The Eikonal Exponentiation in Gravity

Radiation and Tidal Effects

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Based on

P. Di Vecchia, C.H., R. Russo, G. Veneziano [2210.12118]

and on

C.H. [2210.15689]

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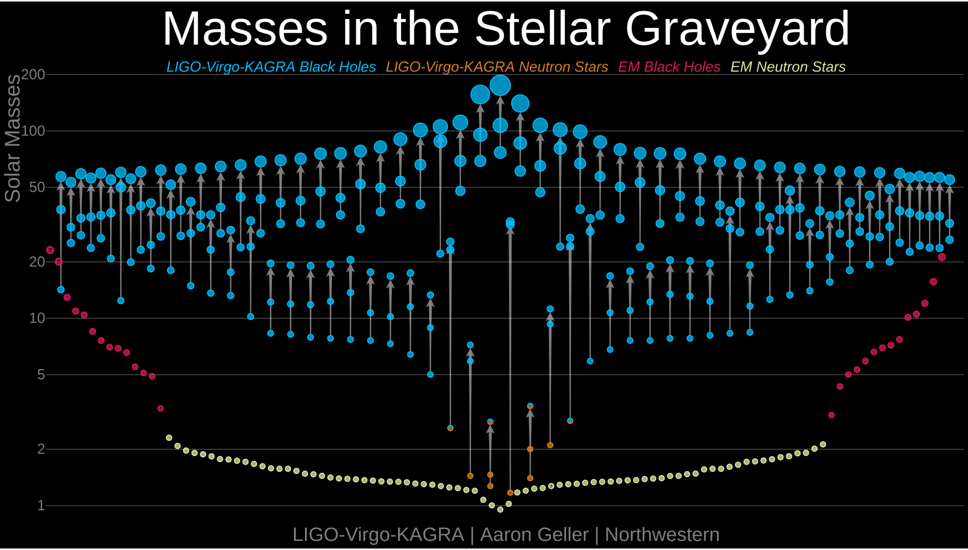
Outline

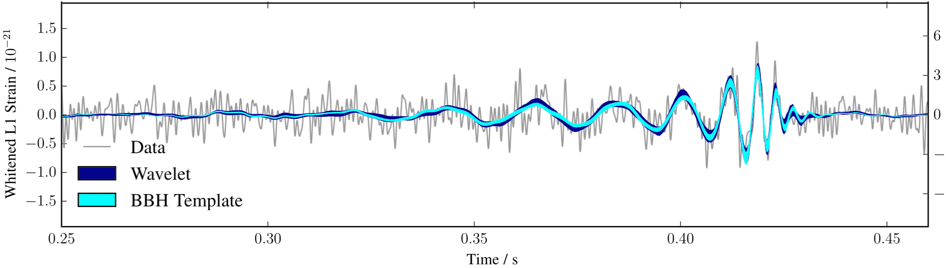
Outline

Gravitational Wave Astronomy



Waveform Templates

[LIGO Scientific Collaboration '16]



Inspiral — Weak gravity

Merger — Strong gravity

Ringdown — Small oscillations

Analytical Approximation Methods

- Post-Newtonian (PN): expansion "for small G and small v"

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1\,.$$

- Post-Minkowskian (PM): expansion "for small G"

$$\frac{Gm}{rc^2} \ll 1\,, \qquad \text{generic } \frac{v^2}{c^2}\,.$$

- Self-Force: expansion in the near-probe limit $m_2 \ll m_1$ or

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \ll 1\,.$$

General Relativity from Scattering Amplitudes

Idea

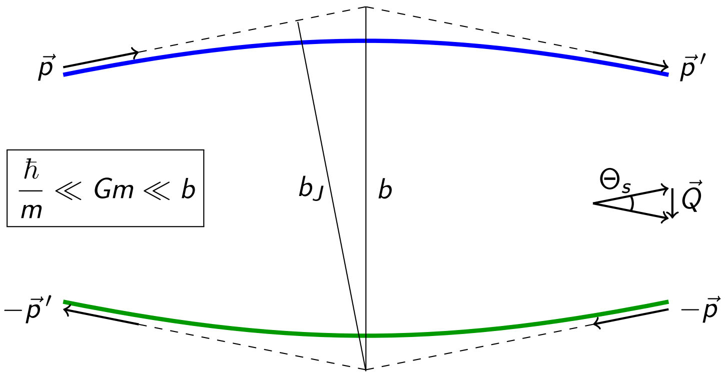
Extract the PM gravitational dynamics from scattering amplitudes.

- Weak-coupling expansion $\leftrightarrow$ PM expansion

Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\mathcal{A}_3 = \mathcal{O}(G^4)$
PM:	1PM	2PM	3PM This talk	4PM State of the art ...

- Lorentz invariance $\leftrightarrow$ generic velocities
- Study scattering events, then export to bound trajectories ($V_{\rm eff}$, analytic continuation...)

Post-Minkowskian (PM) Scattering



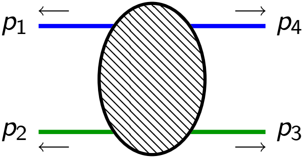
$$\frac{\hbar}{m} \ll Gm \ll b$$

$$Gm^2 \underset{CL}{\gg} \hbar\,, \qquad \frac{Gm}{b} \underset{\mathrm{PM}}{\ll} 1\,.$$

The Elastic Eikonal

[Di Vecchia, C.H., Russo, Veneziano 2104.03256] [and refs. therein]

$\mathcal{A}(s,t) =$



$$s = -(p_1 + p_2)^2 = E^2$$
$$= m_1 + 2m_1 m_2\, \sigma + m_2^2\,,$$
$$t = -(p_1 + p_4)^2 = -q^2\,.$$

- From q to b: Fourier transform

$$\tilde{\mathcal{A}}(b) = \int \frac{d^D q}{(2\pi)^{D-2}} \delta(2p_1 \cdot q)\delta(2p_2 \cdot q)\, e^{ib\cdot q} \mathcal{A}(s, -q^2)\,,$$

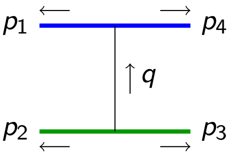
$$\boxed{1 + i\tilde{\mathcal{A}}(b) = e^{2i\delta(b)}} = e^{i(2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots)}$$

- From b to Q: stationary-phase approximation

$$\int d^D b\, e^{-ib\cdot Q} e^{i2\delta(b)} \implies \boxed{Q_\mu = \frac{\partial\, \mathrm{Re}\, 2\delta}{\partial b^\mu}} = Q_\mu^{1PM} + Q_\mu^{2PM} + Q_\mu^{3PM} + \cdots$$

Example: the 1PM Eikonal

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



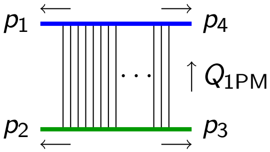
$$\mathcal{A}_0(s,q) = \frac{32\pi G m_1^2 m_2^2 \left(\sigma^2 - \frac{1}{2-2\epsilon}\right)}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0(s,b) = \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2-2\epsilon}\right)}{2\sqrt{\sigma^2-1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} .$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al.'18]

$$e^{2i\delta_0} \underset{\text{“small } G\text{”}}{\to} 1 + i\tilde{\mathcal{A}}_0 \implies 2\delta_0 = \tilde{\mathcal{A}}_0 .$$

- From $Q = \partial_b 2\delta$, we obtain the leading-order deflection

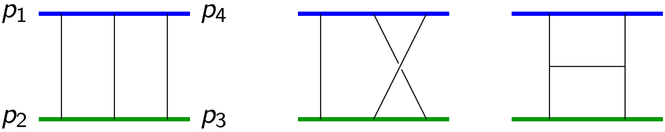


$$Q_{1PM} = \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2-1}}$$

$$\Theta_{1PM} = \frac{4 G E \left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2-1)} .$$

Two-Loop $\mathcal{N} = 8$ Supergravity

- Tackle the problem in a theory with a simpler amplitude integrand.
- Topologies entering the two-loop $\mathcal{O}(G^3)$ calculation (+ crossed topologies): [Caron-Huot, Zahraee '18; Parra-Martinez, Ruf, Zeng '20]

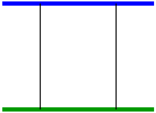


- Same families of integrals as in the GR case.
- We can calculate them via
  Method of Regions + Master Integrals + Differential Equations
  [Parra-Martinez, Ruf, Zeng '20]

Method of Regions: the Soft Region

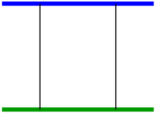
- We are only interested in the large-b/small-q result. Only the non-analytic terms in q^2 matter for long-range effects.

- One-loop example:



$$= \int \frac{d^{4-2\epsilon}\ell}{(-2p_1 \cdot \ell + \ell^2)(2p_2 \cdot \ell + \ell^2)\ell^2(\ell - q)^2}$$

- Hard Region: $\ell \sim p_{1,2} \gg q \implies$ analytic
  Soft Region: $\ell \sim q \ll p_{1,2} \implies$ non-analytic



$$\overset{(s)}{=} \int \frac{d^{4-2\epsilon}\ell}{(-2p_1 \cdot \ell)(2p_2 \cdot \ell)\ell^2(\ell - q)^2} \sim \frac{1}{(q^2)^{1+\epsilon}} \,.$$

Differential Equations and Boundary Conditions

[Parra-Martinez, Ruf, Zeng '20] [Di Vecchia, C.H., Russo, Veneziano 2008.12743, 2104.03256]

- Reduce all *soft* integrals in a given family to a basis $\vec{I} = (I_1, I_2, \dots)$ (master integrals) via integration by parts. (`LiteRed`, `FIRE6`)

- In a "pure" basis (`Epsilon`), they satisfy simple differential equations. M_j constant, ϵ-independent matrices and $x \simeq \sigma - \sqrt{\sigma^2 - 1}$,

$$d\vec{I}(\epsilon; x) = \epsilon \sum_{j=0,\pm 1} M_j \, \vec{I}(\epsilon; x) \, d\log(x - j)$$

- Solve perturbatively $\vec{I}(\epsilon; x) = \vec{I}^{(0)}(x) + \epsilon \, \vec{I}^{(1)}(x) + \cdots$ as $\epsilon \to 0$.

- Boundary conditions fixed in the $x \to 1^-$ (small velocity) limit $\implies$ use method of regions again!

The 3PM Eikonal in $\mathcal{N} = 8$ Supergravity

[Full result: Di Vecchia, C.H., Russo, Veneziano, 2008.12743, 2101.05772, 2104.03256]

[Potential region: Parra-Martinez, Ruf, Zeng '20]

- Eikonal phase:

$$\mathrm{Re}\, 2\delta_2 = \frac{16 G^3 m_1^2 m_2^2}{b^2} \Big[-\frac{\sigma^4}{\sigma^2 - 1} \operatorname{arccosh} \sigma + \frac{\sigma^6}{(\sigma^2-1)^2} + \frac{\sigma^5 (\sigma^2 - 2)}{(\sigma^2-1)^{\frac{5}{2}}} \operatorname{arccosh} \sigma \Big].$$

- Radiation-Reaction part: time-reversal *odd* contributions to Θ_s,

$$\mathrm{Re}\, 2\delta_2^{\mathrm{RR}} = \frac{16 G^3 m_1^2 m_2^2}{b^2} \left[\frac{\sigma^6}{(\sigma^2-1)^2} + \frac{\sigma^5 (\sigma^2 - 2)}{(\sigma^2-1)^{\frac{5}{2}}} \operatorname{arccosh} \sigma \right].$$

- Infrared divergent exponential suppression:

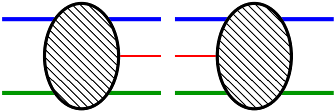
$$\mathrm{Im}\, 2\delta_2 = \frac{1}{\pi} \left[-\frac{1}{\epsilon} + \log(\sigma^2 - 1) \right] \mathrm{Re}\, 2\delta_2^{\mathrm{RR}} + \cdots$$

3PM Radiation-Reaction from Soft Theorems

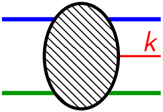
[Di Vecchia, C.H., Russo, Veneziano 2101.05772]

- Analyticity: $i\log(1-\sigma^2 - i0) = i\log(\sigma^2-1)+\pi$
- Unitarity: $\mathrm{Im}\,2\delta_2 = [\mathrm{Im}\,\tilde{\mathcal{A}}_2]_{3p.c.}$ and

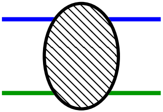
$$[\mathrm{Im}\,2\mathcal{A}]_{3p.c.} = \int d(\mathrm{LIPS})$$



- Soft theorem: [Weinberg '64,'65]



$$\mu\nu \sim \left[\sum_n \frac{\sqrt{8\pi G}\,p_n^\mu p_n^\nu}{p_n\cdot k}\right]$$



For $\mathrm{Im}\,2\delta_2$ this gives $\frac{1}{\pi}\mathrm{Re}\,2\delta_2^{\mathrm{RR}}$ times

$$\int_0^{\omega_{\max}b}\frac{2\,d\omega}{\omega^{1+2\epsilon}} \sim -\frac{1}{\epsilon}+2\log(\omega_{\max}b) \sim -\frac{1}{\epsilon}+\log(\sigma^2-1)\,.$$

3PM Radiation-Reaction from Soft Theorems

[Di Vecchia, C.H., Russo, Veneziano 2101.05772]

The IR divergence in $\text{Im}\,2\delta_2$ determines $\text{Re}\,2\delta_2^{RR}$

$$\text{Re}\,2\delta_2^{RR} = \lim_{\epsilon\to 0}\left[-\pi\epsilon\;\text{Im}\,2\delta_2\right].$$

The 3PM Eikonal in General Relativity

[Di Vecchia, C.H., Russo, Veneziano 2008.12743, 2101.05772, 2104.03256]

[Related work at 3PM: Bern al.'19; Damour '20; Herrmann et al. '21, Bjerrum-Bohr et al.'21; Brandhuber et al.'21]

- Eikonal phase:

$$\mathrm{Re}\,2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left[\frac{s\left(12\sigma^4-10\sigma^2+1\right)}{2m_1 m_2\left(\sigma^2-1\right)^{\frac{3}{2}}} - \frac{\sigma\left(14\sigma^2+25\right)}{3\sqrt{\sigma^2-1}} - \frac{4\sigma^4-12\sigma^2-3}{\sigma^2-1}\,\mathrm{arccosh}\sigma\right] + \mathrm{Re}\,2\delta_2^{\mathrm{RR}}$$

with

$$\mathrm{Re}\,2\delta_2^{\mathrm{RR}} = \frac{G}{2}Q_{1\mathrm{PM}}^2\mathcal{I}(\sigma)\,, \quad \mathcal{I}(\sigma)\equiv\frac{8-5\sigma^2}{3(\sigma^2-1)}+\frac{\sigma\left(2\sigma^2-3\right)}{(\sigma^2-1)^{3/2}}\,\mathrm{arccosh}\,\sigma\,.$$

- Infrared divergent exponential suppression:

$$\mathrm{Im}\,2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon}+\log(\sigma^2-1)\right]\mathrm{Re}\,2\delta_2^{\mathrm{RR}}+\cdots$$

Smoothness and Universality of $\mathrm{Re}\, 2\delta_2$ at High Energy

[Di Vecchia, C.H., Russo, Veneziano 2008.12743, 2101.05772, 2104.03256]

At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit:

- the *complete* eikonal phase is smooth, although the conservative and radiation-reaction parts separately diverge like $\log \sigma$,
- its expression is the same in $\mathcal{N} = 8$ supergravity and in GR,

$$\mathrm{Re}\, 2\delta_2 \sim Gs \frac{\Theta_s^2}{4}, \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

Elastic Final State

- Final state (schematically):

$$\boxed{|\text{out}\rangle = e^{2i\delta(b)}|\text{in}\rangle}$$

- Impulse:

$$Q_\mu = \left(-i\langle\text{out}|\frac{\overleftrightarrow{\partial}}{\partial b^\mu}|\text{out}\rangle\right)/\langle\text{out}|\text{out}\rangle = \frac{\partial\,\text{Re}\,2\delta}{\partial b^\mu}\,.$$

Problems:

1. How do we restore (nonperturbative) unitarity:

$$\langle\text{out}|\text{out}\rangle = e^{-\,\text{Im}\,2\delta}\langle\text{in}|\text{in}\rangle \to 0 \qquad \text{as } D \to 4$$

2. How do we calculate observables associated to the gravitational field?
3. How do we check energy and angular momentum balance?

Outline

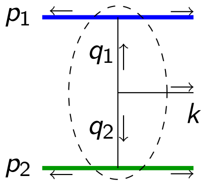
Including Graviton Emissions

[2 → 3 amplitude: Goldberger, Ridgway '17; Luna, Nicholson, O'Connell, White '17]

[Equivalent worldline approaches: Jakobsen, Mogull, Plefka, Steinhoff '21; Mougiakakos, Riva, Vernizzi '21, '22]

- Classical limit of the 2 → 3 amplitude

$$\mathcal{A}^{\mu\nu}(q_1, q_2, k) =$$



- Impact-parameter space 2 → 3 amplitude $(q_1 + q_2 + k = 0)$

$$\tilde{\mathcal{A}}^{\mu\nu}(k) = \int \frac{d^D q_1}{(2\pi)^{D-2}}\, \delta(2p_1 \cdot q_1)\delta(2p_2 \cdot q_2) e^{ib_1\cdot q_1 + ib_2 \cdot q_2} \mathcal{A}^{\mu\nu}(q_1, q_2, k)\,.$$

Including Graviton Emissions

[Mougiakakos, Riva, Vernizzi '21]

$$A^{\mu\nu} = \frac{1}{(p_1\cdot k)^2\,(p_2\cdot k)^2\,q_1^2\,q_2^2}$$
$$\kappa^3\Big(8\,(p_1\cdot k)^4\,(p_2\cdot k)^2\,p_2^\mu\,p_2^\nu + 4\,(p_1\cdot p_2)^2\,(p_2\cdot k)^2\,(q_2\cdot k)\,p_1^\mu\,p_1^\nu\,q_1^2 -$$
$$4\,(p_1\cdot k)\,(p_1\cdot p_2)\,(p_2\cdot k)^2\,\big(-2\,(p_2\cdot k)\,p_1^\mu\,p_1^\nu + (p_1\cdot p_2)\,p_1^{(\mu}q_2^{\nu)}\big)\,q_1^2 -$$
$$8\,(p_1\cdot k)^3\,(p_2\cdot k)\,\Big((p_1\cdot p_2)\,(p_2\cdot k)\,p_2^{(\mu}q_1^{\nu)} -$$
$$(p_1\cdot p_2)\,p_2^\mu\,p_2^\nu\,q_2^2 + (p_2\cdot k)^2\,\big(p_1^{(\mu}p_2^{\nu)} + (p_1\cdot p_2)\,\eta^{\mu\nu}\big)\Big) +$$
$$2\,(p_1\cdot k)^2\,\Big(4\,(p_2\cdot k)^4\,p_1^\mu\,p_1^\nu - 4\,(p_1\cdot p_2)\,(p_2\cdot k)^3\,p_1^{(\mu}q_2^{\nu)} +$$
$$2\,(p_1\cdot p_2)^2\,(q_1\cdot k)\,p_2^\mu\,p_2^\nu\,q_2^2 - 2\,(p_1\cdot p_2)^2\,(p_2\cdot k)\,p_2^{(\mu}q_1^{\nu)}\,q_2^2 - (p_1\cdot p_2)\,(p_2\cdot k)^2$$
$$\Big(-2\,(p_1\cdot p_2)\,\big(q_1^\mu\,q_1^\nu + q_2^\mu\,q_2^\nu\big) + \big(q_1^2 + q_2^2\big)\,\big(2\,p_1^{(\mu}p_2^{\nu)} + (p_1\cdot p_2)\,\eta^{\mu\nu}\big)\Big)\Big) +$$
$$\Big(-2\,(p_1\cdot k)^2\,(q_1\cdot k)\,p_2^\mu\,p_2^\nu\,q_2^2 + 2\,(p_1\cdot k)^2\,(p_2\cdot k)\,p_2^{(\mu}q_1^{\nu)}\,q_2^2 +$$
$$(p_2\cdot k)^2\,\Big(-2\,(q_2\cdot k)\,p_1^\mu\,p_1^\nu\,q_1^2 + (p_1\cdot k)\,\Big(2\,p_1^{(\mu}q_2^{\nu)}\,q_1^2 -$$
$$2\,(p_1\cdot k)\,\big(q_1^\mu\,q_1^\nu + q_2^\mu\,q_2^\nu\big) + (p_1\cdot k)\,\big(q_1^2 + q_2^2\big)\,\eta^{\mu\nu}\Big)\Big)\Big)\,m_1^2\,m_2^2\Big)$$

Here $\kappa = \sqrt{8\pi G}$.

Inelastic Final State

[Di Vecchia, C.H., Russo, Veneziano 2210.12118] [cf. Damgaard, Planté, Vanhove '21, Cristofoli et al.'21]

$\tilde{\mathcal{A}}_j(k) = \varepsilon_{j\mu\nu}(k)^* \, \tilde{\mathcal{A}}^{\mu\nu}(k)$ (polarization j), $\int_k = \int \frac{d^D k}{(2\pi)^D} 2\pi\delta(k^2)\theta(k^0)$.

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation

$$e^{2i\hat{\delta}(b_1,b_2)} = e^{i\,\mathrm{Re}\,2\delta(b)} e^{i\int_k \left[\tilde{\mathcal{A}}_j(k)\hat{a}_j^\dagger(k)+\tilde{\mathcal{A}}_j^*(k)\hat{a}_j(k)\right]}.$$

- Final state (again, schematically):

$$\boxed{|\mathrm{out}\rangle = e^{2i\hat{\delta}(b_1,b_2)}|\mathrm{in}\rangle}$$

- Unitarity:

$$\langle \mathrm{out}|\mathrm{out}\rangle = \langle \mathrm{in}|\mathrm{in}\rangle = 1$$

- Consistency with the elastic exponentiation: by the BCH formula,

$$\langle \mathrm{in}|\mathrm{out}\rangle = e^{i\,\mathrm{Re}\,2\delta(b)} e^{-\,\mathrm{Im}\,2\delta(b)} = e^{2i\delta(b)}$$

L.O. Gravitational Waveform

[Di Vecchia, C.H., Russo, Veneziano 2210.12118]

[cf. worldline approaches: Jakobsen, Mogull, Plefka, Steinhoff '21; Mougiakakos, Riva, Vernizzi '21, '22]

- Metric fluctuation:

$$g_{\mu\nu}(x) - \eta_{\mu\nu} = 2W_{\mu\nu}(x) = 2\sqrt{8\pi G}\,\langle \text{out}|\hat{H}_{\mu\nu}(x)|\text{out}\rangle\,.$$

- L.O. waveform: For a detector at a large distance r, fixed retarded time u and angles $\hat{x}$, i.e. near future null infinity $\mathcal{I}^+$,

$$W^{\mu\nu} \sim \frac{\sqrt{8\pi G}}{4\pi r}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{-i\omega u}\,\tilde{\mathcal{A}}^{\mu\nu}\Big(k = \omega(1,\hat{x})\Big).$$

- $\tilde{\mathcal{A}}^{\mu\nu}(k)$ is a function of $p^{\mu}_{1,2}$, $b^{\mu}_{1,2}$, k^{μ}, and their invariant products. It can be expressed in terms of Bessel functions. Gauge invariance: $k_\mu \tilde{\mathcal{A}}^{\mu\nu}(k) = 0$.

Radiated Energy-Momentum

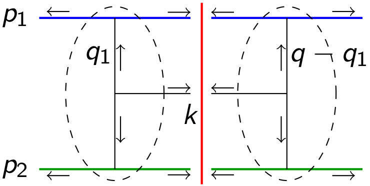
[Di Vecchia, C.H., Russo, Veneziano 2210.12118] [Herrmann, Parra-Martinez, Ruf, Zeng '21]

- $\langle \text{out}|\hat{P}^\alpha|\text{out}\rangle = \boldsymbol{P}^\alpha$
- In terms of the waveform

$$\boldsymbol{P}^\alpha = \int_k k^\alpha \tilde{A}^{\mu\nu}(k)\left(\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\tilde{A}^{*\rho\sigma}(k) \equiv \int_k k^\alpha \tilde{A}\tilde{A}^* \,.$$

- Recast as the FT of a cut in momentum-space (reverse unitarity)

$$\boldsymbol{P}^\alpha = \text{FT}\int d(\text{LIPS})k^\alpha$$



Same integrals that we computed for the $2 \to 2$ amplitude!

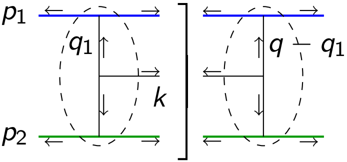
Radiated Angular Momentum

- $\langle \text{out}|\hat{J}^{\alpha\beta}|\text{out}\rangle = \boldsymbol{J}^{\alpha\beta} + \mathcal{J}^{\alpha\beta}$
- $\boldsymbol{J}_{\alpha\beta} = \boldsymbol{J}^{(o)}_{\alpha\beta} + \boldsymbol{J}^{(s)}_{\alpha\beta}$ where [Manohar, Ridgway, Shen '21] [Di Vecchia, C.H., Russo 2203.11915]

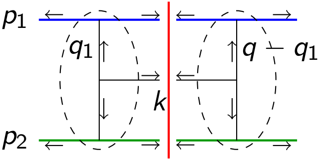
$$i\boldsymbol{J}^{(o)}_{\alpha\beta} = \int_k k_{[\alpha} \frac{\partial \tilde{A}}{\partial k^{\beta]}} \tilde{A}^* \,, \qquad \boldsymbol{J}^{(s)}_{\alpha\beta} = 2i \int_k \tilde{A}^{\mu}_{[\alpha} \tilde{A}^*_{\beta]\mu} \,.$$

- Reverse unitarity: $q_{\|2} = -u_2 \cdot q$ [Di Vecchia, C.H., Russo, Veneziano 2210.12118]

$$i\boldsymbol{J}^{(o)}_{\alpha\beta} = \text{FT} \int k_{[\alpha} \frac{\partial}{\partial k^{\beta]}} \Big[d(\text{LIPS})$$



$$- u_{2[\alpha}\, \text{FT} \frac{\partial}{\partial q_{\|2}} \int d(\text{LIPS}) k_{\beta]}$$



Static Modes as a Soft Dressing [Di Vecchia, C.H., Russo 2203.11915]

[Soft dressing: Bloch, Nordsieck '37; Thirring, Touschek '51; Weinberg '65; Mirbabayi, Porrati '16; Choi, Akhoury '17; Arkani-Hamed et al.'20. Operator exponentiation: Damgaard, Planté, Vanhove '21; Cristofoli et al.'12. Classical soft theorems: Laddha, Sen '18; Sahoo, Sen '18; Saha, Sahoo, Sen '19; Sahoo, Sen '21.]

Operator dressing of the elastic eikonal in b space

$$\hat{S}_{s.r.} = e^{\int_k \left[F_j(k)\hat{a}_j^\dagger(k) - F_j^*(k)\hat{a}_j(k)\right]} .$$

- $F_j(k) = \varepsilon^*_{j\mu\nu}(k) F^{\mu\nu}(k)$

$$F^{\mu\nu}(k) = \sum_n \frac{\sqrt{8\pi G}\, p_n^\mu p_n^\nu}{p_n \cdot k - i0} ,$$

  and $\int_k = \int \frac{d^D k}{(2\pi)^D} 2\pi\delta(k^2)\theta(k^0)\theta(\Lambda - k^0)$, with Λ a cutoff.

- Effectively

$$e^{2i\hat{\delta}(b_1,b_2)} \mapsto \hat{S}_{s.r.} e^{2i\hat{\delta}(b_1,b_2)} .$$

Angular Momentum of the Static Gravitational Field $\mathcal{J}_{\alpha\beta}$

[Di Vecchia, C.H., Russo 2203.11915]

Angular momentum/mass dipole loss due to static modes

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2}\sum_{n,m}\left[\left(\sigma_{nm}^2 - \frac{1}{2}\right)\frac{\frac{\sigma_{nm}\,\text{arccosh}\,\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}} - 1}{\sigma_{nm}^2 - 1} - \frac{2\sigma_{nm}\,\text{arccosh}\,\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}}\right](\eta_n - \eta_m)p_n^{[\alpha}p_m^{\beta]}\,.$$

- Shorthand notation $-\eta_n\eta_m p_n \cdot p_m = m_n m_m \sigma_{nm}$ with $\eta_n = +1$ ($\eta_n = -1$) if n is outgoing (incoming).
- Matches [Damour '20; Manohar, Ridgway, Shen '22] up to $\mathcal{O}(G^3)$ upon expanding

$$Q^\mu = Q^\mu_{\text{1PM}} + Q^\mu_{\text{2PM}} + \cdots$$

Initial State

[Kosower, Maybee, O'Connell '18]

- Incoming state with no radiation

$$|\text{in}\rangle = \int_{-p_1} \int_{-p_2} \Phi_1(-p_1)\Phi_2(-p_2) e^{ib_1\cdot p_1 + ib_2 \cdot p_2} |-p_1, -p_2, 0\rangle \,,$$

where $\int_{p_i} = \int \frac{d^D p_i}{(2\pi)^D} 2\pi\theta(p_i^0)\delta(p_i^2 + m_i^2)$.

- The wavepackets $\Phi_i(-p_i)$ are peaked around the classical incoming momenta.
- $b_J = b_1 - b_2$ is the impact parameter.

Eikonal Final State

[Di Vecchia, C.H., Russo, Veneziano 2210.12118] [cf. Cristofoli et al.'21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation

$$|\text{out}\rangle \simeq \int_{p_3}\int_{p_4} e^{-ib_1\cdot p_4 - ib_2\cdot p_3} \int \frac{d^D Q_1}{(2\pi)^D}\int \frac{d^D Q_2}{(2\pi)^D}\, \Phi_1(p_4-Q_1)\,\Phi_2(p_3-Q_2)$$
$$\times \int d^D x_1 \int d^D x_2\, e^{i(b_1-x_1)\cdot Q_1 + i(b_2-x_2)\cdot Q_2}\, e^{2i\hat{\delta}(x_1,x_2)}|p_4,p_3,0\rangle$$

with

$$e^{2i\hat{\delta}(x_1,x_2)} = e^{i\,\text{Re}\,2\delta(b)} e^{i\int_k \left[\tilde{\mathcal{A}}_j(x_1,x_2,k)a_j^\dagger(k) + \tilde{\mathcal{A}}_j^*(x_1,x_2,k)a_j(k)\right]}.$$

Here

- b is the projection of $x_1 - x_2$ orthogonal to $p_4 - Q_1/2$, $p_3 - Q_2/2$,
- $\tilde{\mathcal{A}}_j = \varepsilon_{j\mu\nu}\,\tilde{\mathcal{A}}^{\mu\nu}$ is the impact-parameter space $2 \to 3$ amplitude.

Mechanical Energy-Momentum

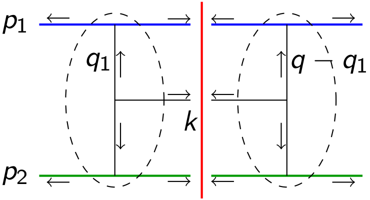
[cf. Cristofoli et al. '21]

- $\langle \text{out}|\hat{P}_1^\mu|\text{out}\rangle - \langle \text{in}|\hat{P}_1^\mu|\text{in}\rangle = Q^\mu + \Delta \boldsymbol{P}_1^\mu$
- In terms of the waveform

$$\Delta \boldsymbol{P}_{1\alpha} = -\frac{i}{2}\int_k \left[\frac{\partial \tilde{A}}{\partial b_1^\alpha}\,\tilde{A}^* - \tilde{A}\,\frac{\partial \tilde{A}^*}{\partial b_1^\alpha}\right].$$

- Reverse unitarity: [Di Vecchia, C.H., Russo, Veneziano 2210.12118]

$$\Delta \boldsymbol{P}_1^\alpha = \text{FT}\int d(\text{LIPS})\left(q_1^\alpha - \frac{1}{2}q^\alpha\right)$$



Energy-Momentum Balance

[Di Vecchia, C.H., Russo, Veneziano 2210.12118] [Herrmann, Parra-Martinez, Ruf, Zeng '21]

- Convenient variables: velocities $u^\alpha_{1,2} = -p^\alpha_{1,2}/m_{1,2}$ and $u^\alpha_1 = \check{u}^\alpha_1 + \sigma\,\check{u}^\alpha_2$, $u^\alpha_2 = \check{u}^\alpha_2 + \sigma\,\check{u}^\alpha_1$.
- Radiated energy-momentum:

$$\boldsymbol{P}^\alpha = \frac{G^3 m_1^2 m_2^2}{b^3}\,(\check{u}^\mu_1 + \check{u}^\mu_2)\,\mathcal{E}\,,$$

- Radiative changes in energy-momentum:

$$\Delta\boldsymbol{P}^\alpha_1 = -\frac{G^3 m_1^2 m_2^2}{b^3}\,\check{u}^\alpha_2\,\mathcal{E}\,, \qquad \Delta\boldsymbol{P}^\alpha_2 = -\frac{G^3 m_1^2 m_2^2}{b^3}\,\check{u}^\alpha_1\,\mathcal{E}\,.$$

$$\boxed{\boldsymbol{P}^\alpha + \Delta\boldsymbol{P}^\alpha_1 + \Delta\boldsymbol{P}^\alpha_2 = 0}$$

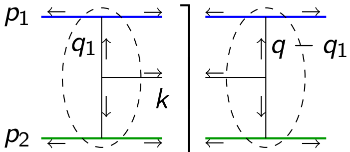
Mechanical Angular Momentum [Di Vecchia, C.H., Russo, Veneziano 2210.12118]

- $\langle \text{out}|\hat{L}_2^{\alpha\beta}|\text{out}\rangle - \langle \text{in}|\hat{L}_2^{\alpha\beta}|\text{in}\rangle = \Delta \boldsymbol{L}_2^{\alpha\beta} + \Delta L_{2\text{cons}}^{\alpha\beta} + \Delta \mathcal{L}_2^{\alpha\beta}$
- $\Delta \boldsymbol{L}_2^{\alpha\beta} = \operatorname{Im} \boldsymbol{J}_2^{\alpha\beta} + b_2^{[\alpha} \Delta \boldsymbol{P}_2^{\beta]}$, where

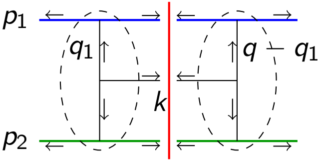
$$\boldsymbol{J}_{2\alpha\beta} = \int_k p_{2[\alpha} \frac{\partial \tilde{\mathcal{A}}}{\partial p_2^{\beta]}} \tilde{\mathcal{A}}^* \,.$$

- Reverse unitarity:

$$\boldsymbol{J}_{2\alpha\beta} = \text{FT} \int u_{2[\alpha} \frac{\partial}{\partial u_2^{\beta]}} \left[d(\text{LIPS}) \right]$$



$$+ \, u_{2[\alpha} \, \text{FT} \frac{\partial}{\partial q_{\|2}} \int d(\text{LIPS}) (q_1 + k)_{\beta]}$$



Angular Momentum Balance

- Convenient functions: $\mathcal{C}\sqrt{\sigma^2-1} = -\mathcal{E}_+ + \sigma\mathcal{E}_-$, $\mathcal{F} = \pm\mathcal{E}_\pm \mp \frac{1}{2}\,\mathcal{E}$.
- Radiated angular momentum [Manohar, Ridgway, Shen '21]

$$\boldsymbol{J}^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3}\,\mathcal{F}\left(b^{[\alpha}\check{u}_1^{\beta]} - b^{[\alpha}\check{u}_2^{\beta]}\right).$$

- Radiative changes in angular momentum [Di Vecchia, C.H., Russo, Veneziano 2210.12118]

$$\Delta\boldsymbol{L}_1^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3}\left[+\frac{\mathcal{E}_+ b^{[\alpha}u_1^{\beta]}}{\sigma-1} - \frac{1}{2}\,\mathcal{E}\,b^{[\alpha}\check{u}_2^{\beta]}\right],$$

$$\Delta\boldsymbol{L}_2^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3}\left[-\frac{\mathcal{E}_+ b^{[\alpha}u_2^{\beta]}}{\sigma-1} + \frac{1}{2}\,\mathcal{E}\,b^{[\alpha}\check{u}_1^{\beta]}\right].$$

$$\boxed{\boldsymbol{J}^{\alpha\beta} + \Delta\boldsymbol{L}_1^{\alpha\beta} + \Delta\boldsymbol{L}_2^{\alpha\beta} = 0}$$

Radiative Functions (Point Particles)

$$\frac{\mathcal{E}}{\pi} = f_1 + f_2 \log\frac{\sigma+1}{2} + f_3 \frac{\sigma \operatorname{arccosh}\sigma}{2\sqrt{\sigma^2-1}}\,, \quad \frac{\mathcal{C}}{\pi} = g_1 + g_2 \log\frac{\sigma+1}{2} + g_3 \frac{\sigma \operatorname{arccosh}\sigma}{2\sqrt{\sigma^2-1}}$$

$$f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48(\sigma^2-1)^{3/2}}$$

$$f_2 = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2-1}}$$

$$f_3 = \frac{(2\sigma^2-3)\left(35\sigma^4 - 30\sigma^2 + 11\right)}{8\left(\sigma^2-1\right)^{3/2}}$$

$$g_1 = \frac{105\sigma^7 - 411\sigma^6 + 240\sigma^5 + 537\sigma^4 - 683\sigma^3 + 111\sigma^2 + 386\sigma - 237}{24(\sigma^2-1)^2}$$

$$g_2 = \frac{35\sigma^5 - 90\sigma^4 - 70\sigma^3 + 16\sigma^2 + 155\sigma - 62}{4(\sigma^2-1)}$$

$$g_3 = -\frac{(2\sigma^2-3)\left(35\sigma^5 - 60\sigma^4 - 70\sigma^3 + 72\sigma^2 + 19\sigma - 12\right)}{4\left(\sigma^2-1\right)^2}$$

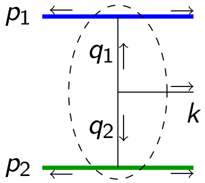
Outline

Including Tidal Effects

[Mougiakakos, Riva, Vernizzi '21, '22]

- Classical limit of the $2 \to 3$ amplitude

$$\mathcal{A}^{\mu\nu}(q_1, q_2, k) =$$



- Impact-parameter space $2 \to 3$ amplitude $(q_1 + q_2 + k = 0)$

$$\tilde{\mathcal{A}}^{\mu\nu}(k) = \int \frac{d^D q_1}{(2\pi)^{D-2}}\, \delta(2p_1 \cdot q_1)\delta(2p_2 \cdot q_2) e^{ib_1 \cdot q_1 + ib_2 \cdot q_2} \mathcal{A}^{\mu\nu}(q_1, q_2, k)\,.$$

- Point-particle + tidal effects:

$$\mathcal{A}^{\mu\nu} = \mathcal{A}^{\mu\nu}_{\text{pp}} + \mathcal{A}^{\mu\nu}_{E_1^2} + \mathcal{A}^{\mu\nu}_{B_1^2} + \cdots$$

Including Tidal Effects

[Mougiakakos, Riva, Vernizzi '21, '22]

$$A_{E_1^2}{}^{\mu\nu} = \frac{1}{(q_2^2)\, m_1^3} c_{E_1^2} \kappa^3 \Big(8 (p_1.k)^4 p_2^\mu p_2^\nu + (p_1.k)^3 \big(-8 (p_2.k) p_1^{(\mu} p_2^{\nu)} + 8 (p_1.p_2) p_2^{(\mu} q_2^{\nu)}\big) - 4 (p_1.k) (p_1.p_2) \big(-2 (p_2.k) p_1^\mu p_1^\nu + (p_1.p_2) p_1^{(\mu} q_2^{\nu)}\big) (q_1^2) + 2 (p_1.p_2)^2 p_1^\mu p_1^\nu (q_1^2)^2 + (p_1.k)^2 \big(8 (p_2.k)^2 p_1^\mu p_1^\nu - 8 (p_1.p_2) (p_2.k) p_1^{(\mu} q_2^{\nu)} - 4 (p_1.p_2) p_1^{(\mu} p_2^{\nu)} (q_1^2) + 8 (p_1.p_2)^2 q_2^\mu q_2^\nu\big) + \big(4 (p_1.k)^3 \big(p_1^{(\mu} q_1^{\nu)} + p_1^{(\mu} q_2^{\nu)} + (p_1.k) \eta^{\mu\nu}\big) - \big(-2 (p_1.k) p_1^{(\mu} q_2^{\nu)} (q_1^2) + p_1^\mu p_1^\nu (q_1^2)^2 + 4 (p_1.k)^2 q_2^\mu q_2^\nu\big) m_1^2\big) m_2^2\Big)$$

$$A_{B_1^2}{}^{\mu\nu} = \frac{1}{(q_2^2)\, m_1^3} c_{B_1^2} \kappa^3 \Big(8 (p_1.k)^4 p_2^\mu p_2^\nu + (p_1.k)^3 \big(-8 (p_2.k) p_1^{(\mu} p_2^{\nu)} + 8 (p_1.p_2) p_2^{(\mu} q_2^{\nu)}\big) - 4 (p_1.k) (p_1.p_2) \big(-2 (p_2.k) p_1^\mu p_1^\nu + (p_1.p_2) p_1^{(\mu} q_2^{\nu)}\big) (q_1^2) + 2 (p_1.p_2)^2 p_1^\mu p_1^\nu (q_1^2)^2 + (p_1.k)^2 \big(8 (p_2.k)^2 p_1^\mu p_1^\nu - 8 (p_1.p_2) (p_2.k) p_1^{(\mu} q_2^{\nu)} - 4 (p_1.p_2) p_1^{(\mu} p_2^{\nu)} (q_1^2) + 8 (p_1.p_2)^2 q_2^\mu q_2^\nu\big) + 4 (p_1.k)^3 \big(p_1^{(\mu} q_1^{\nu)} + p_1^{(\mu} q_2^{\nu)} + (p_1.k) \eta^{\mu\nu}\big) m_2^2 + m_1^2 \Big(\big((p_1.p_2) (q_1^2) \big(-2 (p_2.k) p_1^{(\mu} q_2^{\nu)} + p_1^{(\mu} p_2^{\nu)} (q_1^2)\big)\big) + 4 (p_1.k)^2 \big((p_2.k) p_2^{(\mu} q_2^{\nu)} - p_2^\mu p_2^\nu (q_1^2)\big) + (p_1.k) \big(-4 (p_2.k)^2 p_1^{(\mu} q_2^{\nu)} + 2 (p_2.k) p_1^{(\mu} p_2^{\nu)} (q_1^2) - 2 (p_1.p_2) p_2^{(\mu} q_2^{\nu)} (q_1^2) + 8 (p_1.p_2) (p_2.k) q_2^\mu q_2^\nu\big) - (p_1.k) \big(2 (p_1.k) \big(q_1^{(\mu} q_2^{\nu)} + 2 q_2^\mu q_2^\nu\big) + (q_1^2) \big(p_1^{(\mu} q_1^{\nu)} + p_1^{(\mu} q_2^{\nu)} + 2 (p_1.k) \eta^{\mu\nu}\big)\big) m_2^2\Big)\Big)$$

- It obeys $k_\mu \mathcal{A}^{\mu\nu} = 0$ up to analytic terms in q-space / contact terms in b-space.
- The relation to the Love numbers $k_i^{(2)}$, $j_i^{(2)}$ is $c_{E_i^2} = \frac{1}{6} k_i^{(2)} R_i^5 / G$ and $c_{B_i^2} = \frac{1}{32} j_i^{(2)} R_i^5 / G$ with R_i the radius of object i, roughly of order Gm_i.

Radiated Energy-Momentum Due to Tidal Effects

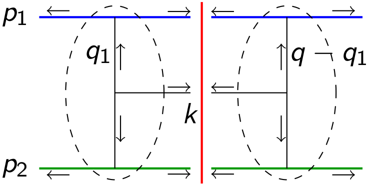
[Mougiakakos, Riva, Vernizzi '22] [C.H. 2210.15689]

- Recall that

$$\boldsymbol{P}^\alpha = \int_k k^\alpha \tilde{A}\tilde{A}^* \,.$$

- We apply again (reverse unitarity)

$$\boldsymbol{P}^\alpha = \text{FT} \int d(\text{LIPS}) k^\alpha$$



- And keep ("cross")-terms linear in the tidal effects.

Radiated Energy-Momentum Due to Tidal Effects

[Mougiakakos, Riva, Vernizzi '22] [C.H. 2210.15689]

$$\boldsymbol{P}^{\alpha}_{\text{tid}} = R_f \sum_X \frac{c_{X_1^2}}{m_1} \left(\mathcal{E}^X \check{u}_1^\alpha + \mathcal{F}^X \check{u}_2^\alpha \right)$$

Here:

- $R_f = 15\pi G^3 m_1^2 m_2^2/(64\, b^7)$
- X can be either E (electric/mass-type) or B (magnetic/current-type)
- $\mathcal{E}^X$ stands for

$$\mathcal{E}^X = f_1^X + f_2^X \log \frac{\sigma+1}{2} + f_3^X \frac{\sigma \operatorname{arccosh} \sigma}{2\sqrt{\sigma^2-1}}$$

with $f_3^X = -\left(\sigma^2 - \frac{3}{2}\right) f_2^X/(\sigma^2-1)$ (no independent "H" topology)

Radiative Functions (E and B)

$$f_1^E = \frac{(\sigma^2-1)^{-\frac{1}{2}}}{2(\sigma+1)^3}[937\sigma^9 + 1551\sigma^8 - 2463\sigma^7 - 5645\sigma^6 + 20415\sigma^5 + 65965\sigma^4 - 349541\sigma^3 + 535057\sigma^2 - 360356\sigma + 92160]$$

$$f_2^E = 30\sqrt{\sigma^2-1}(21\sigma^4 - 14\sigma^2 + 9)$$

$$\mathcal{F}^E = \frac{3(\sigma^2-1)^{\frac{3}{2}}}{(\sigma+1)^5}[42\sigma^8 + 210\sigma^7 + 315\sigma^6 - 105\sigma^5 - 944\sigma^4 - 1528\sigma^3 + 22011\sigma^2 - 33201\sigma + 16272]$$

$$f_1^B = \frac{\sqrt{\sigma^2-1}}{4(\sigma+1)^4}[1559\sigma^8 + 3716\sigma^7 - 1630\sigma^6 - 11660\sigma^5 + 28288\sigma^4 + 155292\sigma^3 - 543442\sigma^2 + 535212\sigma - 180775]$$

$$f_2^B = 210(\sigma^2-1)^{\frac{3}{2}}(3\sigma^2+1)$$

$$\mathcal{F}^B = \frac{-3(105\sigma^5 + 1630\sigma^4 + 1840\sigma^3 + 3690\sigma^2 - 17769\sigma + 15984)}{(\sigma+1)^6(\sigma^2-1)^{-\frac{5}{2}}}$$

Radiated Angular Momentum Due to Tidal Effects

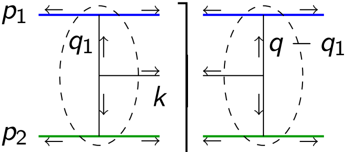
[C.H. 2210.15689]

- Recall that: $\boldsymbol{J}_{\alpha\beta} = \boldsymbol{J}^{(o)}_{\alpha\beta} + \boldsymbol{J}^{(s)}_{\alpha\beta}$,

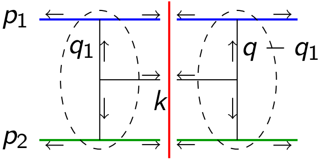
$$i\boldsymbol{J}^{(o)}_{\alpha\beta} = \int_k k_{[\alpha} \frac{\partial \tilde{\mathcal{A}}}{\partial k^{\beta]}} \tilde{\mathcal{A}}^* \,, \qquad \boldsymbol{J}^{(s)}_{\alpha\beta} = 2i \int_k \tilde{\mathcal{A}}^{\mu}_{[\alpha} \tilde{\mathcal{A}}^*_{\beta]\mu} \,.$$

- We apply again (reverse unitarity)

$$i\boldsymbol{J}^{(o)}_{\alpha\beta} = \text{FT} \int k_{[\alpha} \frac{\partial}{\partial k^{\beta]}} \Big[d(\text{LIPS})$$



$$- u_{2[\alpha}\, \text{FT} \frac{\partial}{\partial q_{\|2}} \int d(\text{LIPS}) k_{\beta]}$$



- And keep ("cross")-terms linear in the tidal effects.

Radiated Angular Momentum Due to Tidal Effects

[C.H. 2210.15689]

In a frame where $b_1^\alpha = 0$,

$$\boldsymbol{J}^{\alpha\beta}_{\text{tid}} = R_f \sum_X \frac{c_{X_1^2}}{m_1} \left(\mathcal{C}^X b^{[\alpha} u_1^{\beta]} + \mathcal{D}^X u_2^{[\alpha} b^{\beta]} \right)$$

where

$$\mathcal{C}^X = g_1^X + g_2^X \log\frac{\sigma+1}{2} + g_3^X \frac{\sigma \ \text{arccosh}\,\sigma}{2\sqrt{\sigma^2-1}}\,,$$

$$\mathcal{D}^X = h_1^X + h_2^X \log\frac{\sigma+1}{2} + h_3^X \frac{\sigma \ \text{arccosh}\,\sigma}{2\sqrt{\sigma^2-1}}\,,$$

with $g_3^X = -\left(\sigma^2 - \frac{3}{2}\right) g_2^X/(\sigma^2-1)$ and $h_3^X = -\left(\sigma^2 - \frac{3}{2}\right) h_2^X/(\sigma^2-1)$.

Radiative Functions (E) [C.H. 2210.15689]

$$g_1^E = \frac{(\sigma^2-1)^{-\frac{3}{2}}}{10(\sigma+1)^3}\big[2573\sigma^9 + 9819\sigma^8 + 13143\sigma^7 + 1845\sigma^6$$
$$-897603\sigma^5 + 3221239\sigma^4 - 5046195\sigma^3 + 4203751\sigma^2$$
$$-1862318\sigma + 351826\big]$$

$$g_2^E = -6(35\sigma^4 - 50\sigma^2 - 1)/\sqrt{\sigma^2-1}$$

$$h_1^E = \frac{4(\sigma^2-1)^{-\frac{3}{2}}}{5(\sigma+1)^2}[492\sigma^7 + 564\sigma^6 - 609\sigma^5 - 722\sigma^4$$
$$-4636\sigma^3 + 13478\sigma^2 - 14143\sigma + 5096]$$

$$h_2^E = 48\sigma(7\sigma^2+1)/\sqrt{\sigma^2-1}$$

Radiative Functions (B) [C.H. 2210.15689]

$$g_1^B = \frac{20(\sigma^2-1)^{-\frac{1}{2}}}{(\sigma+1)^4}[4495\sigma^8+22180\sigma^7+46630\sigma^6+50020\sigma^5$$
$$-1748636\sigma^4+4687932\sigma^3-5397990\sigma^2+3026428\sigma-681459]$$

$$g_2^B = -30\sqrt{\sigma^2-1}(7\sigma^2-3)$$

$$h_1^B = \frac{2(\sigma^2-1)^{-\frac{1}{2}}}{5(\sigma+1)^3}[879\sigma^6+1797\sigma^5-492\sigma^4-2908\sigma^3$$
$$-10491\sigma^2+18815\sigma-9280]$$

$$h_2^B = 336\sigma\sqrt{\sigma^2-1}$$

PN Limit and Cross-Check [C.H. 2210.15689]

Nonrelativistic limit, $\sigma = \sqrt{1 + p_\infty^2}$ and $p_\infty \to 0$,

$$\begin{aligned}
\mathcal{C}^E &= \frac{1056}{5} p_\infty - \frac{349}{35} p_\infty^3 + \mathcal{O}(p_\infty^5) \\
\mathcal{D}^E &= \frac{1056}{5} p_\infty - \frac{324}{7} p_\infty^3 + \mathcal{O}(p_\infty^5) \\
\mathcal{C}^B &= 40 p_\infty^3 + \frac{3833}{35} p_\infty^5 + \mathcal{O}(p_\infty^7) \\
\mathcal{D}^B &= -\frac{168}{5} p_\infty^3 + \frac{1471}{10} p_\infty^5 + \mathcal{O}(p_\infty^7)\,.
\end{aligned}$$

Quantitative cross-check of the results:

- Expand[1] $\mathcal{A}^{\mu\nu}(k)$ for small p_∞ and $k^\alpha \sim \mathcal{O}(p_\infty)$.
- Perform the Fourier transform to obtain $\tilde{\mathcal{A}}^{\mu\nu}(k)$ in the frame $b_1^\alpha = 0$.
- Substitute into the expression for $\boldsymbol{J}^{\alpha\beta}$ and integrate over k^μ.
- Perfect agreement with the above expansion of the PM result.

[1] [Leading PN waveform for E contributions $\mathcal{O}(p_\infty)$,

first subleading correction for point-particle $\mathcal{O}(p_\infty^{-1}) + \mathcal{O}(p_\infty^0)$ and B contributions $\mathcal{O}(p_\infty^2) + \mathcal{O}(p_\infty^3)$.]

UR Limit and KT Bound [C.H. 2210.15689]

We consider the high-energy limit $\sigma \gg 1$, in which

$$\frac{\boldsymbol{J}_{\text{tid}}}{R_f J} = \frac{c_{E_1^2}}{m_1^2} 63\sigma^5 - \frac{c_{E_1^2} + c_{B_1^2}}{2m_1 m_2} 315\sigma^4 \log\sigma + \mathcal{O}(\sigma^4).$$

- For small deflection angle $\Theta_s \sim Gm\sqrt{\sigma}/b$, since $c_{E_1^2} \sim G^4 m^5$,

$$\boldsymbol{J}_{\text{tid}}/J \sim \Theta_s^4 (\sqrt{\sigma}\Theta_s)^3$$

- This is a well-known issue

$$\boldsymbol{P}_{\text{pp}}/p \sim \boldsymbol{J}_{\text{pp}}/J \sim \Theta_s^2 (\sqrt{\sigma}\Theta_s)$$

- The perturbative PM expansion is limited to $\sqrt{\sigma}\Theta_s \lesssim 1$ (KT bound)

[Di Vecchia, C.H., Russo, Veneziano 2204.02378] [and refs. therein]

Angular Momentum of the Static Gravitational Field $\mathcal{J}_{\alpha\beta}$

[C.H. 2203.11915]

Angular momentum/mass dipole loss due to static modes

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2}\sum_{n,m}\left[\left(\sigma_{nm}^2 - \frac{1}{2}\right)\frac{\frac{\sigma_{nm}\,\text{arccosh}\,\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}} - 1}{\sigma_{nm}^2 - 1} - \frac{2\sigma_{nm}\,\text{arccosh}\,\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}}\right](\eta_n - \eta_m)\, p_n^{[\alpha} p_m^{\beta]} \,.$$

- Shorthand notation $-\eta_n\eta_m p_n \cdot p_m = m_n m_m \sigma_{nm}$ with $\eta_n = +1$ ($\eta_n = -1$) if n is outgoing (incoming).
- Includes tidal effects via [Bern et al. '20; Cheung, Solon '20]

$$Q^\mu = Q^\mu_{1\text{PM}} + Q^\mu_{2\text{PM}} + Q^\mu_{E_1^2} + Q^\mu_{B_1^2} + \cdots$$

- No tidal corrections to the 2PM result, only to the 3PM result.

Summary and Outlook

- We calculated the 3PM eikonal $2\delta_2$ and checked that $\mathrm{Re}\, 2\delta_2$ is smooth and universal at high energy.
- Eikonal operator gives a unitary description including radiation:
  - L.O. emitted waveform
  - 3PM variations of energy-momentum
  - 3PM variations of angular momentum
- Tidal effects can be easily included by incorporating them in the $2 \to 3$ amplitude.

For the future:

- Learning more about RR effects at 4PM?

  [cf. Manohar, Ridgway, Shen '22, Dlapa et al. '22]
- N.L.O. waveform?
- Inclusion of spin effects?
- "Energy crisis"? $E_{\mathrm{rad}}/E \sim \Theta_s^3 \sqrt{\sigma}$

ADDITIONAL MATERIAL

Two Leftover Problems

Problem 1

$$1 + i\tilde{\mathcal{A}} = e^{2i\delta} = e^{i\,\mathrm{Re}\,2\delta}e^{-\,\mathrm{Im}\,2\delta} \to 0,$$

as $\epsilon \to 0$ the elastic $2 \to 2$ process is infinitely suppressed.

The ZFL of the energy emission spectrum is given by

$$W \equiv \lim_{\omega\to 0} \frac{dE_{\text{rad}}}{d\omega} = \lim_{\epsilon\to 0} \left[-4\epsilon\,\mathrm{Im}\,2\delta_2\right].$$

Problem 2

The energy radiated in the window $\Delta\omega = 1/b$ divided by the CM energy is

$$\frac{\Delta E_{\text{rad}}}{E} = \frac{W\Delta\omega}{\sqrt{s}} \approx \Theta_s^3 \log\sigma\,, \quad \text{as } \sigma \to \infty\,.$$

The system can emit more energy that it initially has!

Eikonal Operator in the ZFL [Di Vecchia, C.H., Russo 2203.11915]

[Soft dressing: Bloch, Nordsieck '37; Thirring, Touschek '51; Weinberg '65; Mirbabayi, Porrati '16; Choi, Akhoury '17; Arkani-Hamed et al.'20. Operator exponentiation: Damgaard, Planté, Vanhove '21; Cristofoli et al.'12. Classical soft theorems: Laddha, Sen '18; Sahoo, Sen '18; Saha, Sahoo, Sen '19; Sahoo, Sen '21.]

Operator dressing of the elastic eikonal in b space

$$S_{s.r.} = e^{\int_k \left[f^{\mu\nu}(k) a^\dagger_{\mu\nu}(k) - f^{*\mu\nu}(k) a_{\mu\nu}(k) \right]} e^{i \operatorname{Re} 2\delta} .$$

- $f^{\mu\nu}(k) = F^{\mu\nu}_{TT}(k)$ [Weinberg '64,'65]

$$F^{\mu\nu}(k) = \sum_n \frac{\sqrt{8\pi G}\, p_n^\mu p_n^\nu}{p_n \cdot k - i0} ,$$

and $\int_k = \int \frac{d^D k}{(2\pi)^D} 2\pi\delta(k^2)\theta(k^0)\theta(\Lambda - k^0)$, with Λ a cutoff.

- Key identification: $p_1 + p_4 = Q = -p_2 - p_3$ with

$$Q_\mu = e^{-i \operatorname{Re} 2\delta} \left(-i \frac{\partial}{\partial b^\mu} \right) e^{i \operatorname{Re} 2\delta} = \frac{\partial \operatorname{Re} 2\delta}{\partial b^\mu} .$$

Using the Soft Eikonal Operator [Di Vecchia, C.H., Russo, Veneziano 2204.02378]

- Infrared divergences: $\langle 0|S_{s.r.}|0\rangle = e^{2i\delta}$,

$$\operatorname{Im} 2\delta = \frac{1}{2}\int_k F^{\mu\nu}\left(\eta_{\mu\rho}\eta_{\nu\sigma} - \tfrac{1}{D-2}\,\eta_{\mu\nu}\eta_{\rho\sigma}\right)F^{\rho\sigma} + \mathcal{O}(\epsilon^0)\,.$$

Problem 1 Solved ✓

- Energy and momentum in the ZFL:

$$\langle 0|S^\dagger_{s.r.}P^\mu S_{s.r.}|0\rangle = \mathcal{P}^\mu_{\text{rad}}\,, \qquad W \equiv \lim_{\omega\to 0}\frac{dE_{\text{rad}}}{d\omega} = \lim_{\epsilon\to 0}\left[-4\epsilon\,\operatorname{Im} 2\delta\right].$$

ZFL of the Spectrum for a $2 \to 2$ Process

[Di Vecchia, C.H., Russo, Veneziano 2204.02378]

ZFL of the spectrum, with exact dependence on σ, Q and $m_{1,2}$

$$W = \frac{4G}{\pi}\Bigg\{2m_1 m_2\left(\sigma^2 - \tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}} - 2m_1 m_2\left(\sigma_Q^2 - \tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma_Q}{\sqrt{\sigma_Q^2-1}}$$

$$+\sum_{j=1,2}\left[\frac{m_j^2}{2} - m_j^2\left(\left(1+\frac{Q^2}{2m_j^2}\right)^2 - \frac{1}{2}\right)\frac{\operatorname{arccosh}\left(1+\frac{Q^2}{2m_j^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_j^2}\right)^2-1}}\right]\Bigg\}.$$

$$= \lim_{\epsilon\to 0}\left[-4\epsilon \,\operatorname{Im} 2\delta\right]$$

(Shorthand $\sigma_Q = \sigma - \dfrac{Q^2}{2m_1 m_2}$)

PM Expansion at Low and High Energy

[Di Vecchia, C.H., Russo, Veneziano 2204.02378]

- Standard PM regime: When $Q^2 \ll m_{1,2}^2$, we recover the previous results. *Bonus:* W, $\text{Im}\, 2\delta$ up to $\mathcal{O}(G^5)$ for any background hard process, including the spinning case [Alessio, Di Vecchia '22].

- Sharp convergence radius $\boxed{Q < 2m_{1,2}}$ [Kovacs, Thorne '77,'78; D'Eath '78].

- UR regime: Expanding for $m_{1,2}^2 \ll Q^2 \ll s$, because $Q \sim \frac{\sqrt{s}}{2}\,\Theta_s$ and $\Theta_s \sim \frac{4G\sqrt{s}}{b} \ll 1$, [Sahoo, Sen '21]

$$W \sim \frac{4G}{\pi}\left(\frac{\Theta_s}{2}\right)^2\left[\log\left(\frac{2}{\Theta_s}\right)^2 + 1\right] \implies \frac{\Delta E_{\text{rad}}}{E} \approx \Theta_s^3 \log\frac{1}{\Theta_s}\,.$$

Energy emission is finite ... but not analytic in G!

Problem 2 Solved ✓

Warm-Up: Memory Effect

[Di Vecchia, C.H., Russo 2203.11915; Di Vecchia, C.H. Russo, Veneziano 2204.02378]

[Kovacs, Thorne '77,'78; Strominger, Zhiboedov '14; Jakobsen et al.'21; Mougiakakos, Riva, Vernizzi '21; Cristofoli et al.'21]

- Waveform: $\sqrt{8\pi G}\,\langle 0|S^{\dagger}_{s.r.}H_{\mu\nu}(x)S_{s.r.}|0\rangle = W_{\mu\nu}(x)$ with

$$H_{\mu\nu}(x) = \int_k \left[a_{\mu\nu}(k)e^{ikx} + a^{\dagger}_{\mu\nu}(k)e^{-ikx}\right].$$

- Send $r \to \infty$ for fixed u, $\hat{x}$, letting $p_n = \eta_n(E_n, \vec{k}_n)$, ($\eta_n = +1$ if n is outgoing, $\eta_n = -1$ if n is incoming)

$$W^{\mu\nu} \sim \frac{2G}{r}\sum_n \theta\left(\eta_n u\right)\frac{(p_n^{\mu}p_n^{\nu})_{TT}}{E_n - \vec{k}_n\cdot\hat{x}}\,.$$

- The $-i0$ fixes the BMS ambiguity [cf. Damour '20; Veneziano, Vilkovisky '22]

$$\int_{-\infty}^{+\infty}\frac{d\omega}{i2\pi}\,\frac{e^{-i\omega u}}{-\eta_n\omega - i0} = \int_{-\infty}^{+\infty}\frac{d\omega}{i2\pi}\,\frac{e^{i\omega\eta_n u}}{\omega - i0} = \theta\left(\eta_n u\right).$$

$\mathcal{J}^{\text{sc}}_{\alpha\beta}$ for a Massless Scalar [Di Vecchia, C.H., Russo 2203.11915]

- Loss of angular momentum/mass dipole due to soft scalars, $\mathcal{J}^{\text{sc}}_{\alpha\beta} = \langle 0|S^\dagger_{s.r.} J^{\text{sc}}_{\alpha\beta} S_{s.r.}|0\rangle$ gives

$$i\mathcal{J}^{\text{sc}}_{\alpha\beta} = \frac{1}{2}\int_k \left(f^* k_{[\alpha} \frac{\partial f}{\partial k^{\beta]}} - k_{[\alpha} \frac{\partial f^*}{\partial k^{\beta]}} f \right), \quad f = \sum_n \frac{g_n}{p_n \cdot k - i0}\,.$$

- The $-i0$ prescription is important and the result localizes to $\omega = 0$,

$$\int_{-\Lambda}^{\Lambda} \frac{\omega d\omega}{(-\eta_n\omega + i0)(-\eta_m\omega - i0)} = -\frac{i\pi}{2}(\eta_n - \eta_m)$$
$$\int_{-\Lambda}^{\Lambda} \frac{\omega d\omega}{(-\eta_m\omega - i0)^2} = -i\pi\eta_m\,.$$

$\mathcal{J}_{\alpha\beta}$ for the Graviton [Di Vecchia, C.H., Russo 2203.11915]

Loss of angular momentum/mass dipole due to soft gravitons, $\mathcal{J}_{\alpha\beta} = \langle 0 | S^\dagger_{s.r.} J_{\alpha\beta} S_{s.r.} | 0 \rangle$ gives

$$i\mathcal{J}_{\alpha\beta} = \int_k F^*_{\mu\nu} \Big[\Big(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\eta^{\mu\nu}\eta^{\rho\sigma} \Big) k_{[\alpha} \frac{\overleftrightarrow{\partial}}{\partial k^{\beta]}} + 2\eta^{\mu\rho}\delta^\nu_{[\alpha}\delta^\sigma_{\beta]} \Big] F_{\rho\sigma}$$

in agreement with [Manohar, Ridgway, Shen '22].

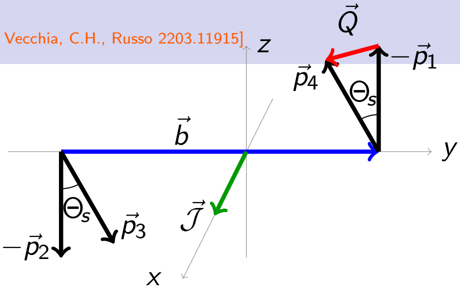
Angular momentum/mass dipole loss due to soft gravitons

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2} \sum_{n,m} \left[\left(\sigma_{nm}^2 - \tfrac{1}{2}\right) \frac{\frac{\sigma_{nm} \operatorname{arccosh} \sigma_{nm}}{\sqrt{\sigma_{nm}^2 - 1}} - 1}{\sigma_{nm}^2 - 1} - \frac{2\sigma_{nm} \operatorname{arccosh} \sigma_{nm}}{\sqrt{\sigma_{nm}^2 - 1}} \right] (\eta_n - \eta_m) p_n^{[\alpha} p_m^{\beta]} \,.$$

(Shorthand notation $-\eta_n \eta_m p_n \cdot p_m = m_n m_m \sigma_{nm}$)

Angular Momentum Loss [Di Vecchia, C.H., Russo 2203.11915]

The formula captures the loss of mechanical angular momentum $\vec{\mathcal{J}}$ completely up to $\mathcal{O}(G^2)$ [Damour '20; Jakobsen et al.'21; Mougiakakos, Riva, Vernizzi '21; Gralla, Lobo '21; Manohar, Ridgway, Shen '22]



Analyticity vs Linear Response:

$$\mathcal{J}^{yz} \sim \frac{4p}{Q} \lim_{\epsilon\to 0}[-\pi\epsilon \,\mathrm{Im}\, 2\delta] + \mathcal{O}\left(G^4\right)$$

ensures that/explains why [Di Vecchia, C.H., Russo, Veneziano 2101.05772]

$$\Theta^{\mathrm{RR}}_{\mathrm{3PM}} = -\frac{1}{p}\frac{\partial\, \mathrm{Re}\, 2\delta_2^{\mathrm{RR}}}{\partial b} = \frac{2}{pb}\lim_{\epsilon\to 0}\left[-\pi\epsilon\, \mathrm{Im}\, 2\delta_2\right]$$

agrees with the linear-response link [Bini, Damour '12; Damour '20]

$$\Theta^{\mathrm{RR}}_{\mathrm{3PM}} \simeq -\frac{1}{2p}\frac{\partial \Theta_{1\mathrm{PM}}}{\partial b}\mathcal{J}^{yz} \simeq \frac{Q}{2p^2 b}\mathcal{J}^{yz}\,.$$

Angular Momentum Loss [Di Vecchia, C.H., Russo 2203.11915]

- The formula captures the loss of mechanical angular momentum to $\mathcal{O}(G^2)$ also for spinning particles, for generic spin alignments [Alessio, Di Vecchia '22].

- It also captures the $\mathcal{O}(G^n)$ loss due to zero-frequency gravitons attached to the elastic process. Cross-checked to $\mathcal{O}(G^3)$ against [Manohar, Ridgway, Shen '22].

- In the high-energy limit $m_i^2 \ll Q^2 = s \sin^2 \frac{\Theta_s}{2}$,

$$\mathcal{J}^{yz} \sim 2Gs \sin\Theta_s \log \frac{\cos\frac{\Theta_s}{2}}{\sin\frac{\Theta_s}{2}}$$

  and for small Θ_s

$$\mathcal{J}^{yz} \sim Gs\Theta_s \log \frac{4}{\Theta_s^2} \implies \frac{\mathcal{J}^{yz}}{pb} \approx \Theta_s^2 \log \frac{1}{\Theta_s}\,.$$

Mass-Dipole Loss [Di Vecchia, C.H., Russo 2203.11915]

- We find an $\mathcal{O}(G^2)$ loss for the *ty* component

$$\frac{\mathcal{J}_{ty}}{b\,(E_1 - E_2)} \sim \frac{\mathcal{J}_{yz}}{2bp}$$

in agreement with [Manohar, Ridgway, Shen '22]. Solving

$$\Delta\,(b_1 E_1 - b_2 E_2) = -\mathcal{J}_{ty}\,, \qquad \Delta\,(b_1 + b_2)\,p = -\mathcal{J}_{yz}$$

yields

$$\Delta b_1 p = \Delta b_2 p = -\mathcal{J}_{yz}/2\,.$$

- There is an $\mathcal{O}(G^3)$ loss for the *tz* component

$$\frac{\mathcal{J}_{tz}}{b\,(E_1 - E_2)} \sim \frac{\Theta_s}{8}\frac{\mathcal{J}_{yz}}{bp}\,.$$

- Our formula and [Manohar, Ridgway, Shen '22] do not find some $\mathcal{O}(G)$ and $\mathcal{O}(G^2)$ terms in the mass-dipole loss in [Gralla, Lobo '21].