phi+\phi_c]}$$

— Split physical fields in bckgrnd + quantum fluctuations.
— Gauge-fixing can be chosen to preserve bckgrnd gauge-invariance:
— Then: the symmetry acts linearly on the 1PI effective action ➔ easier.

$$\mathcal{L}_{\mathrm{g.f.}} = -\frac{1}{2\xi}(D^\mu_c A_\mu)_a (D^\mu_c A_\mu)_a$$

Alternatively:
Gauge-fixing leaves BRST ➔ (non-linear) Slavnov-Taylor Identities.

Martin-SanchezRuiz (2000)
SanchezRuiz (2003)
BeluscaMaito et al. (2020-2021)

Chiral gauge theories at 1-loop

Use MSbar: introduce divergent counterterms (symmetric & evanescent non-symmetric!) to make the regularized 1PI effective action finite. At 1-loop:

$$e^{i\Gamma^{(d)}_{\text{fin},1}[\phi_c]} = \int_{\text{1PI}} \mathcal{D}\phi \; e^{iS^{(d)}[\phi+\phi_c]+iS^{(d)}_{\text{div},1}[\phi+\phi_c]}$$

$$L_a^{\text{bckgrd}}\Gamma^{(d)}_{\text{fin},1} = \frac{\int_{\text{1PI}} \mathcal{D}\phi \; e^{iS^{(d)}+iS^{(d)}_{\text{div},1}} L_a \left[S^{(d)} + S^{(d)}_{\text{div},1}\right]}{\int_{\text{1PI}} \mathcal{D}\phi \; e^{iS^{(d)}+iS^{(d)}_{\text{div},1}}}$$

$$= \langle L_a S^{(d)} \rangle_1 + L_a S^{(d)}_{\text{div},1}$$

Cancels the divergent part of the 1-loop matrix element of classical anomaly.

$$= \langle L_a S^{(d)} \rangle_1 \Big|_{\text{finite part}}$$

Combining evanescent & divergent (local) we get a local finite quantum anomaly.

In summary, at 1-loop:
— the quantum anomaly is given by the finite part of the matrix element of the classical anomaly
— it is local because LS is evanescent: to survive in 4-dim it must be multiplied by a divergence.

$$L_a^{\mathrm{bckgrd}} \Gamma^{(d)}_{\mathrm{fin},1} = \left. \frac{\int_{\mathrm{1PI}} \mathcal{D}\phi \; e^{iS^{(d)}} \, L_a S^{(d)}}{\int_{\mathrm{1PI}} \mathcal{D}\phi \; e^{iS^{(d)}}} \right|_{\mathrm{finite}}$$

Note it is trivial to automatize:
Introduce η Anomaly=η LS as a new vertex and evaluate finite part of diagrams with 1 external η.

The explicit form of the gauge-restoring counterterm (up to gauge-invariant terms) is:

$$\mathcal{L}_{\rm ct}\big|_{(1)} = \frac{\epsilon^{\mu\nu\alpha\beta}}{16\pi^2}{\rm Tr}\left\{\frac{8}{3}\partial_\mu \mathcal{V}_\nu \left\{\mathcal{V}_\alpha, \mathcal{A}_\beta\right\} + 4i\mathcal{V}_\mu\mathcal{V}_\nu\mathcal{V}_\alpha\mathcal{A}_\beta + \frac{4}{3}i\mathcal{V}_\mu\mathcal{A}_\nu\mathcal{A}_\alpha\mathcal{A}_\beta\right\}$$

$$+\frac{1}{16\pi^2}{\rm Tr}\left\{-\frac{4}{3}(D^{\mathcal{V}}_\mu\mathcal{A}_\nu)^2 + 2(D^{\mathcal{V}}_\mu\mathcal{A}^\mu)^2 - \frac{4}{3}[\mathcal{A}_\mu,\mathcal{A}_\nu]^2 + \frac{4}{3}(\mathcal{A}_\mu\mathcal{A}_\nu)^2 + \mathcal{A}^2_{\mu\nu}\right\}$$

$$-\frac{2}{16\pi^2}\left(1+\frac{\xi-1}{6}\right)G_{aa}\bar{f}\gamma_5\gamma^\mu T^a\mathcal{A}_\mu T^a f$$

Holds for most general fermion + gauge theory

Very compact: CP and (spurious) P are manifest.

Cornella-Feruglio-LV (2022)

In the Standard Model:

— QCD & QED are vector-like and manifest

— no terms with Levi-Civita, peculiarity of SU(2)xU(1)

— Contains all interactions that respect QCD & QED but violate SU(2)xU(1)

VVDD: $D_\mu W^-_\nu D^\mu W^{+\nu}$ $\quad \partial_\mu Z_\nu \partial^\mu Z^\nu$

VVVD: $iF^{\mu\nu} W^+_\mu W^-_\nu$ $\quad iD^\mu W^-_\mu W^+_\nu Z^\nu$ $\quad iD^\nu W^-_\mu W^+_\nu Z^\mu$ $\quad iD_\nu W^-_\mu W^{+\mu} Z^\nu$ $\quad$ +hc

VVVV: $(W^-_\mu W^{+\mu})^2$ $\quad (W^-_\mu W^{-\mu})(W^+_\nu W^{+\nu})$ $\quad (Z_\mu Z^\mu)^2$ $\quad (W^+_\mu Z^\mu)(W^-_\nu Z^\nu)$ $\quad (W^+_\mu W^{-\mu})(Z_\nu Z^\nu)$

ffW: $W^+_\mu \overline{f_u} \gamma^\mu P_L f_d$ $\quad W^+_\mu \overline{f_u} \gamma^\mu P_R f_d$ $\quad$ +hc

ffZ: $Z_\mu \overline{f} \gamma^\mu P_L f$ $\quad Z_\mu \overline{f} \gamma^\mu P_R f$ $\quad$ +hc

Finally…

Previous slide

$$L_a^{\text{bckgrd}}\Gamma^{(d)}_{\text{fin},1} = -L_a^{\text{bckgrd}} S^{(d)}_{\text{ct},1} + \text{finite evanescent}$$

$$e^{i\Gamma^{(d)}_{\text{fin,invariant},1}[\phi_c]} = \int_{1\text{PI}} \mathcal{D}\phi \; e^{iS^{(d)}[\phi+\phi_c]+iS^{(d)}_{\text{div},1}[\phi+\phi_c]+iS^{(d)}_{\text{ct},1}[\phi+\phi_c]}$$

At 1-loop
(local and global symmetries)

Renormalized

$$L_a^{\text{bckgrd}}\Gamma^{(4)}_{\text{fin,invariant},1} = +\frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\alpha\beta}\,\text{tr}\left(\left[T^a_L\left\{T^b_L, T^c_L\right\}\right] - \left[T^a_R\left\{T^b_R, T^c_R\right\}\right]\right)$$

Standard result (Bardeen)

Conclusions

- ☑ In concrete calculations, counterterms needed to restore chiral invariance
- ☑ **1-loop counterterm in dim-reg & BMHV for general fermionic reps**
  - (i) Non-trivial check of explicit calculations
  - (ii) Useful for automation
- ☐ Our results should be extended to Yukawa sector and SM-EFT: **in progress**.
- ☐ Inertia: for QED & QCD our renormalization functions are different!
- ☐ **RG equations, anomalous dimensions, etc all expected to differ!**
- ☐ …

Conceptual Issues

- Since "everybody else" uses "alternative approaches to γ_5"… lots of inertia!
- Must address conceptual questions on these "alternative approaches":
  (i) Ambiguity or inconsistency? An ambiguity may be resolved by a more complete prescription.
  (ii) If inconsistent, in which sense: Is it a different theory at $n>N$ loops? Is it flawed?
  (iii) If it is flawed for $n>N$ loops? Are $n<N$ loops correct? RG- and IR-resummation?
  (iv) Are there instances in which such approaches CANNOT give the right answer? (θ angle?)
- It would be very useful to study a toy model (in the SM we need 4-loops…)