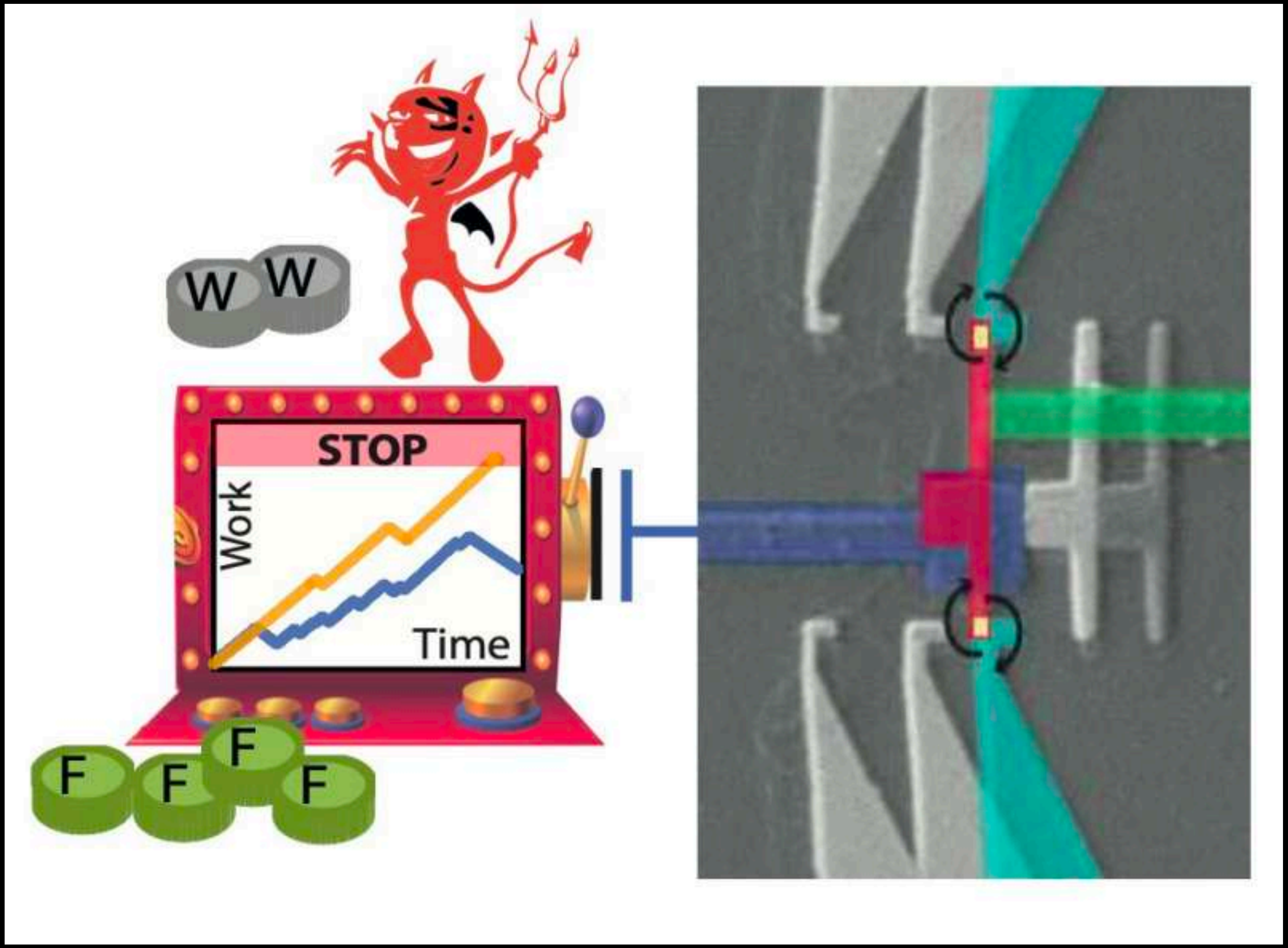


Martingales and Gambling in stochastic thermodynamics

Édgar Roldán

www.edgarroldan.com

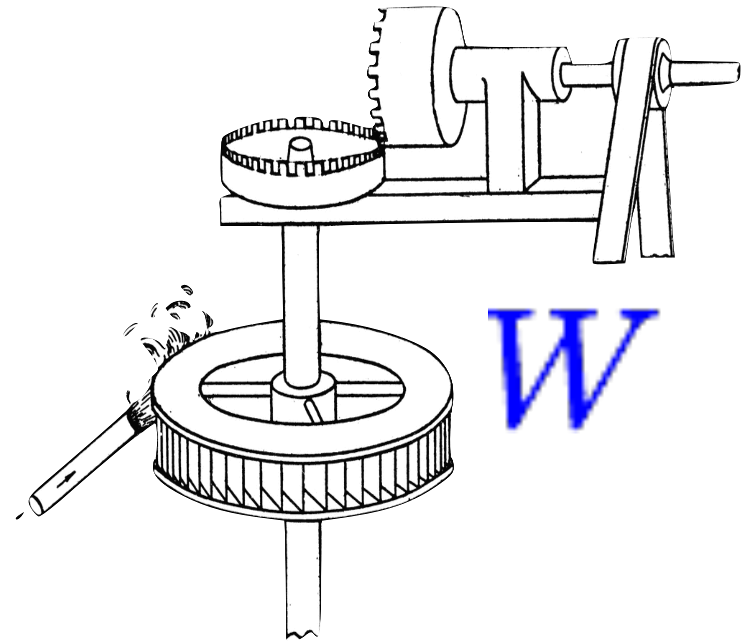
Universita’ di Padova, 22.10.21

Second Law and *stochastic* entropy production for small systems

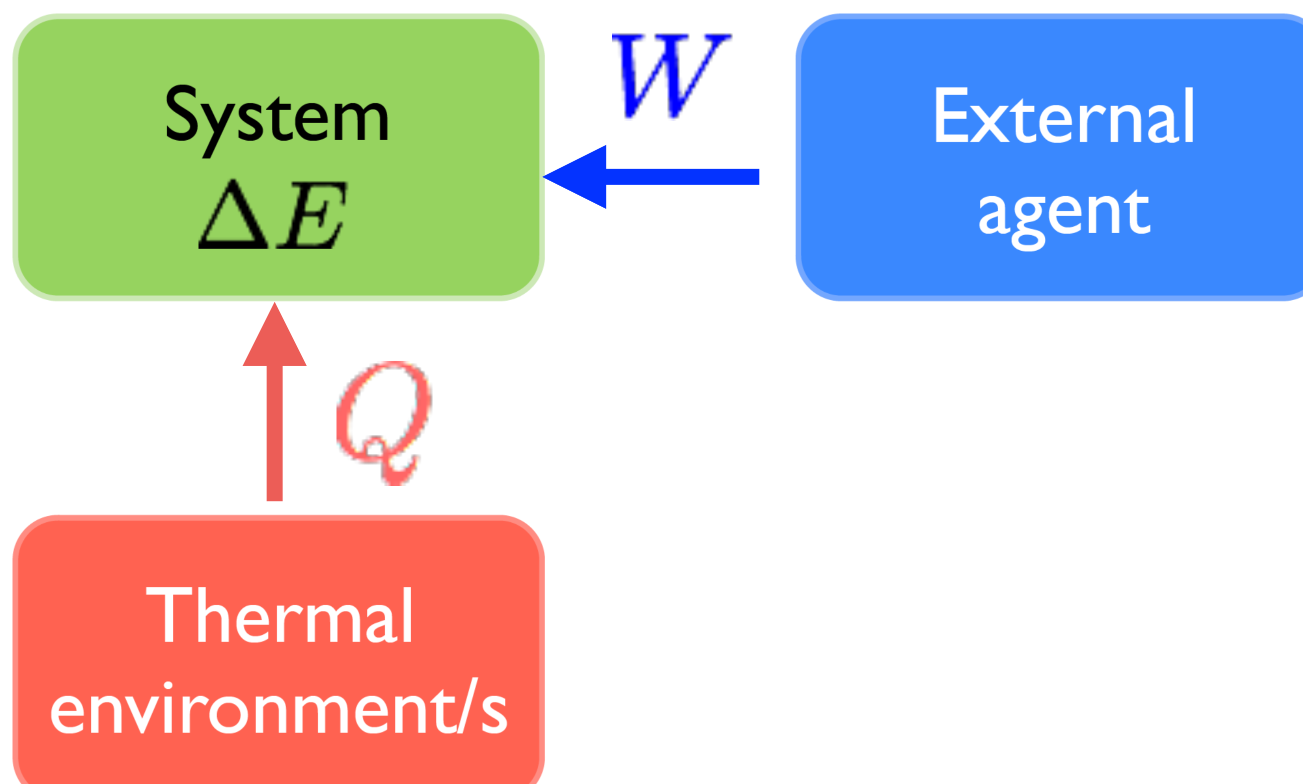
First Law of Thermodynamics

Heat

Work



$$\Delta E = Q + W$$



Second Law

Second Law of Thermodynamics and the arrow of time

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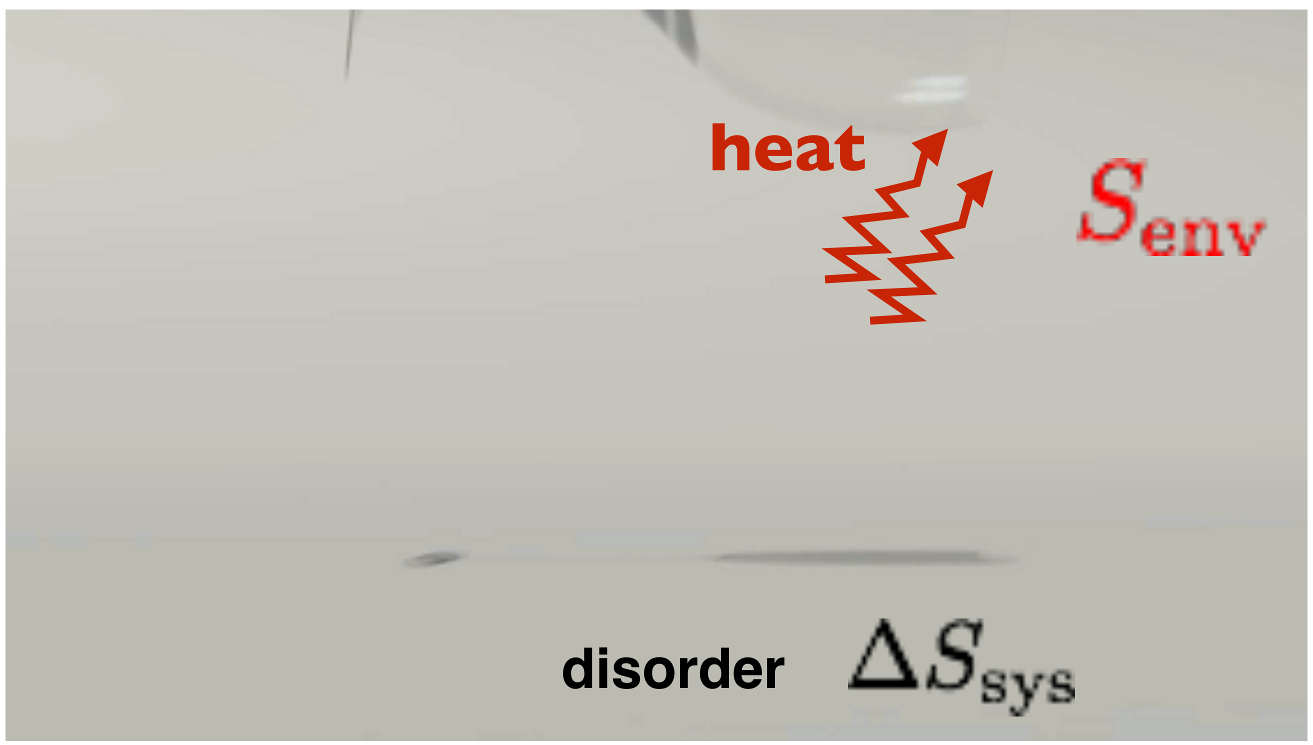
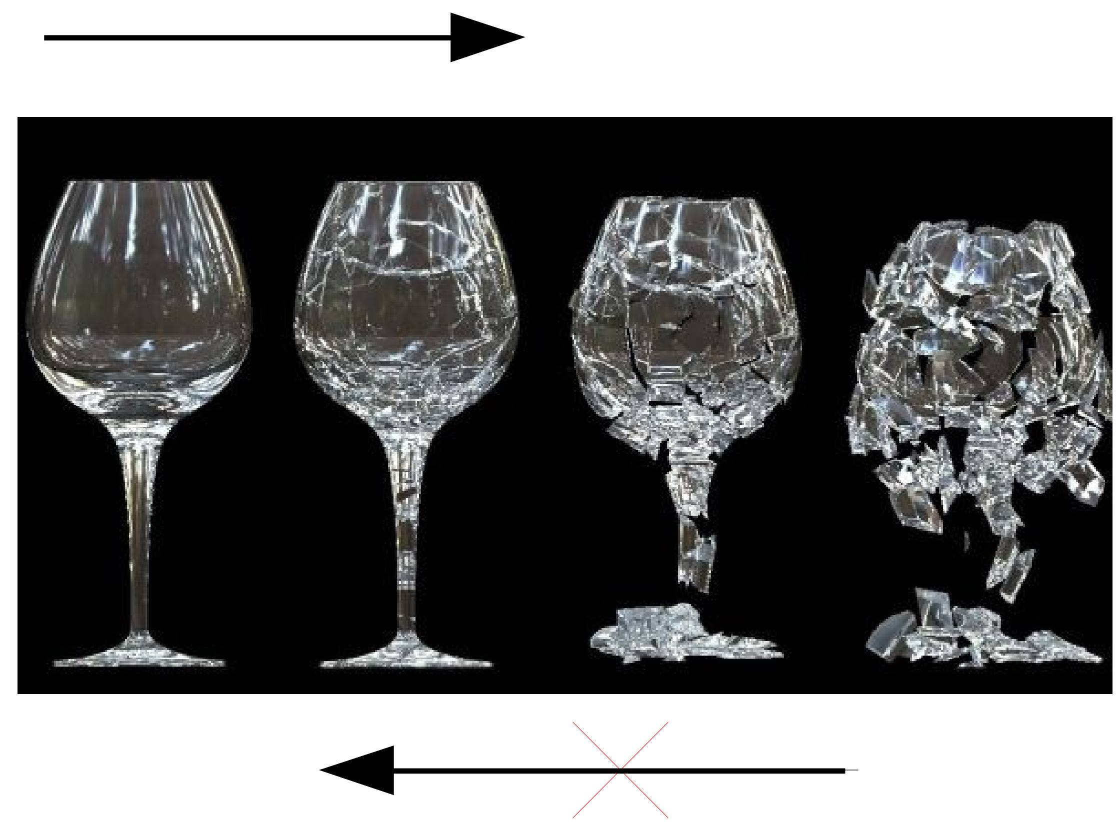
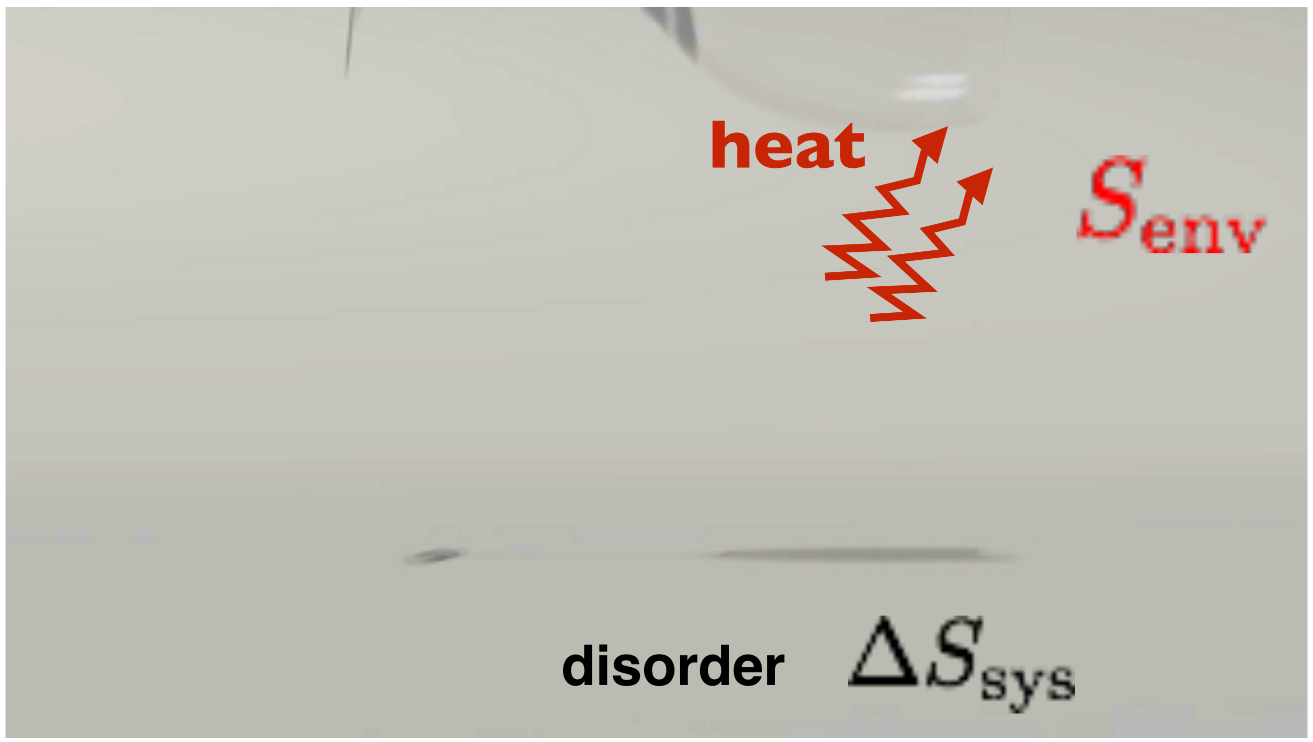
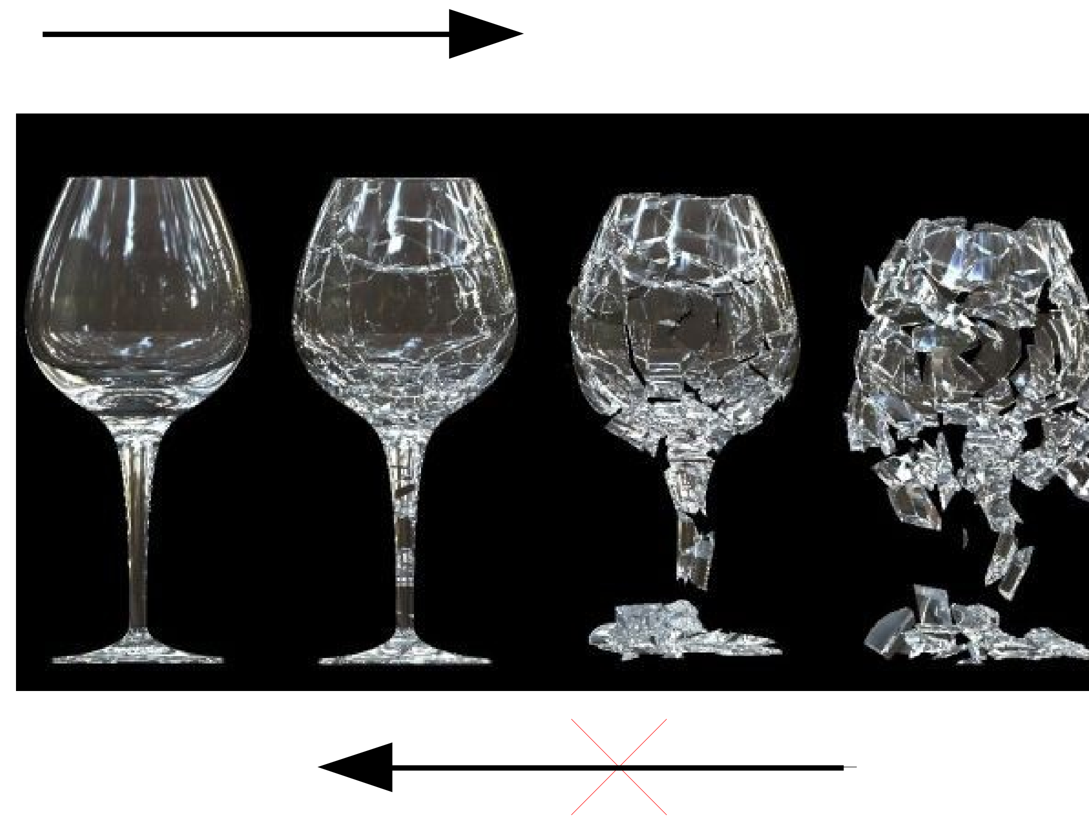




Second Law

Second Law of Thermodynamics and the arrow of time

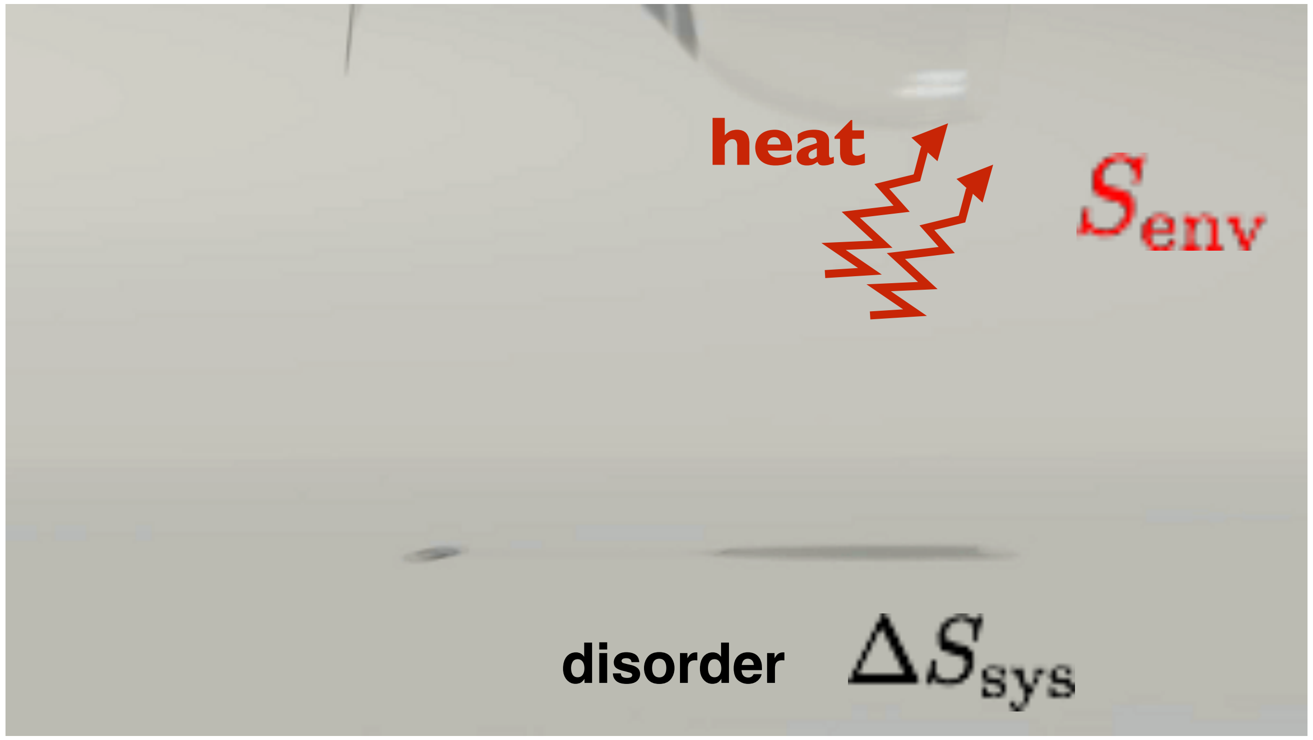




$$S(t) = S_{\rm env}(t) + \Delta S_{\rm sys}(t) \geq 0$$

Second Law

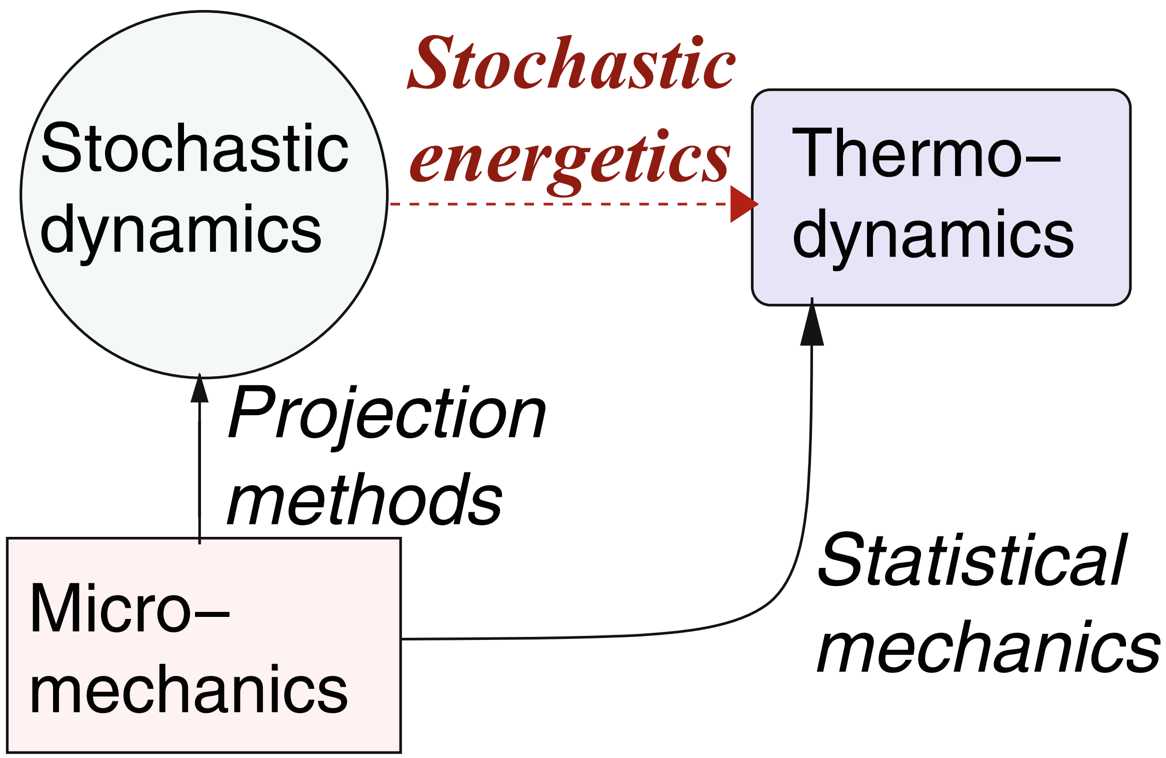
Second Law of Thermodynamics and the arrow of time

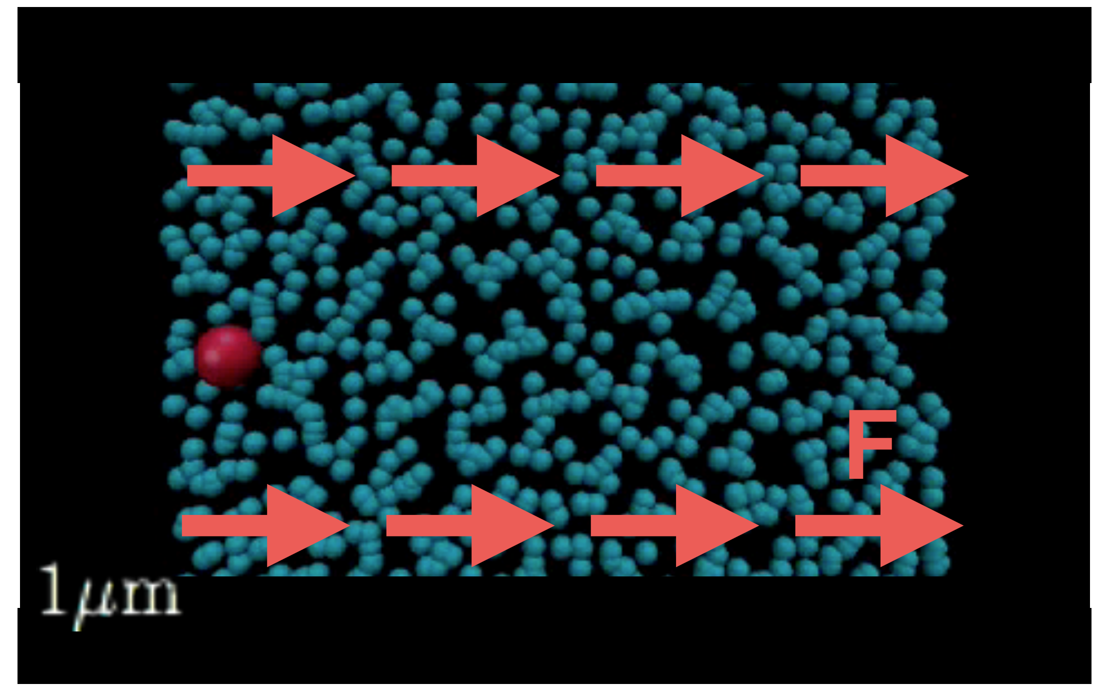


$$S(t) = S_{\mathrm{env}}(t) + \Delta S_{\mathrm{sys}}(t) \geq 0$$

You can't even break even

Stochastic thermodynamics





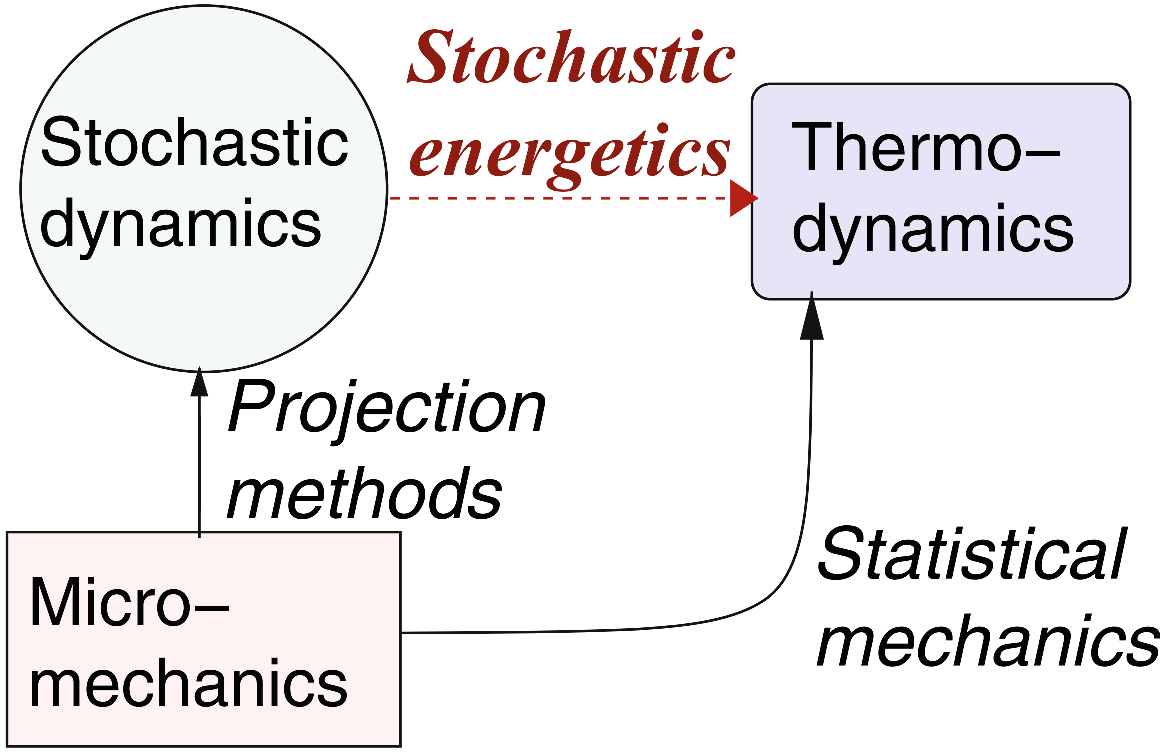
First Law: $\Delta U(t) = Q(t) + W(t)$

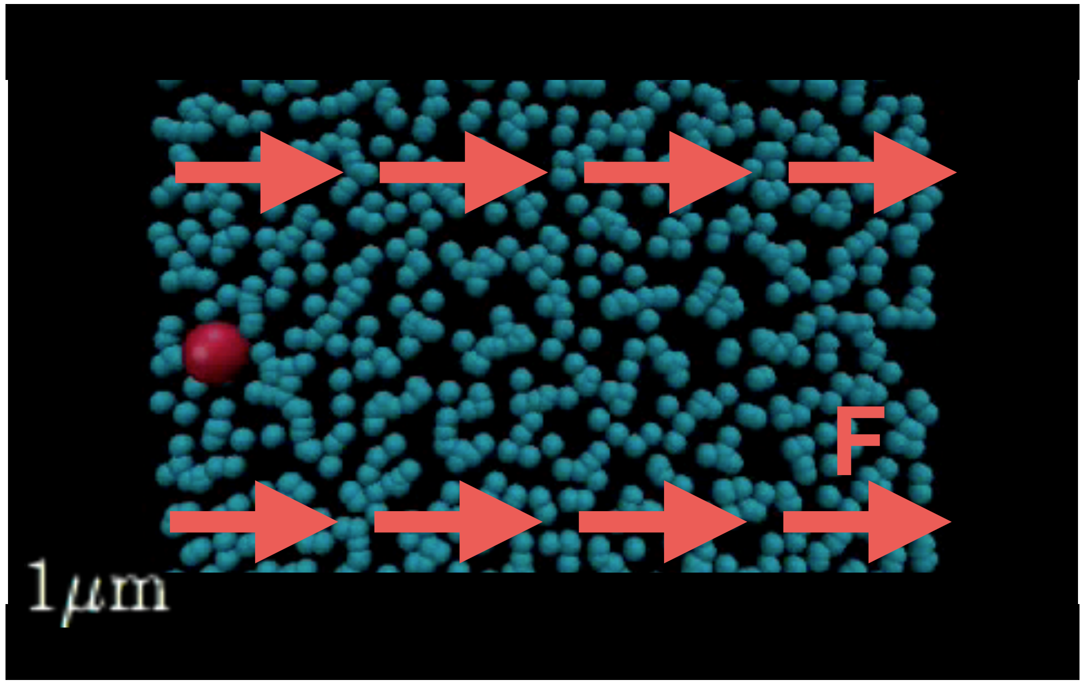
Second Law: $\langle S_{\text{tot}}(t) \rangle \geq 0$

K. Sekimoto, Prog. Theor. Phys. Suppl. 130, 17 (1998)

U. Seifert, Phys. Rev. Lett. **95**(4) 040602 (2005)

Stochastic thermodynamics





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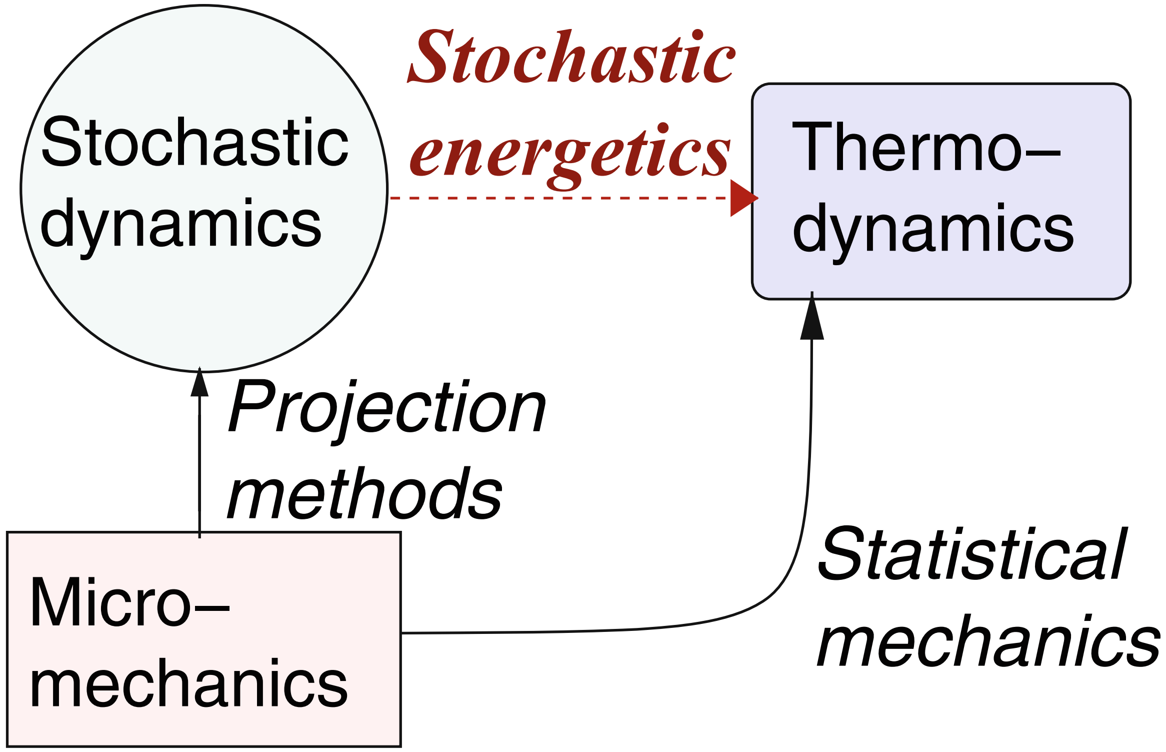

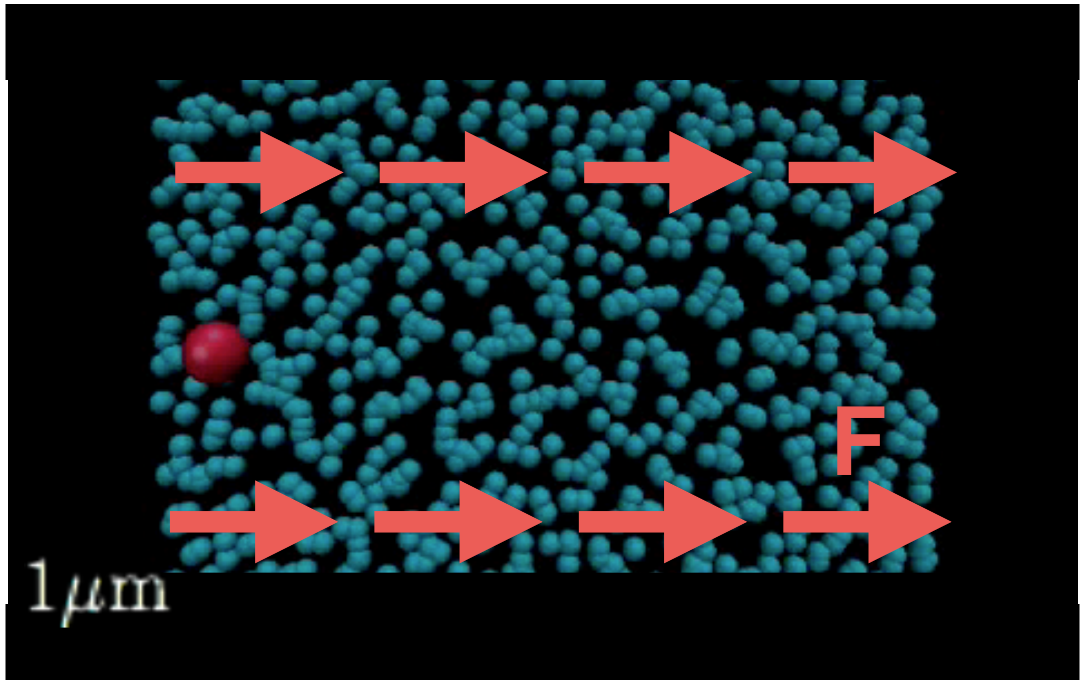

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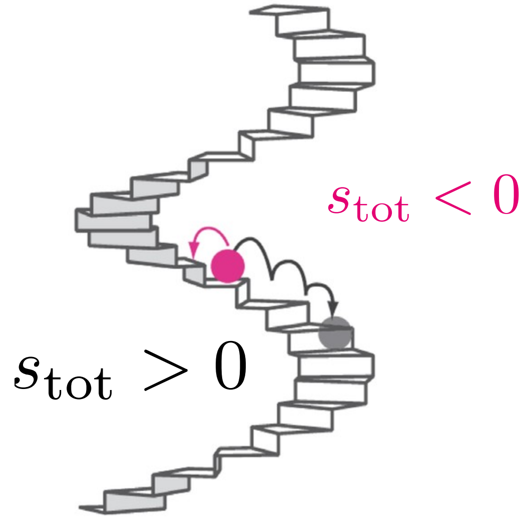

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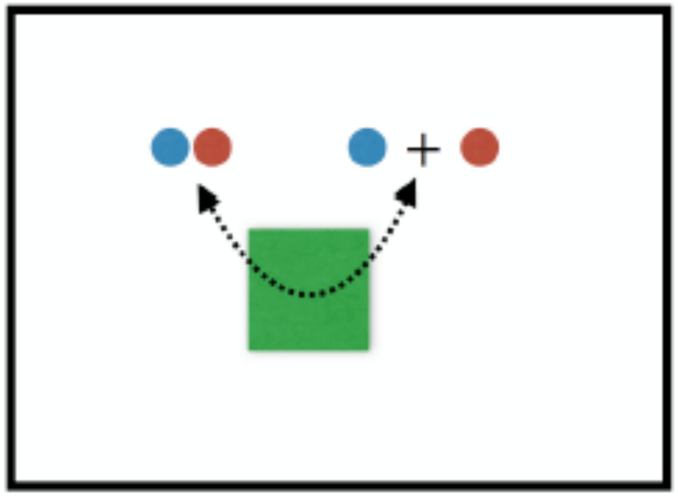
A typical example



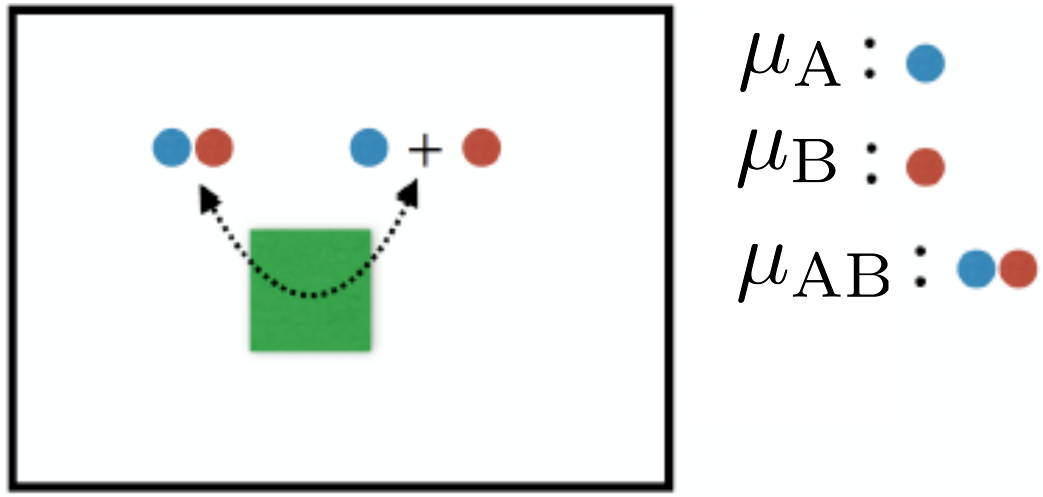
μ_{A} :

μ_{B} :

μ_{AB} :

Thermodynamics: $J(t)\,(\mu_{\mathrm{AB}} - \mu_A - \mu_B) \geq 0$

A typical example



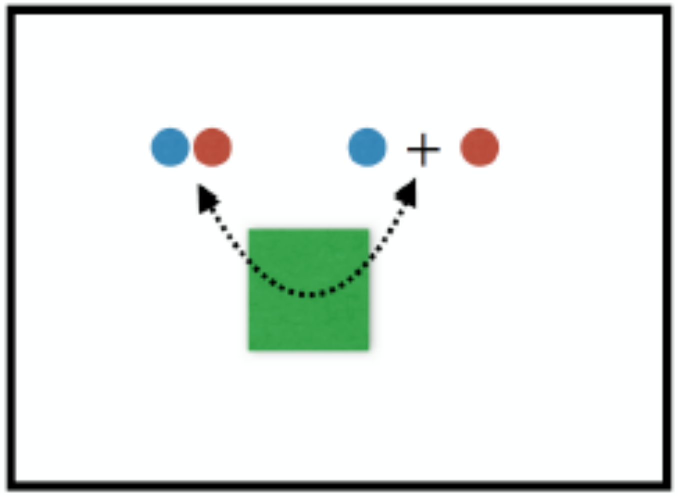
Thermodynamics: $J(t)\,(\mu_{\rm AB} - \mu_A - \mu_B) \geq 0$

Stochastic thermodynamics:

$$\frac{k_{{\color{blue}\bullet}+{\color{red}\bullet}\to{\color{blue}\bullet}{\color{red}\bullet}}}{k_{{\color{blue}\bullet}{\color{red}\bullet}\to{\color{blue}\bullet}+{\color{red}\bullet}}} = e^{S_{\rm env}({\color{blue}\bullet}+{\color{red}\bullet}\to{\color{blue}\bullet}{\color{red}\bullet})} = e^{\frac{\mu_{{\color{blue}\bullet}{\color{red}\bullet}} - \mu_{\color{blue}\bullet} - \mu_{\color{red}\bullet}}{T_{\rm env}}}$$

Stochastic entropy production

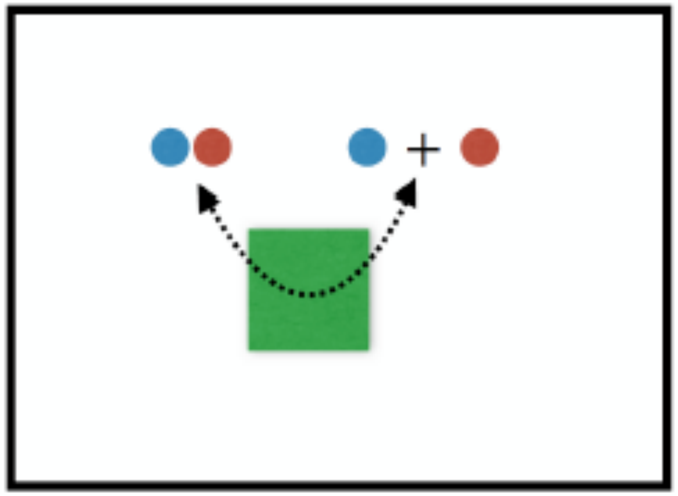
Local detailed balance



$$\frac{P(x(1), x(2), \ldots, x(t)|x(0))}{P(x(t-1), x(t-2), \ldots, x(0)|x(t))} = e^{S_{\mathrm{env}}(t)/k_{\mathrm{B}}}$$

U. Seifert, Phys. Rev. Lett. **95**(4) 040602 (2005)

Stochastic entropy production



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Nonequilibrium system entropy

$$S_{\mathrm{sys}}(t) \equiv -k_{\mathrm{B}} \ln P_{\mathrm{st}}(x(t))$$

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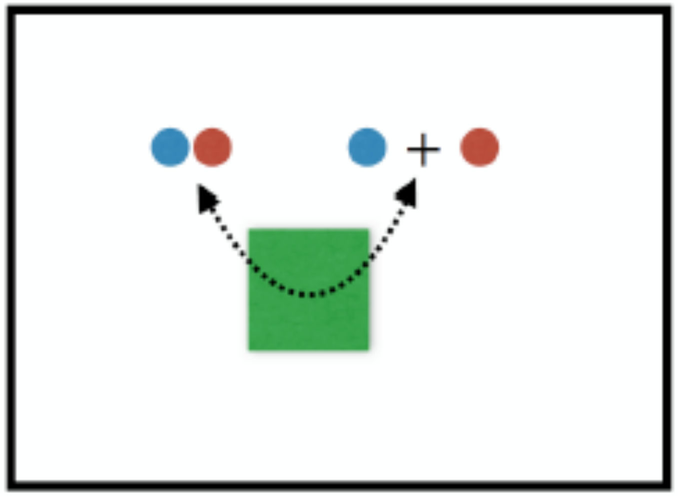
Stochastic entropy production



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Nonequilibrium system entropy

$$S_{\text{sys}}(t) \equiv -k_{\text{B}} \ln P_{\text{st}}(x(t))$$

Stochastic entropy production

$$S_{\text{tot}}(t) = \Delta S_{\text{sys}}(t) + S_{\text{env}}(t) = k_{\text{B}} \ln \frac{P(x(0), x(1), x(2), \ldots, x(t))}{P(x(t), x(t-1), x(t-2), \ldots, x(0))}$$

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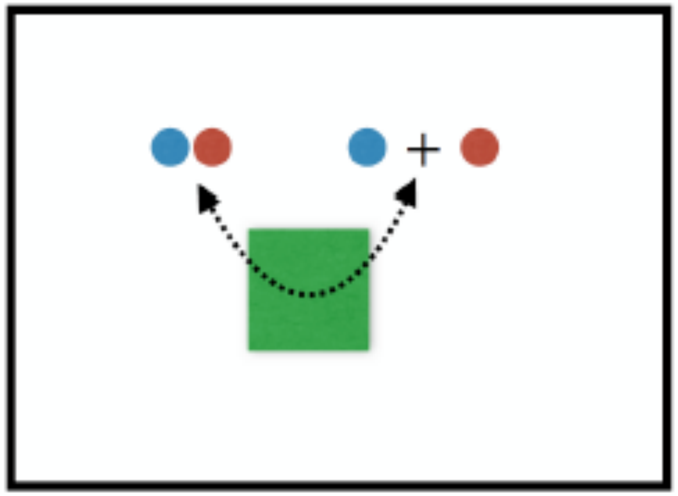
Stochastic entropy production

Local detailed balance



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Stochastic entropy production

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$$S_{\text{tot}}(t) = k_{\text{B}} \ln \frac{P(X_{[0,t]})}{P(\Theta X_{[0,t]})}$$

Time-reversed measure

$$\tilde{P}(X_{[0,t]}) \equiv P(\Theta X_{[0,t]})$$

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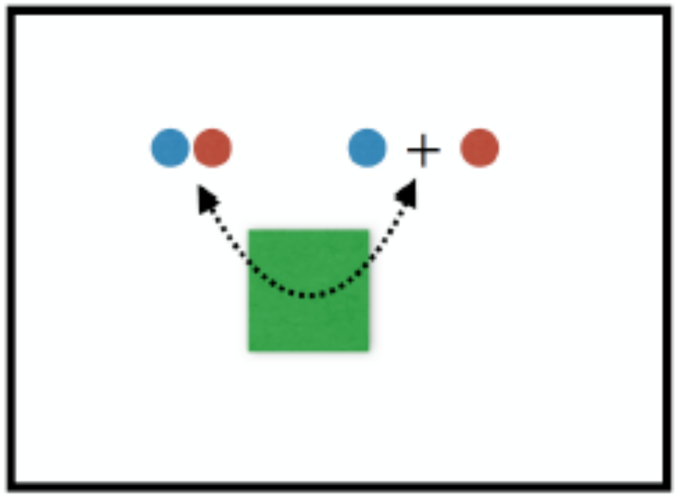
Stochastic entropy production



Local detailed balance

$$\frac{P(x(1),x(2),\dots,x(t)|x(0))}{P(x(t-1),x(t-2),\dots,x(0)|x(t))} = e^{S_{\mathrm{env}}(t)/k_{\mathrm{B}}}$$

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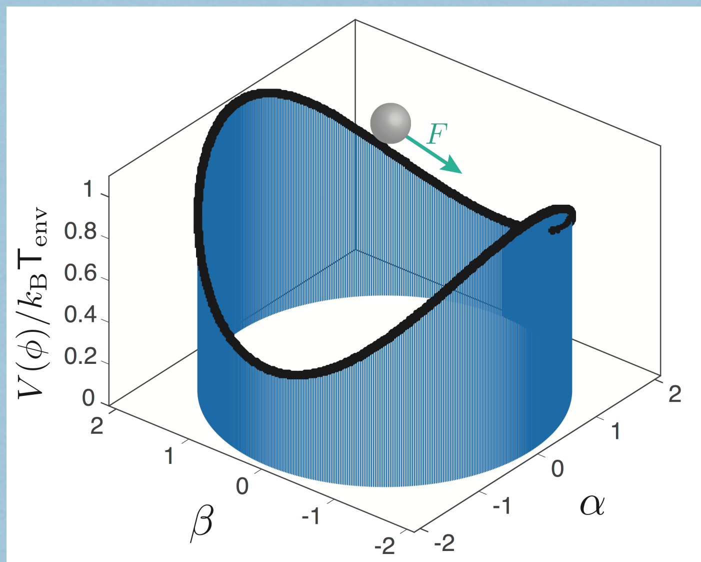
$$S_{\mathrm{tot}}(t) = \Delta S_{\mathrm{sys}}(t) + S_{\mathrm{env}}(t) = k_{\mathrm{B}} \ln \frac{P(x(0),x(1),x(2),\dots,x(t))}{P(x(t),x(t-1),x(t-2),\dots,x(0))}$$

Second law of stochastic thermodynamics

$$\langle S_{\mathrm{tot}}(t)\rangle = \int dx(0)\cdots\int dx(t) P(X_{[0,t]}) \ln \frac{P(X_{[0,t]})}{\tilde{P}(X_{[0,t]})} = D[P(X_{[0,t]})||\tilde{P}(X_{[0,t]})] \geq 0$$

U. Seifert, Phys. Rev. Lett. **95**(4) 040602 (2005)

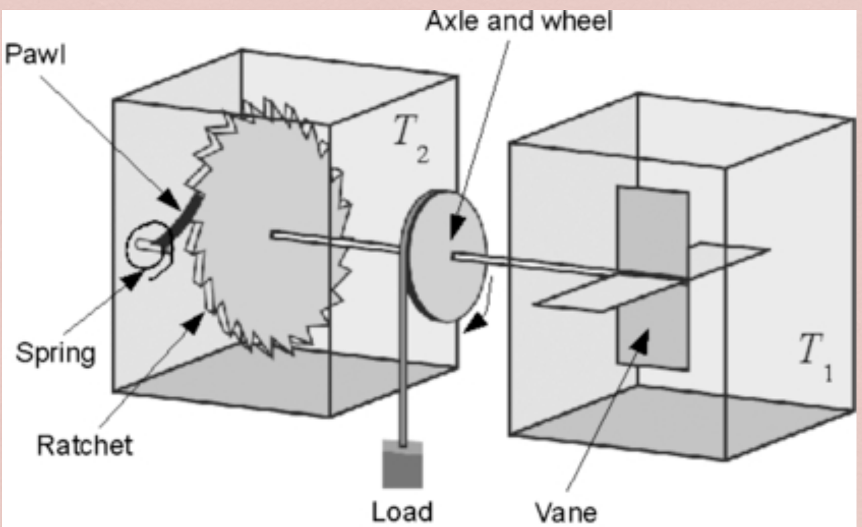
Examples



Isothermal non-equilibrium steady states

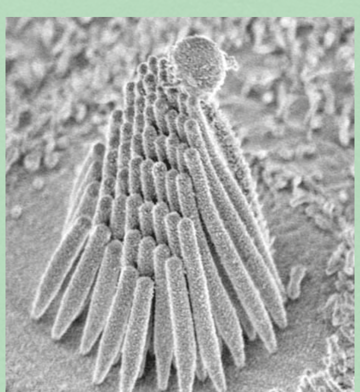
$$S_{tot}(t) = \frac{W(t) - \Delta F(t)}{T}$$

Noneq. free energy $\quad P_{st}(x) = e^{-\beta(U(x) - F(x))}$



Non-isothermal steady states
(e.g. autonomous heat engines)

$$S_{tot}(t) = \Delta S_{sys}(t) - \frac{Q_h(t)}{T_h} - \frac{Q_c(t)}{T_c}$$



Active matter
systems with hidden nonequilibrium degrees of freedom

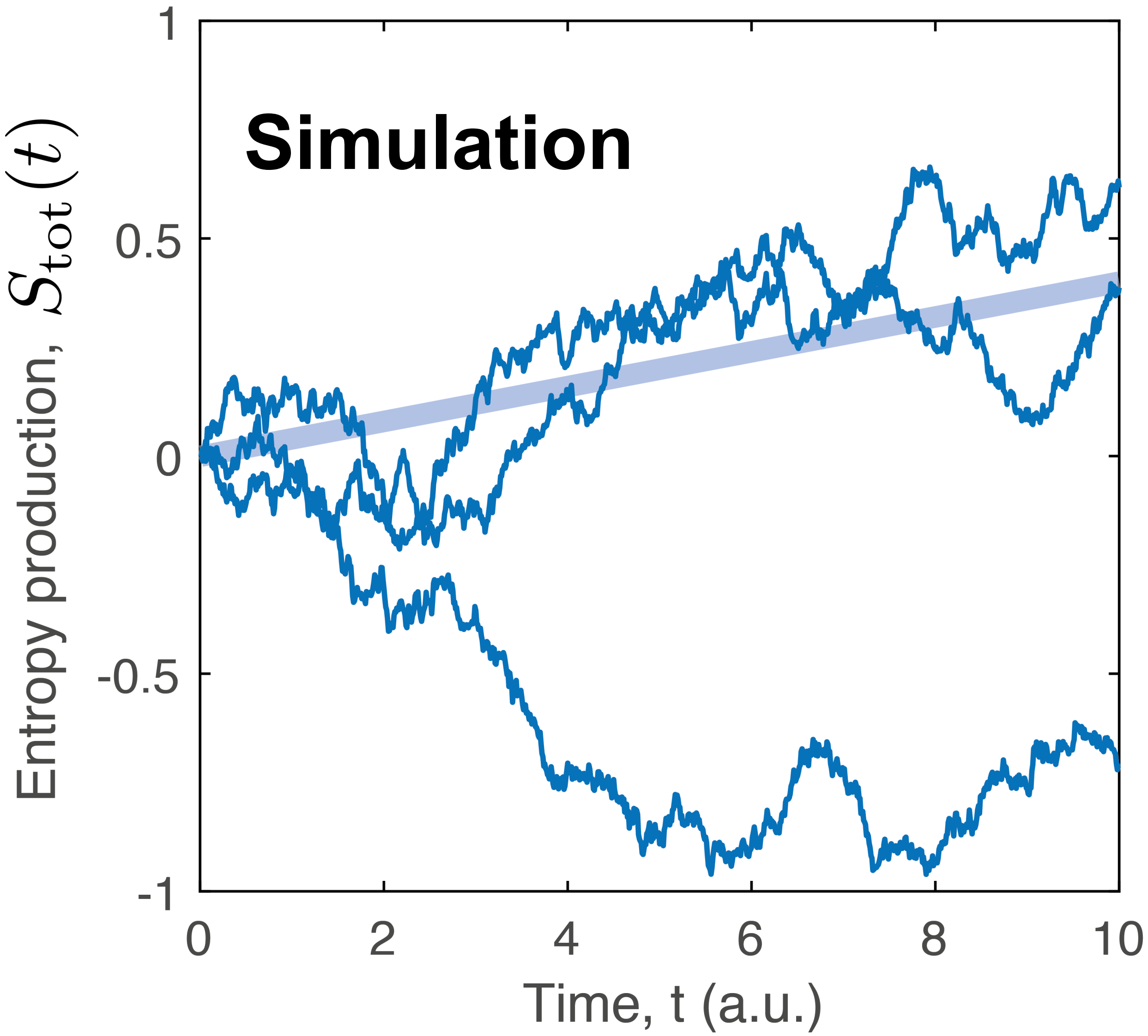
$$S_{tot}(t) = \text{Irreversibility}$$

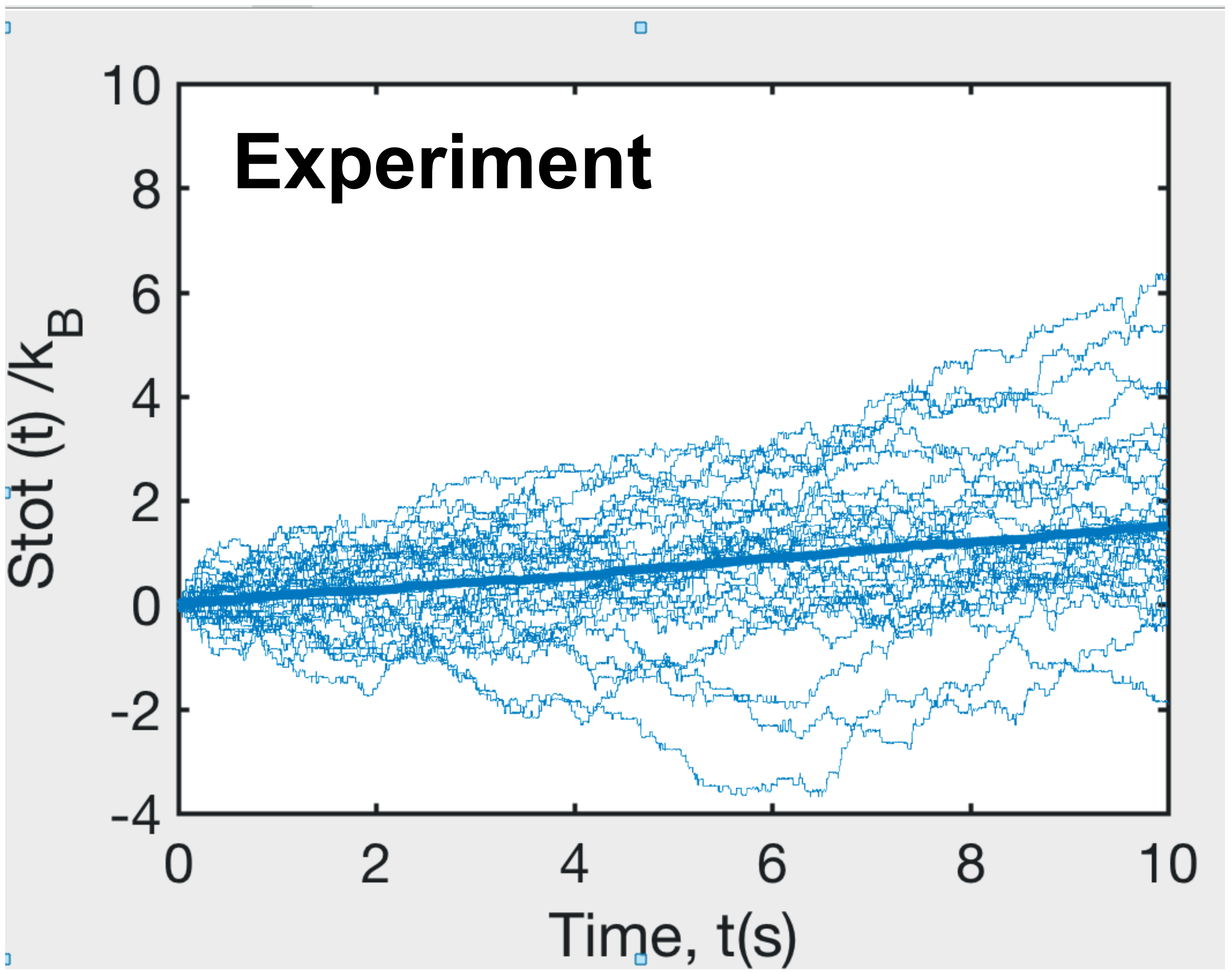
$$-\frac{\langle Q(t)\rangle}{T} \geq \langle S_{tot}(t)\rangle = k_B D[P(X_{[0,t]})||\tilde{P}(X_{[0,t]})]$$

Basic knowledge on stochastic entropy

Equilibrium : $S_{\rm tot}(t) = 0 \Rightarrow \langle S_{\rm tot}(t) \rangle = 0$

Nonequilibria: $\langle S_{\rm tot}(t) \rangle \geq 0$





Shilpi Singh (Jukka Pekola Lab)

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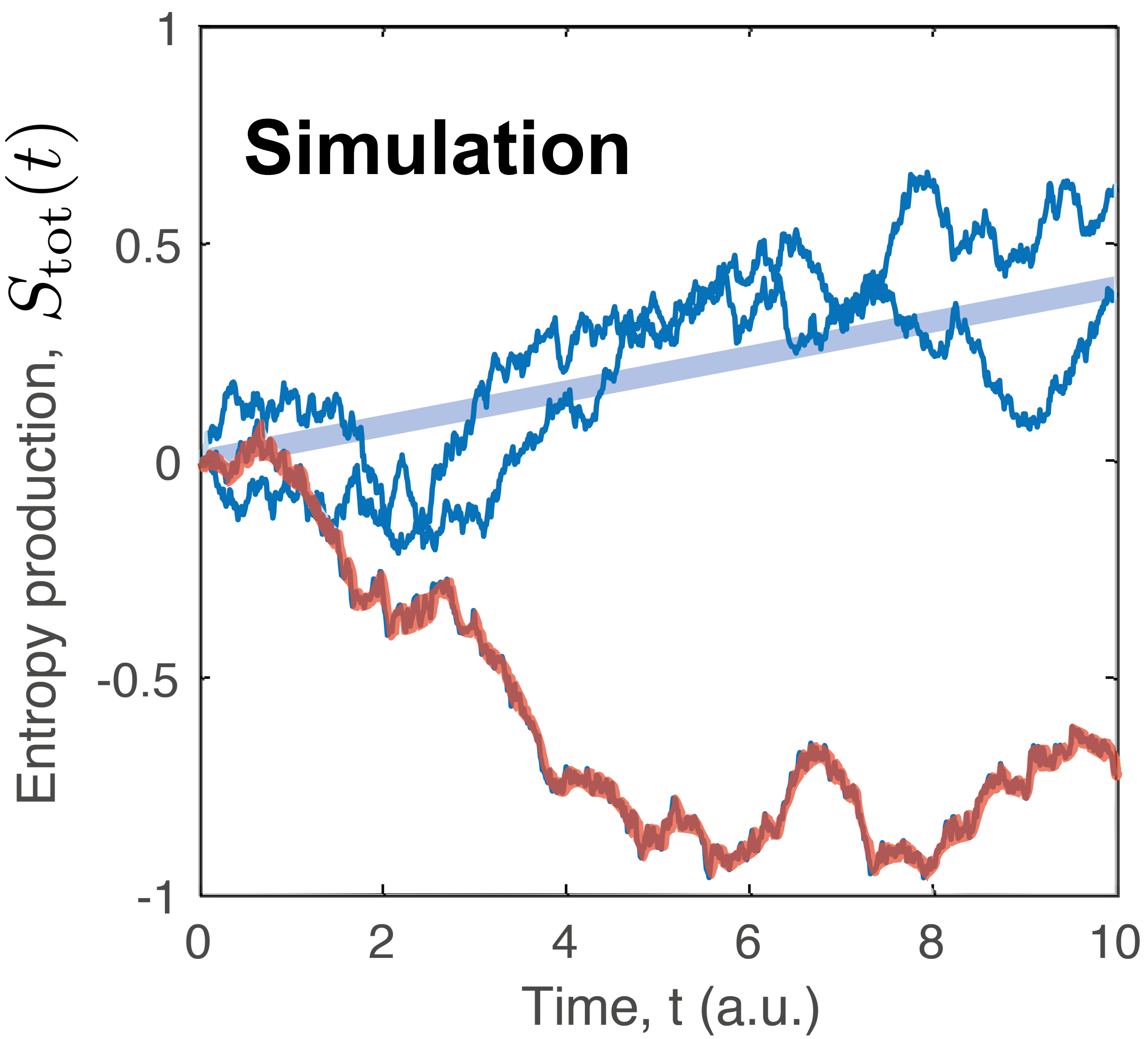

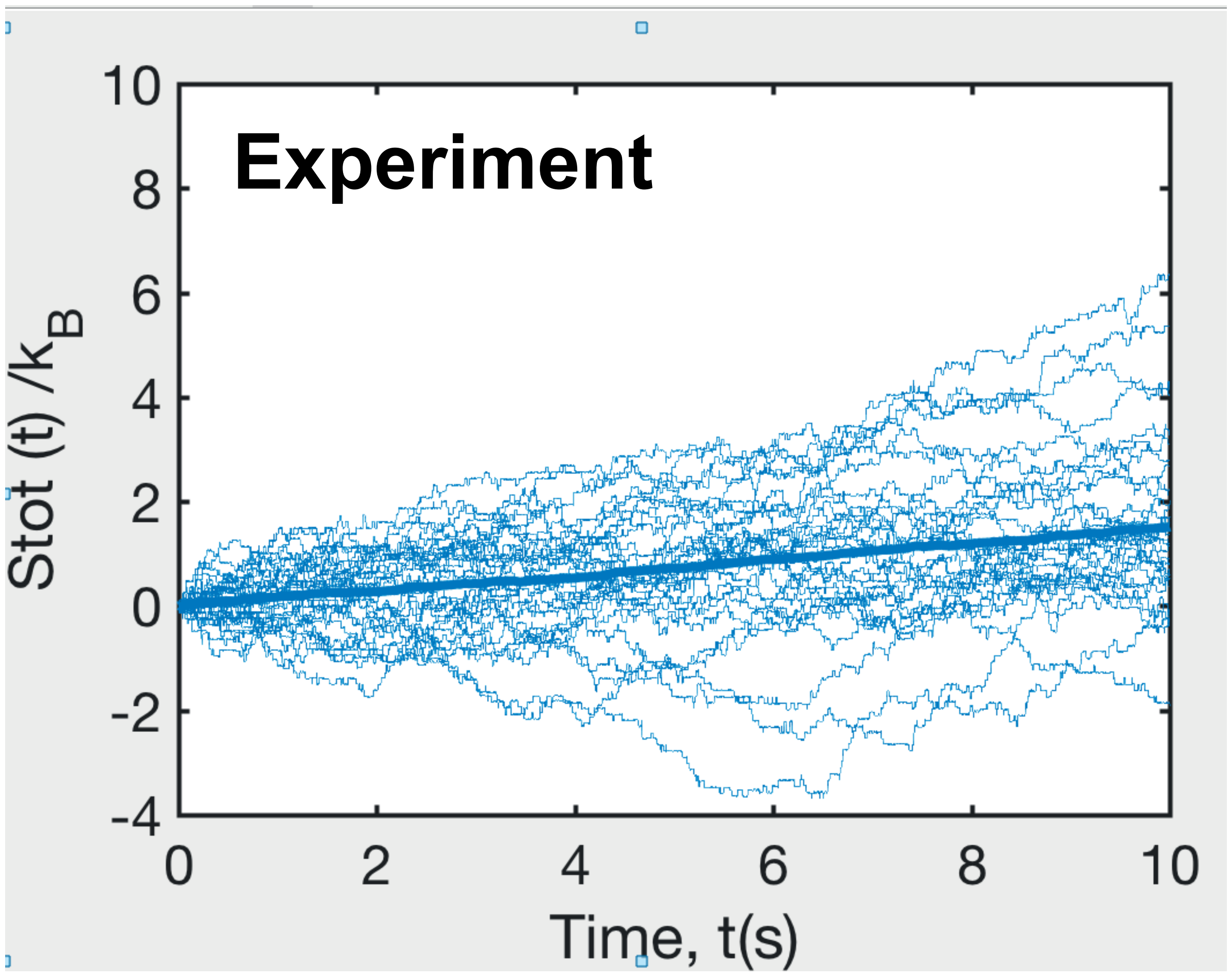

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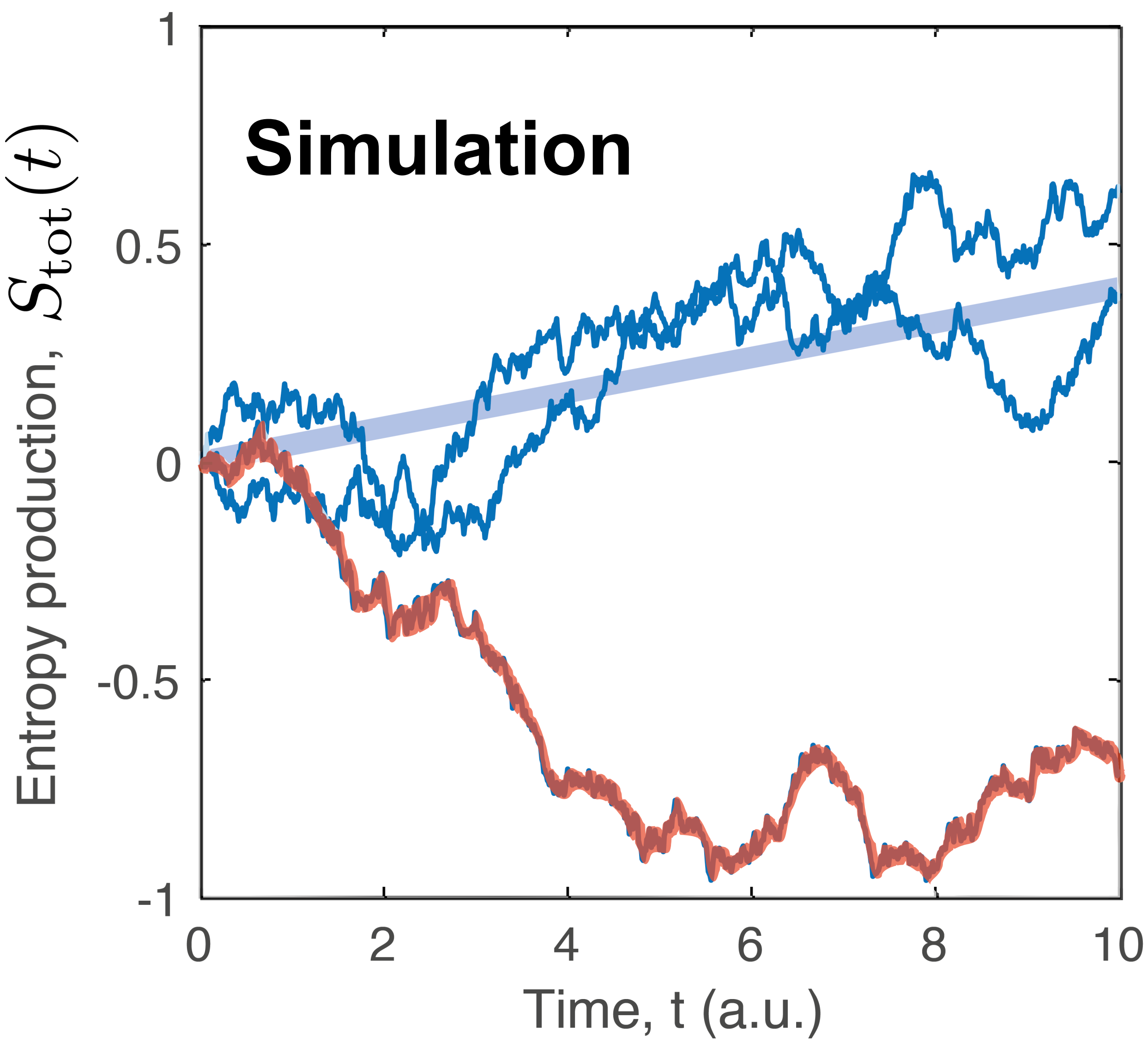
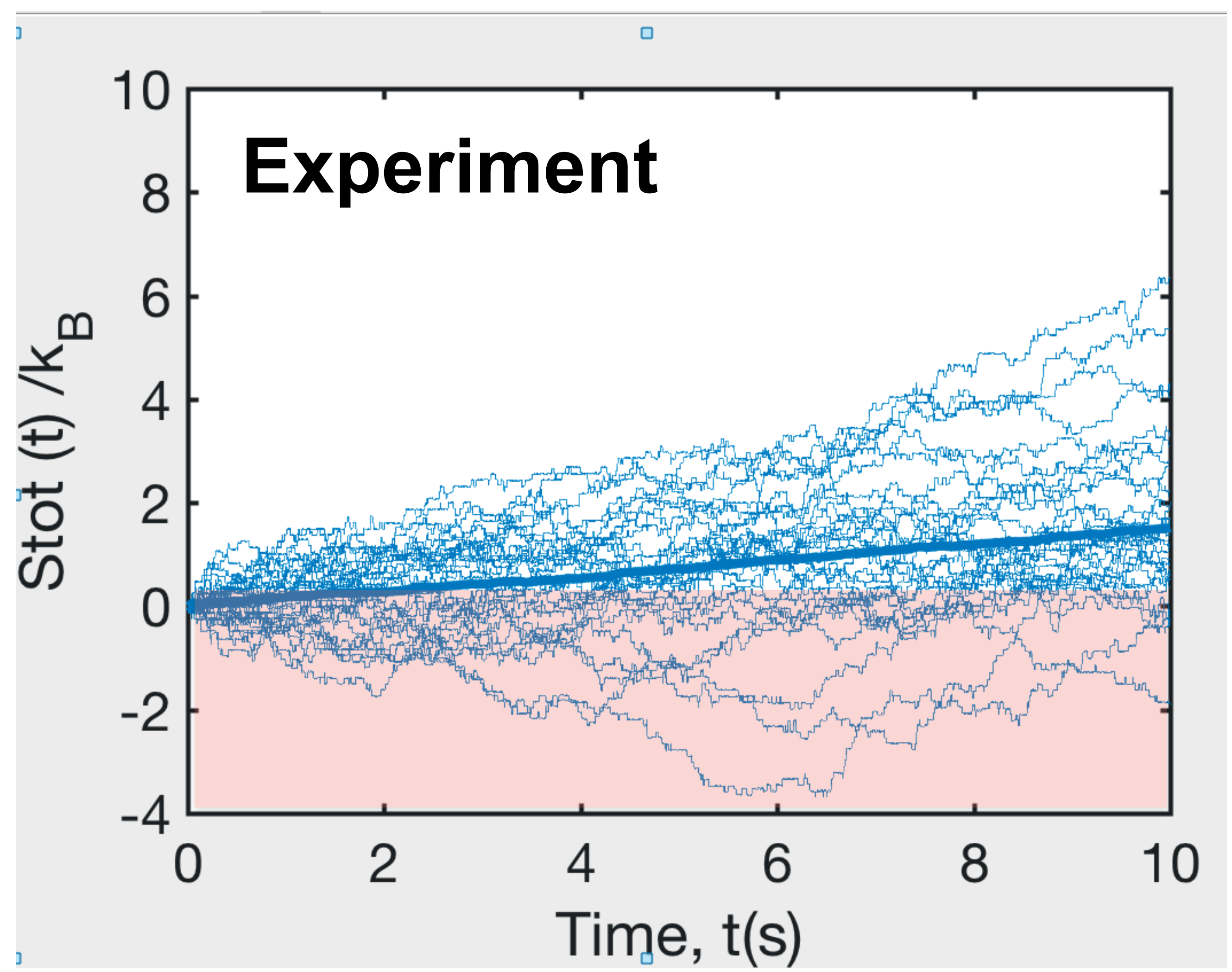
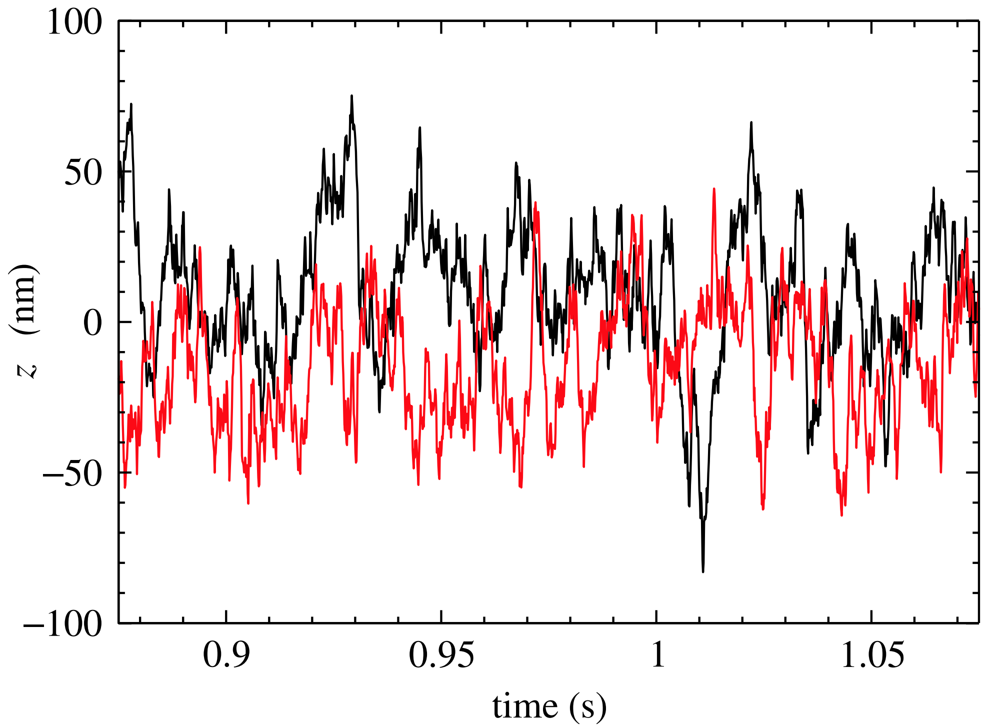


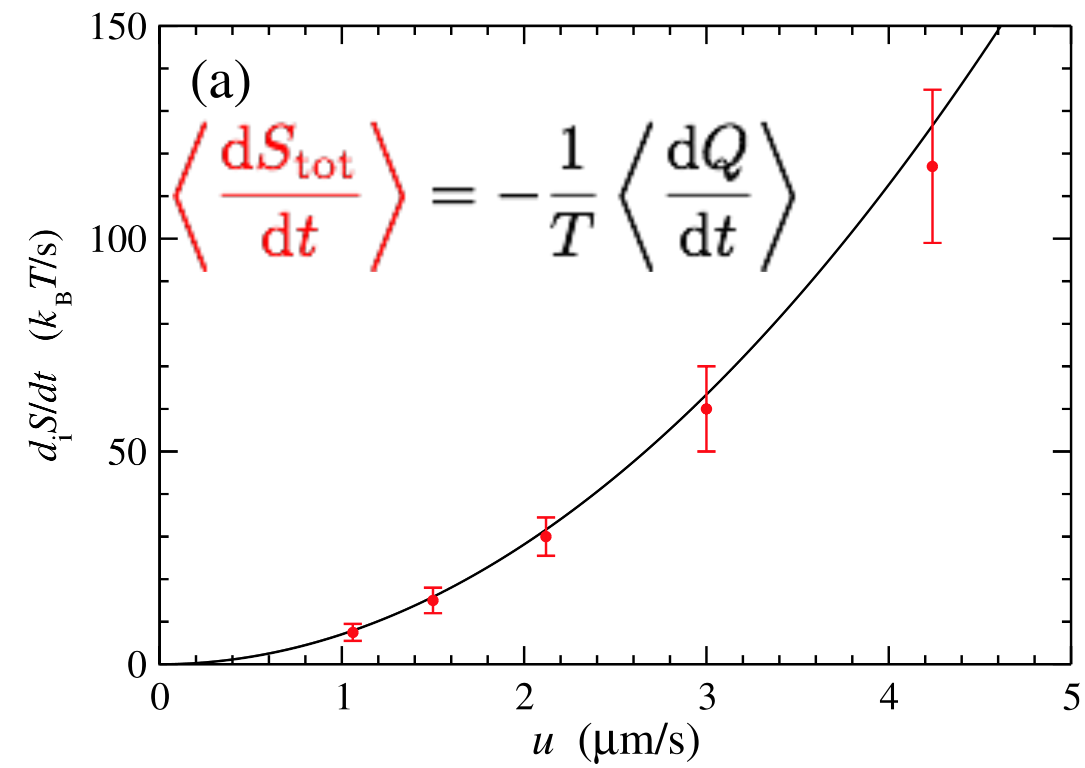


Shilpi Singh (Jukka Pekola Lab)

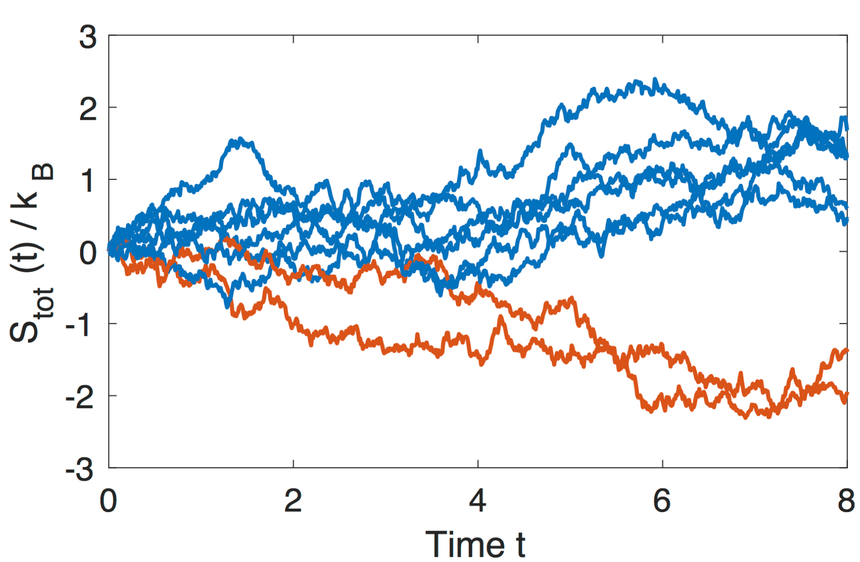
Fluctuation theorems



Andrieux, Gaspard, Ciliberto, Garnier, Joubaud, Petrosyan, PRL 2007



<u>Fluctuation theorems</u> for stochastic entropy production



Detailed Fluctuation theorem

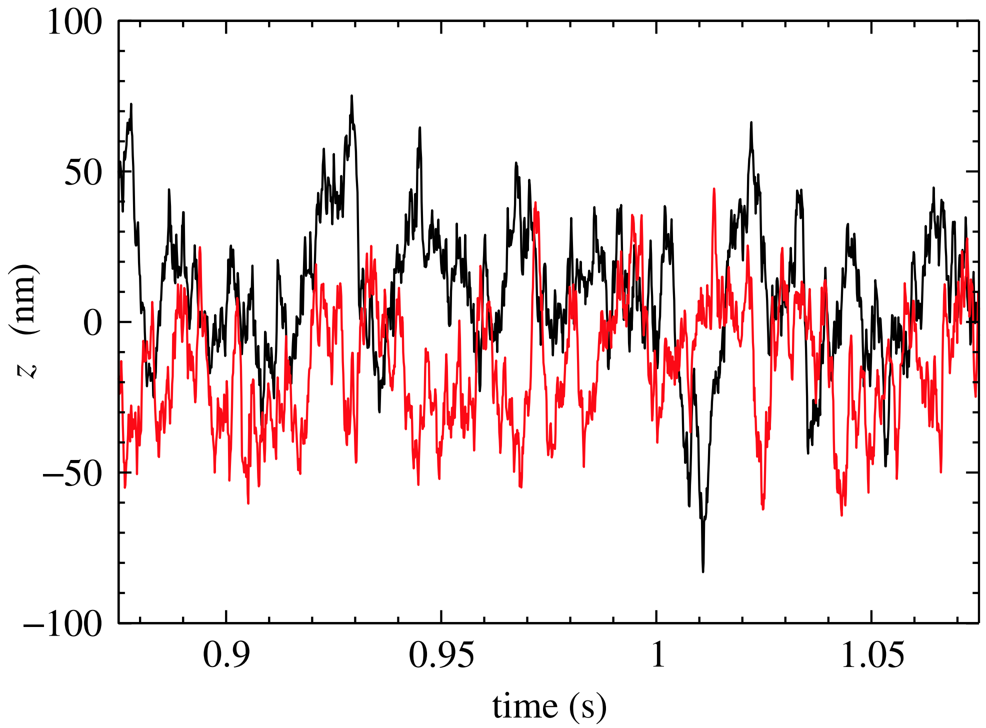
$$\frac{p_S(s;t)}{p_S(-s;t)} = e^{s/k_B}$$

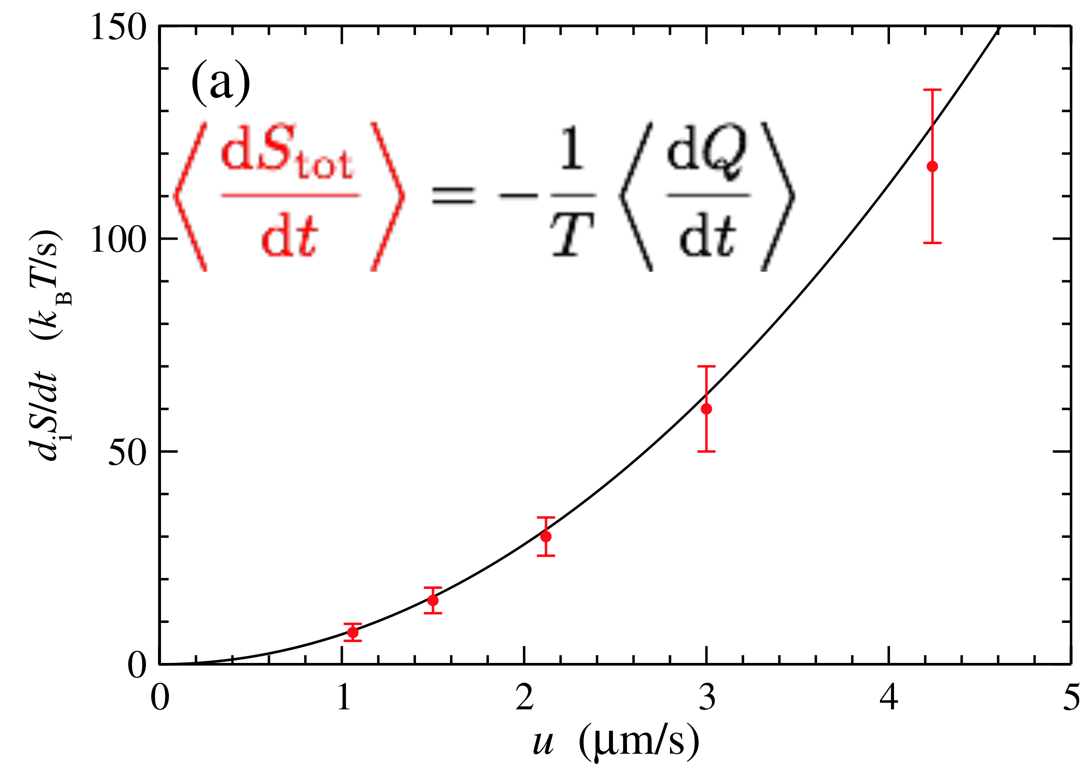
Jarzynski's equality (Integral FT)

$$\langle e^{-S(t)/k_B} \rangle = 1$$

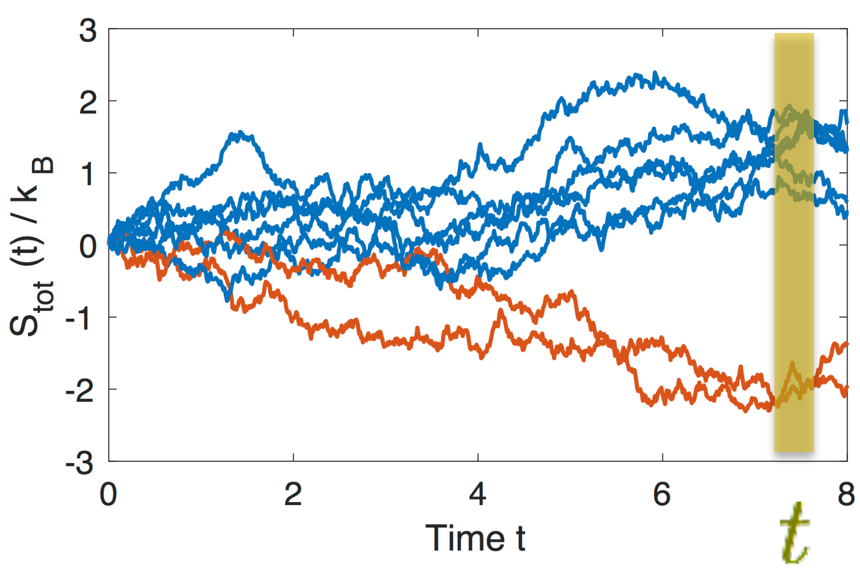
Fluctuation theorems

Andrieux, Gaspard, Ciliberto, Garnier, Joubaud, Petrosyan, PRL 2007





Fluctuation theorems for stochastic entropy production



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(Integral FT)

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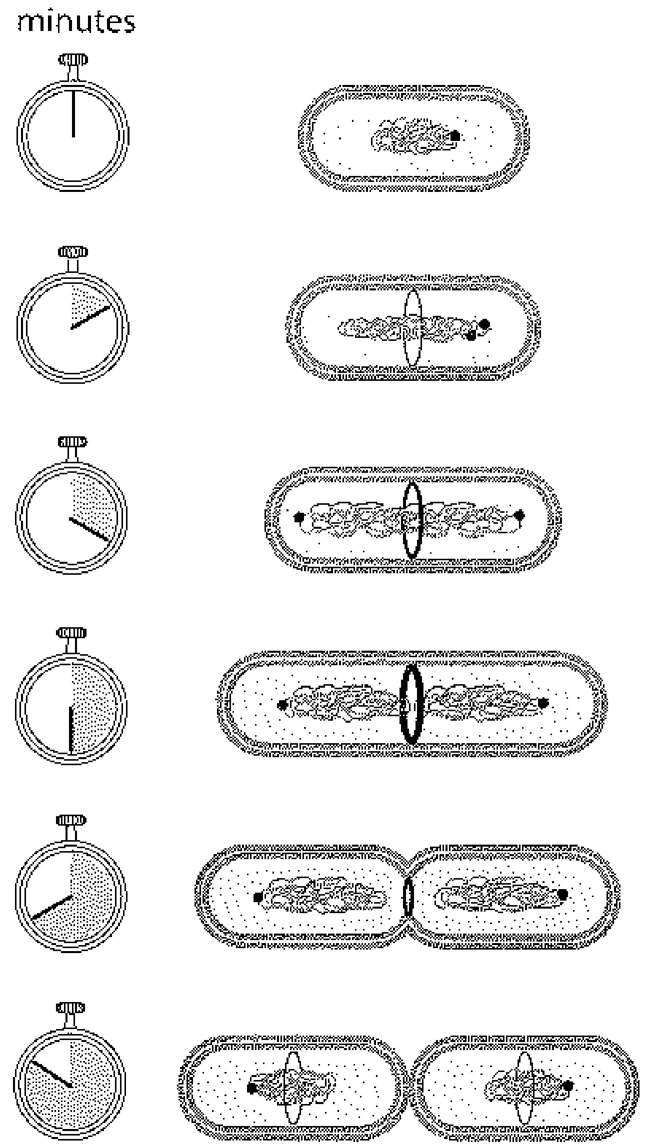
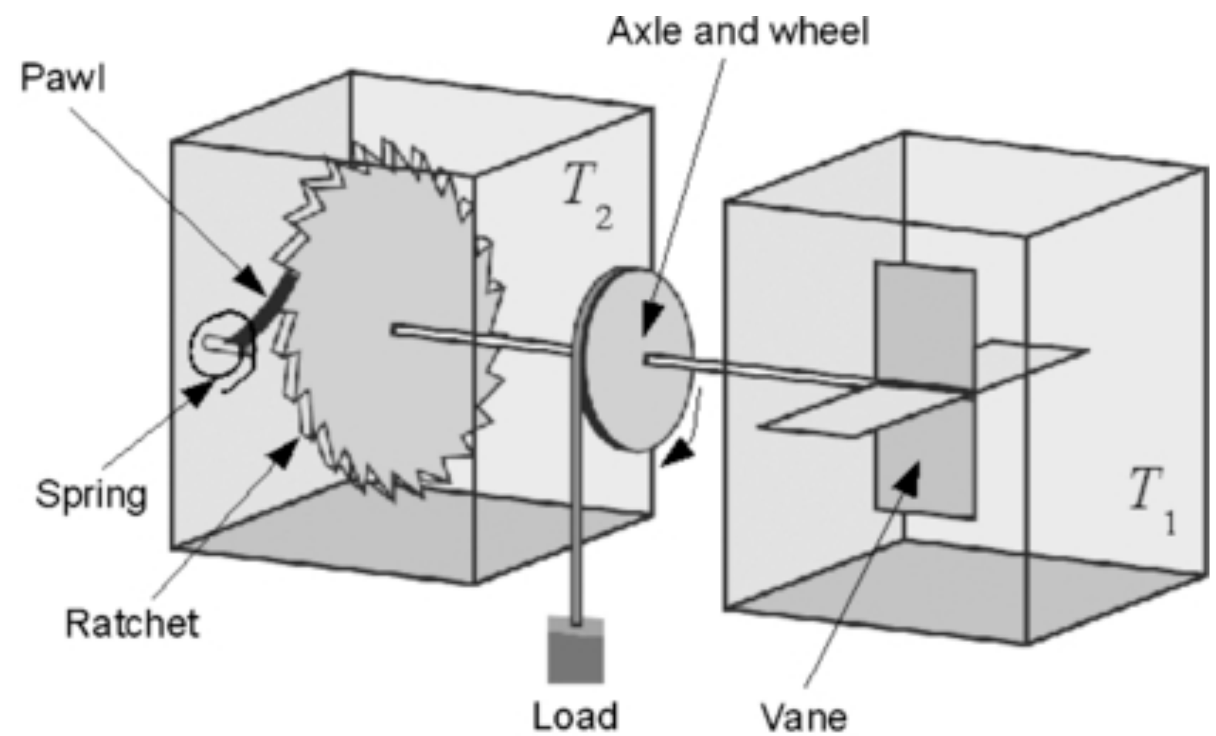
Fixed time properties

Why martingales?

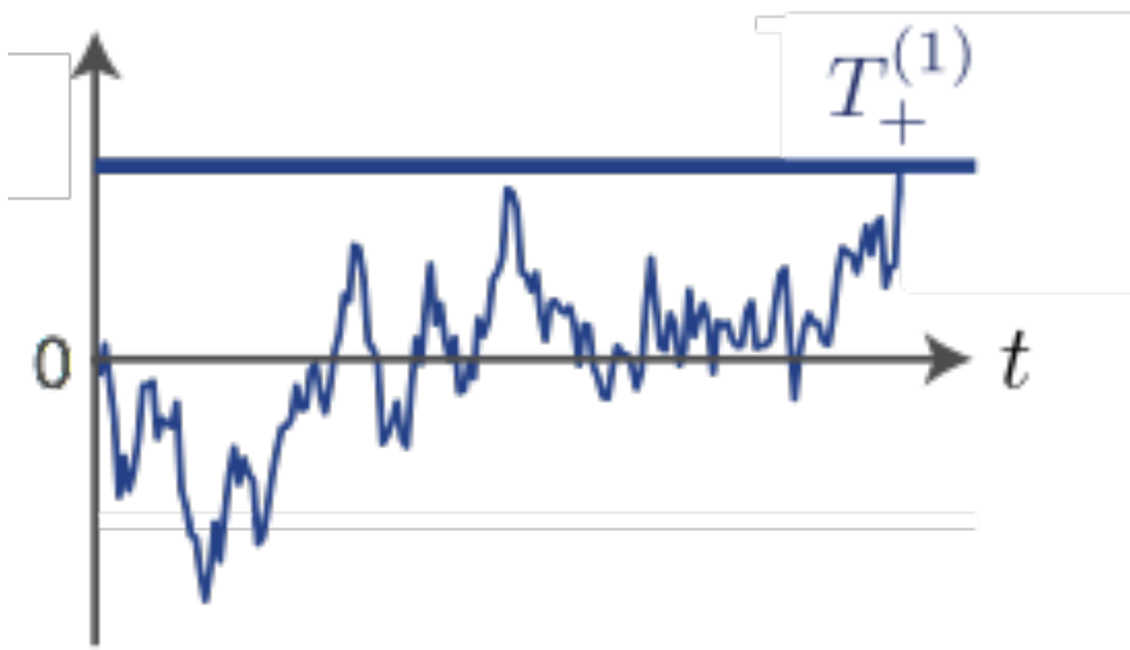
Most fluctuation theorems concern events that take place **at a fixed time**

However, interesting phenomena take place at ***stochastic times***

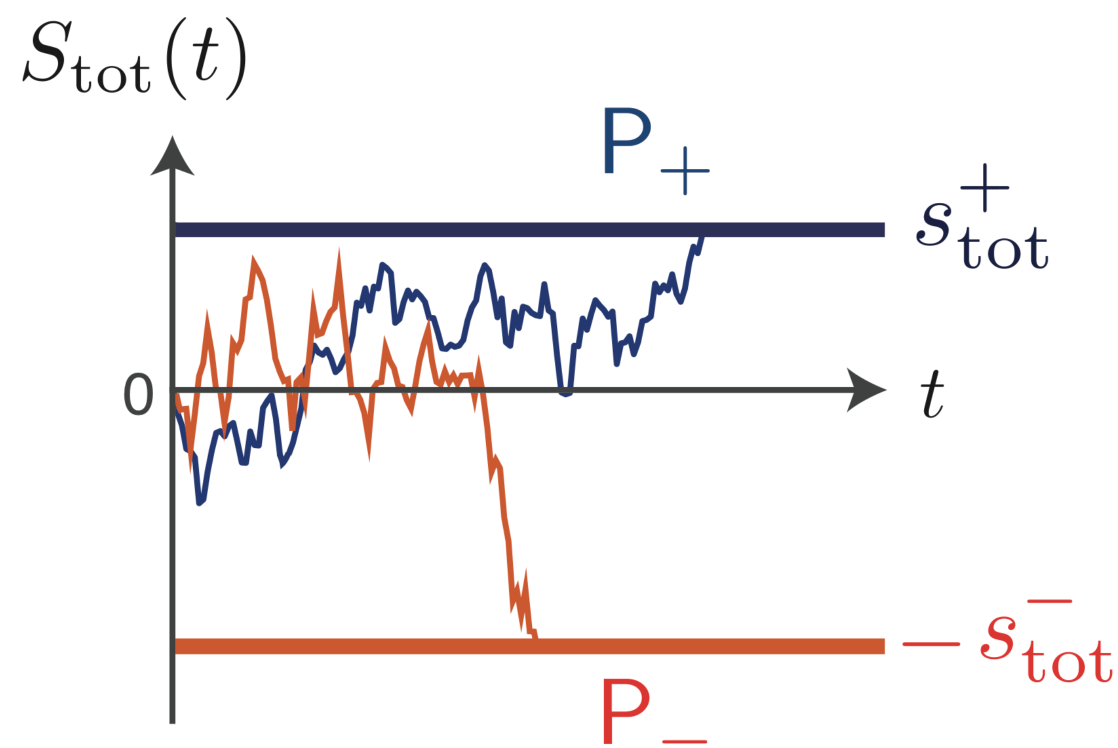




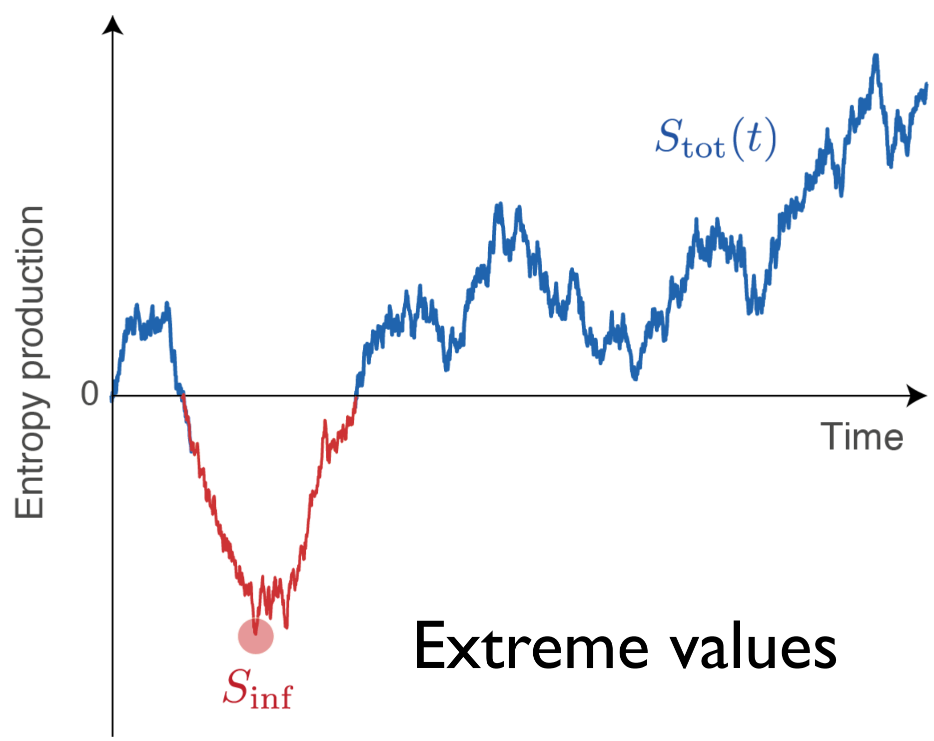
Unknown properties of entropy production



(Random) time to reach a certain threshold

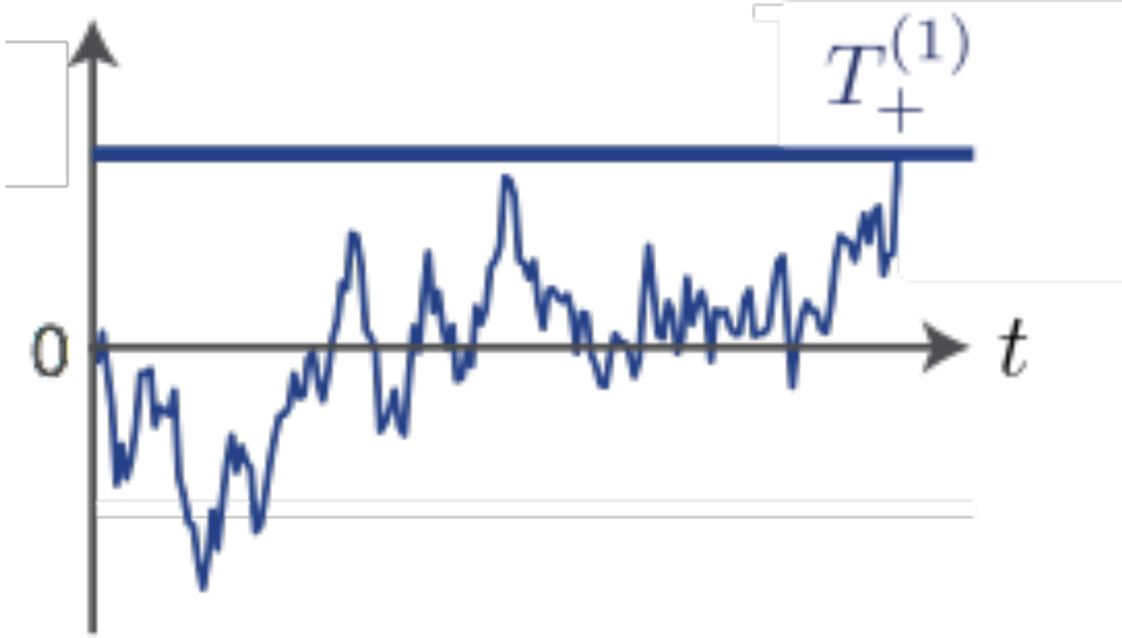


Splitting probabilities



Extreme values

Unknown properties of entropy production



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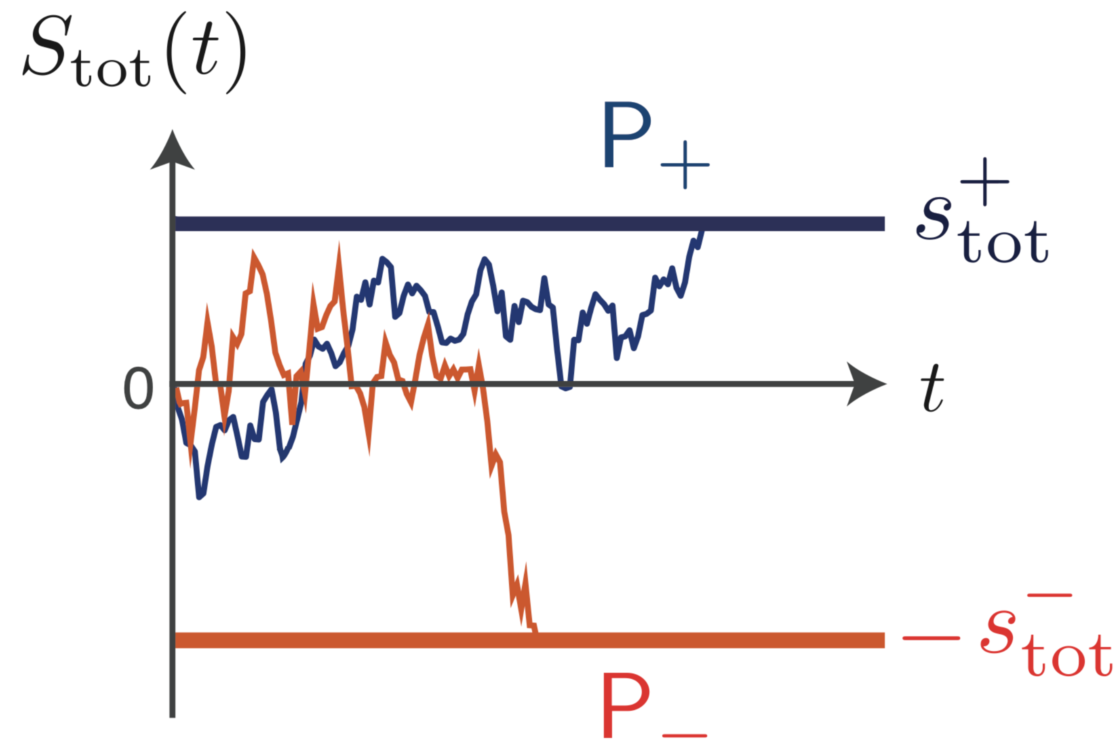

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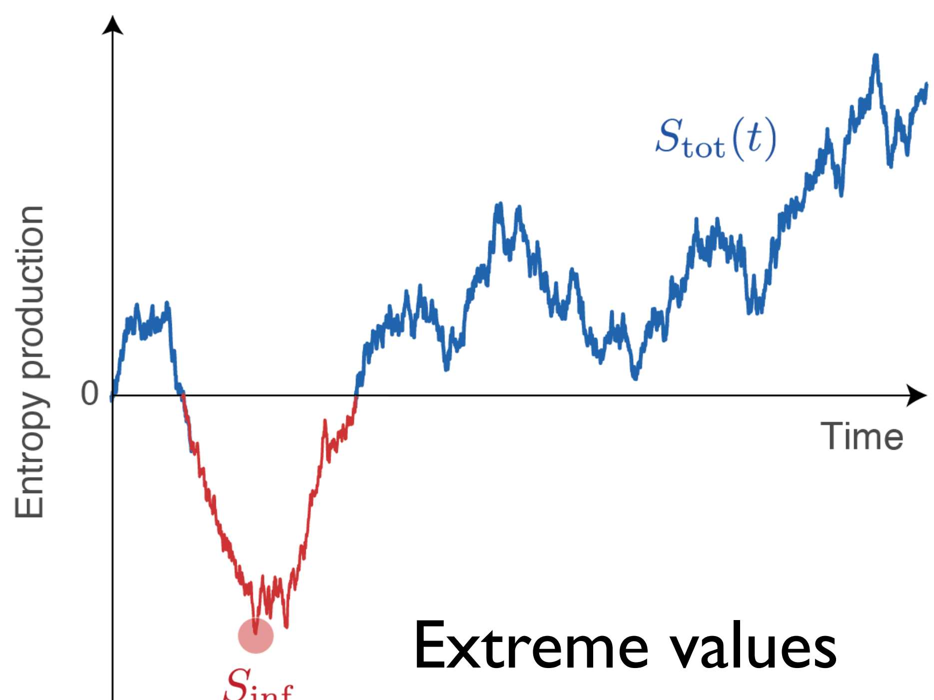
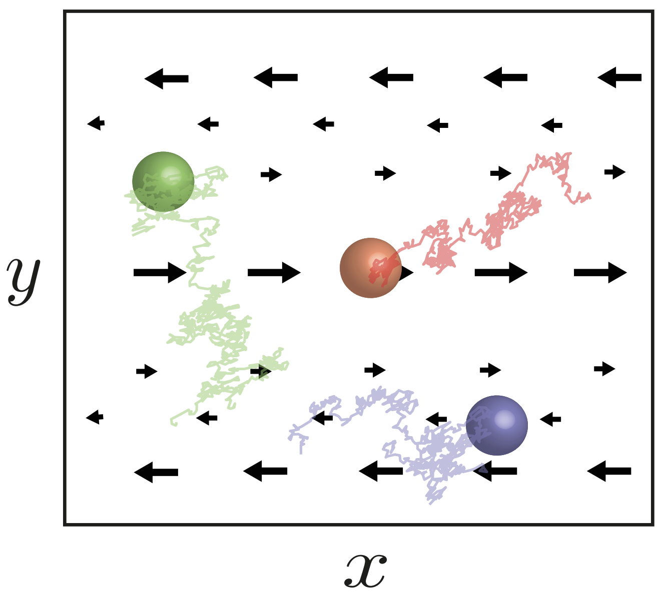


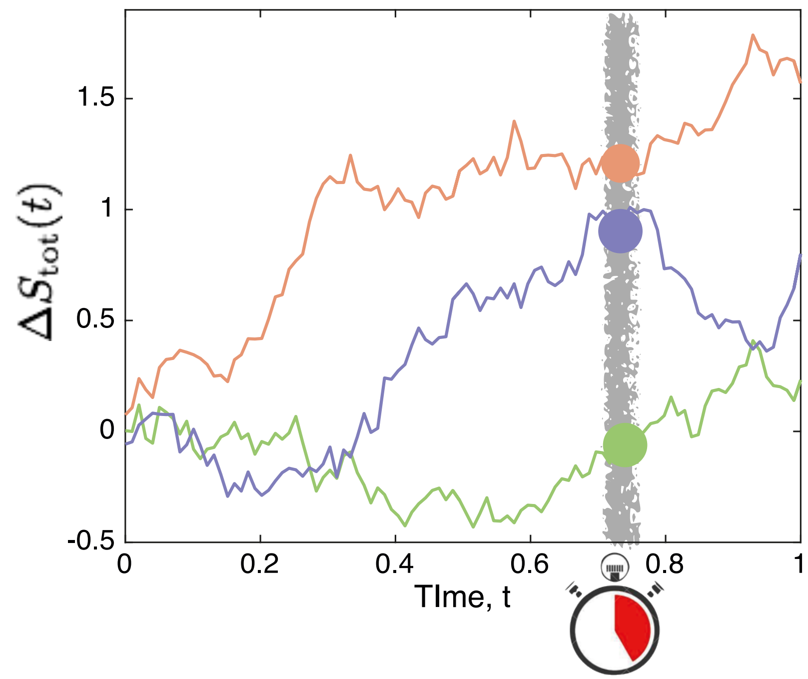
Extreme values

New universal properties ?

Martingale theory for stochastic thermodynamics

Stochastic thermodynamics without martingales





$$\langle e^{-\Delta S_{\rm tot}(t)/k_{\rm B}} \rangle = 1$$

Jarzynski's equality (1997)

Integral Fluctuation theorem (Seifert 2005)

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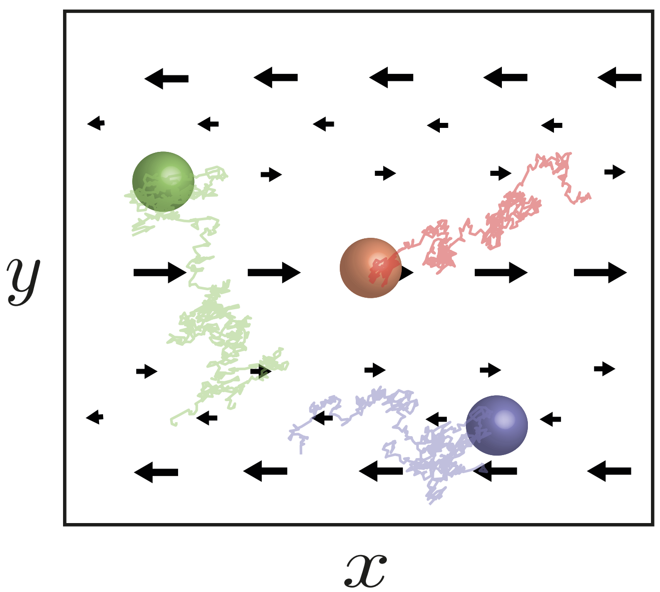
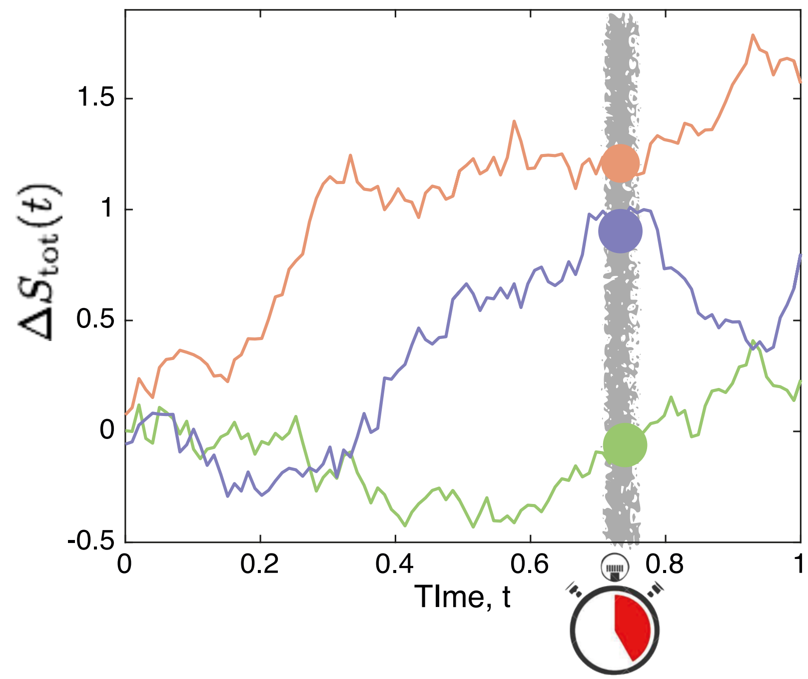
Second law of thermodynamics

Martingale theory for stochastic thermodynamics

Stochastic thermodynamics without martingales





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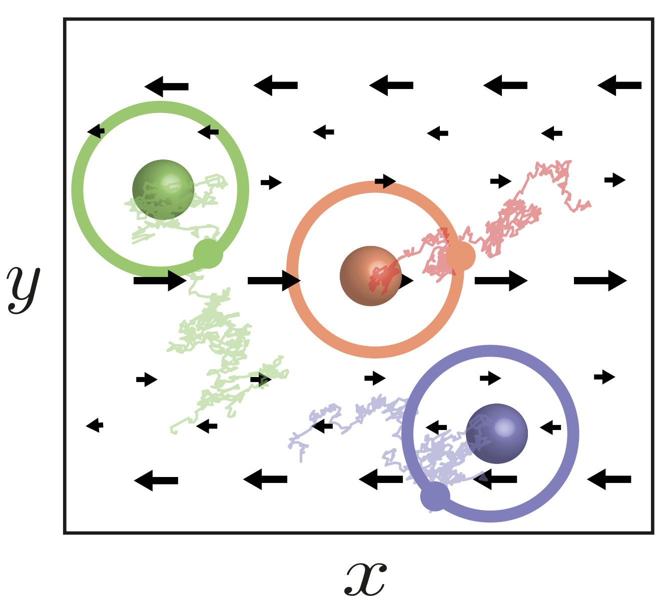
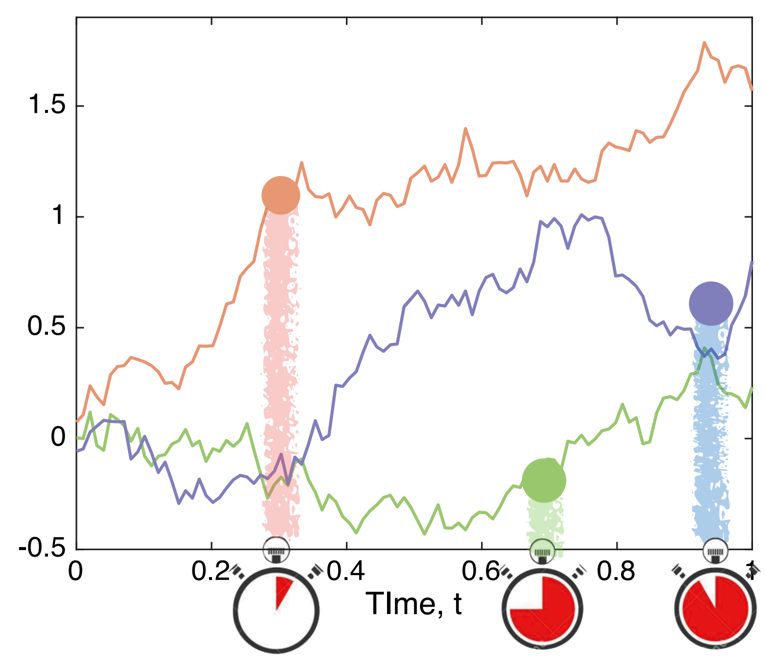
Second law of thermodynamics

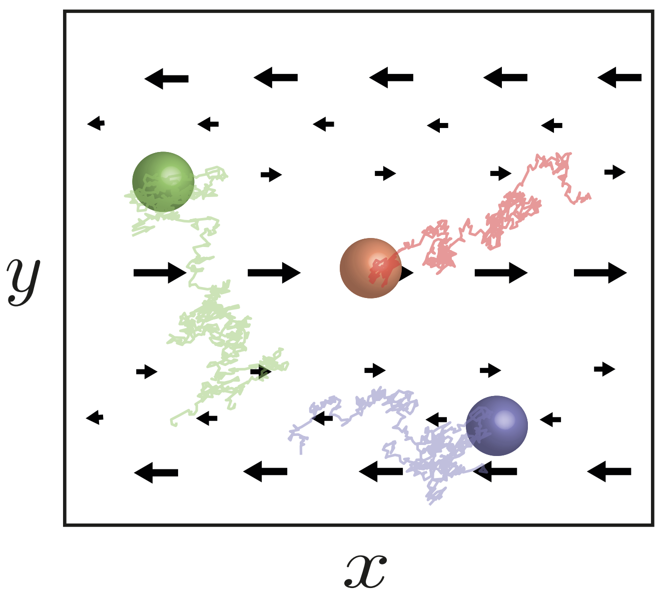
Stochastic thermodynamics with Martingales

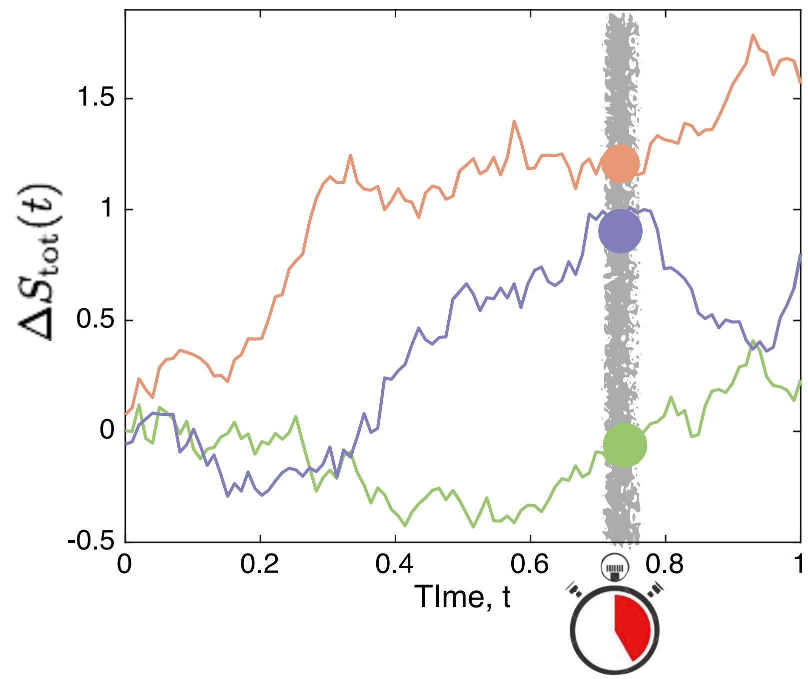




Martingale theory for stochastic thermodynamics

Stochastic thermodynamics without martingales





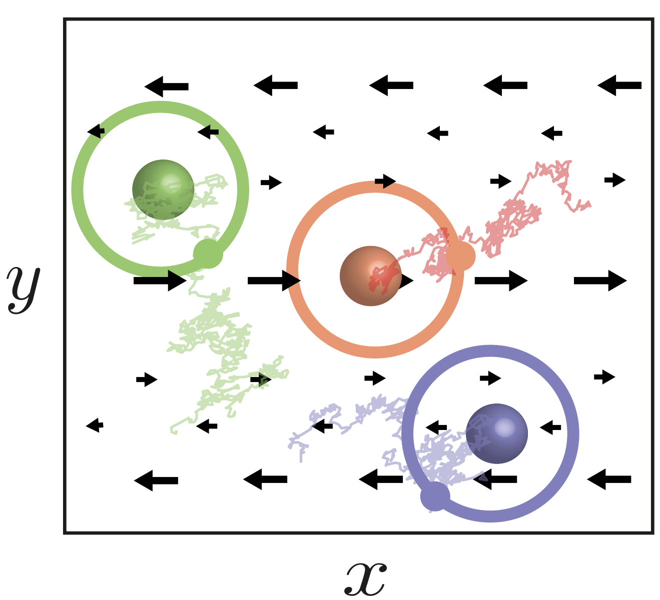
$$\langle e^{-\Delta S_{\rm tot}(t)/k_{\rm B}} \rangle = 1$$

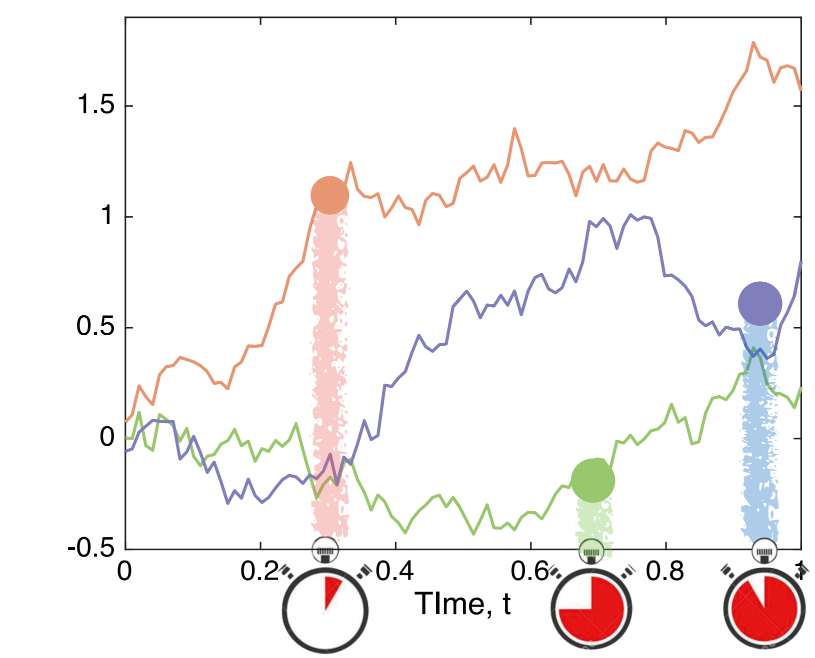
Jarzynski's equality (1997)
Integral Fluctuation theorem (Seifert 2005)

$$\langle \Delta S_{\rm tot}(t) \rangle \geq 0$$

Second law of thermodynamics

Stochastic thermodynamics with Martingales

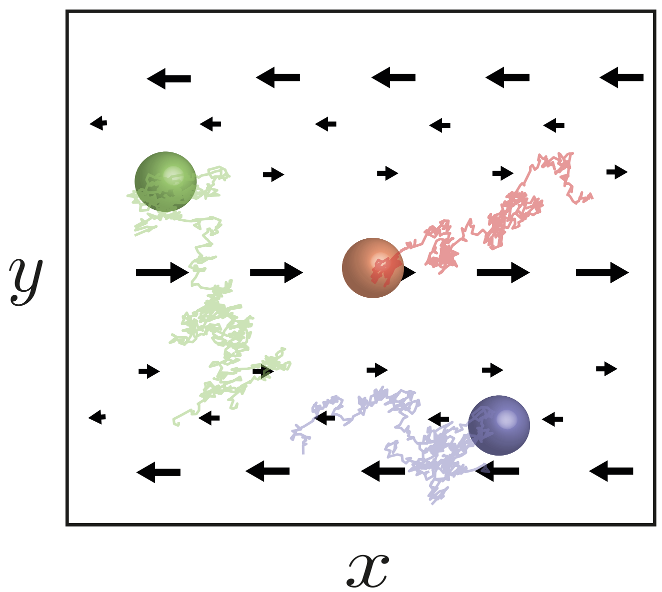


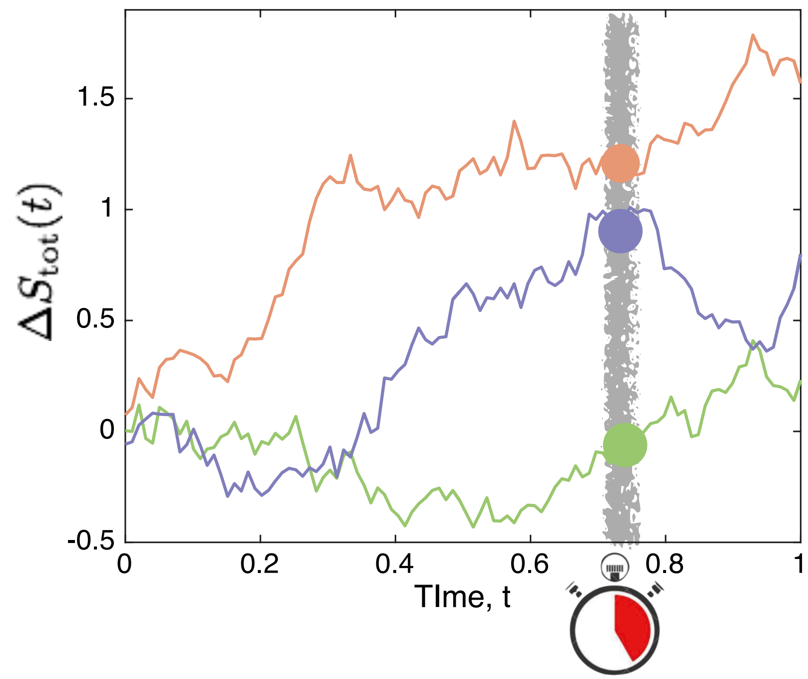


Stopping-time fluctuation theorems ?

Martingale theory for stochastic thermodynamics

Stochastic thermodynamics without martingales





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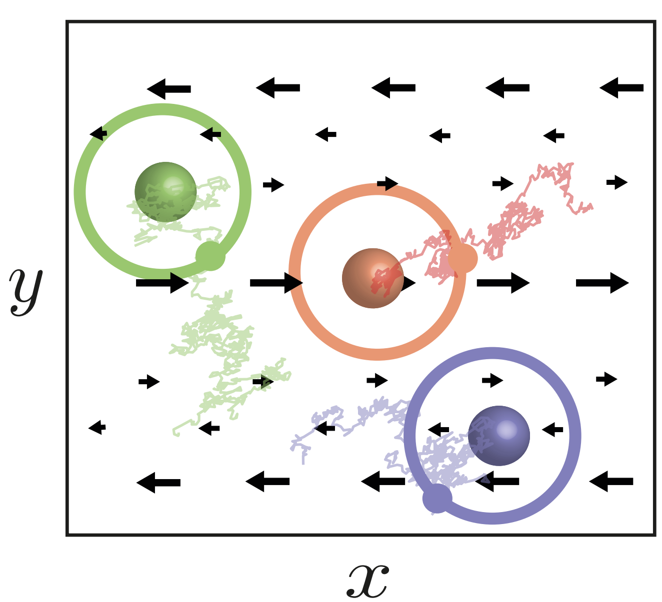
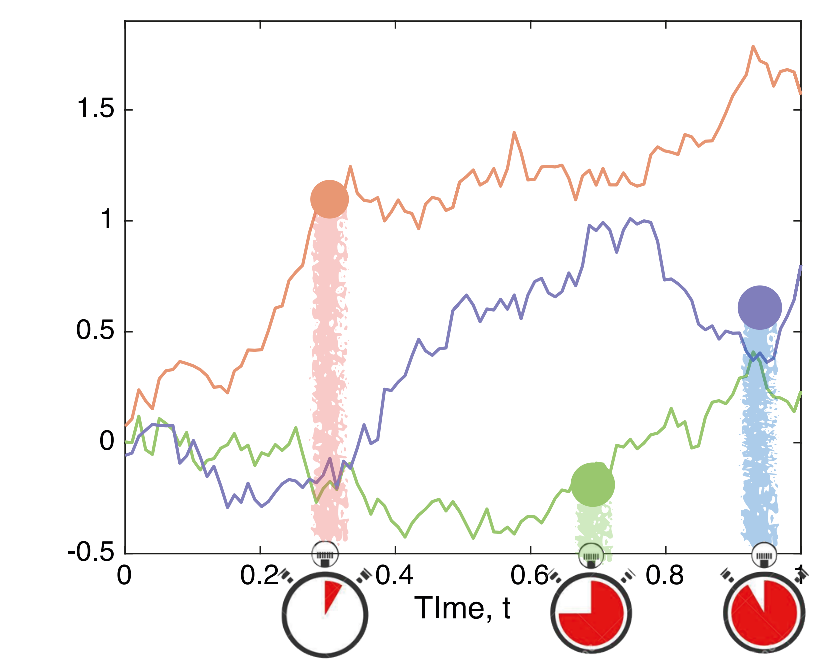


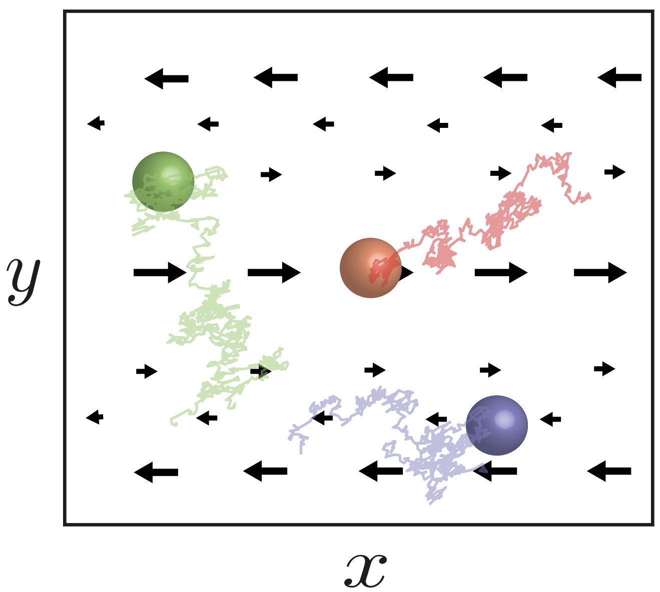


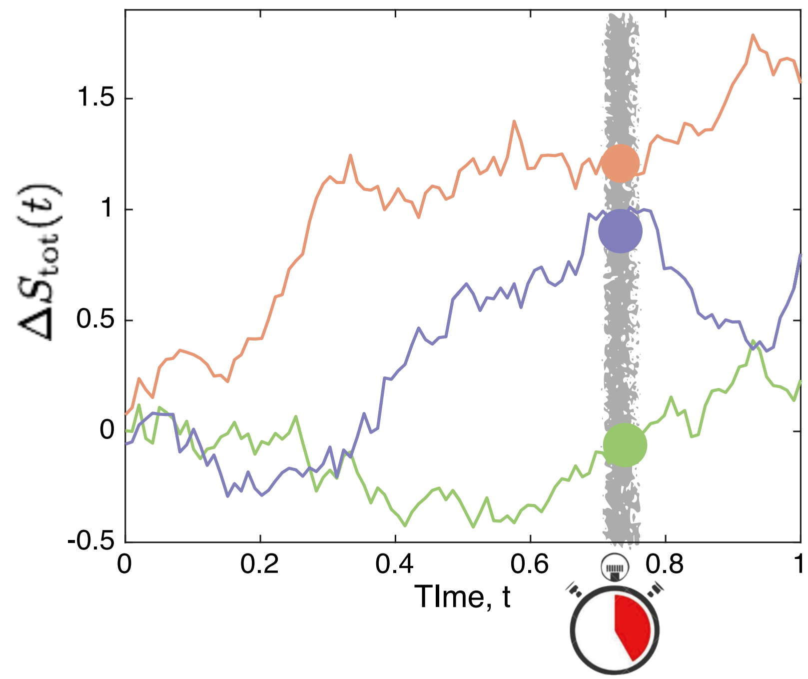
Stopping-time fluctuation theorems ?

Extrema ?

Martingale theory for stochastic thermodynamics

Stochastic thermodynamics without martingales





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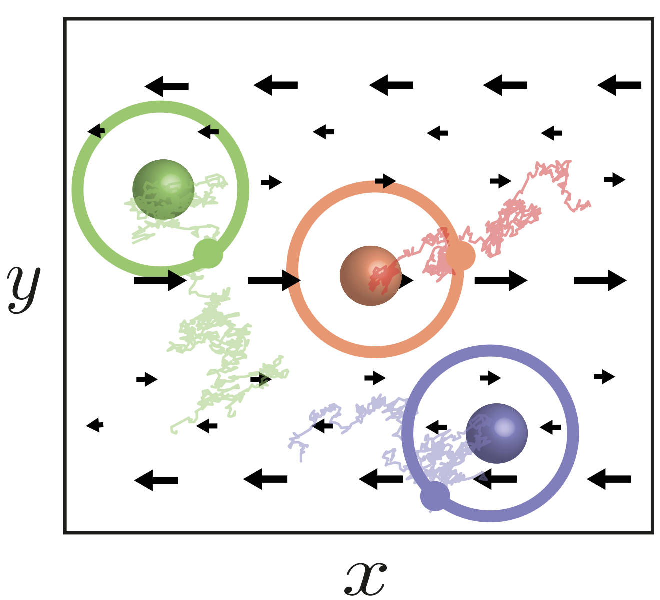
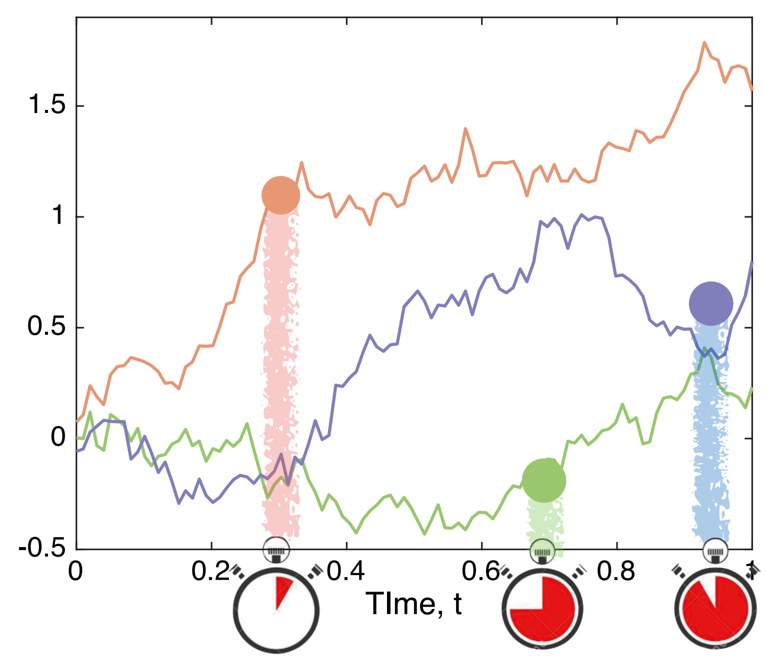




Stopping-time fluctuation theorems ?

Extrema ?

Gambling ?

Martingale theory for entropy production:

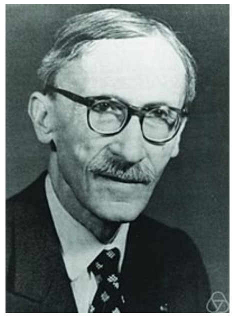
nonequilibrium steady states

Martingales

M(t) is a martingale with respect to *X(t)* if:

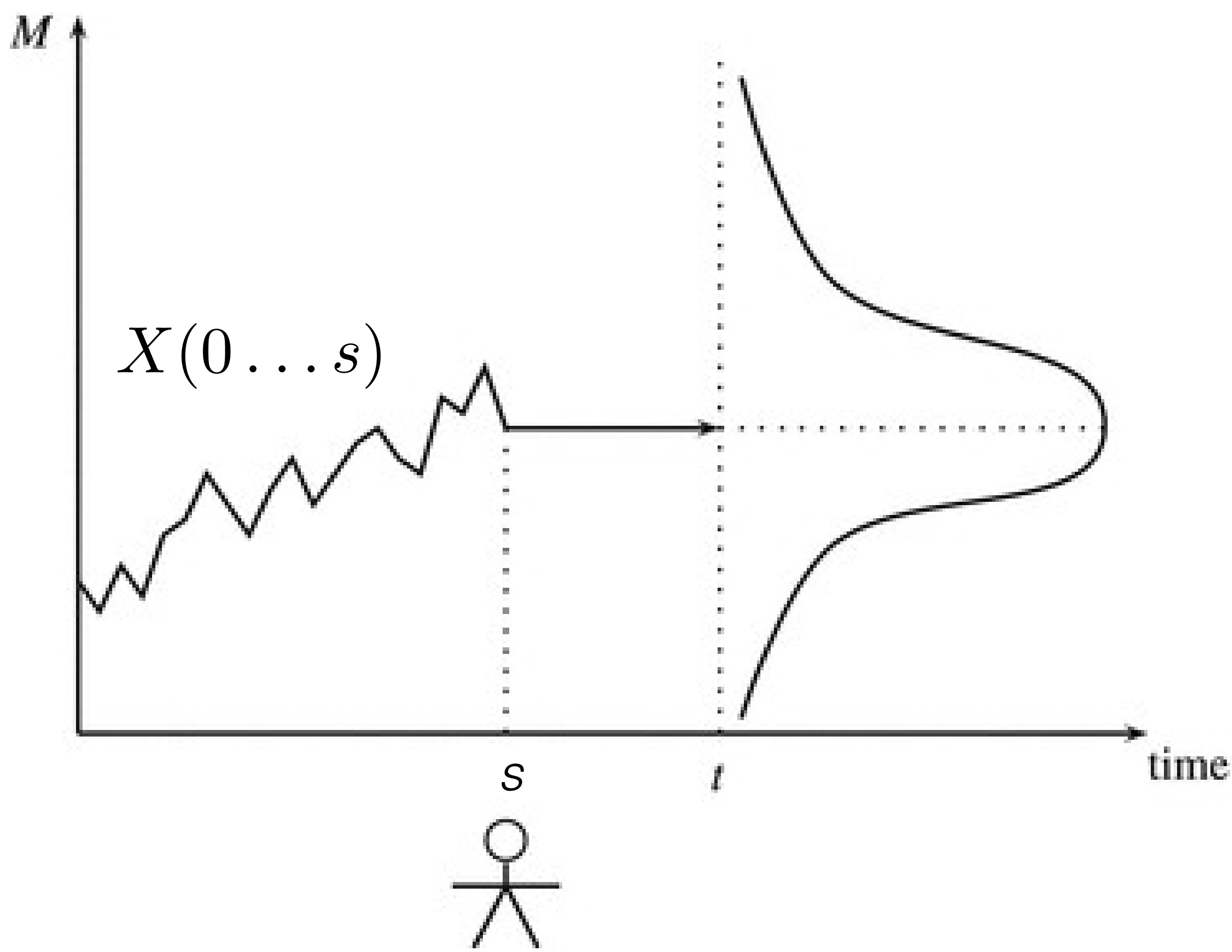
- *M(t)* is a real-valued function on *X(0…t)*
- $\langle |M(t)| \rangle < \infty$
- $\langle M(t) | X(0 \ldots s) \rangle = M(s)$,   for all *s<t*

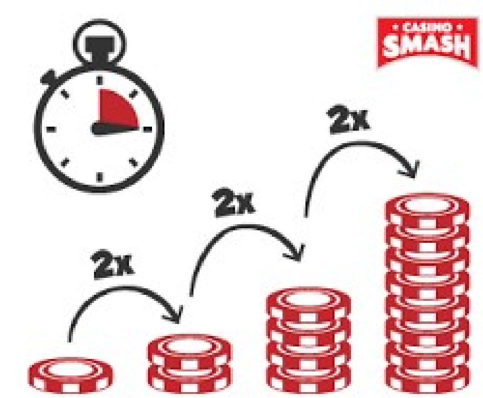
Martingales



Introduced in probability theory by Paul Lévy in 1934 and named by Ville (1939)

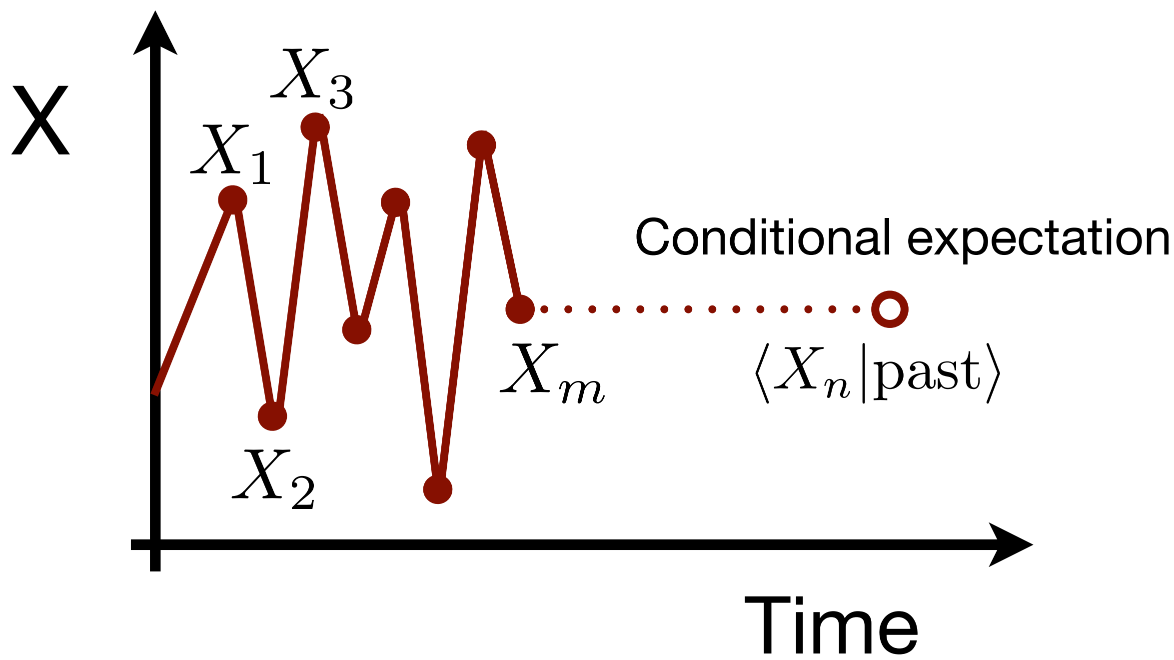
Popular meaning of "martingale": Double-up strategy in gambling





$$\langle M(t) | X(0 \ldots s) \rangle = M(s) \,, \qquad \text{for all } s<t$$

Martingales and Submartingales



Martingale $n \geq m$

$$\langle X_n | X_1, \ldots, X_m \rangle = X_m$$

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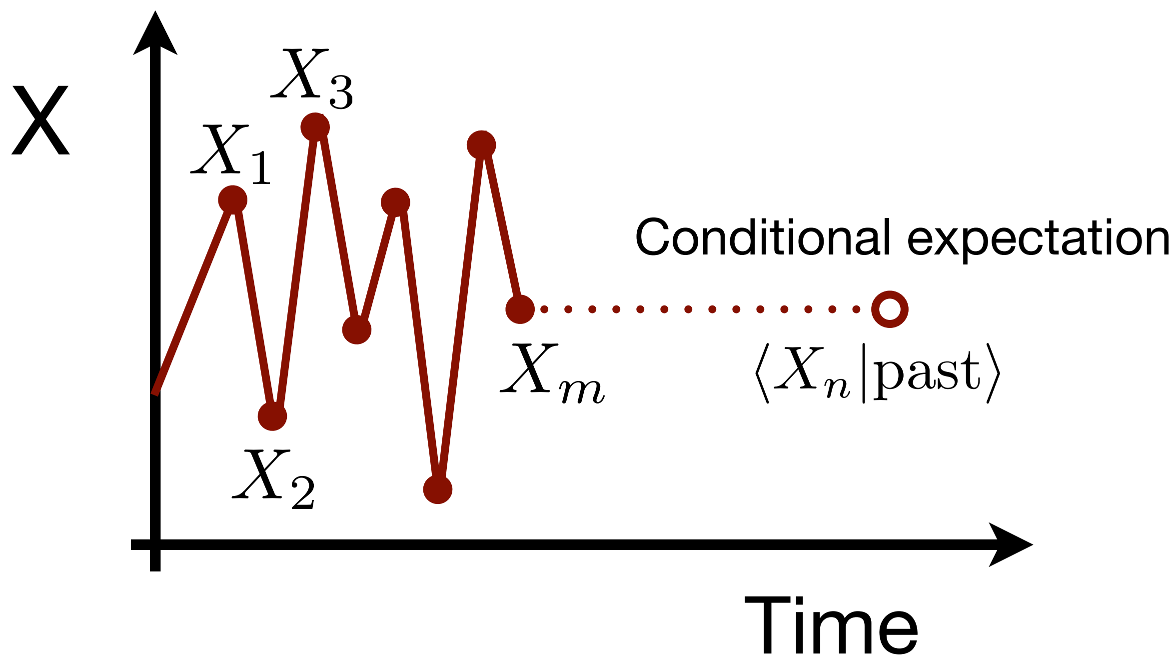

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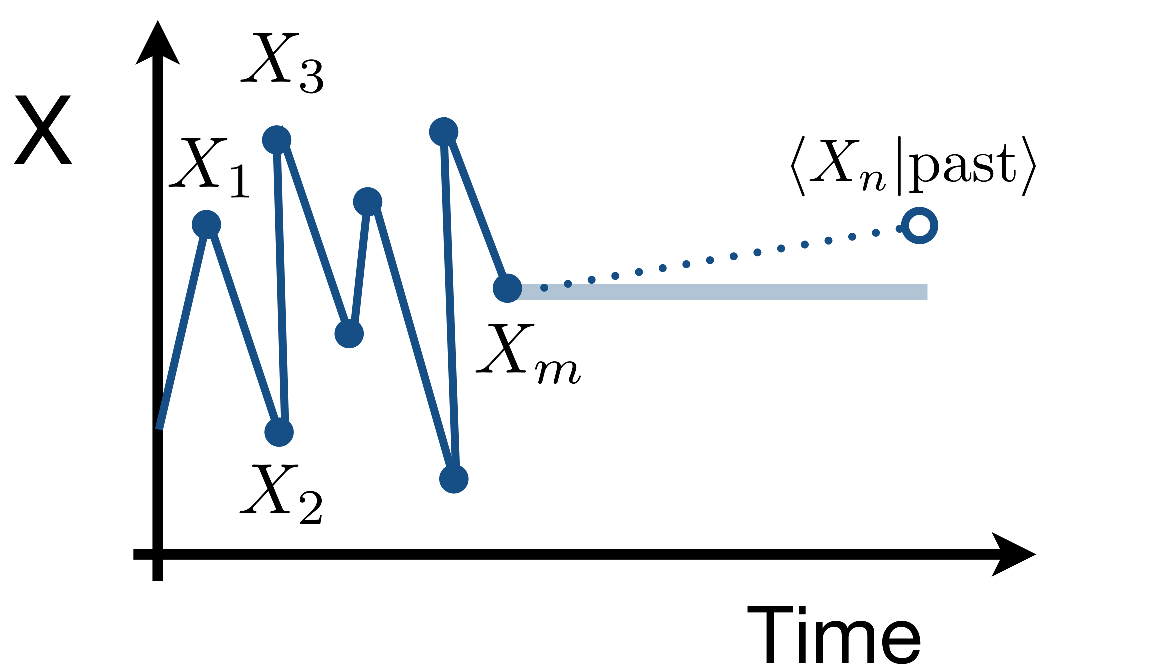

Submartingale $n \geq m$

$$\langle X_n | X_1, X_2, \ldots, X_m \rangle \geq X_m$$

Quick examples of martingales

ER, et al., *Physics with Martingales* (in preparation)

- **Random walks**

Drifted-diffusion $X(t) = vt + B(t)$ submartingale if v>0
supermartingale if v<0

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Poisson processes $N(t) \in \mathbb{N}$ with rate λ are submartingales

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Gambler's fortune in a fair game of chance, progeny in branching processes, path probability ratios, etc.

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The exponentiated negative entropy is a martingale

$$\langle e^{-S_{\rm tot}(t)/k_{\rm B}} | x(0 \ldots s) \rangle =$$

I. Neri, ER, F. Jülicher, PRX **7,** 011019 (2017)

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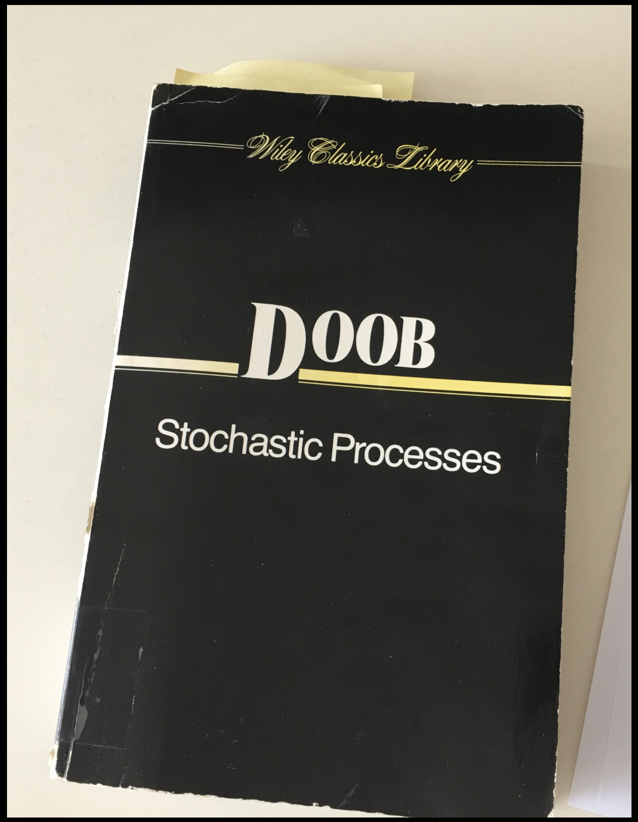
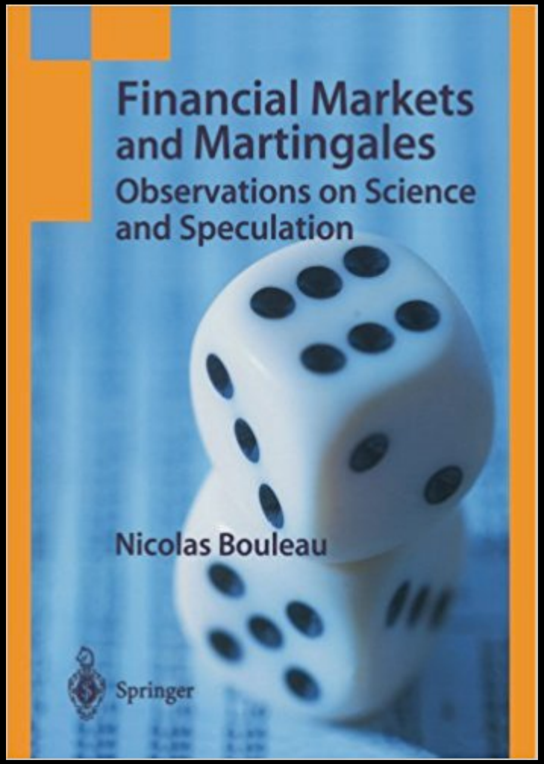
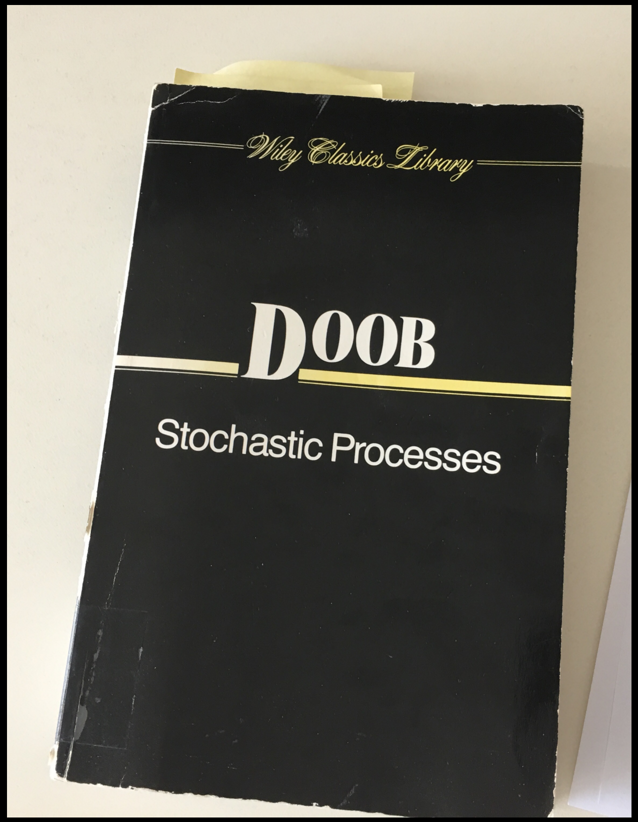
- Assumptions: (1) non-equilibrium steady state
- Valid for: continuous, discrete, Markovian and non-Markovian processes
- Also for other reference measures (“action functionals”):

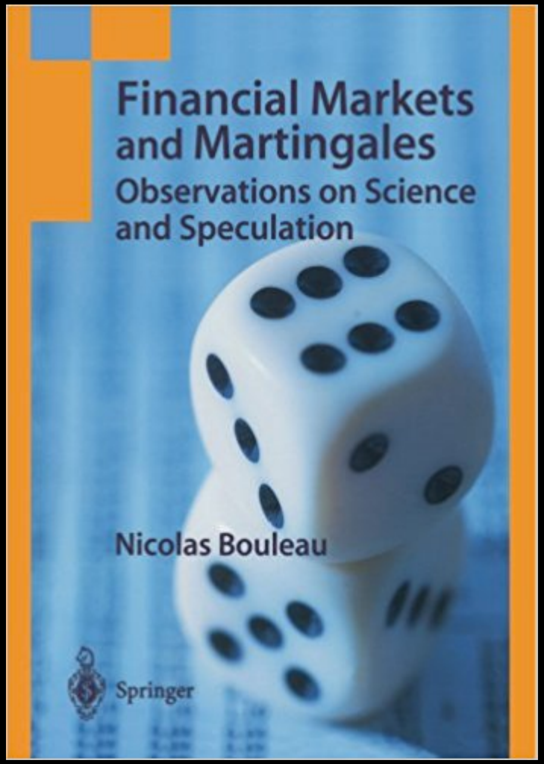
R. Chetrite, S. Gupta, J. Stat. Phys. **143,** 543 (2011) I. Neri, ER, F. Jülicher, PRX **7,** 011019 (2017)





Unveiling new generic thermodynamic properties using Martingales



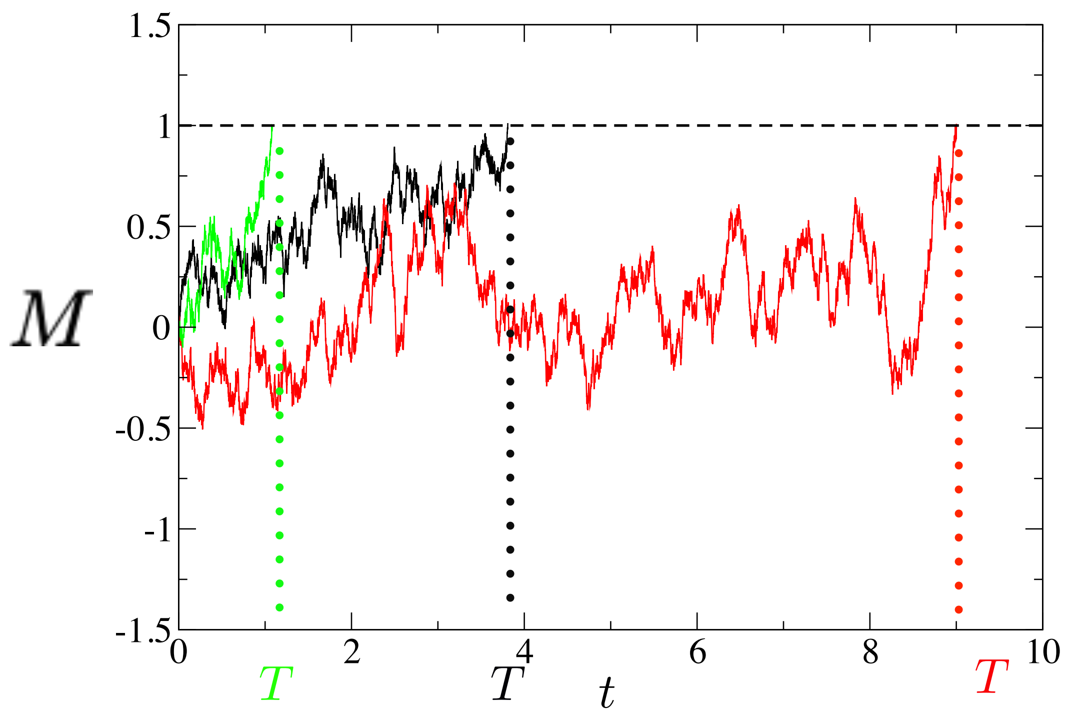


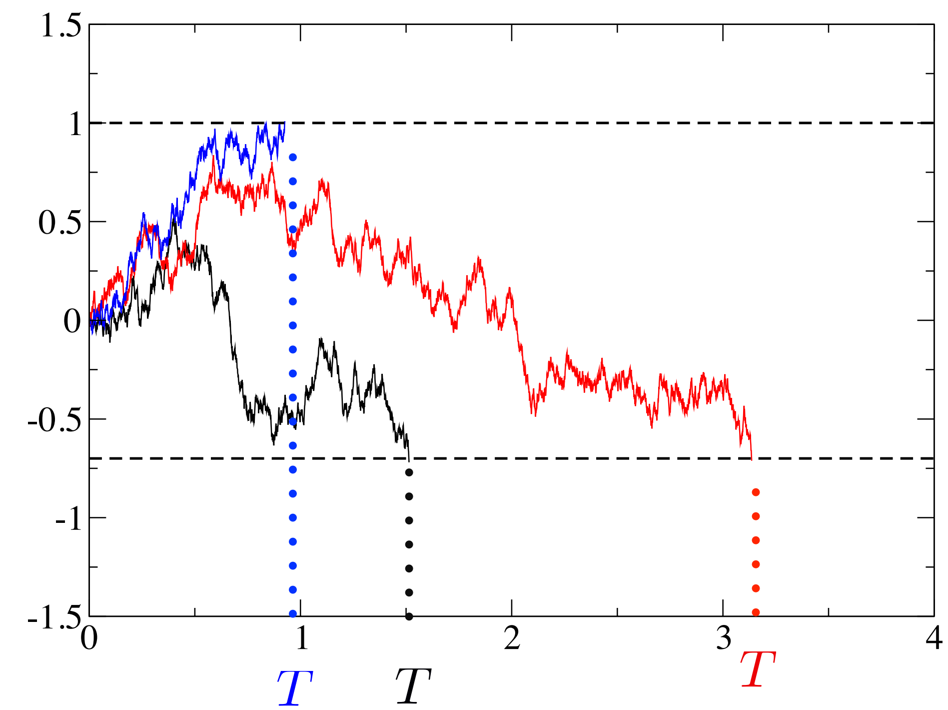
Unveiling new generic thermodynamic properties using Martingales

Thermodynamic laws at stopping times

Can a gambler make fortune in a fair game by quitting at an intelligently chosen moment?

Gambling with Martingales





$$\langle M(T)\rangle = 1 \neq \langle M(0)\rangle = 0$$

$$\langle M(T)\rangle = \langle M(0)\rangle = 0$$

Gambler makes profit

Gambler on average makes no profit

Can a gambler make fortune in a fair game by quitting at an intelligently chosen moment?

No, if the gambler cannot foresee the future, cannot cheat, and has access to a finite budget

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Doob's optional stopping theorem

$$\langle M(\mathcal{T})|M(0)\rangle = M(0)$$

if $M(t)$ is a uniformly integrable martingale, and

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stopping time: random time, functional on the stochastic trajectory

" to answer $\mathcal{T} < t?$ one only needs information in [0,t] "

Integral fluctuation theorems at stopping times

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Second law
at stopping times

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Any gambling strategy
cannot achieve negative entropy

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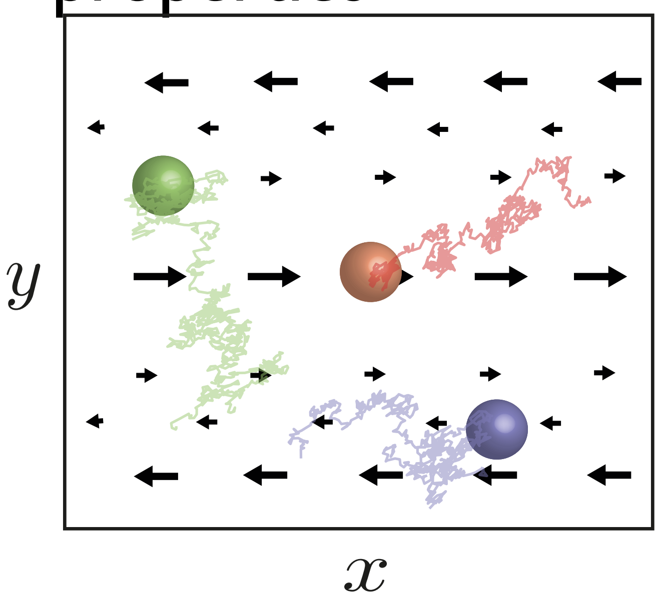
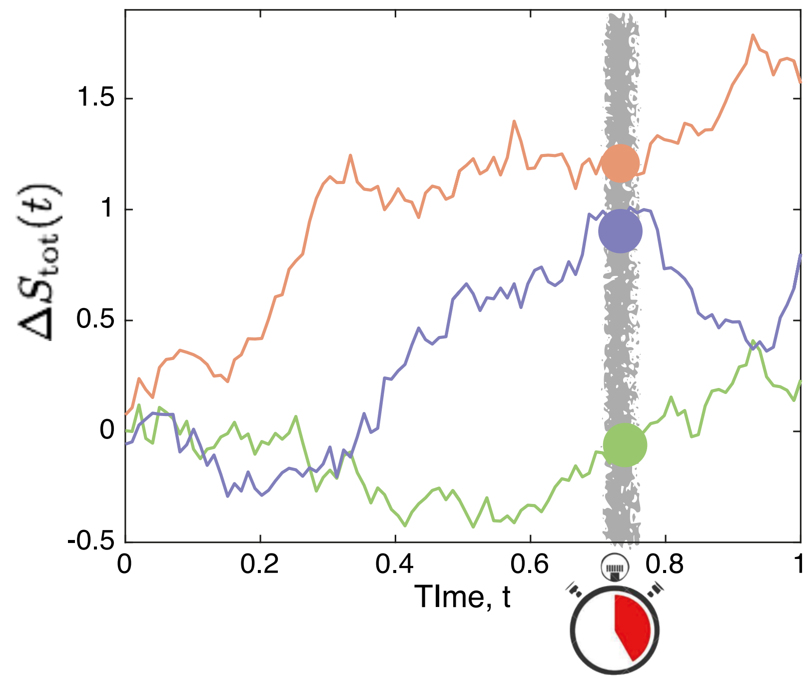
Martingale theory for stochastic thermodynamics

Fluctuation theorems for non-equilibrium steady states : fixed-time properties





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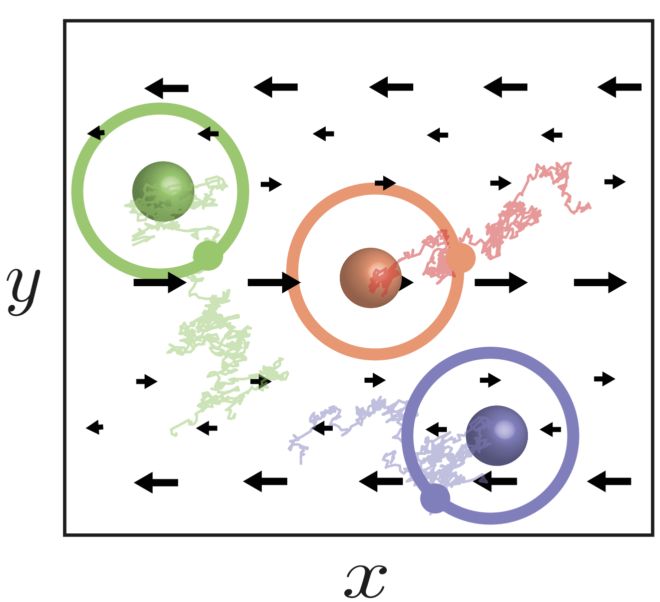
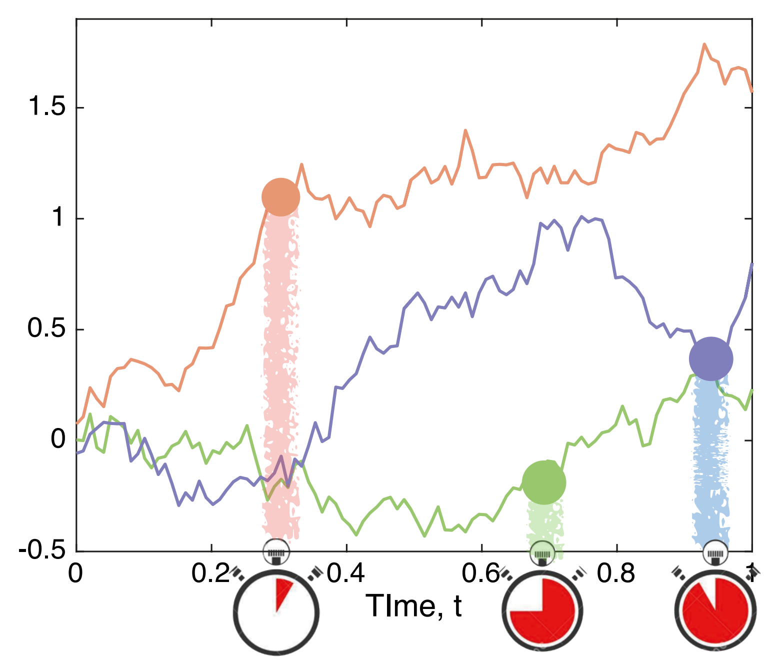
Martingale theory: stopping-time statistics





Stopping-time fluctuation theorems ?

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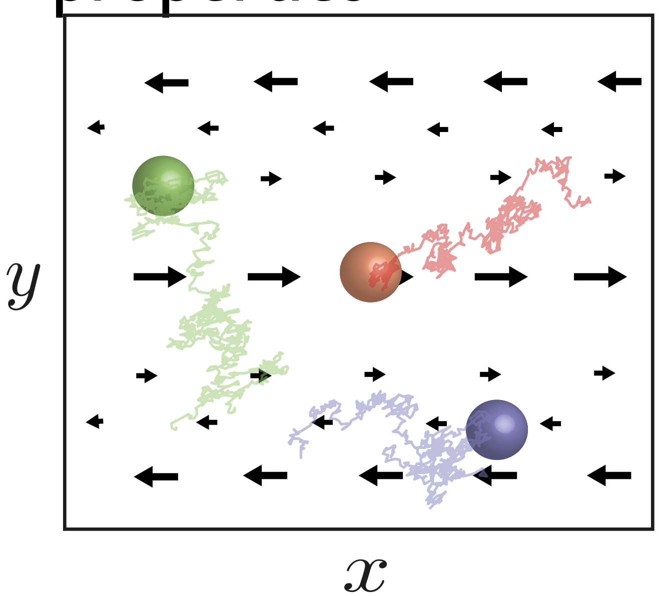

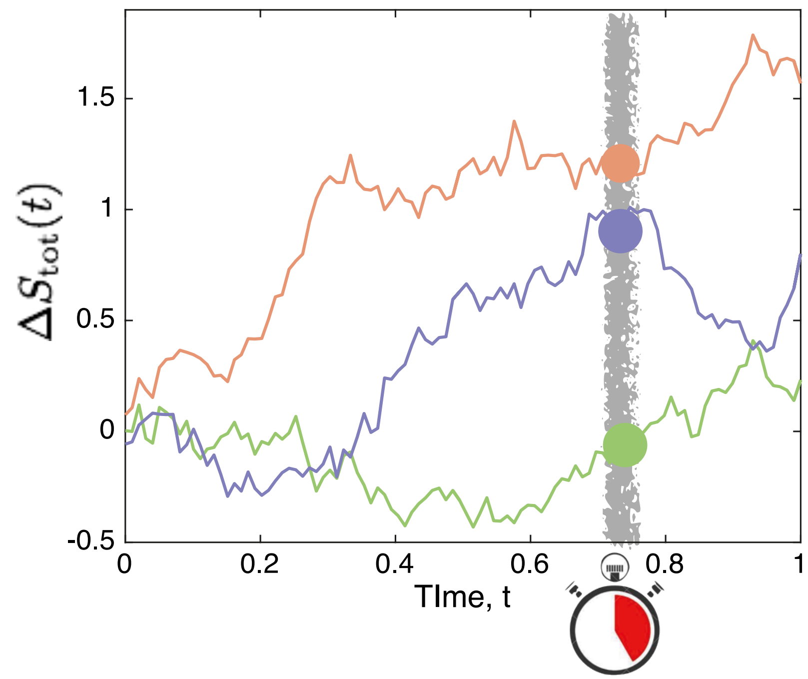

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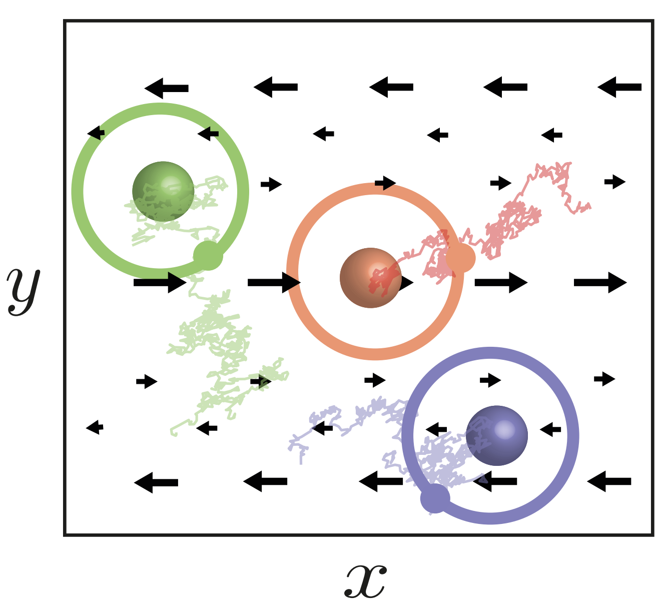
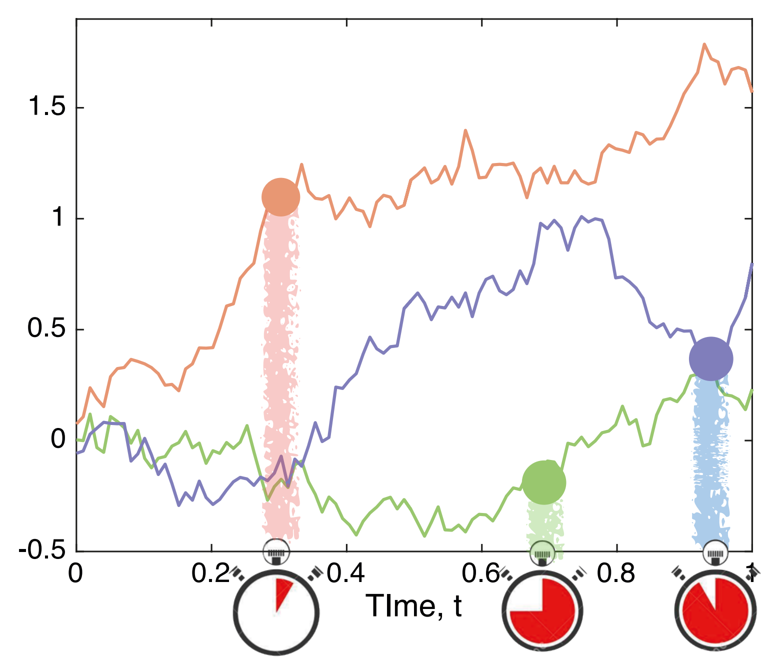
Second law of thermodynamics

Martingale theory: stopping-time statistics

I. Neri, ER, F. Jülicher, PRX **7,** 011019 (2017)

I. Neri, ER, S. Pigolotti, F. Jülicher, arXiv 1903.08115 (2019)





$$\langle e^{-\Delta S_{\mathrm{tot}}(\mathcal{T})/k_{\mathrm{B}}} \rangle = 1$$

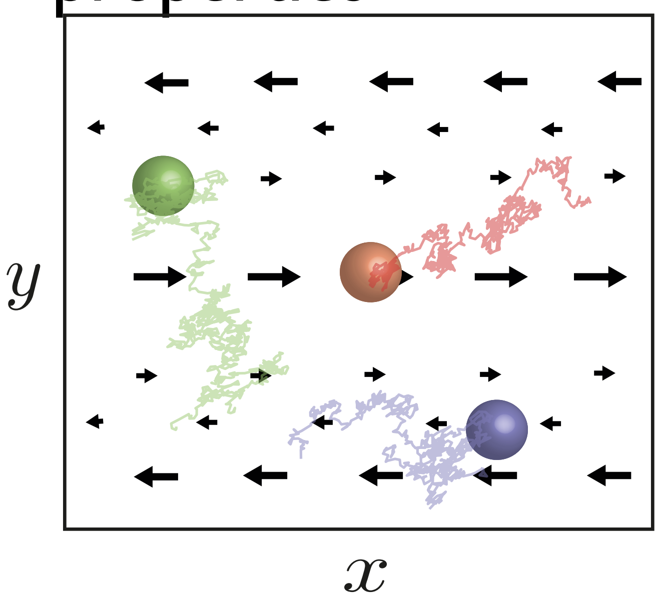
Integral Fluctuation theorem at stopping times

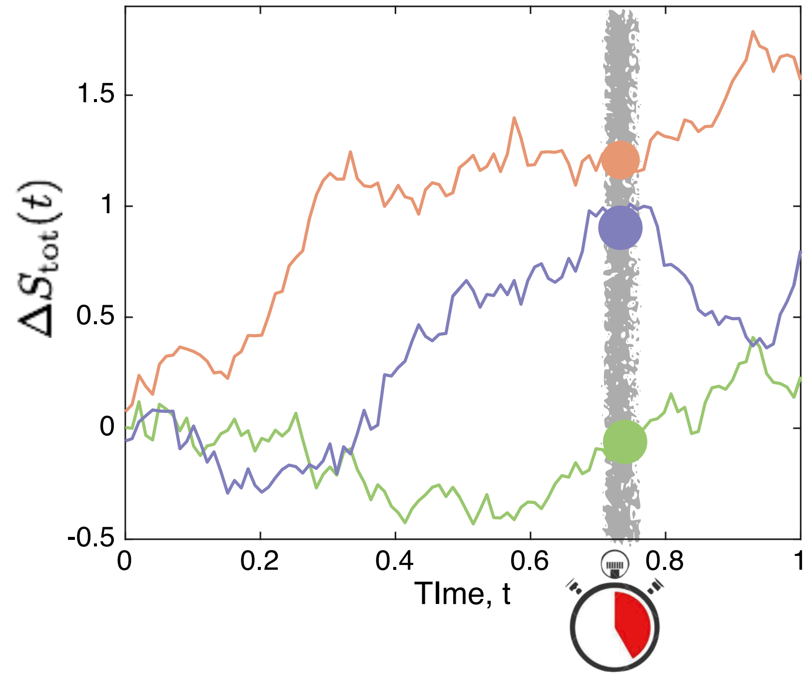
$$\langle \Delta S_{\mathrm{tot}}(\mathcal{T}) \rangle \geq 0$$

Second law at stopping times

Martingale theory for stochastic thermodynamics

Fluctuation theorems for non-equilibrium steady states : fixed-time properties





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Jarzynski's equality (1997)

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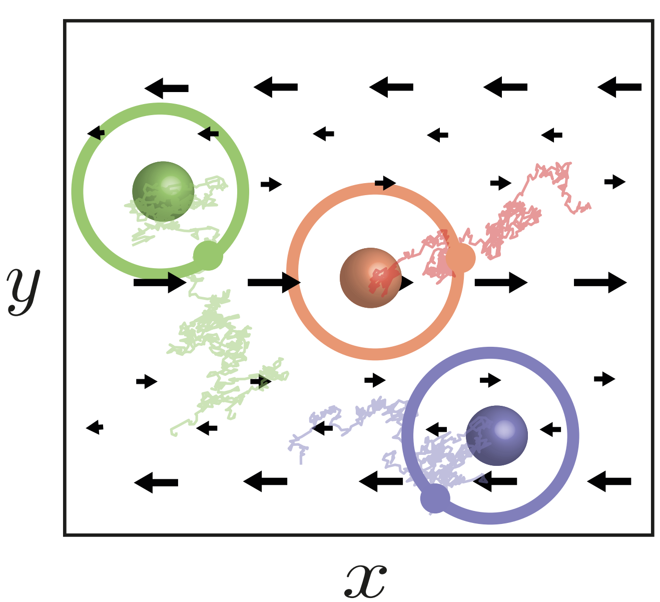

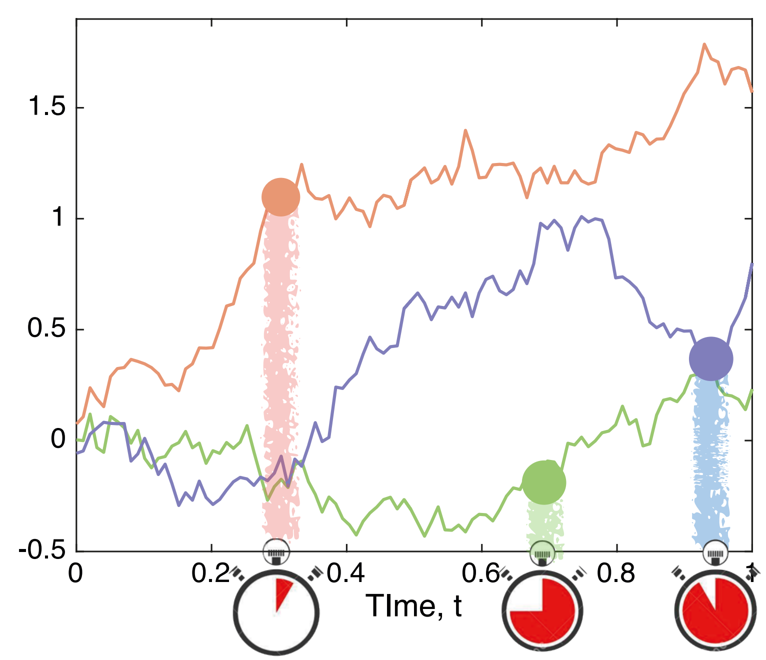

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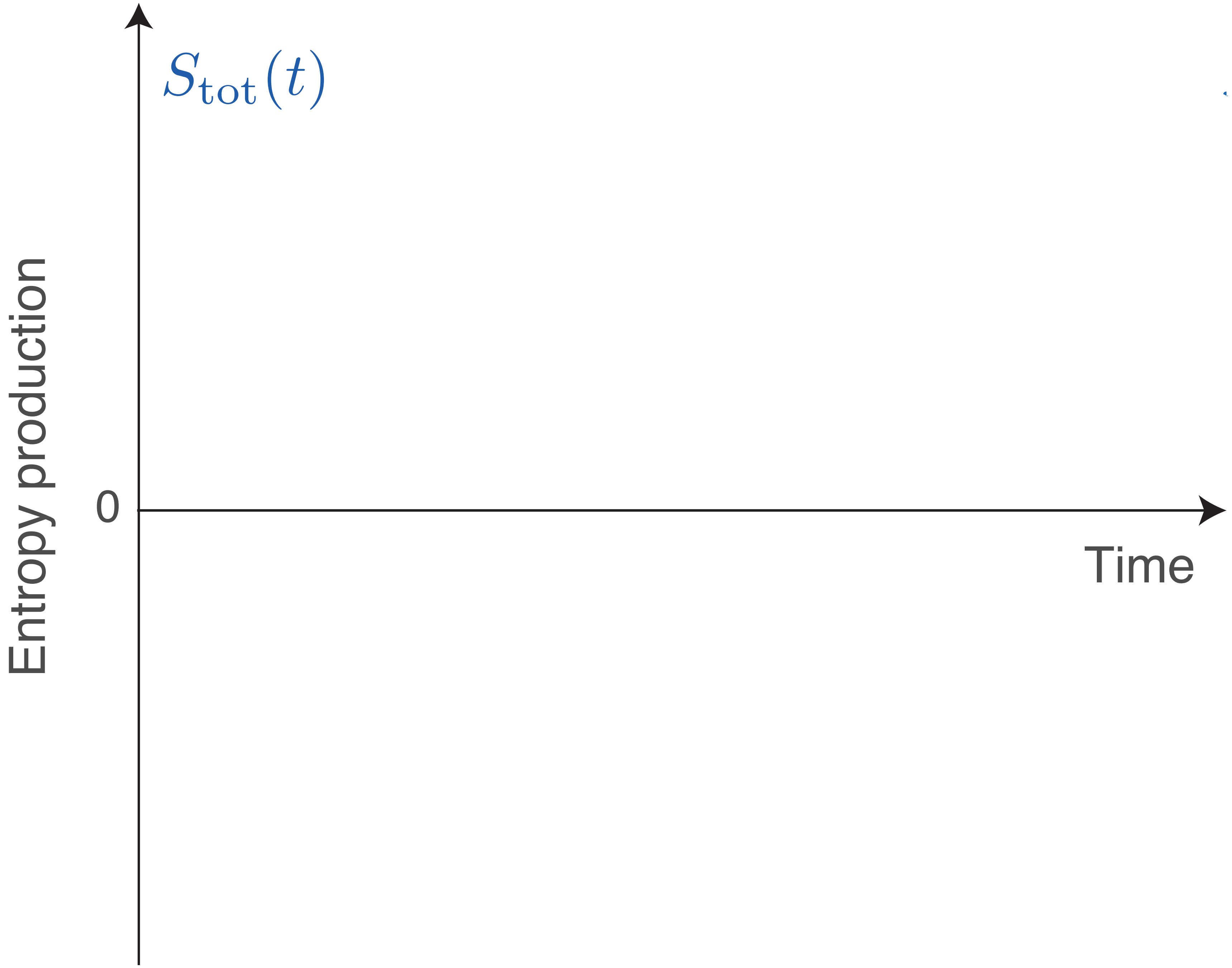
Second law at stopping times

Extreme-value statistics

Extreme values

What´s the negative record of entropy production?



Extreme values

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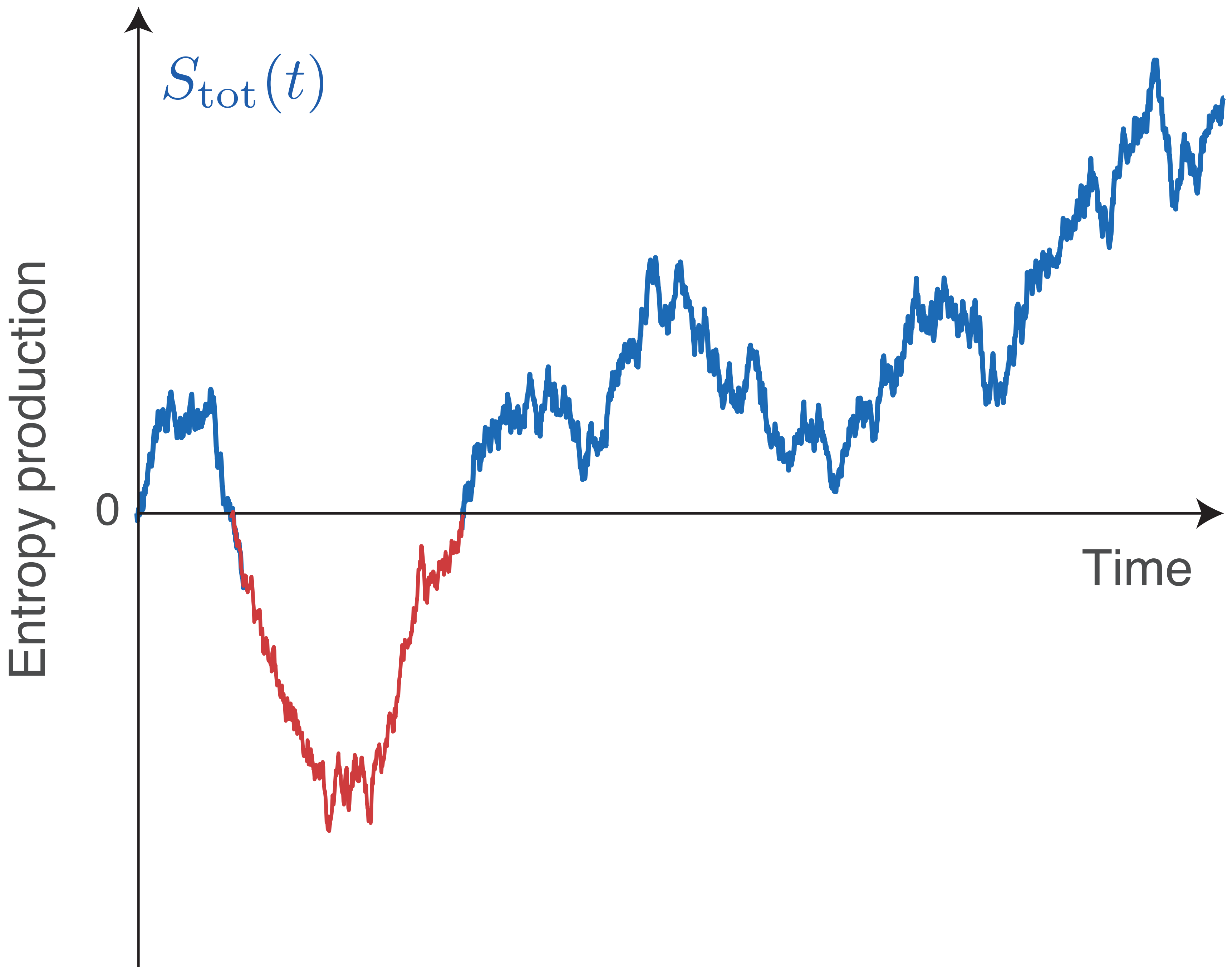

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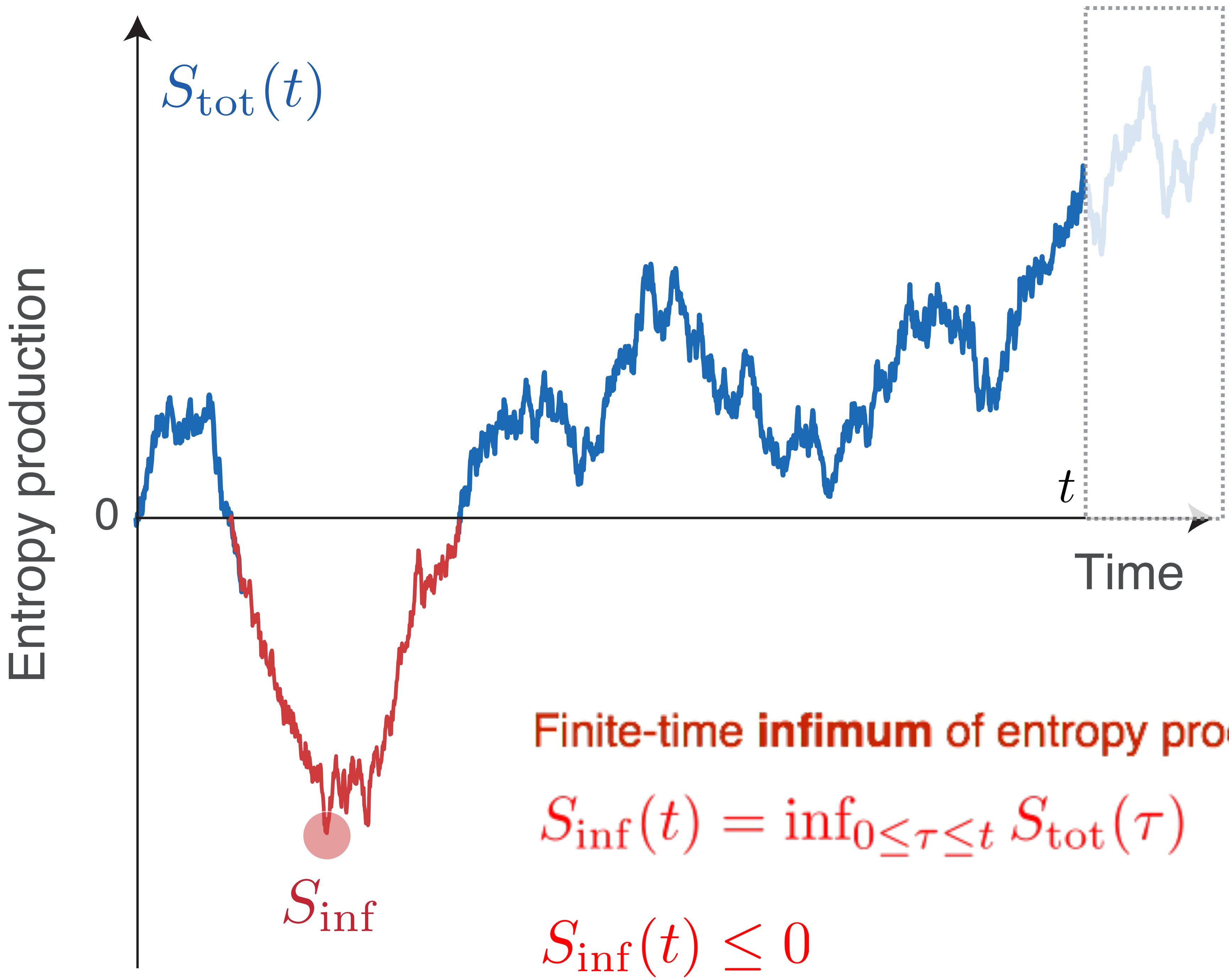


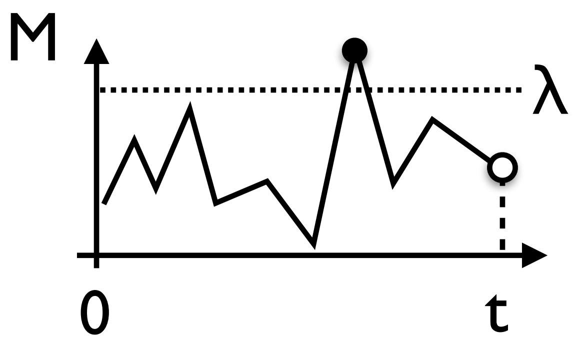
Finite-time **infimum** of entropy production

$$S_{\mathrm{inf}}(t) = \inf_{0\leq\tau\leq t} S_{\mathrm{tot}}(\tau)$$

$$S_{\mathrm{inf}}(t) \leq 0$$

Extreme values: infimum law

- Doob´s maximal inequality for positive submartingales (Jean Ville’s PhD Thesis)



$$\Pr\left(\sup_{\tau\in[0,t]} M(\tau) \geq \lambda\right) \leq \frac{\langle M(t)\rangle}{\lambda}$$

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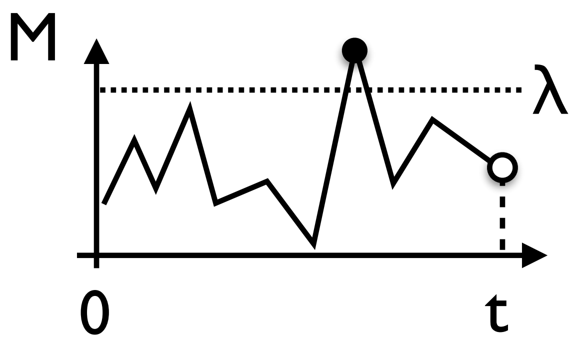

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- Apply theorem to the positive martingale $M = e^{-S_{\rm tot}/k_{\rm B}}$, and the fluctuation theorem

$$\Pr\left(\sup_{\tau\in[0,t]} e^{-S_{\rm tot}(\tau)/k_{\rm B}} \geq \lambda\right) \leq \frac{\overbrace{\langle e^{-S_{\rm tot}(t)/k_{\rm B}}\rangle}^{=1}}{\lambda}$$

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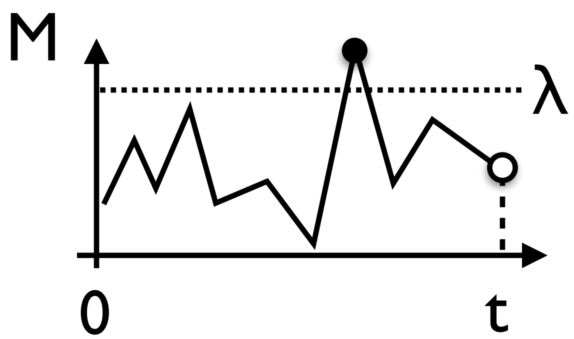

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$$\Pr\Big(\underbrace{\inf_{\tau\in[0,t]} S_{\mathrm{tot}}(\tau)}_{\equiv S_{\mathrm{inf}}(t)} \leq -k_{\mathrm{B}}\ln\lambda\Big) \leq \frac{1}{\lambda} \;\Rightarrow\; \Pr\Big(S_{\mathrm{inf}}(t) \leq -s\Big) \leq e^{-s/k_{\mathrm{B}}}$$

$$\Pr\Big(S_{\mathrm{inf}}(t) \geq -s\Big) \geq 1 - e^{-s/k_{\mathrm{B}}}$$

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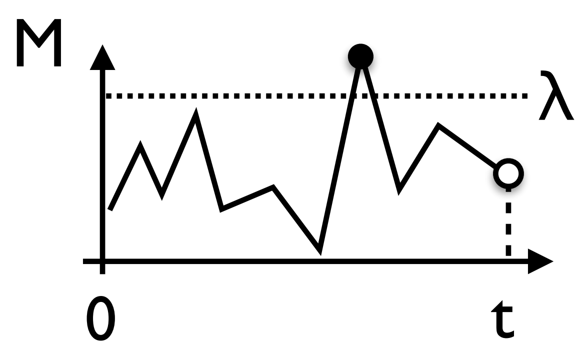

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- Stochastic dominance (s>0)

$$\Pr\big(-S_{\rm inf}(t) \leq s\big) \geq 1 - e^{-s/k_{\rm B}} \quad \Rightarrow \langle -S_{\rm inf}(t)\rangle \leq k_{\rm B} \quad \Rightarrow \langle S_{\rm inf}(t)\rangle \geq -k_{\rm B}$$

CDF of -Infimum   CDF of Exp r.v. with mean kB

Extreme values: infimum law

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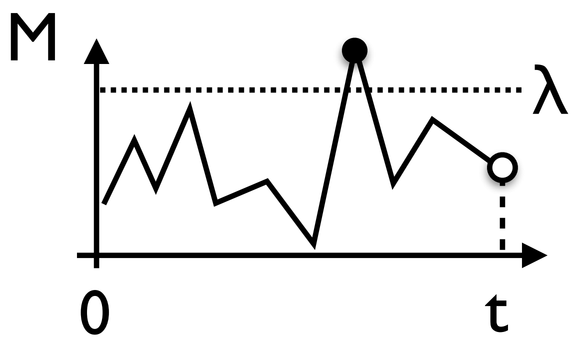

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CDF of -Infimum CDF of Exp r.v. with mean kB

Numerical and experimental tests

$$\Pr\Big(-S_{\rm inf}(t) \leq s\Big) \geq 1 - e^{-s/k_{\rm B}}$$

$$\langle S_{\rm inf}(t)\rangle \geq -k_{\rm B}$$

hold for "any" **nonequilibrium steady state:**

- Discrete-time processes (e.g. Markov chains)
- Continuous processes (e.g. Langevin dynamics)
- Continuous-time processes with jumps
  (e.g. Markov-jump processes)

Numerical test: Langevin dynamics in a tilted periodic potential

I. Neri, É. Roldán, F. Jülicher, Phys. Rev. X **7** , 011019 (2017)

Numerical and experimental tests

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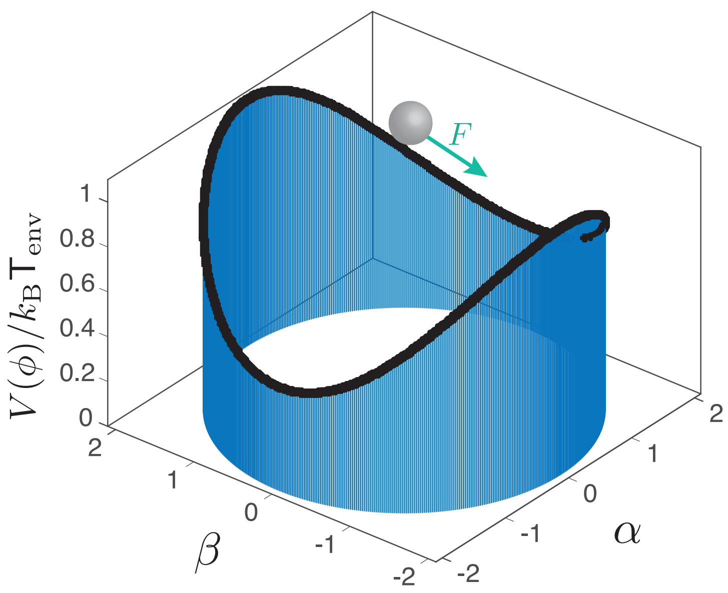

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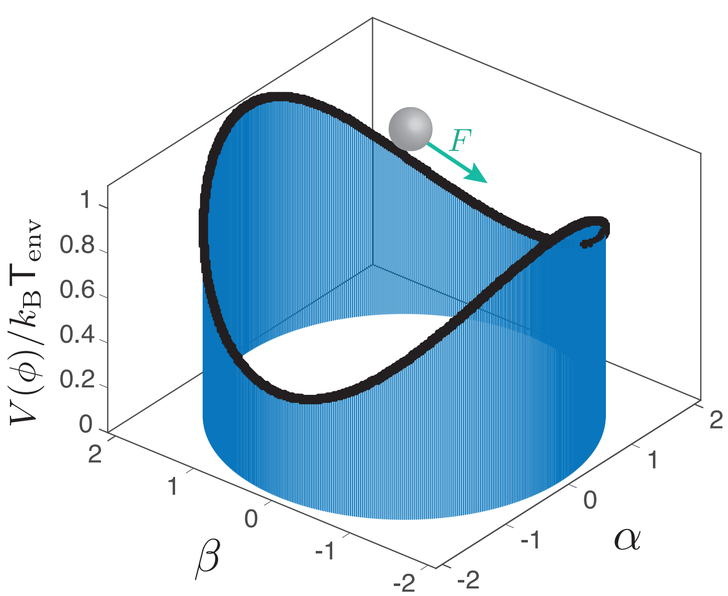

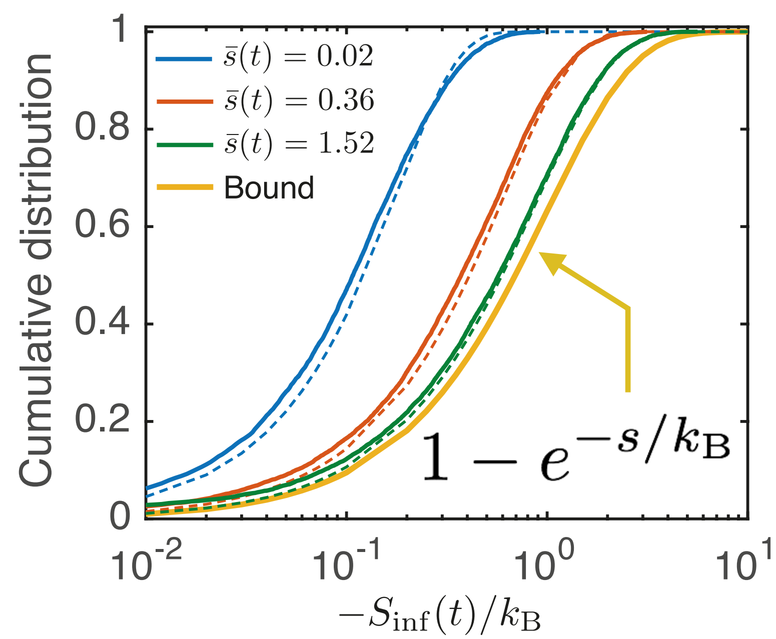

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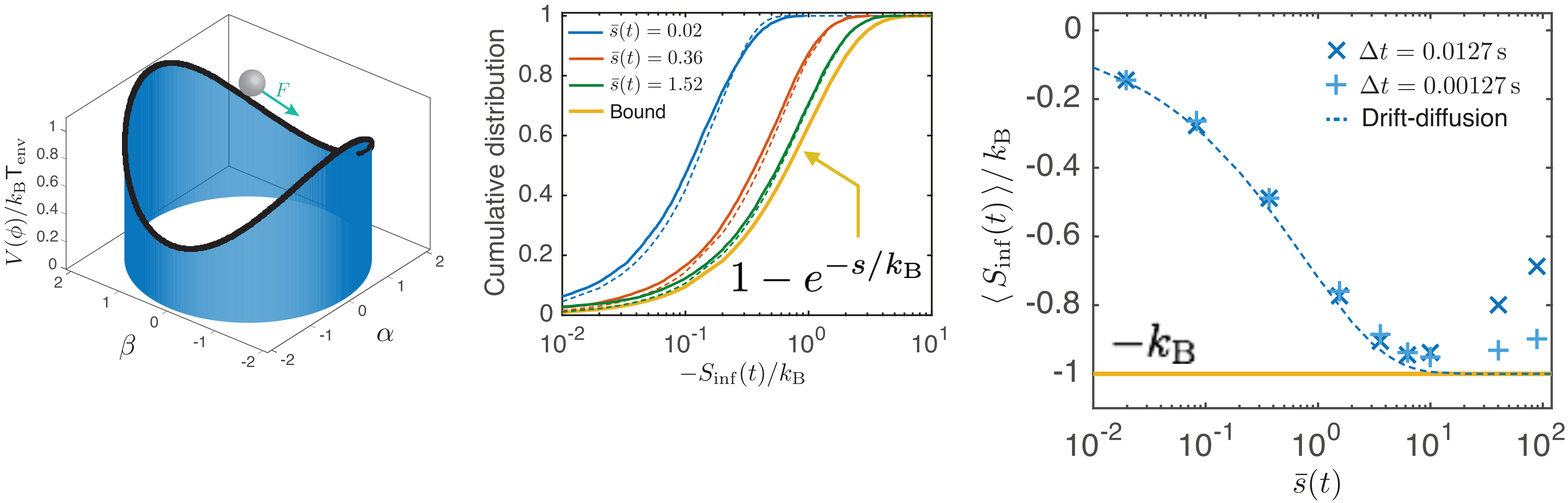

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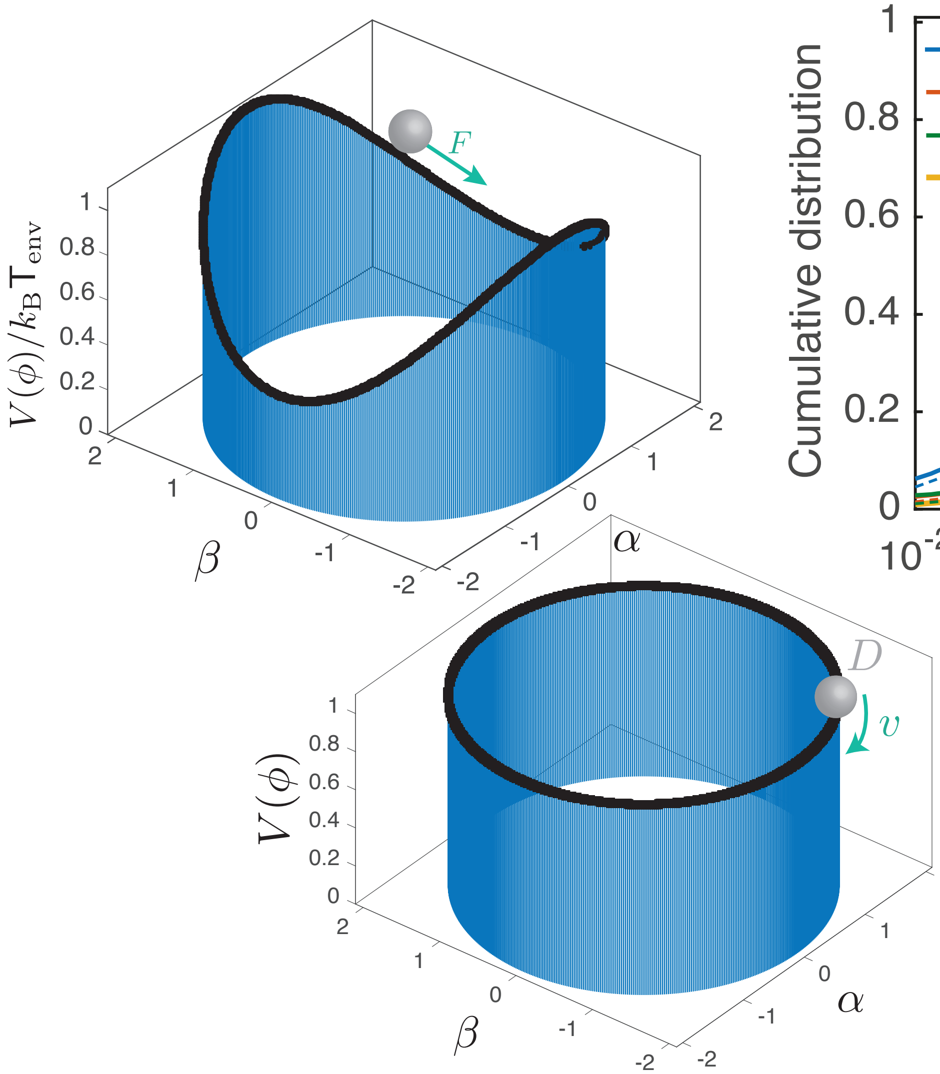
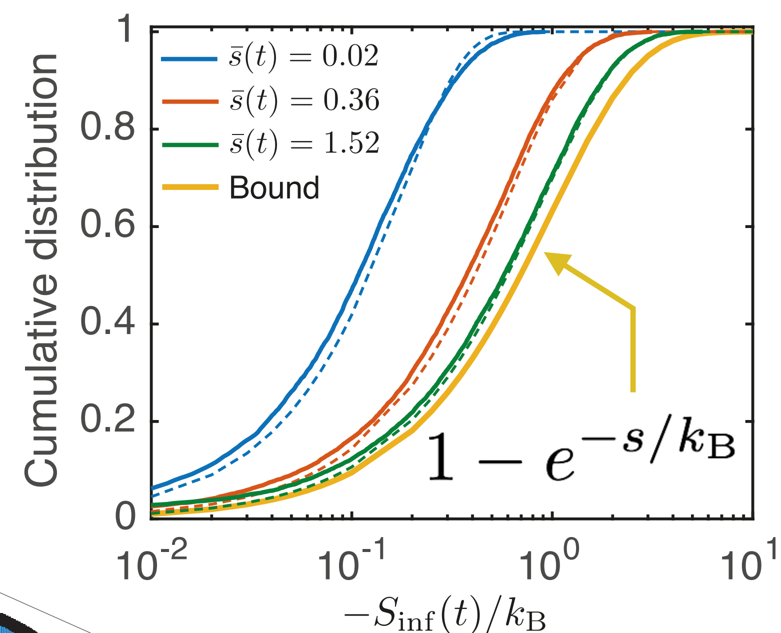
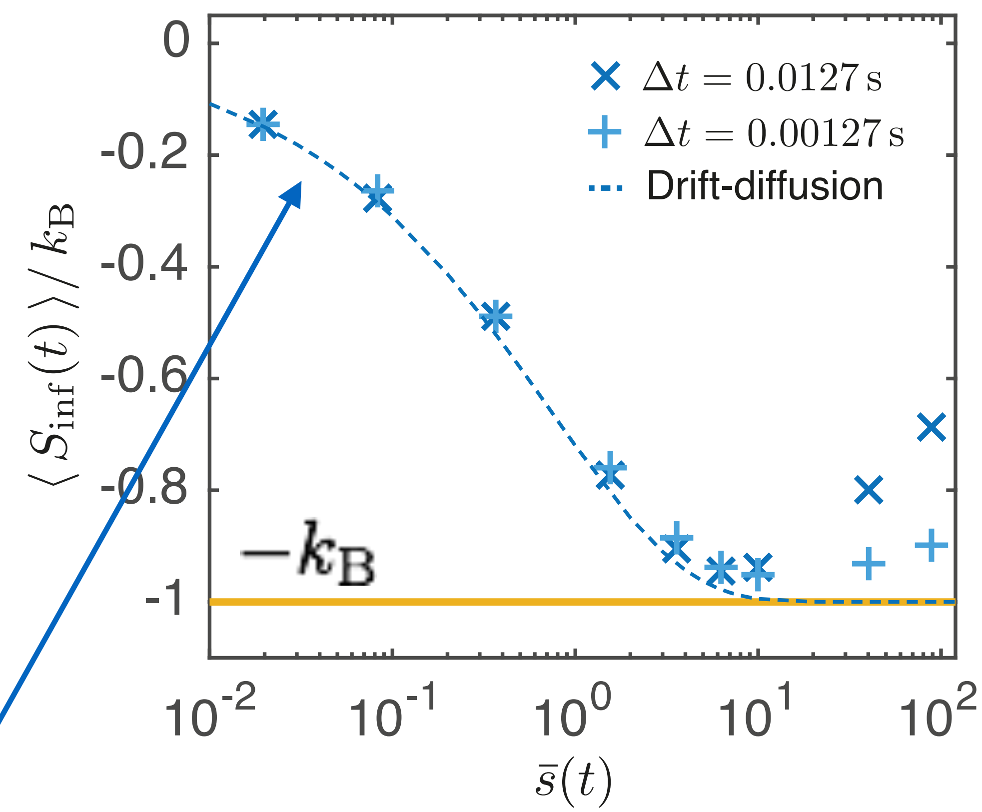


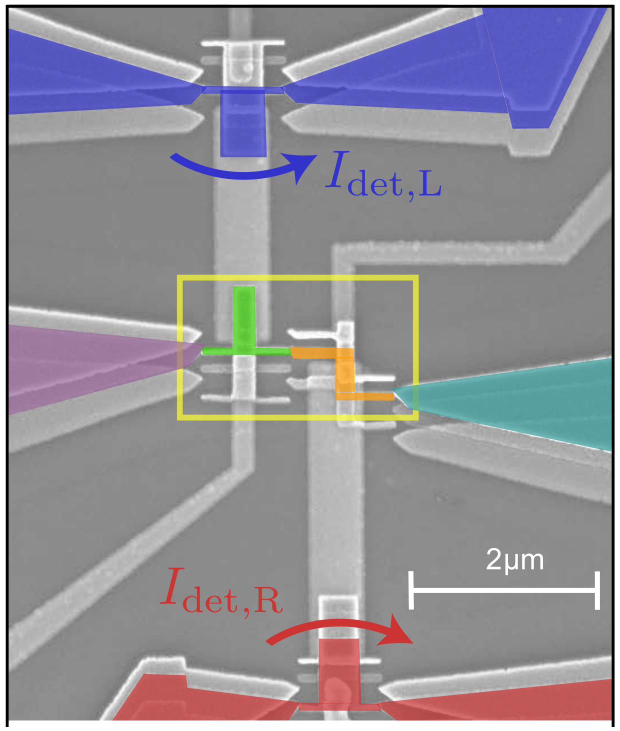


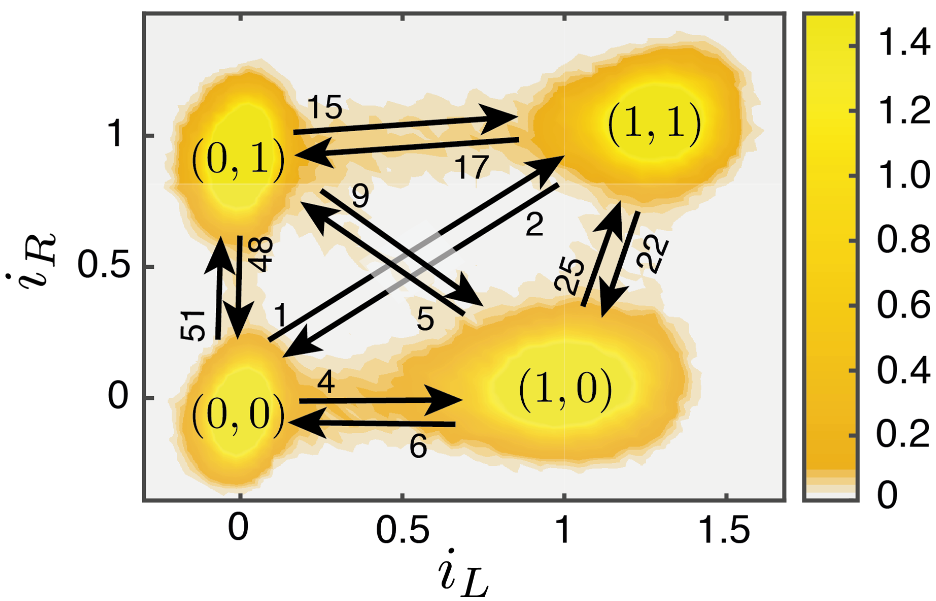


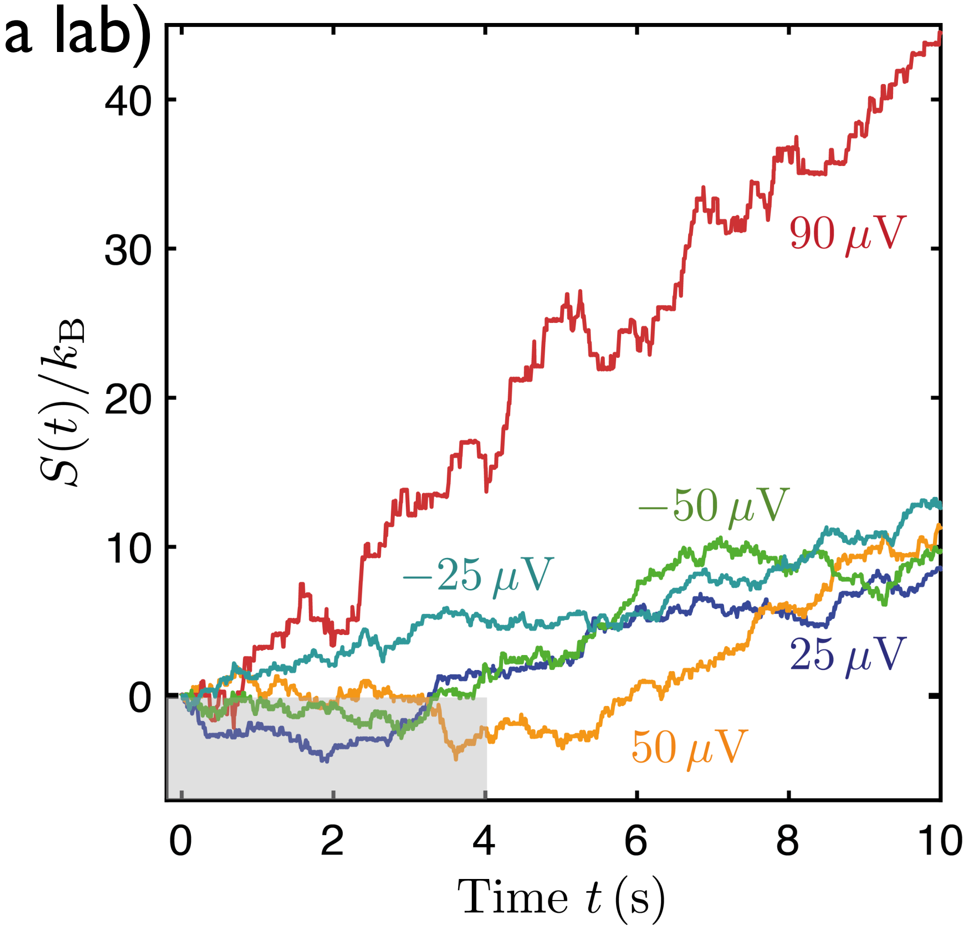
I. Neri, É. Roldán, F. Jülicher, Phys. Rev. X **7** , 011019 (2017)

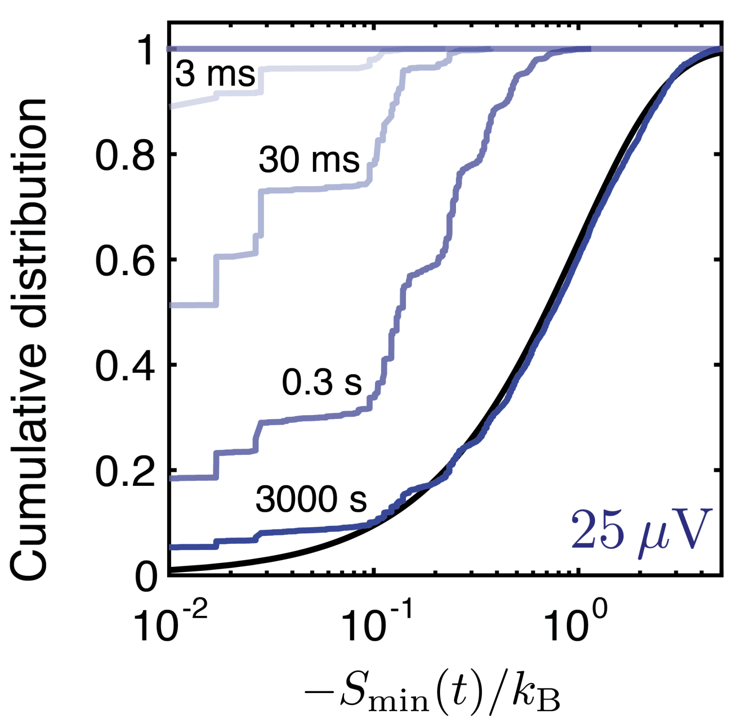
Numerical and experimental tests

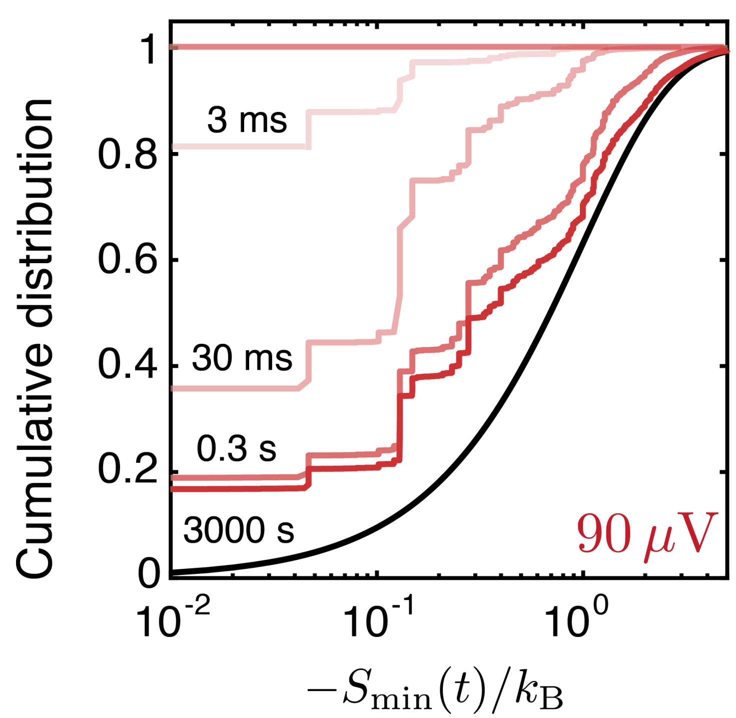
Experimental test: Electronic double dot (Jukka Pekola lab)

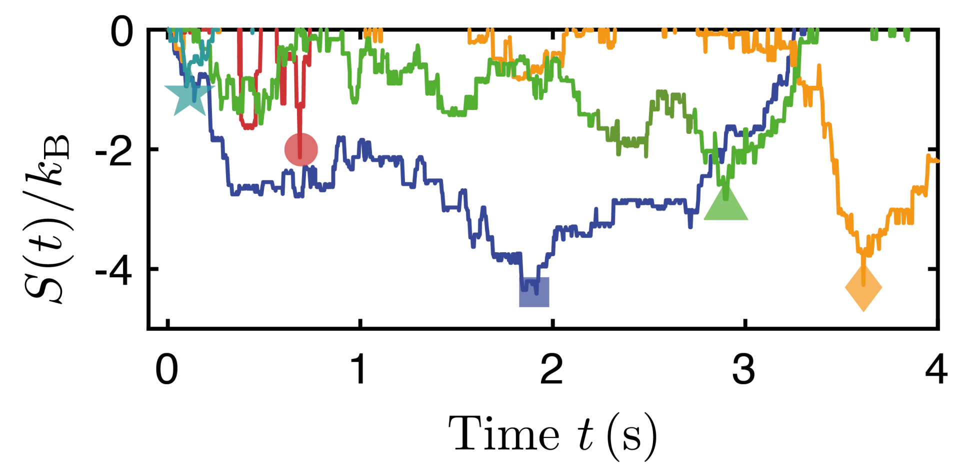












S. Singh et al., Phys. Rev. B **99**, 115422 (2019)

Langevin dynamics:

universal (martingale) properties of entropy production

Langevin processes

Ito Langevin equation with multiplicative noise

$$\frac{\mathrm{d}\vec{X}}{\mathrm{d}t} = \mu \cdot \vec{F} + \vec{\nabla} \cdot D + \sqrt{2}\,\sigma \cdot \vec{\xi}$$

Force

$$\vec{F} = -\vec{\nabla} U(\vec{X}(t), t) + \vec{f}(\vec{X}(t), t)$$

conservative non-conservative

Diffusion coefficient

$$\sigma(\vec{X}(t))\sigma(\vec{X}(t))^{\mathsf{T}} = D(\vec{X}(t))$$

$$D(\vec{X}(t)) = \mu(\vec{X}(t))\, k_{\mathrm{B}} T$$ Einstein relation

Smoluchowski's Equation

$$\partial_t P = -\vec{\nabla} \cdot \vec{J}$$

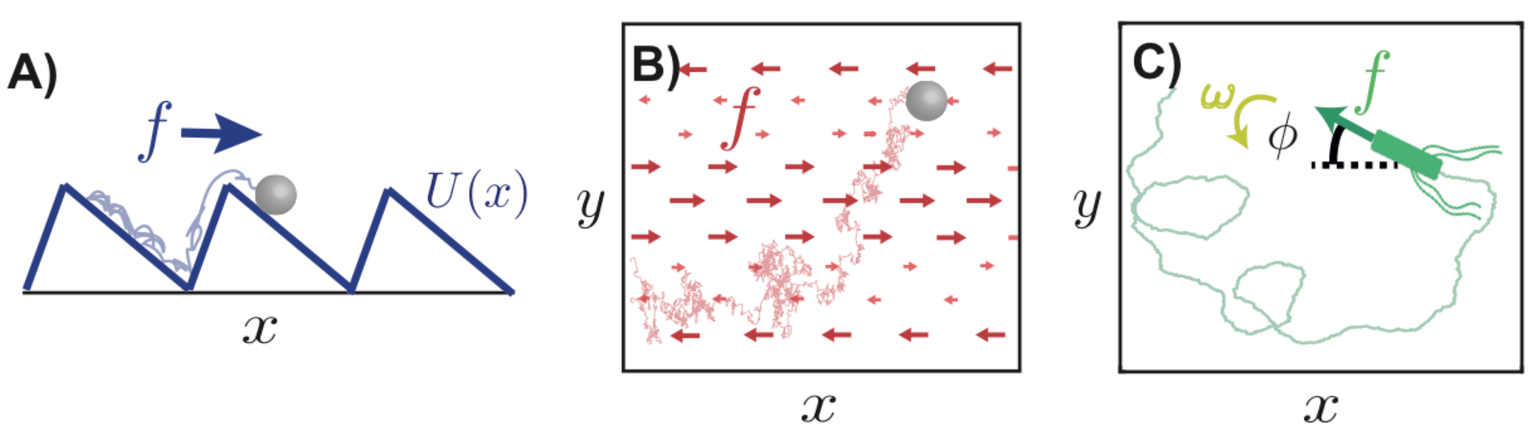
$$\vec{J} = \mu \cdot \vec{F}\, P - D \cdot \vec{\nabla} P$$

Noise

$$\langle \xi_i(t) \rangle = 0$$ Gaussian white noise

$$\langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} \delta(t - t')$$

M. v. Smoluchowski, Ann. d. Phys. **21,** 756 (1906)



S. Pigolotti, I. Neri, É. Roldán, F. Jülicher, Phys. Rev. Lett. **119** (14), 140604 (2017)

Martingality in Langevin processes

Ito Langevin equation for stochastic entropy production

non-steady state
$$\frac{\mathrm{d}S_{\mathrm{tot}}}{\mathrm{d}t} = -2k_{\mathrm{B}}\partial_t \ln P + v_{\mathrm{S}} + \sqrt{2k_{\mathrm{B}}v_{\mathrm{S}}}\,\xi_S$$

steady states
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Entropic drift

$$v_S = k_{\mathrm{B}}\frac{\vec{J}\cdot D^{-1}\cdot\vec{J}}{P^2}$$

Entropic noise

$$\langle \xi_S(t)\rangle = 0$$
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$e^{-S_{\mathrm{tot}}(t)/k_{\mathrm{B}}}$ is a Geometric Brownian motion with zero drift (Martingale)

$$\frac{\mathrm{d}e^{-S_{\mathrm{tot}}/k_{\mathrm{B}}}}{\mathrm{d}t} = -\sqrt{2\frac{v_S}{k_{\mathrm{B}}}}\,e^{-S_{\mathrm{tot}}/k_{\mathrm{B}}}\,\xi_S$$

S. Pigolotti, I. Neri, É. Roldán, F. Jülicher, Phys. Rev. Lett. **119** (14), 140604 (2017)

Langevin fluctuations and time

$$\frac{\mathrm{d}\vec{X}}{\mathrm{d}t} = \mu \cdot \vec{F} + \vec{\nabla} \cdot D + \sqrt{2}\,\sigma \cdot \vec{\xi}$$

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S. Pigolotti, I. Neri, É. Roldán, F. Jülicher, Phys. Rev. Lett. **119** (14), 140604 (2017)

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“Entropic” time

$$v_S = k_{\mathrm{B}} \frac{\vec{J} \cdot D^{-1} \cdot \vec{J}}{P^2} \qquad \tau(t) = \frac{1}{k_{\mathrm{B}}} \int_0^t v_S(X(t'))\mathrm{d}t'$$

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Random-time change $t \to \tau(t)$

S. Pigolotti, I. Neri, É. Roldán, F. Jülicher, Phys. Rev. Lett. **119** (14), 140604 (2017)

Langevin fluctuations and time

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$$\frac{\mathrm{d}S_{\mathrm{tot}}}{\mathrm{d}t} = v_{\mathrm{S}} + \sqrt{2k_{\mathrm{B}}v_{\mathrm{S}}}\,\xi_S \longrightarrow \frac{1}{k_{\mathrm{B}}}\frac{\mathrm{d}S_{\mathrm{tot}}}{\mathrm{d}\tau} = 1 + \sqrt{2}\,\eta$$

"Entropic" time

$$v_S = k_{\mathrm{B}}\frac{\vec{J} \cdot D^{-1} \cdot \vec{J}}{P^2} \qquad \tau(t) = \frac{1}{k_{\mathrm{B}}}\int_0^t v_S(X(t'))\mathrm{d}t'$$

Random-time change $t \to \tau(t)$

S. Pigolotti, I. Neri, É. Roldán, F. Jülicher, Phys. Rev. Lett. **119** (14), 140604 (2017)

Langevin fluctuations and time

$$\frac{\mathrm{d}\vec{X}}{\mathrm{d}t} = \mu \cdot \vec{F} + \vec{\nabla} \cdot D + \sqrt{2}\,\sigma \cdot \vec{\xi}$$

$$\frac{\mathrm{d}S_{\mathrm{tot}}}{\mathrm{d}t} = v_{\mathrm{S}} + \sqrt{2 k_{\mathrm{B}} v_{\mathrm{S}}}\, \xi_S \longrightarrow \frac{1}{k_{\mathrm{B}}} \frac{\mathrm{d}S_{\mathrm{tot}}}{\mathrm{d}\tau} = 1 + \sqrt{2}\, \eta$$

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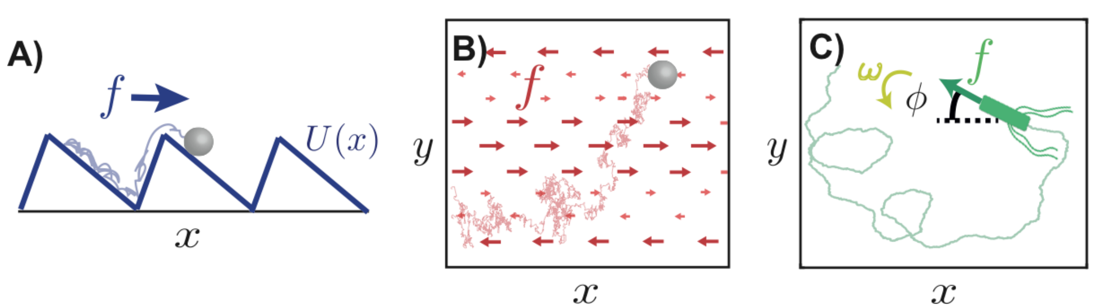

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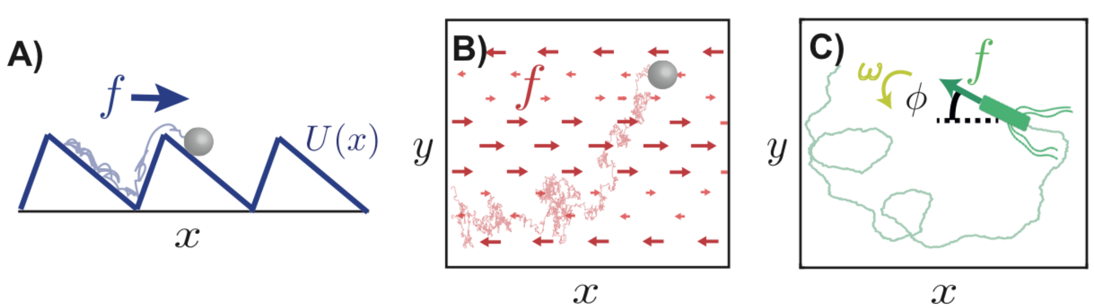

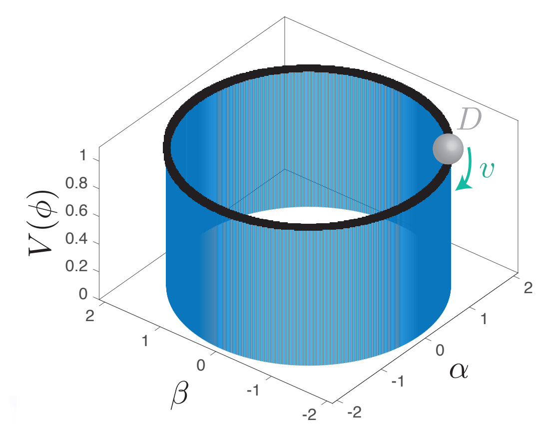

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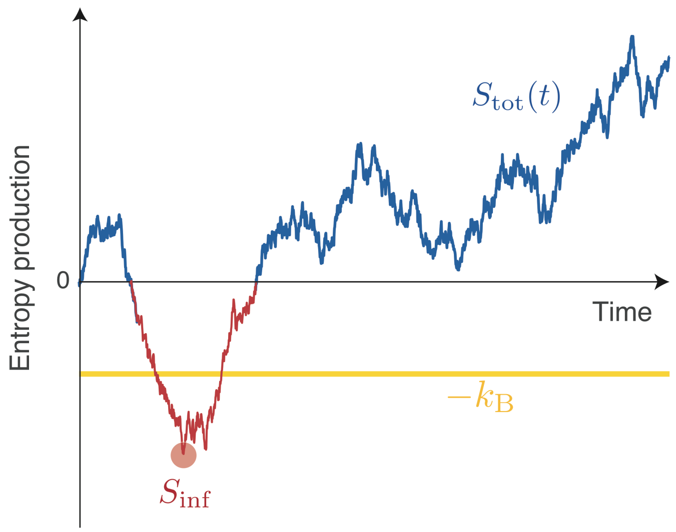
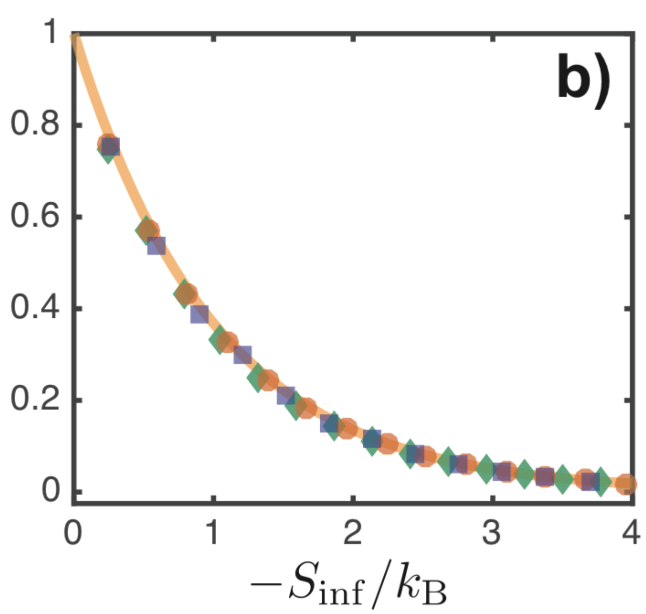
Generic properties of entropy production

Global infimum





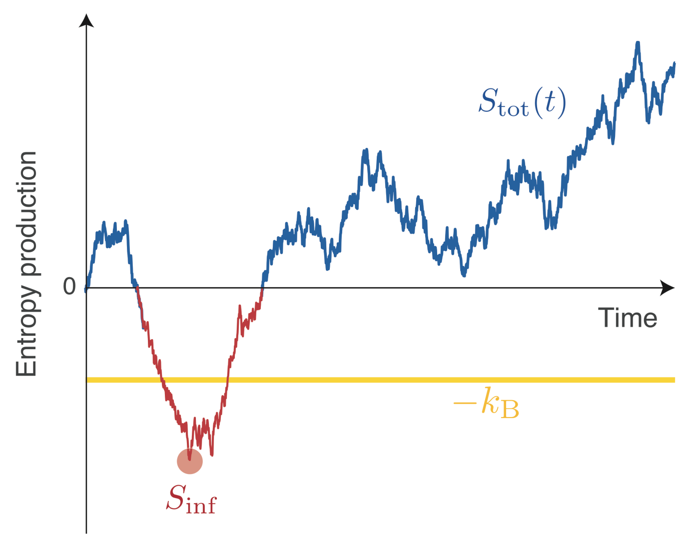
$$P(S_{\rm inf}) = e^{S_{\rm inf}/k_{\rm B}}/k_{\rm B}$$

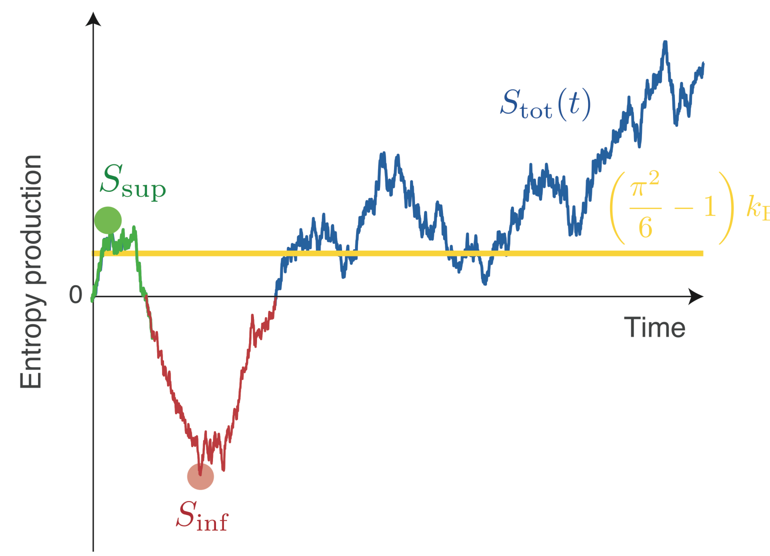
S. Pigolotti, I. Neri, É. Roldán, F. Jülicher, Phys. Rev. Lett. **119** (14), 140604 (2017)

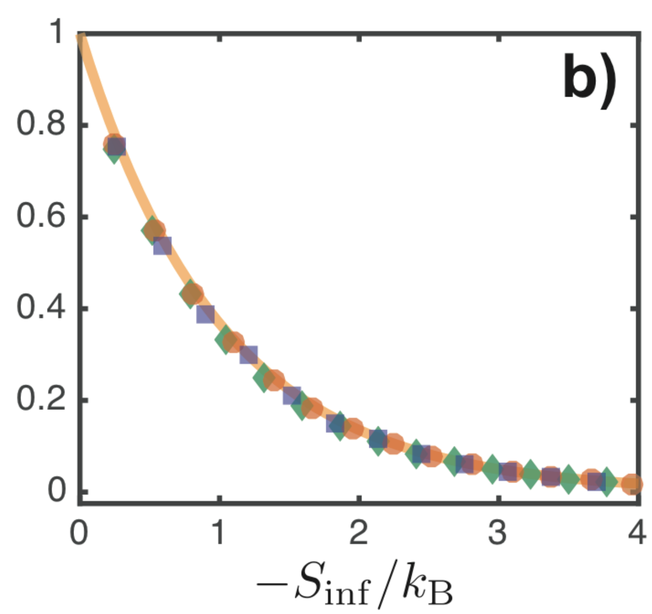
Generic properties of entropy production

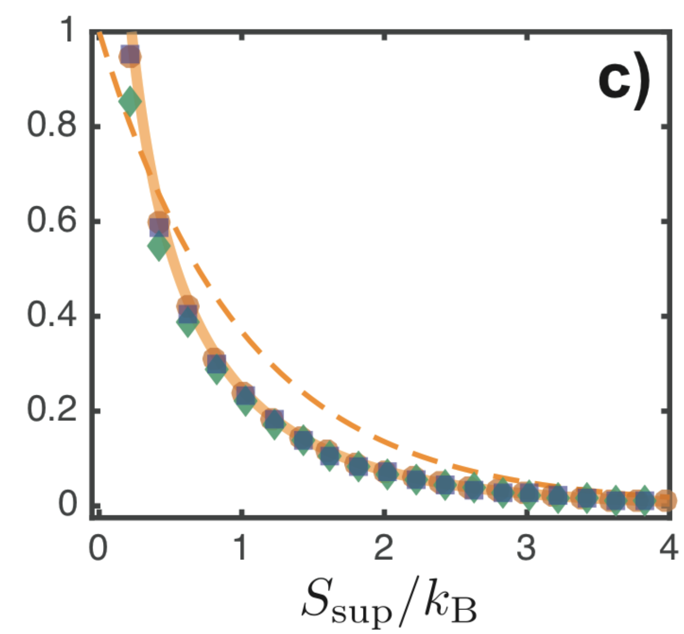
Global infimum

Supremum before the global infimum









$$P(S_{\rm inf}) = e^{S_{\rm inf}/k_{\rm B}}/k_{\rm B}$$

$$P(S_{\rm sup}) = 2e^{S_{\rm sup}/k_{\rm B}}\,{\rm acoth}(2e^{S_{\rm sup}/k_{\rm B}}-1)-1$$

S. Pigolotti, I. Neri, É. Roldán, F. Jülicher, Phys. Rev. Lett. **119** (14), 140604 (2017)

Generic properties of entropy production

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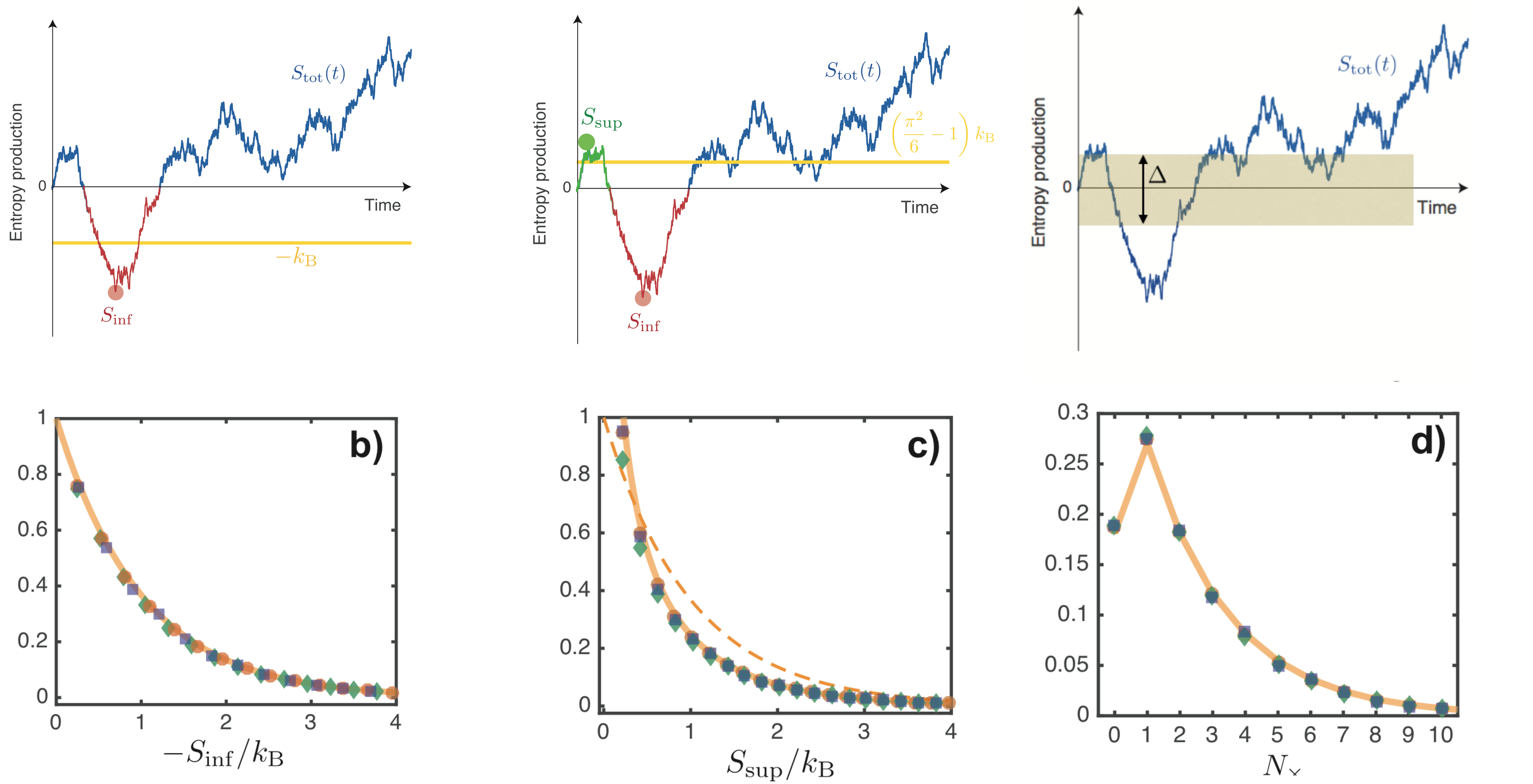
Supremum before the global infimum

Number of crossings



$$P(S_{\rm inf}) = e^{S_{\rm inf}/k_B}/k_B$$

$$P(N_\times;\Delta) = \begin{cases} 1 - e^{-\Delta/k_B} & N_\times = 0 \\ 2\sinh(\Delta/k_B)e^{-2N_\times\Delta/k_B} & N_\times \geq 1 \end{cases}$$

$$P(S_{\rm sup}) = 2e^{S_{\rm sup}/k_B}\mathrm{acoth}(2e^{S_{\rm sup}/k_B} - 1) - 1$$

S. Pigolotti, I. Neri, É. Roldán, F. Jülicher, Phys. Rev. Lett. **119** (14), 140604 (2017)

Non-generic properties: Uncertainty e**quality**

Finite-time **uncertainty** **equality** for entropy production
of nonequilibrium steady-state* Langevin processes

$$\frac{1}{k_{\mathrm{B}}}\frac{\sigma^2_{S_{\mathrm{tot}}}(t)}{\langle S_{\mathrm{tot}}(t)\rangle} = 2 + \frac{\sigma^2_{\tau}(t)}{\langle \tau(t)\rangle}$$

* also for non-steady state

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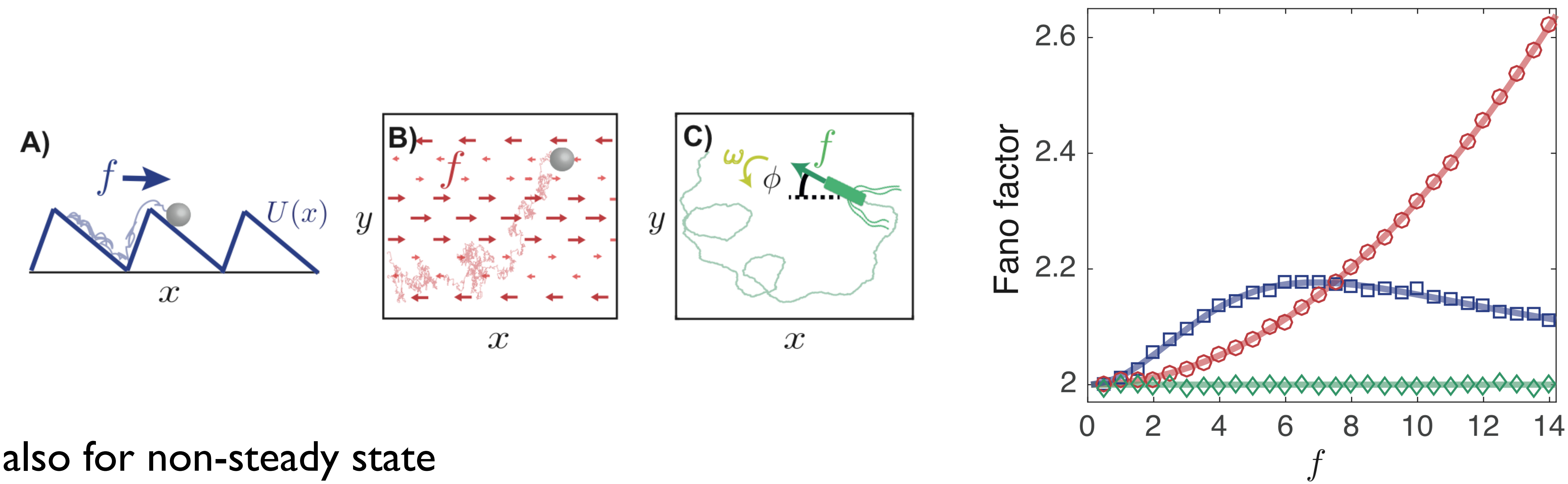

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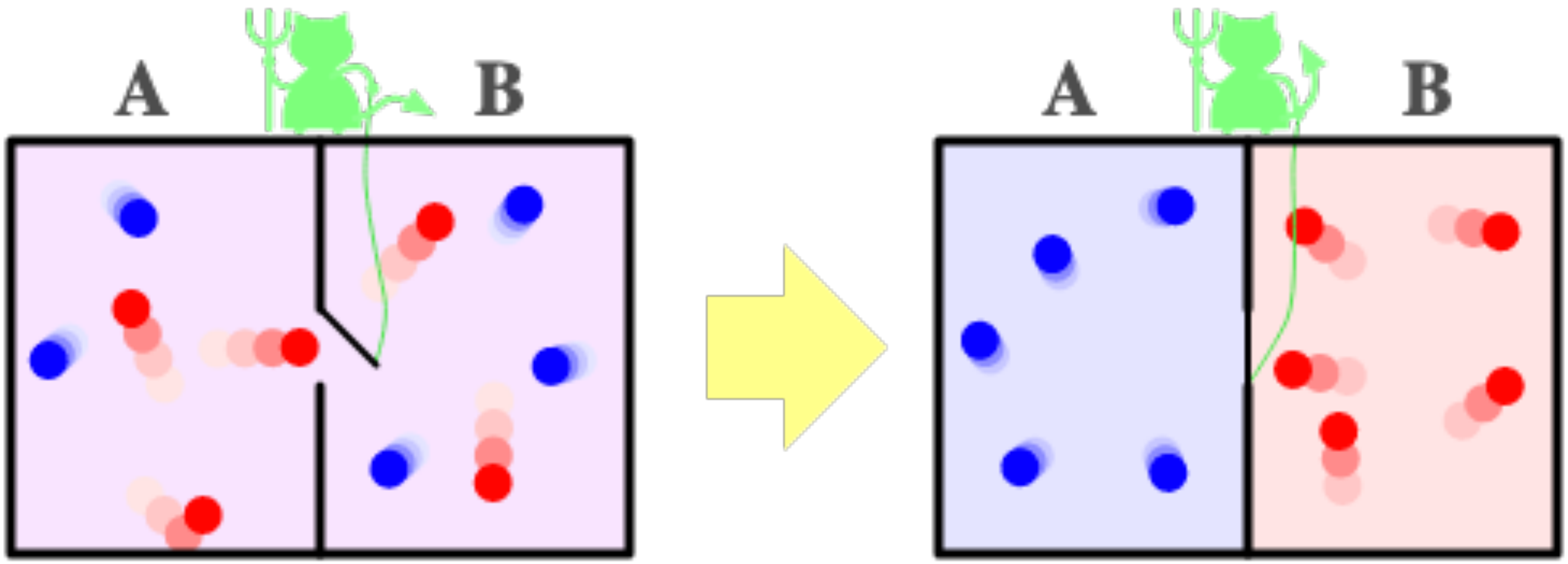
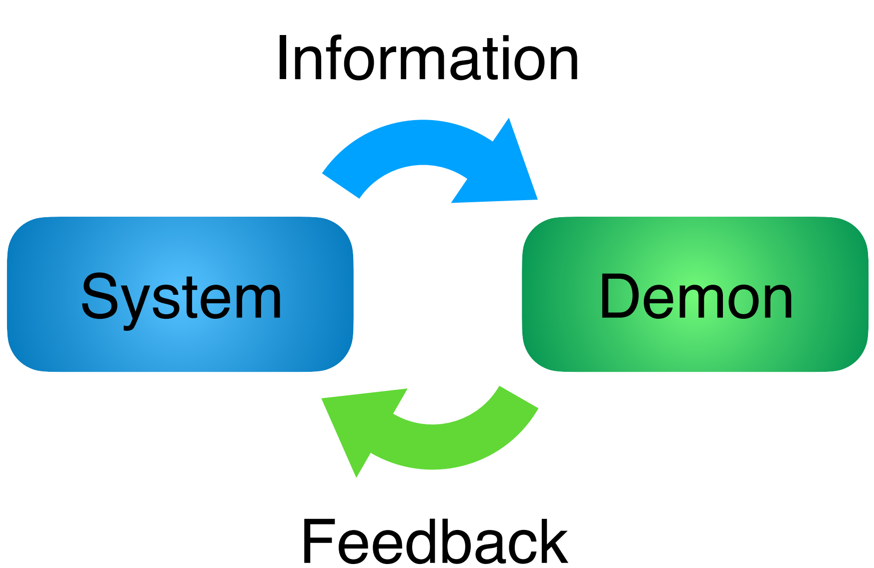
S. Pigolotti, I. Neri, É. Roldán, F. Jülicher, Phys. Rev. Lett. **119** (14), 140604 (2017)

Martingales out of steady states:

Gambling demons

Revisiting Maxwell's demons





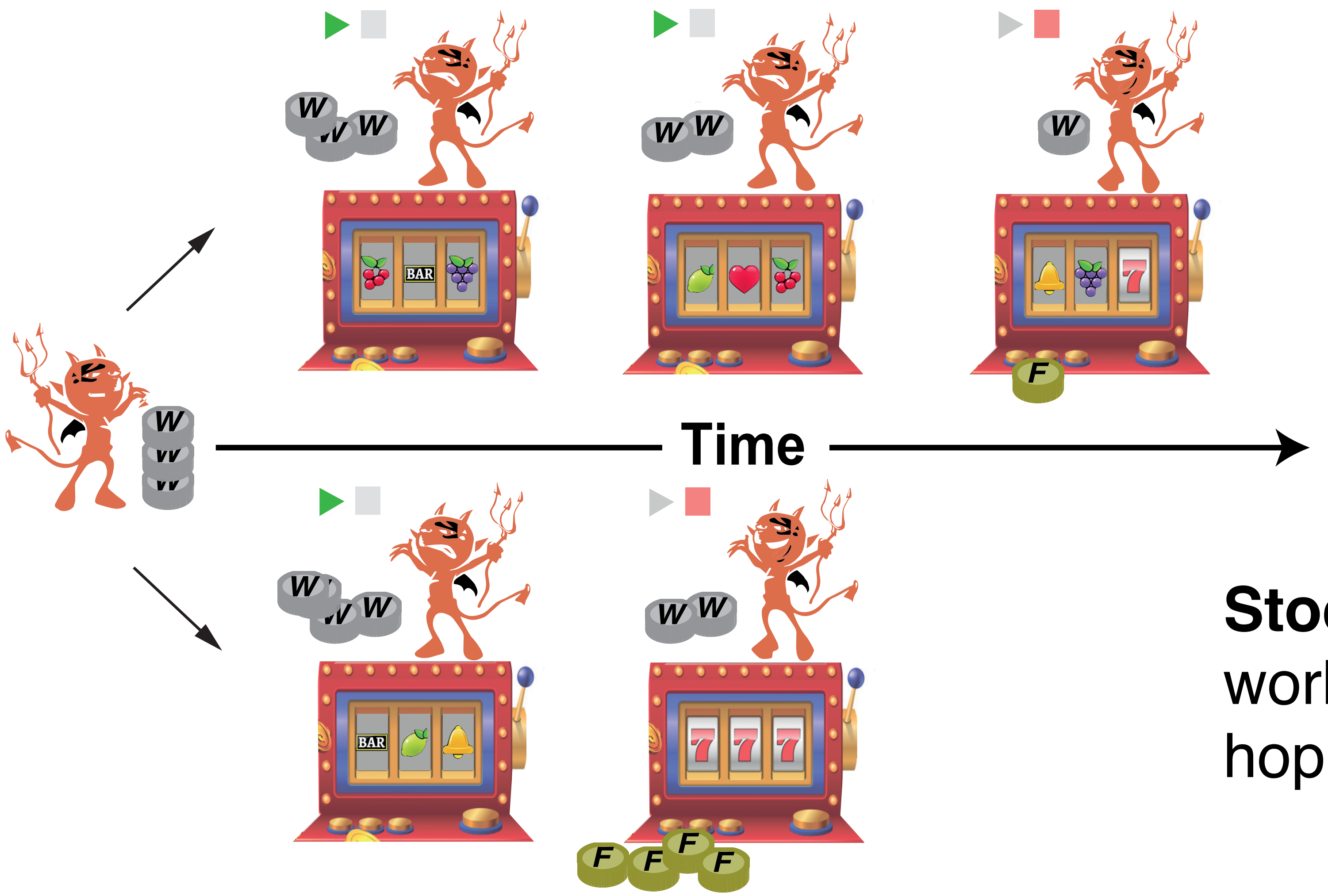
"*Seemingly-violation*" of the second law

<u>Feedback</u> control depending on measurement outcome

<u>Stochastic times</u> for opening/closing of the gate

Gambling demons

A "demon" invests work performing a nonequilibrium process and decides whether to stop the process or not a stopping (gambling) strategy



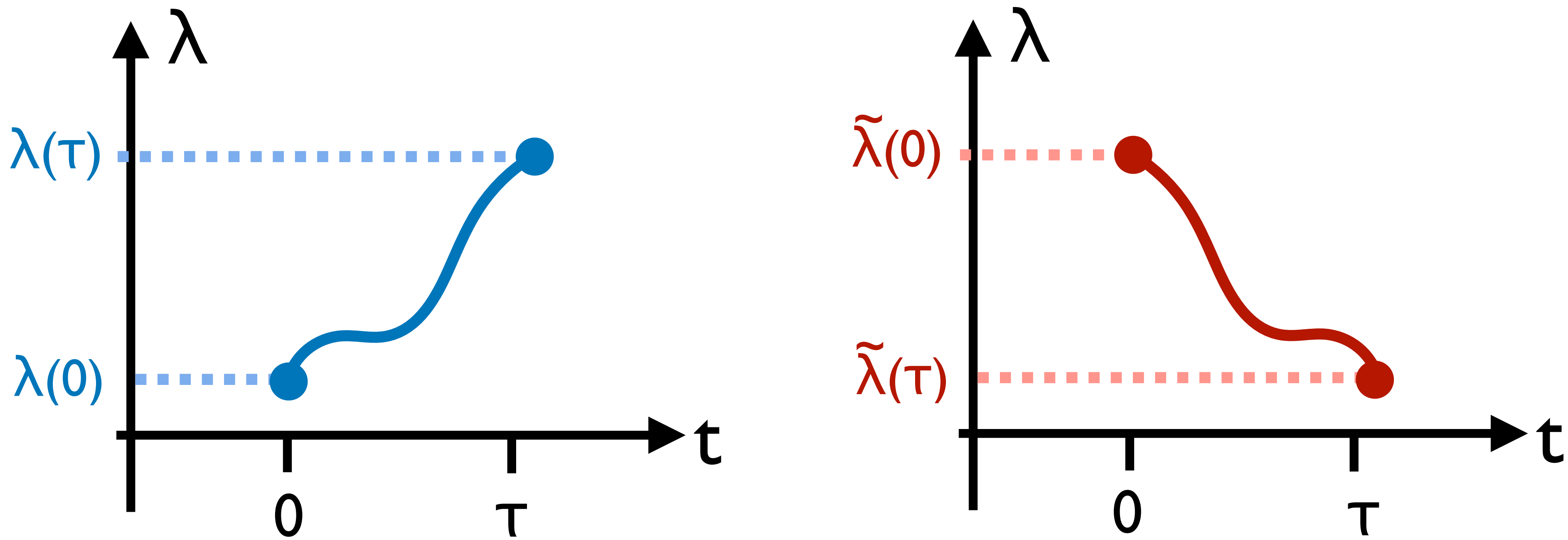
No feedback control of system's dynamics

Stochastic times: the gambler invests work (coins) in a slot machine (system) hoping to obtain a positive payoff

Goal: Extract heat from a thermal bath by conditionally stopping the system with a clever strategy

G Manzano, D Subero, O Maillet, R Fazio, J Pekola, ER, Physical Review Letters **126** (8), 080603 (2021)

Martingale theory for non-stationary driving



"Forward" process: Drive $\lambda(t)$, initial state $\varrho(x,0)$, final $\varrho(x,\tau)$

"Backward" process: Drive $\tilde{\lambda}(t) = \lambda(\tau - t)$, initial $\tilde{\varrho}(x,0)$, final $\tilde{\varrho}(x,\tau)$

G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)

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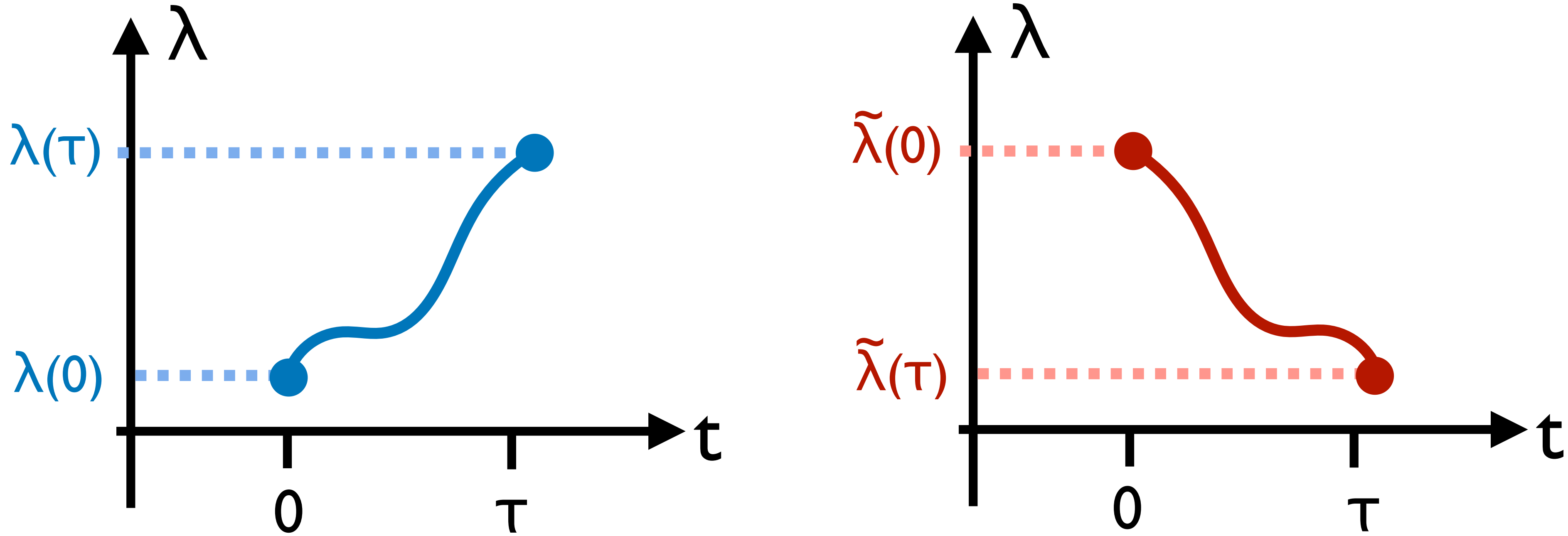

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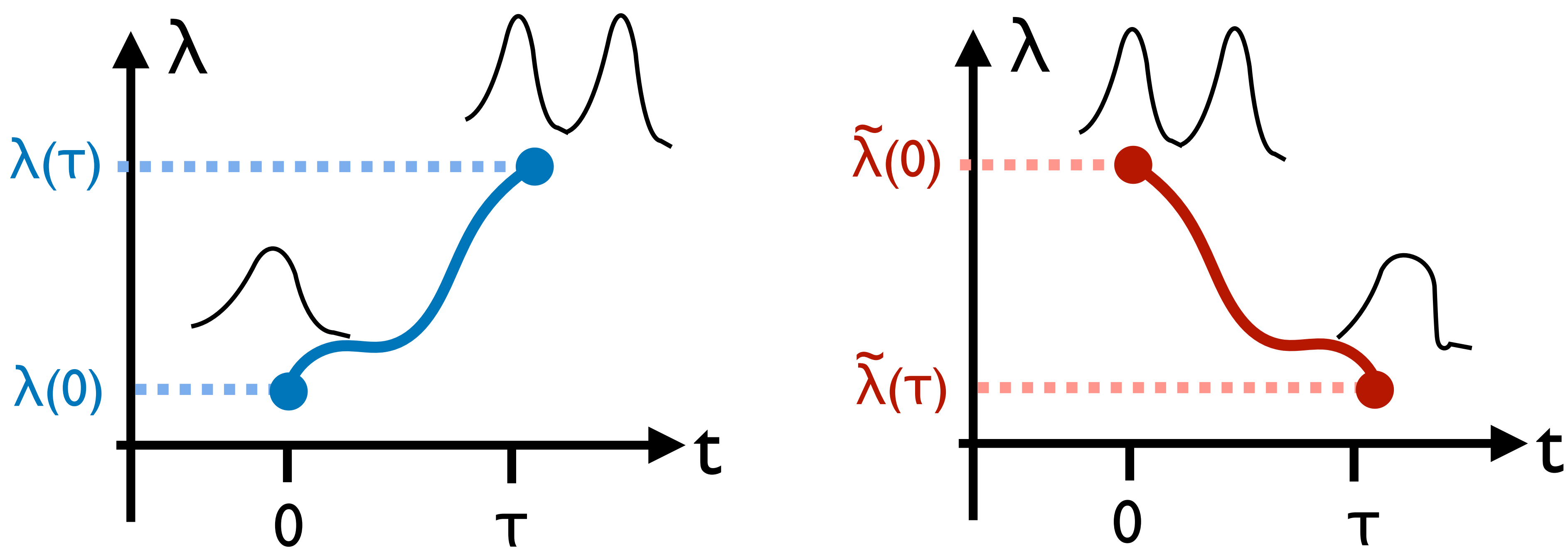

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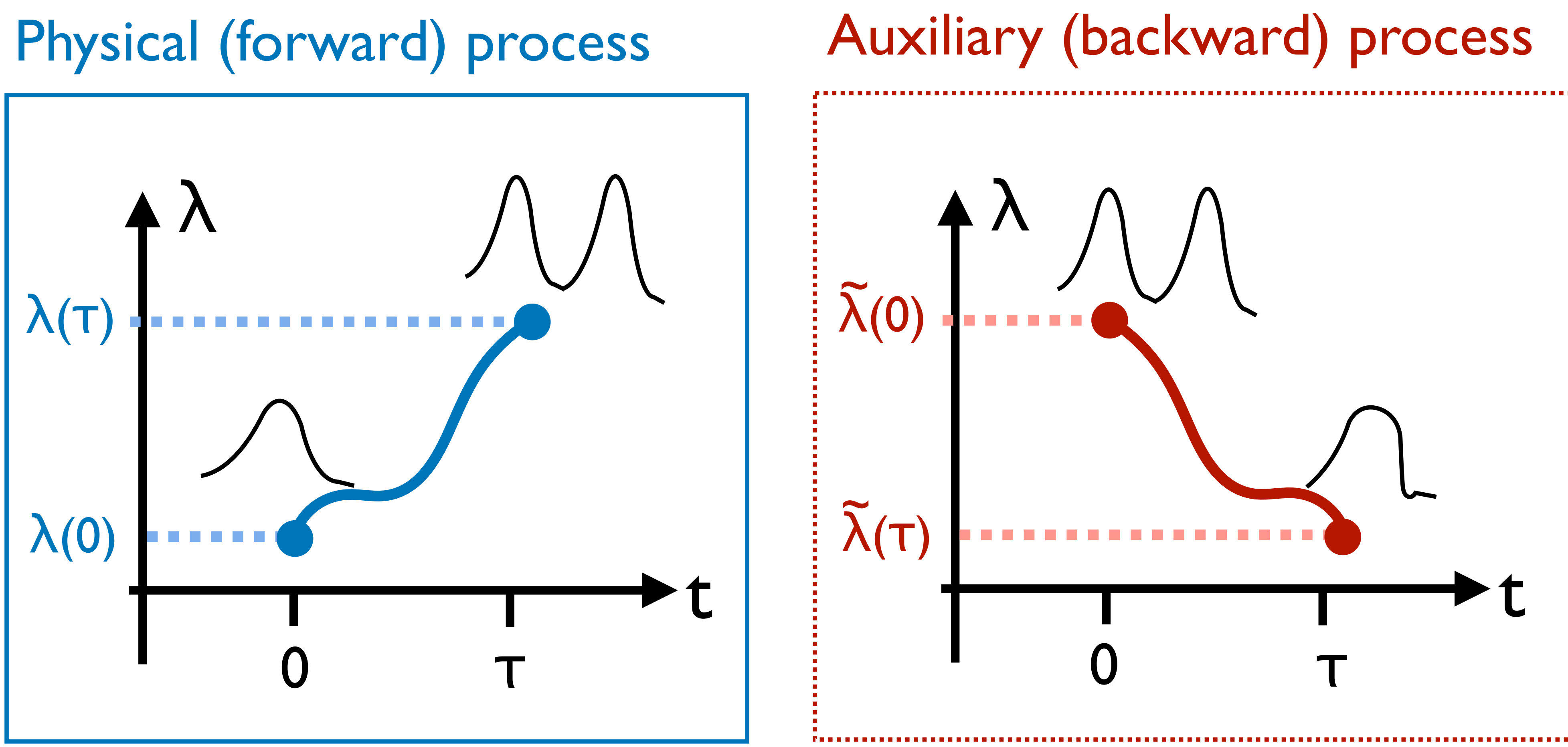

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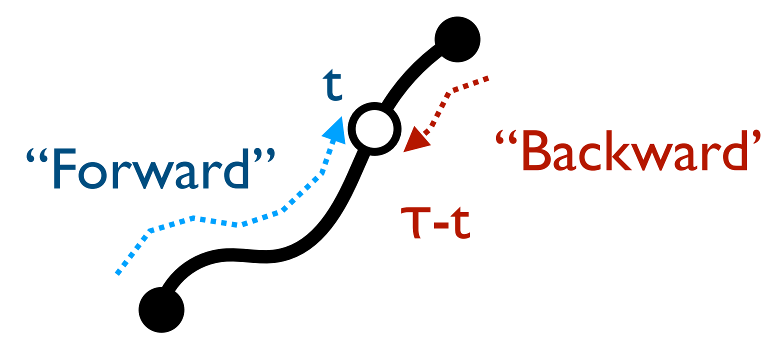
Entropy production is **not** exponential martingale for non-stationary processes:

$$S_{\mathrm{tot}}(t) = k_{\mathrm{B}} \ln \frac{P[X_t]}{\tilde{P}[\tilde{X}_t]}$$

$$\langle e^{-S_{\mathrm{tot}}(t)/k_{\mathrm{B}}} | X_s \rangle = e^{-S_{\mathrm{tot}}(s)/k_{\mathrm{B}}} \frac{\varrho(x(s),s)}{\tilde{\varrho}(x(s),\tau - s)} = e^{-S_{\mathrm{tot}}(t)/k_{\mathrm{B}} + \delta(s)}$$

Stochastic distinguishability

$$\delta(t) \equiv \ln \left[\frac{\varrho(x(t),t)}{\tilde{\varrho}(x(t),\tau - t)} \right]$$



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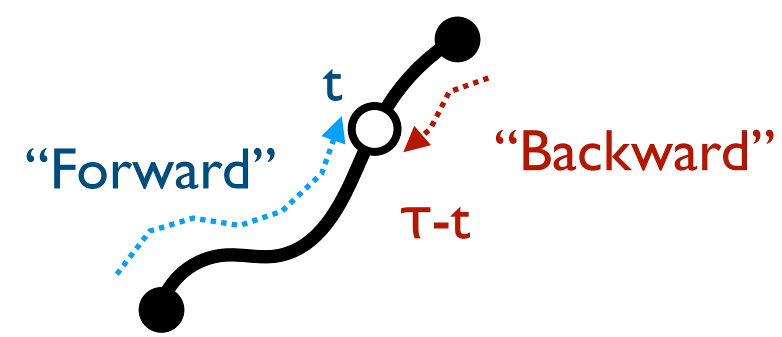

Martingalization of entropy production

$$\langle e^{-S_{\rm tot}(t)/k_{\rm B} - \delta(t)} | X_s \rangle = e^{-S_{\rm tot}(s)/k_{\rm B} - \delta(s)}$$

G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)

Second law inequality at *stopping times*

Work and free energy in a time interval of (finite) stochastic duration $[0, \mathcal{T}]$

$$W(\mathcal{T}) = \int_0^{\mathcal{T}} \partial_t H(x(t), t) dt$$

$$F(\mathcal{T}) = H(x(\mathcal{T}), \mathcal{T}) - S(\mathcal{T})$$
$$\Delta F(\mathcal{T}) = F(\mathcal{T}) - F(0)$$

Integral FT at stopping times

$$\langle e^{-S_{\rm tot}(\mathcal{T})/k_{\rm B} - \delta(\mathcal{T})} \rangle = 1$$

G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)

Initial equilibrium: I. Neri, Phys. Rev. Lett. **124** (4) 040601(2020)

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Jarzynski-like relation at stopping times

$$\boxed{\langle e^{-\beta[W(\mathcal{T}) - \Delta F(\mathcal{T})] - \delta(\mathcal{T})} \rangle = 1}$$

G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)

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G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)
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Second law at stopping times

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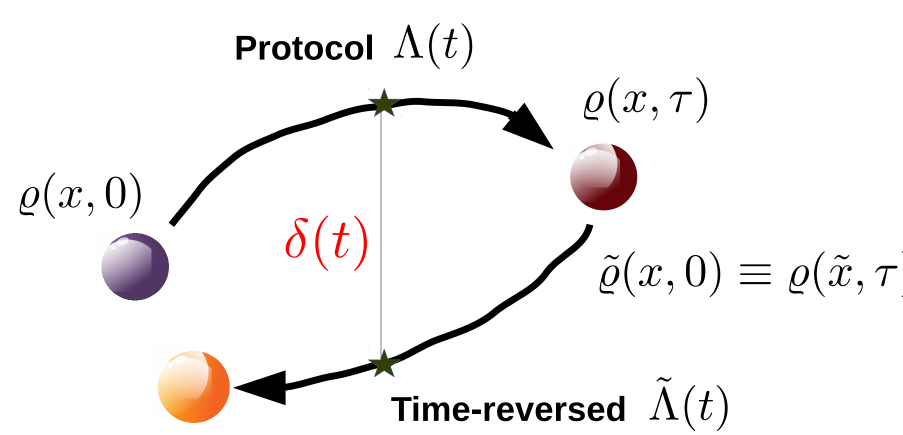
Average distinguishability at stopping times

$$\langle \delta(\mathcal{T}) \rangle = \int d\mathcal{T} \int dx\, \varrho(x, \mathcal{T}) \ln \frac{\varrho(x, \mathcal{T})}{\tilde{\varrho}(x, \tau - \mathcal{T})} \geq 0$$



G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)

Initial equilibrium: I. Neri, Phys. Rev. Lett. **124** (4) 040601(2020)

Second law at fixed *times*

$$\varrho(\mathcal{T}) = \delta(\mathcal{T} - \tau)$$

Integral FT [Seifert, PRL 2005]

$$\langle e^{-S_{\rm tot}(\tau)/k_{\rm B}} \rangle = 1$$

Jarzynski equality [Jarzynski, PRL 1997]

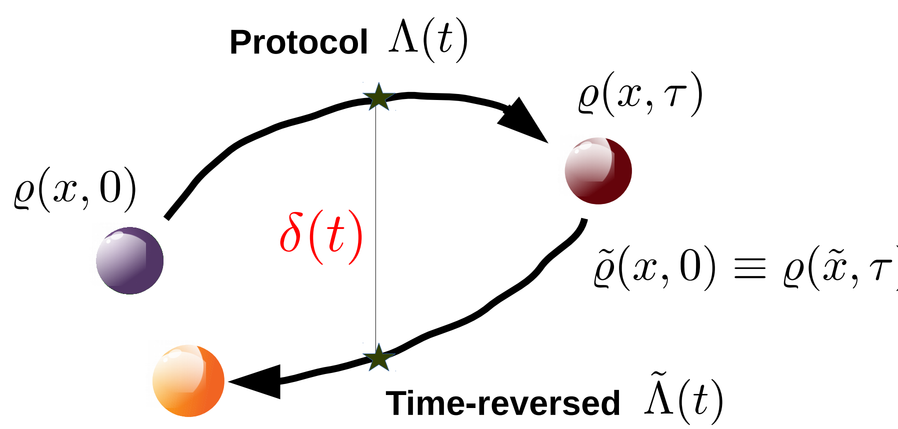
$$\langle e^{-\beta[W(\tau) - \Delta F(\tau)]} \rangle = 1$$

Second law at fixed times

$$\langle W(\mathcal{T}) \rangle - \langle \Delta F(\mathcal{T}) \rangle \geq 0$$

Distinguishability at fixed times

$$\langle \delta(\tau) \rangle = \int dx\, \varrho(x,\tau) \ln \frac{\varrho(x,\tau)}{\tilde{\varrho}(x,0)} = 0$$



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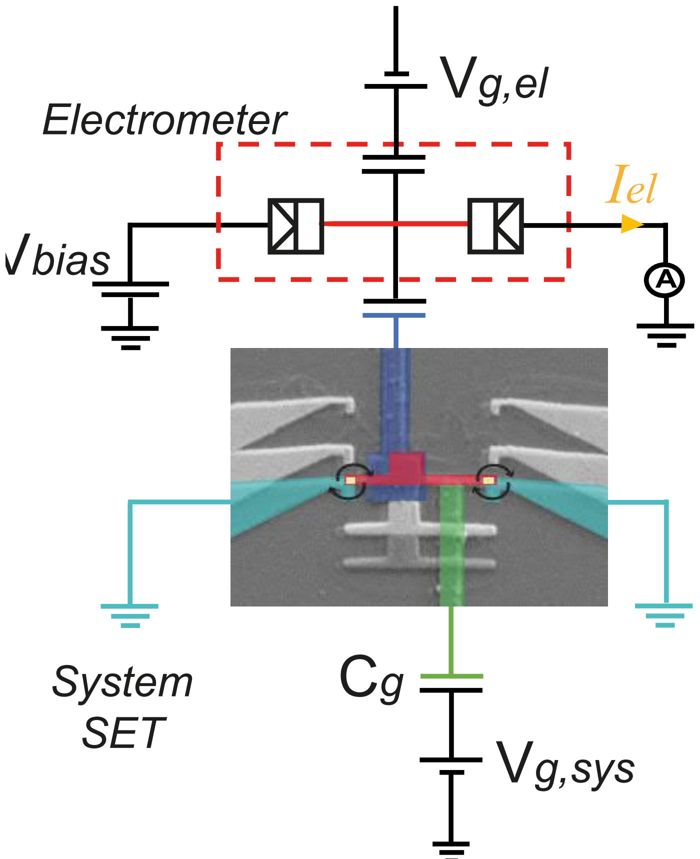
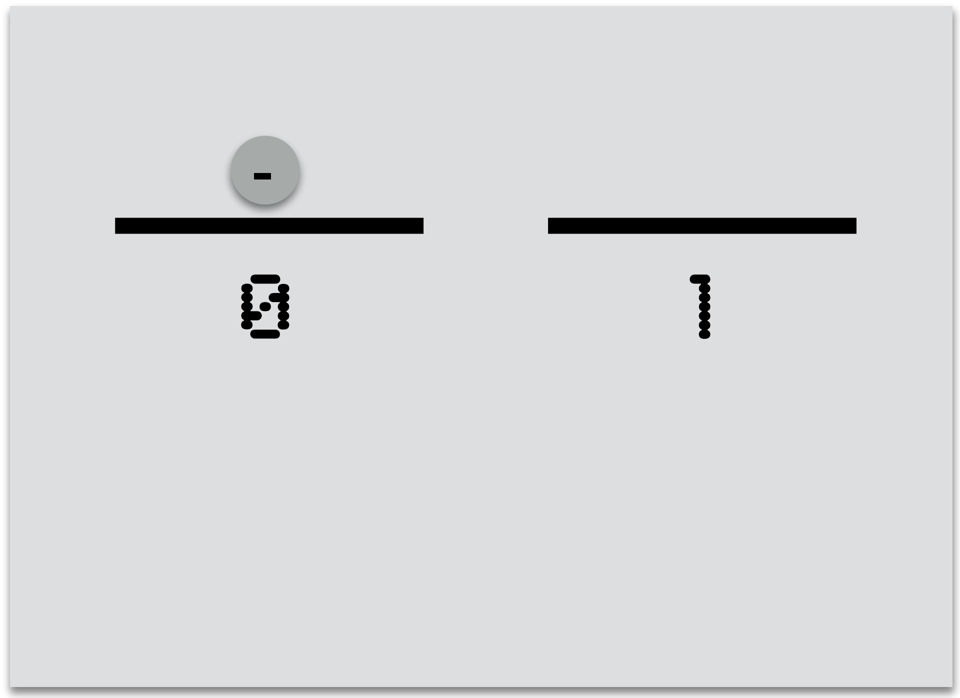
Gambling with a single electron



Experimental realization: Single-electron box (Pekola lab)



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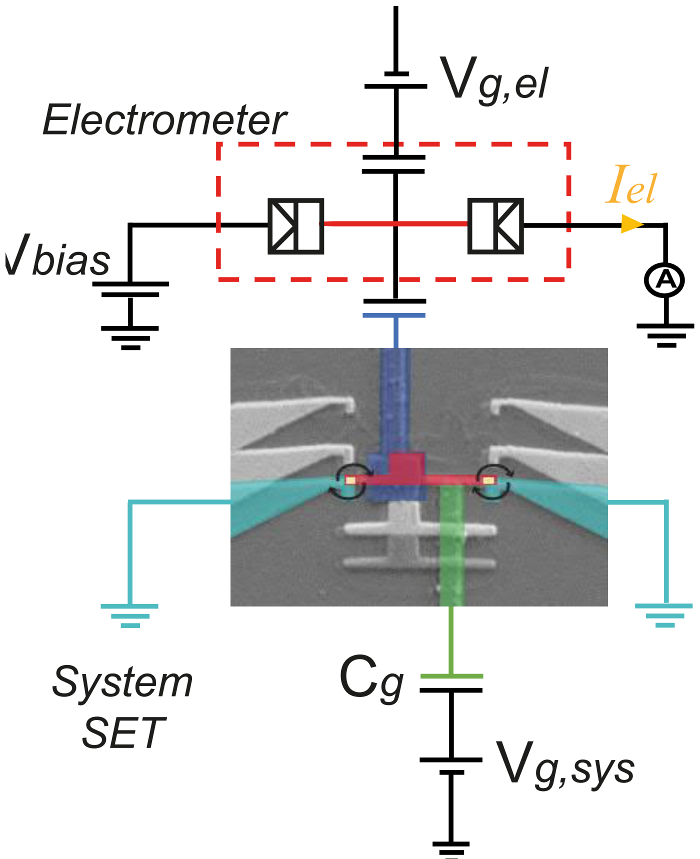

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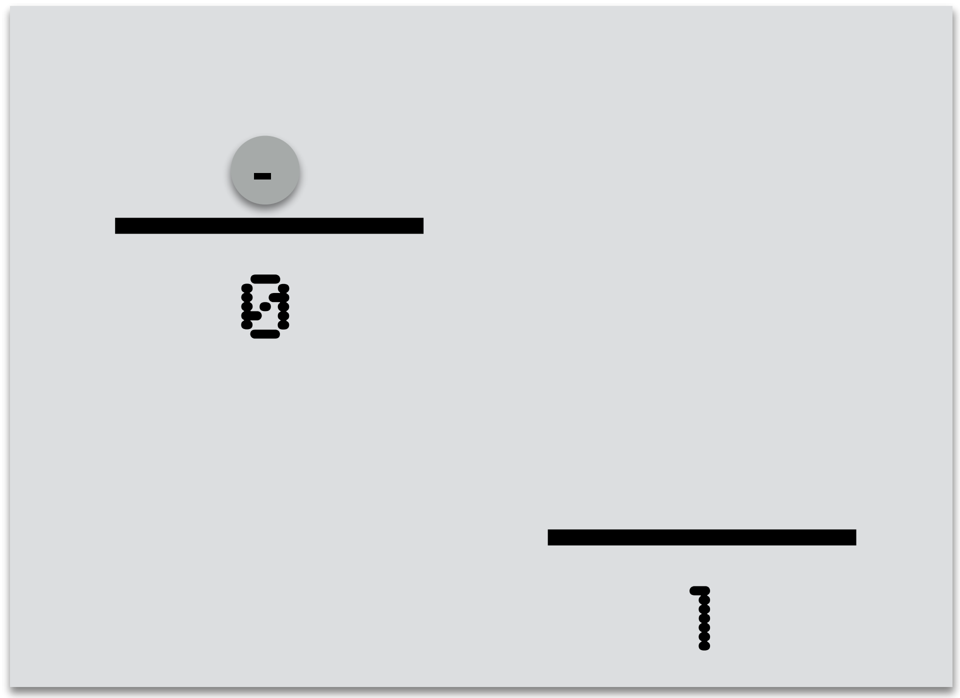

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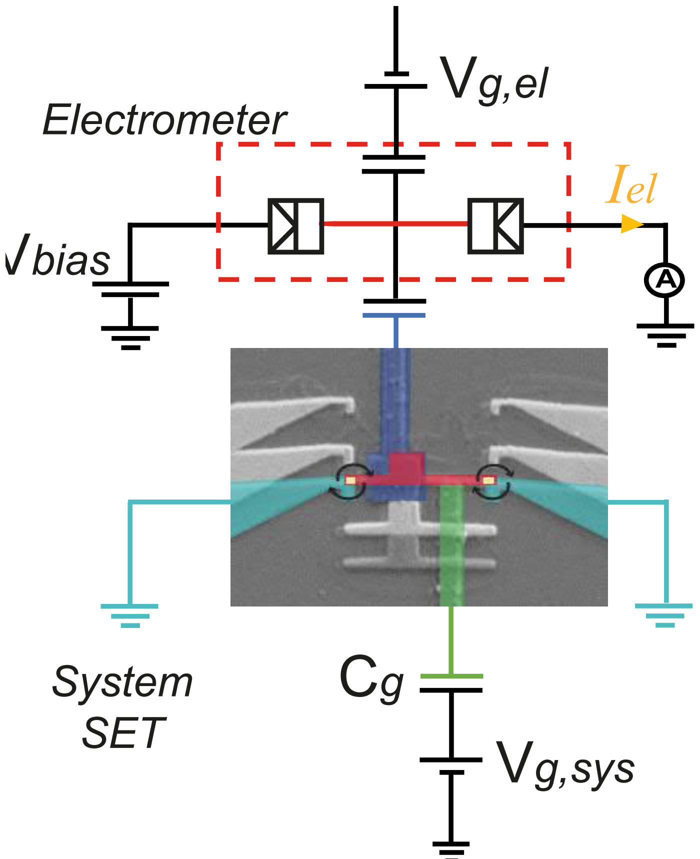

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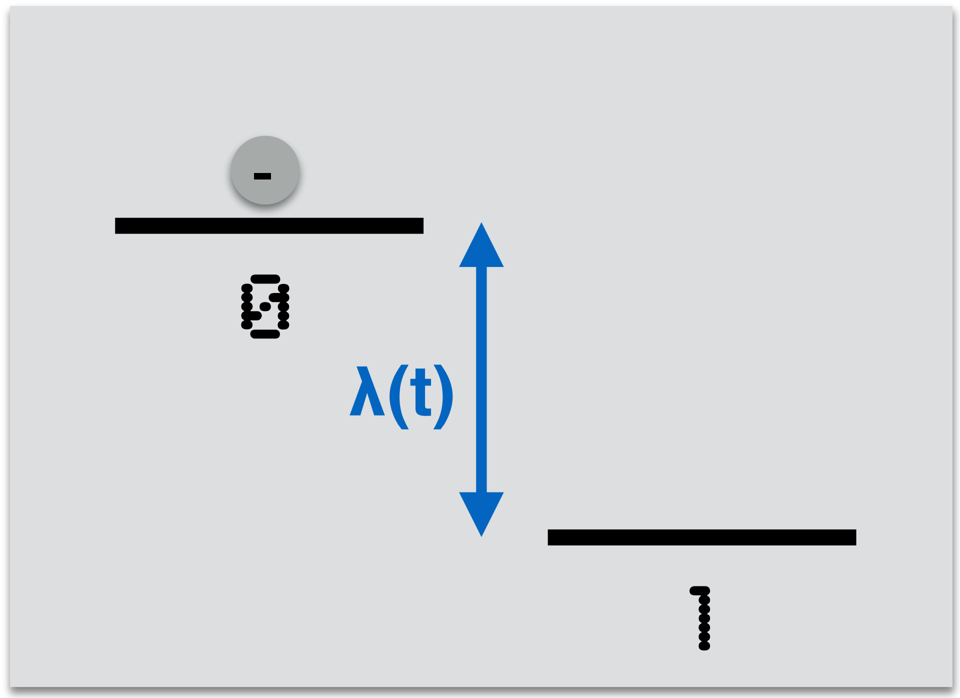
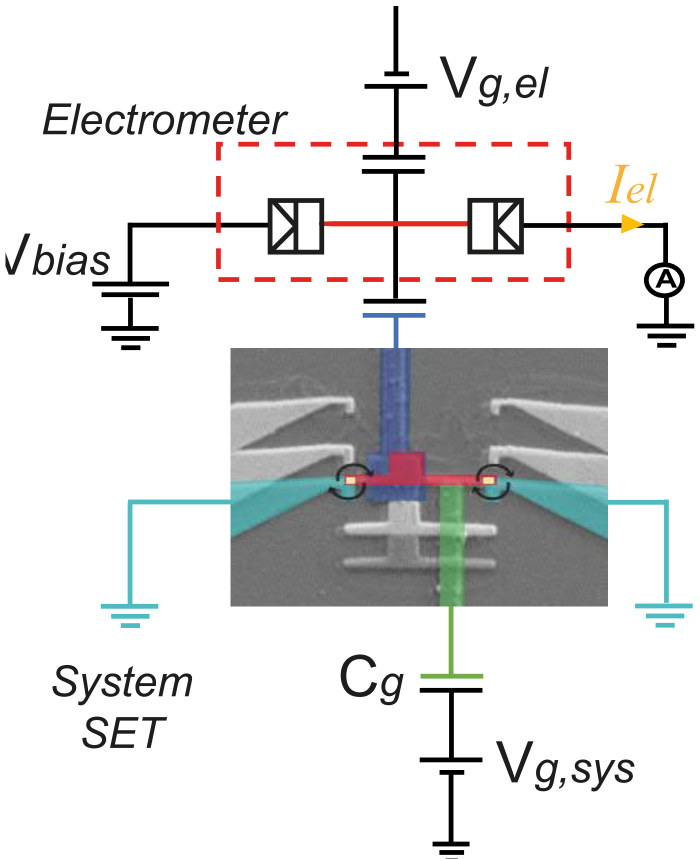
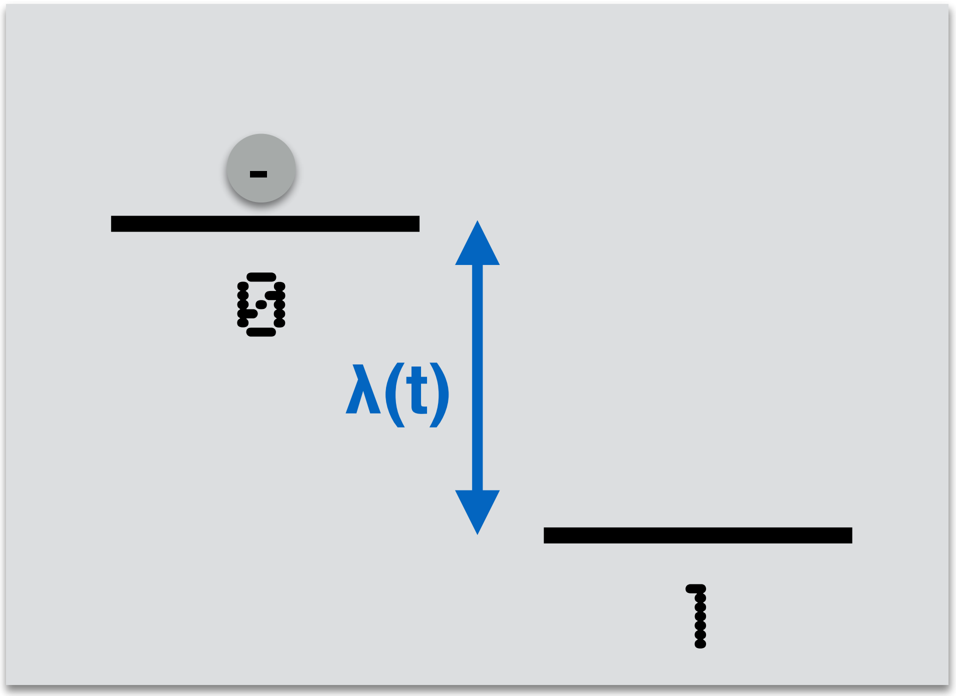


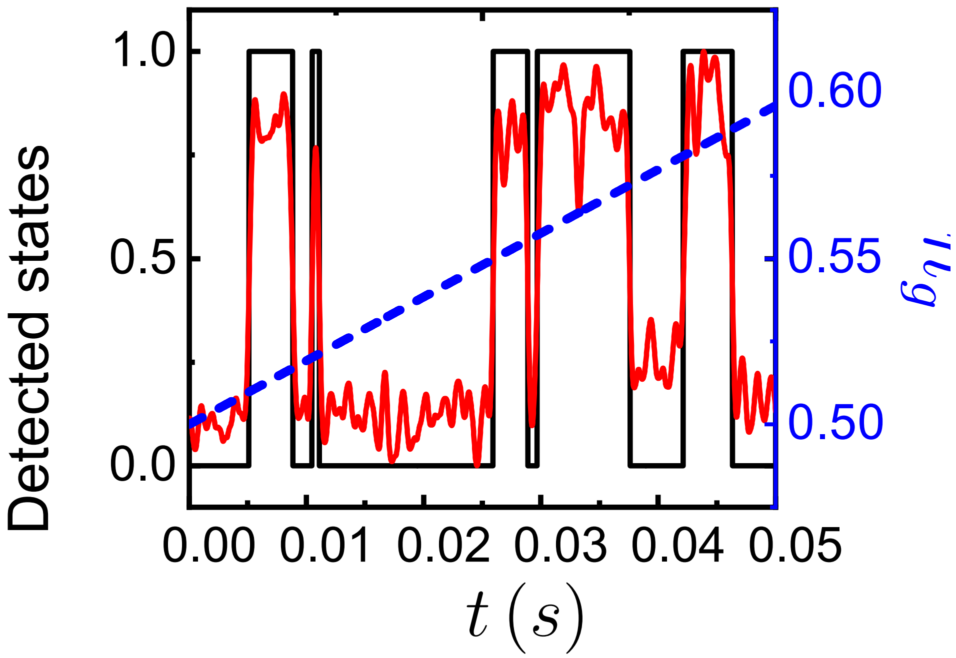
G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)

Gambling with a single electron

Experimental realization: Single-electron box (Pekola lab)







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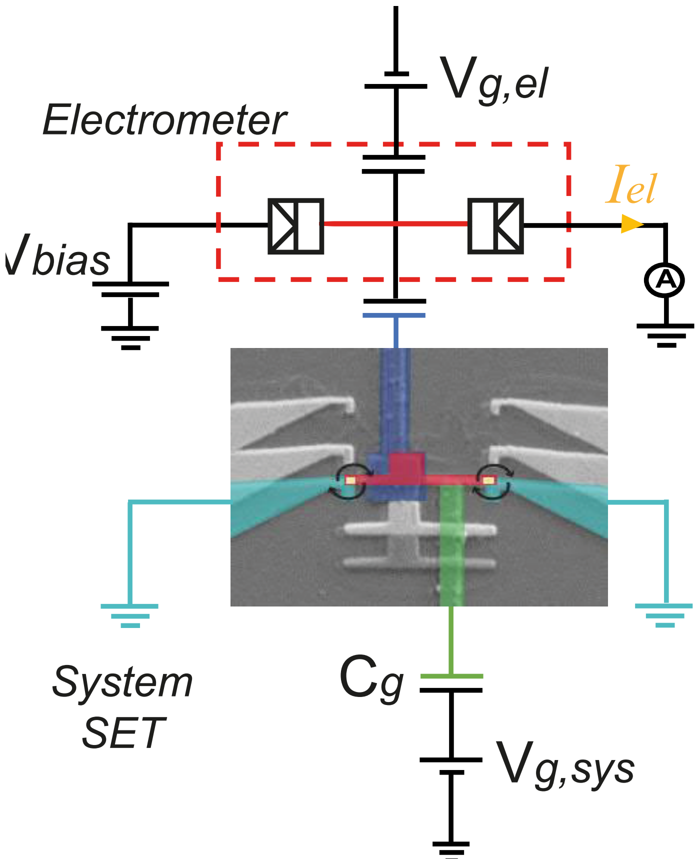
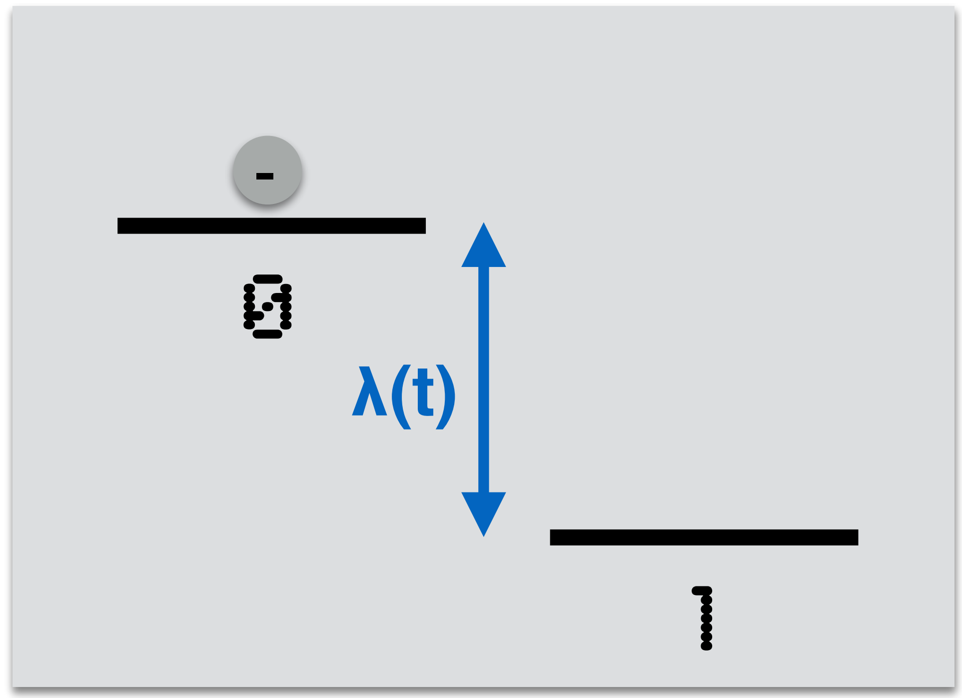
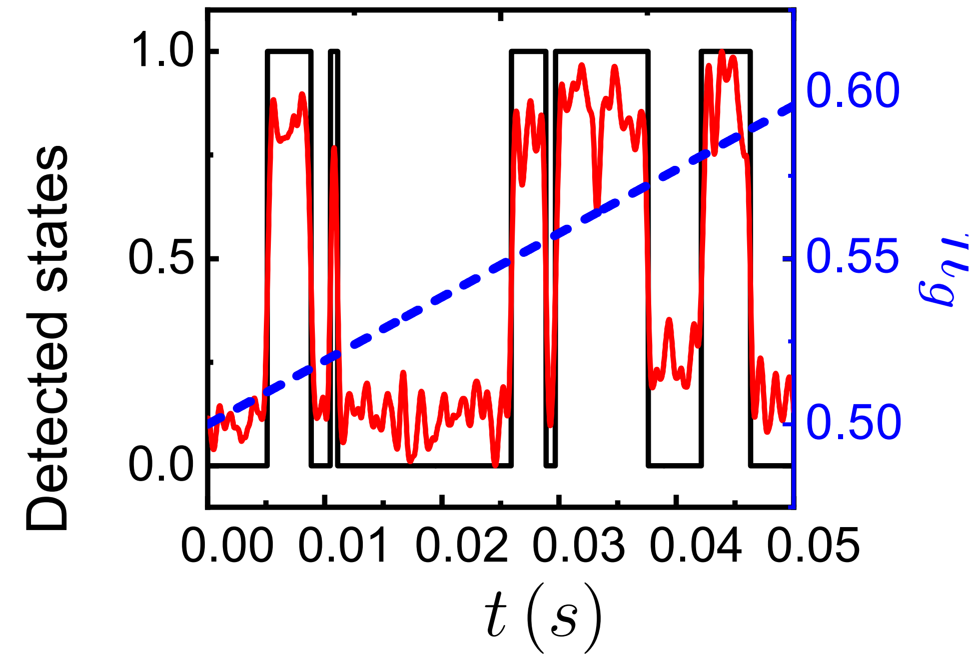
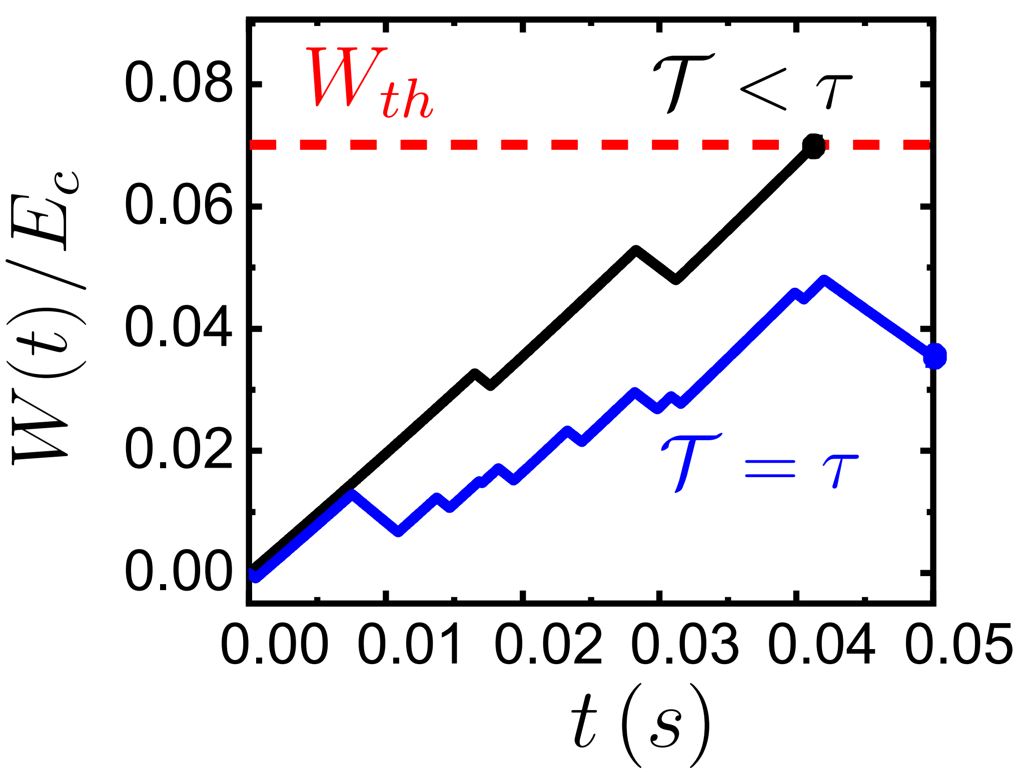








Gambling strategy: stopping time for the work

$$\mathcal{T} = \min(\tau_{\rm th}, \tau)$$

Conditionally stop the process before τ if the work exceeds a threshold value $W_{\rm th}$

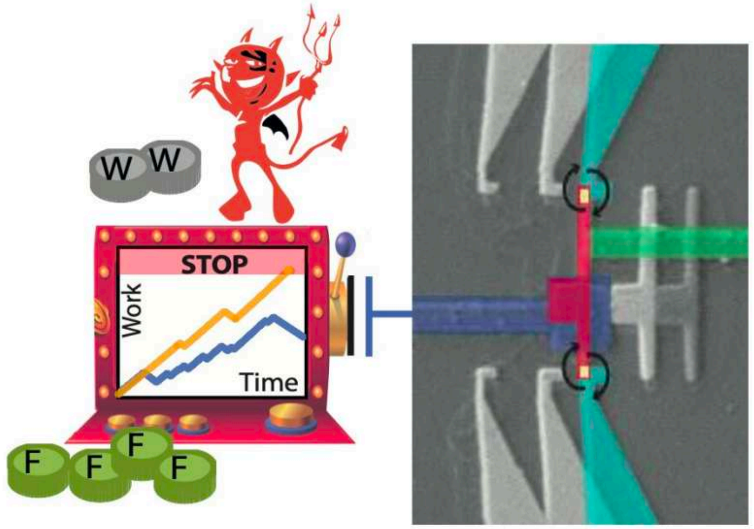
*No exp. feedback: trajectory post-processing

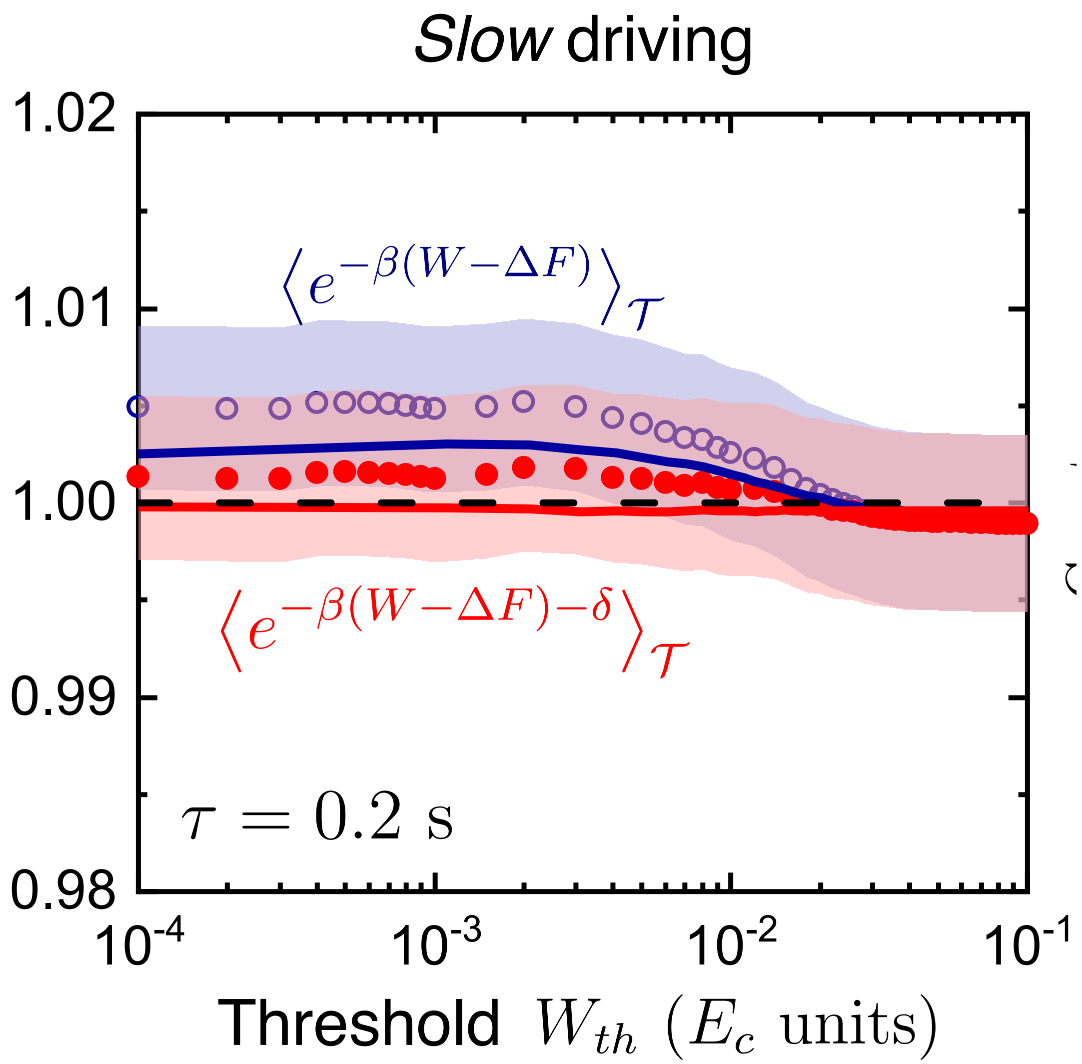
G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)

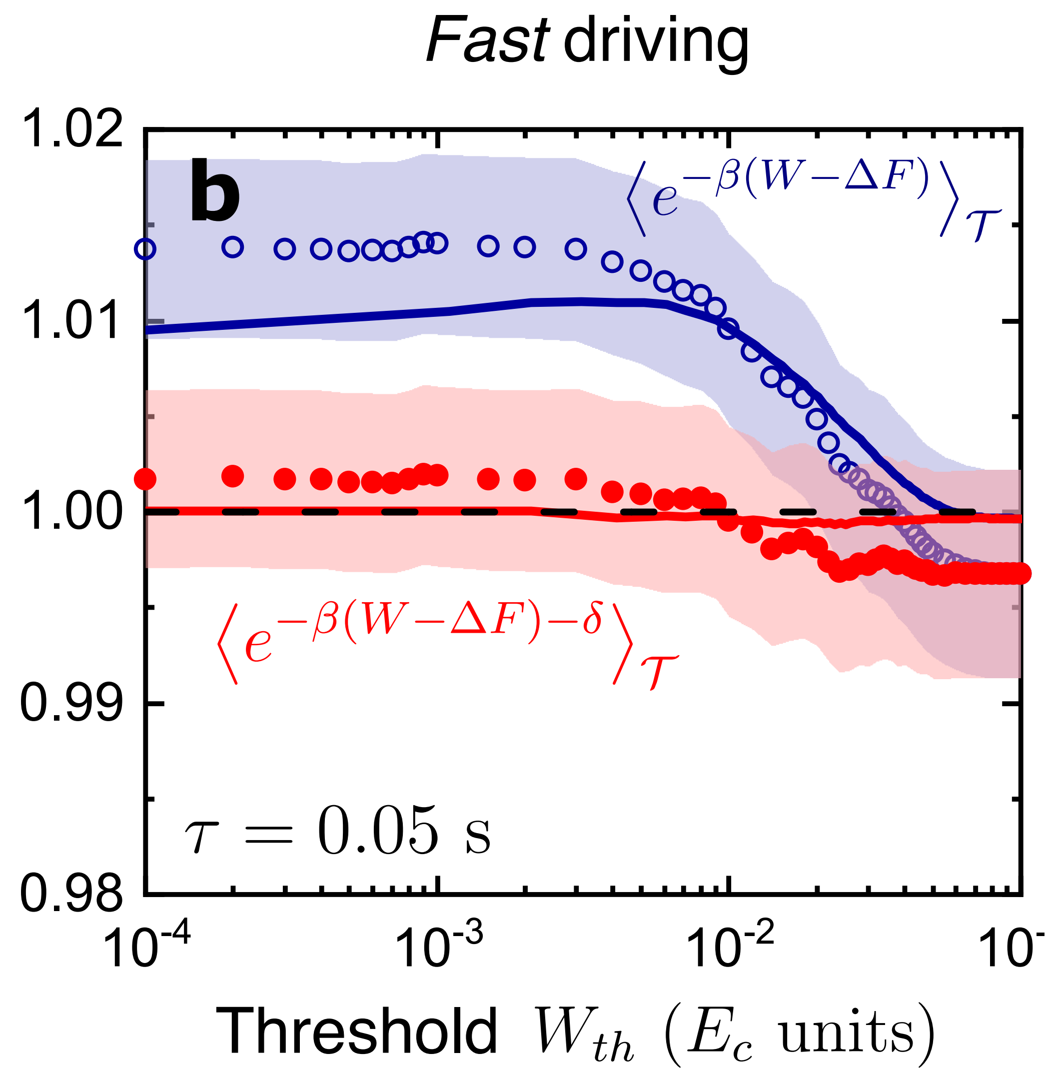
Experimental test

Theory: valid for any driving and any gambling strategy (new level of universality)

Experiment: Driven single-electron box



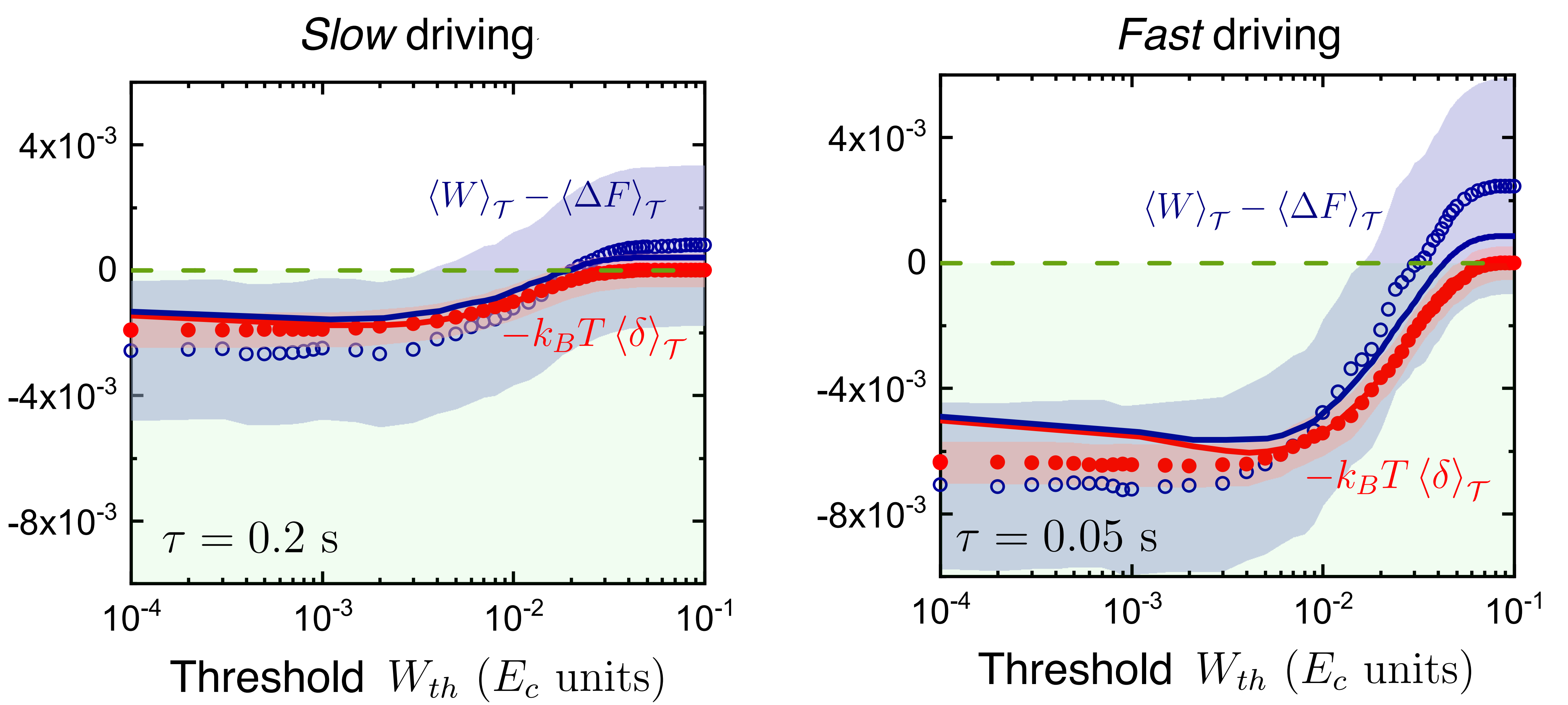




G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)

Second law at stopping times

$$\langle W(\mathcal{T})\rangle - \langle \Delta F(\mathcal{T})\rangle \geq -k_{\rm B}T\,\langle \delta(\mathcal{T})\rangle$$



Ingredients for "negative dissipation"

- ***gambling*** (stopping times)
- **nonequilibrium** (time asymmetric protocols)

G Manzano, et al., Phys. Rev. Lett. **126** (8), 080603 (2021)

Thermodynamic Martingales and Stopping times

I. Neri, PRL **124** (4) 040601(2020)

K. Hiura, S.-i. Sasa, arXiv 2102.06398 (2021)

C. Moslonka, K. Sekimoto, PRE **101** (6), 062139 (2020)

Y.-J. Yang, H. Qian, PRE **101** (2), 022129 (2020)

H. Ge et al., arXiv 1811.04259 (2018)

K. Cheng et al., PRE **102** (6) 062127 (2020)

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FOCUS

The Gambler, Maxwell's New Demon

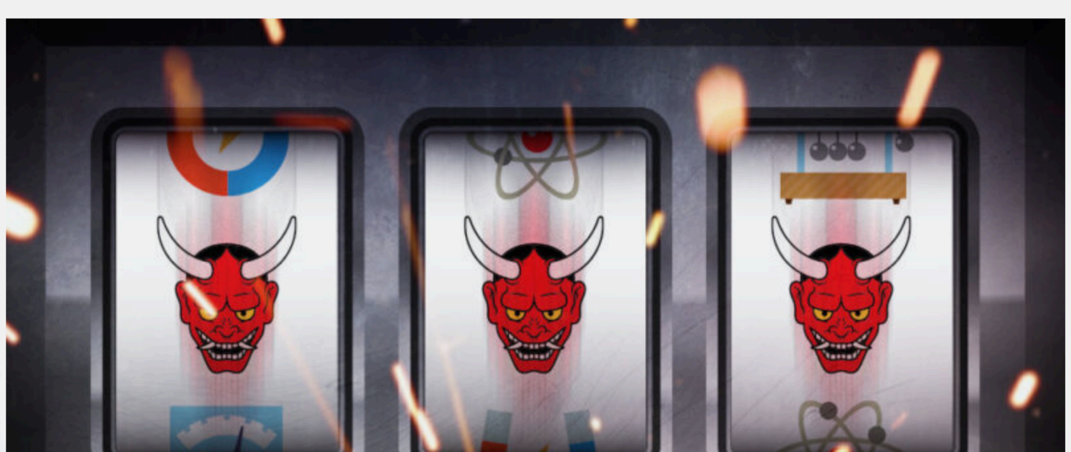
ars TECHNICA

KNOW WHEN TO WALK AWAY, KNOW WHEN TO RUN —

Meet Maxwell's gambling demon—smart enough to quit while it's ahead

Physicists even demonstrated the basic principle in a nanoscale electronic device.

JENNIFER OUELLETTE - 3/4/2021, 11:39 PM



SCIENCE

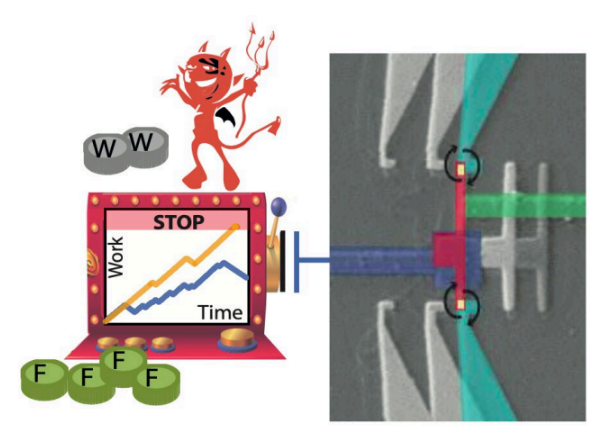
A New 'Information Demon' Thrives on Chaos

'Chaos reigns.'

PHYS ORG

The realization of a new type of information demon that profits from gambling strategies

15 March 2021, by Ingrid Fadelli



explored the properties of a unique family of stochastic processes known as martingales in the context of thermodynamics.

Martingales are paradigmatic examples of stochastic processes that have been used in a variety of fields, including finance and mathematics. Manzano, Roldan and their colleagues applied knowledge of martingales to the study of thermodynamics with the aim of unveiling new universal thermodynamic laws.

"Our paper concerns the following questions: What happens when one gambles with the information acquired about the response of a small system

Summary and discussion

- Martingales are not scary! Useful for new developments in stochastic thermodynamics: stopping times, extrema, and beyond

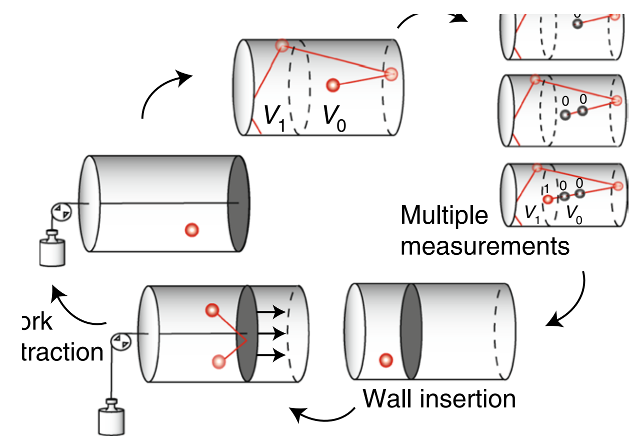
- Work extraction beyond the free energy change without feedback control, by stopping a driven mesoscopic system at stochastic times

$$\langle W(\mathcal{T})\rangle - \langle \Delta F(\mathcal{T})\rangle \geq -k_{\rm B}T\,\langle \delta(\mathcal{T})\rangle$$

- No violation of the Second Law with information: ∞

$$\langle W(\mathcal{T})\rangle - \langle \Delta F(\mathcal{T})\rangle \geq -k_{\rm B}T\langle \delta(\mathcal{T})\rangle - k_{\rm B}T\,\langle I \rangle$$

continuous measurements



Ribezzi, Ritort, N Phys (2019)

- Optimal work extraction requires:
  patience (first-passage phenomena)
  nonequilibrium (time-asymmetric protocols)
  and wisdom (suitable stopping times)

- Further applications: biophysics, computation, frenesy, etc.

-

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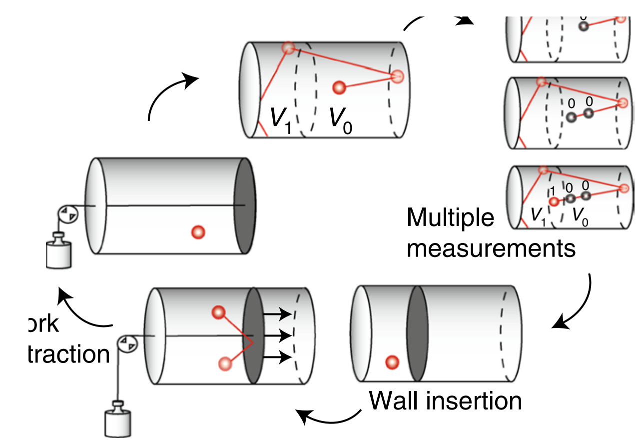

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- Further applications: biophysics, computation, frenesy, etc.

- *Physics with Martingales*, ER, I Neri, R Chetrite, S Gupta, S Pigolotti, F Julicher, K Sekimoto (coming soon 2022)

Acknowledgments

Theory collaborators

- Gonzalo Manzano (IFISC)
- Izaak Neri (KCL)
- Raphael Chetrite (U Nice)
- Shamik Gupta (RKMVERI)
- Simone Pigolotti (OIST)
- Rosario Fazio (ICTP)
- Frank Julicher (MPIPKS)

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- Olivier Maillet (Aalto)
- Shilpi Singh (Aalto)
- Ivan Khaymovich (MPIPKS)
- Jukka Pekola (Aalto)

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- Sarah Loos
- Gennaro Tucci
- Pierluigi Muzzeddu

Thanks for your attention !

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