Tropical Feynman Integration

Henrik J. Munch

Advanced Calculus for Fundamental Interactions

July 10 2023

Work in collaboration with

Michael Borinsky

Felix Tellander

- This talk is based on *Tropical Feynman integration in the Minkowski regime [2302.08955]*
- Continuation of research program started by M. Borinsky in *Tropical Monte Carlo quadrature for Feynman integrals [2008.12310]*

Motivation

From model to prediction

Quantum Field Theory model: $\mathcal{L} \sim -\frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{4!}\varphi^4$

$\downarrow$

Cross section: $\sigma \sim \int |\mathcal{A}|^2$

From model to prediction

Quantum Field Theory model: $\mathcal{L} \sim -\frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{4!}\varphi^4$

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Perturbative amplitude: $\mathcal{A} \sim \sum$ Feynman diagrams

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Quantum Field Theory model: $\mathcal{L} \sim -\frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{4!}\varphi^4$

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Bottleneck: Diagrams with several loops and many scales

$\downarrow$

Cross section: $\sigma \sim \int |\mathcal{A}|^2$

Computing loop diagrams

Analytic/symbolic approach

- Express solution via special functions

$$\Gamma(x),\ \log(x),\ \mathrm{Li}_2(x),\ K(x),\ {}_2F_1(x),\ \ldots$$

- While this is the ideal solution, it is generally very hard!

Numerical approach

- Input: Diagram + numerical phase space point
- Output: Numerical value for Feynman integral

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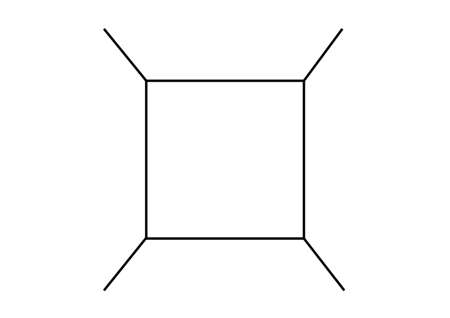
Feynman integrals

Loop diagram example



Feynman integral in momentum space:

$$\mathcal{I} \quad = \quad \int_{\mathbb{R}^{1,D-1}} \frac{\mathrm{d}^D k}{k^2 \ (k+p_1)^2 \ (k+p_1+p_2)^2 \ (k+p_1+p_2+p_3)^2}$$

Feynman representation

- Numerical integration is easier in Feynman representation
- Given a Feynman loop diagram $G \implies$ two polynomials $\mathcal{U}(x), \mathcal{F}(x)$
- Feynman integral becomes

$$\mathcal{I} \quad " = " \quad \int_0^\infty \mathcal{U}(x)^{a(D)} \mathcal{F}(x)^{b(D)} \mathrm{d}^n x$$

$a(D)$ and $b(D)$ depend on spacetime dimension D

- Dimensional regularization: $D = D_0 - 2\epsilon, \quad \epsilon \ll 1$

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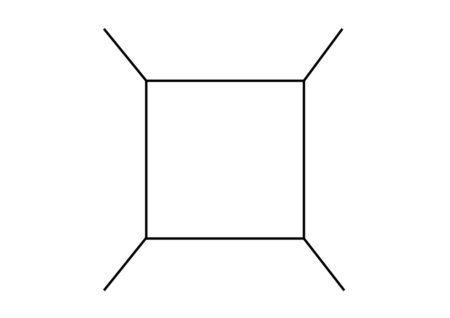

Feynman representation

$$\mathcal{I} \quad = \quad \int_0^\infty \left(x_1 + x_2 + x_3 + x_4\right)^{a(D)} \left(s\,x_1 x_3 + t\,x_2 x_4\right)^{b(D)} \mathrm{d}^4 x$$

Main goal for the rest of the talk

- Want fast numerical evaluation of the Feynman integral

$$\mathcal{I} = \int_0^\infty \mathcal{U}(x)^{a(D)} \mathcal{F}(x)^{b(D)} \mathrm{d}^n x$$

- More precisely, for amplitude calculations we only need a few orders in ϵ:

$$\mathcal{I} = \sum_k \mathcal{I}_k \, \epsilon^k$$

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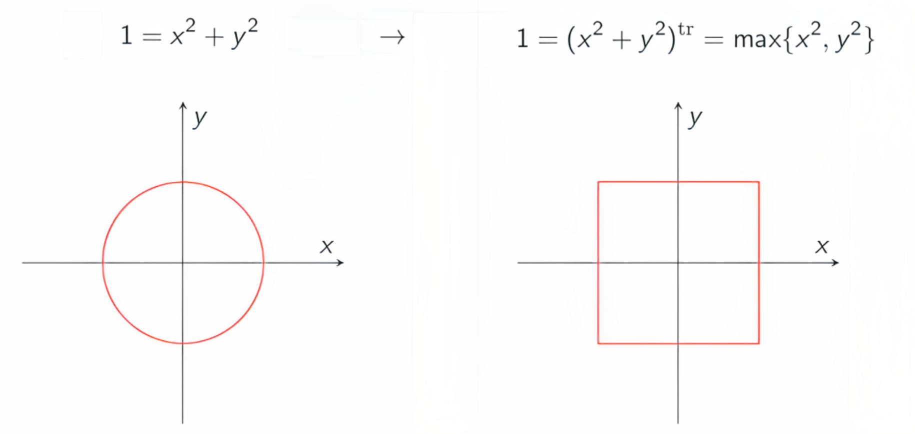
Tropical integration

Tropical geometry

Idea: Deform "smooth" geometries into "flat" pieces



Tropicalized polynomials

- Polynomial

$$p(x) = \sum_a c_a \, x^a \,, \quad x^a = x_1^{a_1} \cdots x_n^{a_n}$$

- Tropicalized version ignores coefficients and only cares about the largest monomial:

$$p^{\mathrm{tr}}(x) = \max_a \{x^a\}$$

- Theorem [Borinsky]: There exist constants $C_1, C_2 > 0$ such that

$$C_1 \leq \frac{|p(x)|}{p^{\mathrm{tr}}(x)} \leq C_2 \quad \text{for all} \quad x$$

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Tropical Monte Carlo

- Tropicalize the graph polynomials:

$$\mathcal{U}^{\mathrm{tr}}(x) = \max_a \{x^a\}\,, \quad \mathcal{F}^{\mathrm{tr}}(x)$$

- Monte Carlo sampling from the tropical probability measure

Tropical Monte Carlo

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$$\mathcal{U}^{\mathrm{tr}}(x) = \max_a \{x^a\}\,, \quad \mathcal{F}^{\mathrm{tr}}(x)$$

- Take the Feynman integral

$$\mathcal{I} = \int \mathrm{d}^n x\, \mathcal{U}(x)^{a(D)} \mathcal{F}(x)^{b(D)}$$

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- Tropicalize the graph polynomials:

$$\mathcal{U}^{\mathrm{tr}}(x) = \max_a \{x^a\}\,, \quad \mathcal{F}^{\mathrm{tr}}(x)$$

- Take the Feynman integral and multiply by a "1":

$$\mathcal{I} = \int \mathrm{d}^n x\, \mathcal{U}(x)^{a(D)} \mathcal{F}(x)^{b(D)} \;\times\; \left(\frac{\mathcal{U}^{\mathrm{tr}}(x)}{\mathcal{U}^{\mathrm{tr}}(x)}\right)^{a(D)} \left(\frac{\mathcal{F}^{\mathrm{tr}}(x)}{\mathcal{F}^{\mathrm{tr}}(x)}\right)^{b(D)}$$

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- Rearrange factors:

$$\mathcal{I} = \int \underbrace{\mathrm{d}^n x\, \mathcal{U}^{\mathrm{tr}}(x)^{a(D)}\ \mathcal{F}^{\mathrm{tr}}(x)^{b(D)}}_{\text{Tropical probability measure}} \times \underbrace{\left(\frac{\mathcal{U}(x)}{\mathcal{U}^{\mathrm{tr}}(x)}\right)^{a(D)} \left(\frac{\mathcal{F}(x)}{\mathcal{F}^{\mathrm{tr}}(x)}\right)^{b(D)}}_{\text{Bounded because } C_1 \leq \frac{|p(x)|}{p^{\mathrm{tr}}(x)} \leq C_2}$$

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Feynman's $i\varepsilon$ prescription

- Required for integration with momenta in Minkowski signature
- Infinitesimal $i\varepsilon$ not suited for numerics

- Consider the "Schwinger" representation:

$$\mathcal{I} = \int_0^\infty \ldots \exp\left[i\left(-\mathcal{F}(x)/\mathcal{U}(x) + i\varepsilon\sum_{m=1}^{n} x_m\right)\right]$$

- Can drop $i\varepsilon$ and still have convergence if

$$\mathrm{Im}[-\mathcal{F}(X)/\mathcal{U}(X)] > 0$$

- Idea: deformation of x_m such that the above becomes true

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$i\varepsilon$ via analytic continuation

- Analytic continuation [Mizera, Telen] [Hannesdottir, Mizera]

$$X_m = x_m \exp\left[-i\lambda \frac{\partial(\mathcal{F}/\mathcal{U})}{\partial x_m}\right]$$

- Taylor expansion in λ:

$$\begin{aligned} -\mathcal{F}(X)/\mathcal{U}(X) = &- \mathcal{F}(x)/\mathcal{U}(x) \\ &+ i\lambda \times \text{something positive} \\ &+ O(\lambda^2) \end{aligned}$$

- I.e. $\mathrm{Im}\left[-\mathcal{F}(X)/\mathcal{U}(X)\right] > 0,$ as wanted

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feyntrop package

feyntrop

- Tropical Monte Carlo algorithm implemented in `feyntrop`

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- C++ code with python interface
- Includes ✓
- Limitations ✗

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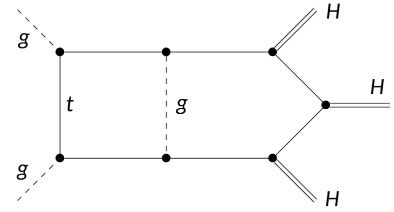
$gg \to HHH$ with internal top quark loop

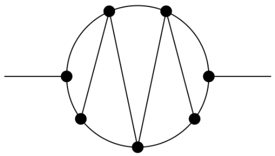


$$(m_H^2, m_t^2) = (1, 1.8995)$$

$$(s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}) = (17.5, -2.3, -2.4, -2.5, -2.6, -2.7)$$

```
* N = 10^10.
* Lambda = 0.29.
* Finished in 10.2 minutes.
-- eps^0: [0.000469553 +/- 0.000000063]  +  i * [-0.000721636 +/- 0.000000062]
-- eps^1: [0.00056055  +/- 0.00000025 ]  +  i * [ 0.00385763  +/- 0.00000024 ]
-- eps^2: [-0.00680350 +/- 0.00000054 ]  +  i * [-0.00516286  +/- 0.00000055 ]
-- eps^3: [0.01194325  +/- 0.00000089 ]  +  i * [-0.00211739  +/- 0.00000089 ]
-- eps^4: [-0.0064124  +/-  0.0000012 ]  +  i * [ 0.0109338   +/-  0.0000012 ]
```

5-loop 2-point graph with 11 different masses



$$p_1^2 = 100, \quad m_1^2 = 1, m_1^2 = 2, \ldots, m_{11}^2 = 11$$

```
* N = 10^12.
* Lambda = 0.02.
* Finished in 20 hours.
-- eps^0: [0.000196885 +/- 0.000000032]  +  i * [0.000140824 +/- 0.000000034]
-- eps^1: [-0.00493791 +/- 0.00000040 ]  +  i * [-0.00079691 +/- 0.00000038 ]
-- eps^2: [ 0.0491933  +/-  0.0000025 ]  +  i * [-0.0154647  +/-  0.0000025 ]
-- eps^3: [ -0.253458  +/-  0.000012  ]  +  i * [ 0.246827   +/-  0.000012  ]
-- eps^4: [ 0.587258   +/-  0.000046  ]  +  i * [ -1.720213  +/-  0.000046  ]
-- eps^5: [  1.05452   +/-   0.00015  ]  +  i * [  7.38725   +/-   0.00015  ]
-- eps^6: [ -14.66144  +/-   0.00047  ]  +  i * [ -20.86779  +/-   0.00046  ]
-- eps^7: [  65.8924   +/-   0.0013   ]  +  i * [  35.0793   +/-   0.0013   ]
-- eps^8: [ -190.9702  +/-   0.0036   ]  +  i * [  -4.4620   +/-   0.0034   ]
-- eps^9: [ 393.2522   +/-   0.0092   ]  +  i * [ -183.7431  +/-   0.0087   ]
-- eps^10:[ -558.202   +/-    0.023   ]  +  i * [  688.556   +/-    0.021   ]
```

Conclusion and outlook

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Conclusion:

- Efficient numerical evaluation of Feynman integrals via tropical Monte Carlo
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- Canonical choice of analytic continuation parameter λ
- Remove "quasi-finite" condition via the protocol of [Berkesch, Forsgaard, Passare]
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