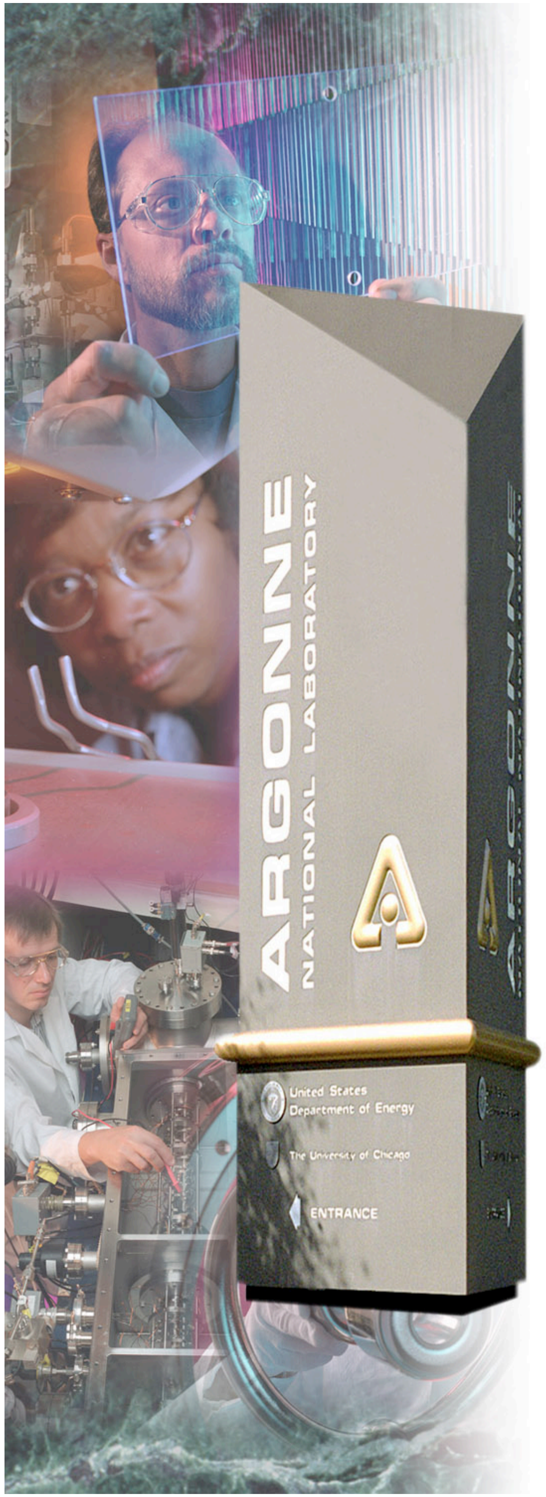

Two analytic formulas of Heavy-ion Fusion Cross sections

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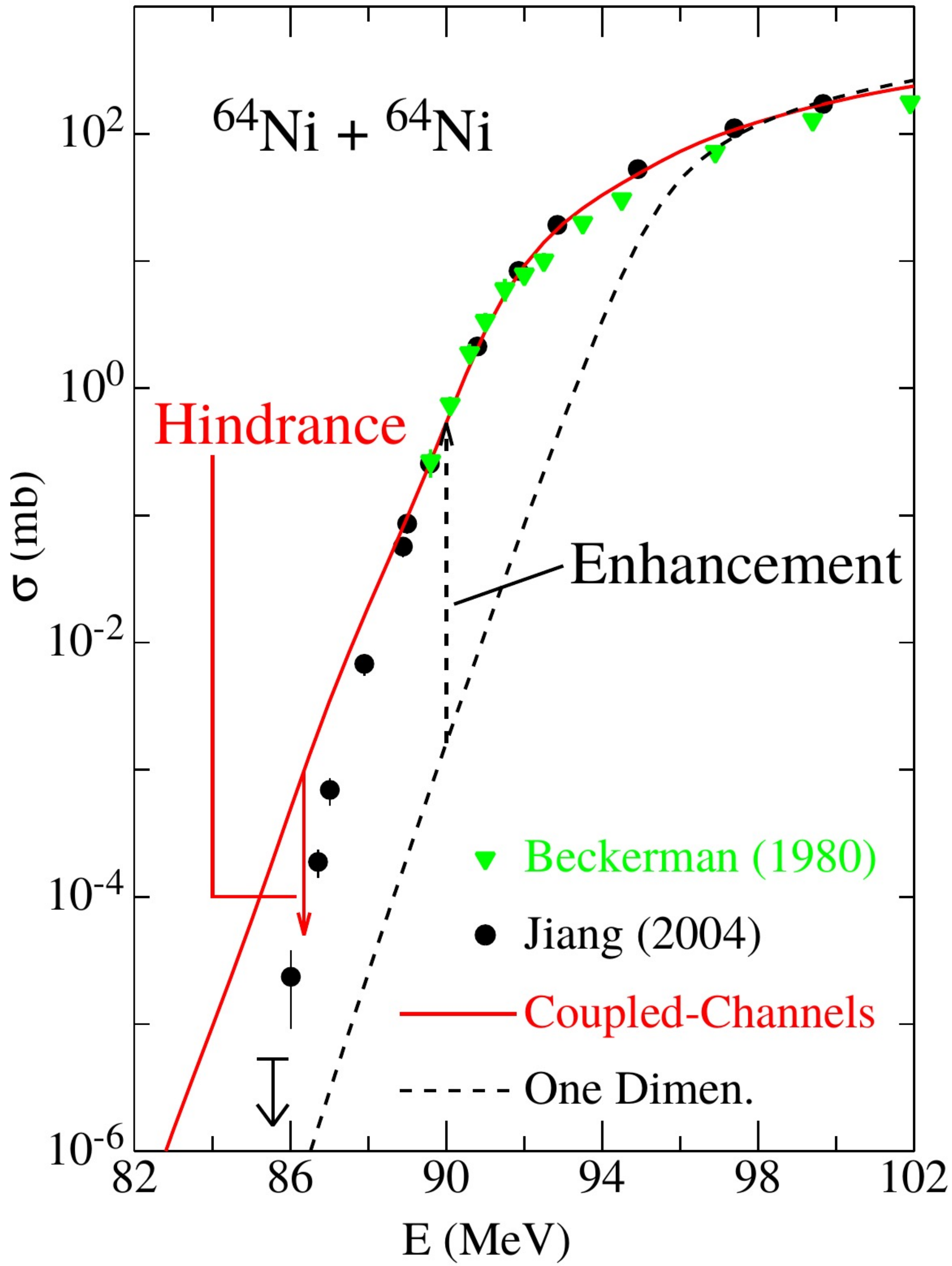


Fusion hindrance at extreme sub-barrier energies discovered twenty years ago.

We will demonstrate two simple, analytic cross section formulas which can reproduce the hindrance behavior very well

EPJ A57, 235 (2021)

Wong formula: Includes the effects of quantum mechanical tunneling through the Coulomb potential

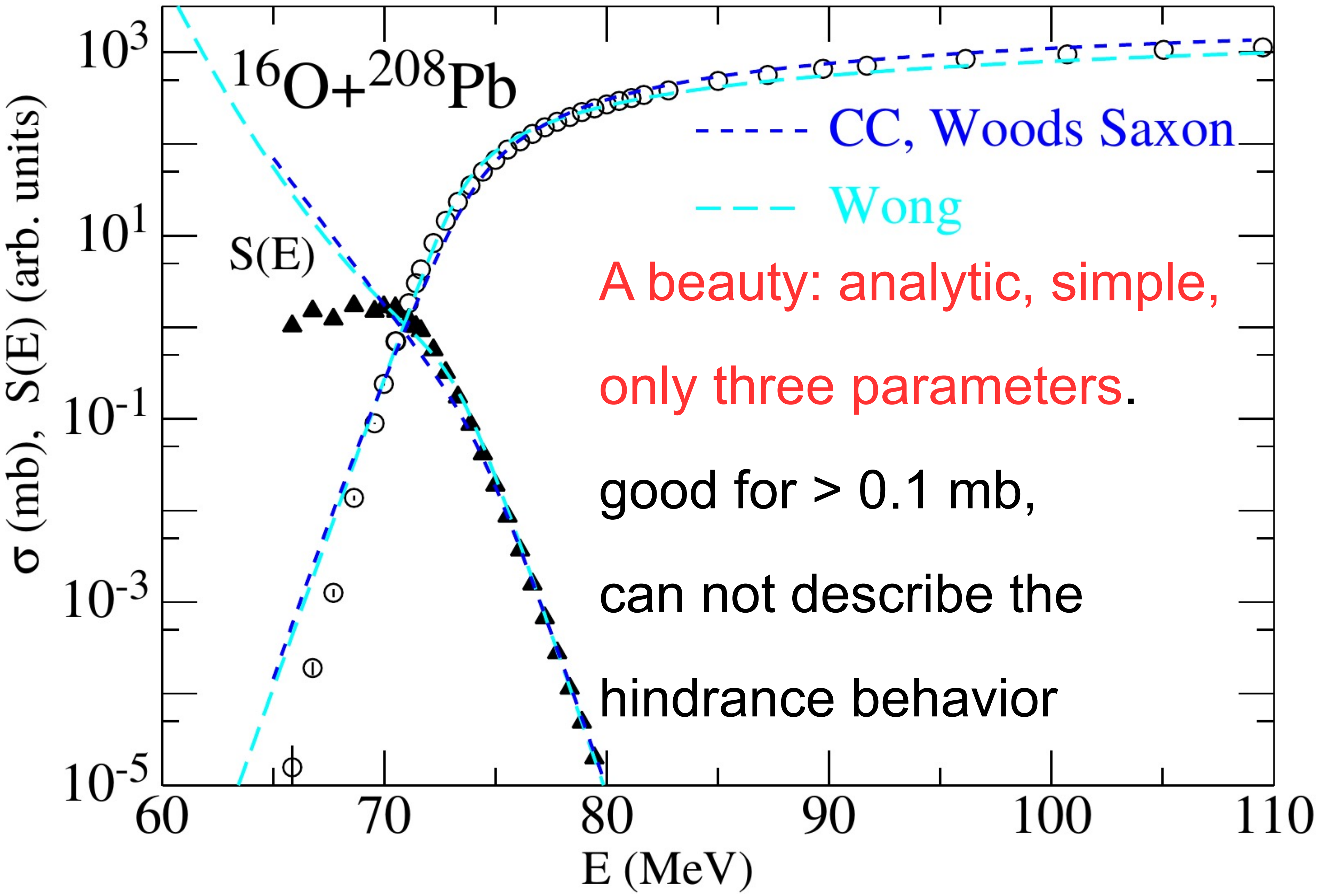
$$\sigma_W(E) = \frac{R^2}{2E}\hbar\omega \ln[1 + \exp((2\pi/\hbar\omega)(E - V))]$$

R : radius

V : potential barrier height

$\hbar\omega$: curvature of the potential barrier

All three parameters, are L independent



$^{16}O+^{208}Pb$
CC, Woods Saxon
Wong
S(E)
A beauty: analytic, simple,
only three parameters.
good for > 0.1 mb,
can not describe the
hindrance behavior
σ (mb), S(E) (arb. units)
E (MeV)
10^3
10^1
10^{-1}
10^{-3}
10^{-5}
60
70
80
90
100
110

Rowley and Hagino, PRC 91, 044617 (2015)

The R, V and $\hbar\omega$ are sensitive to angular momentum L, they suggested:

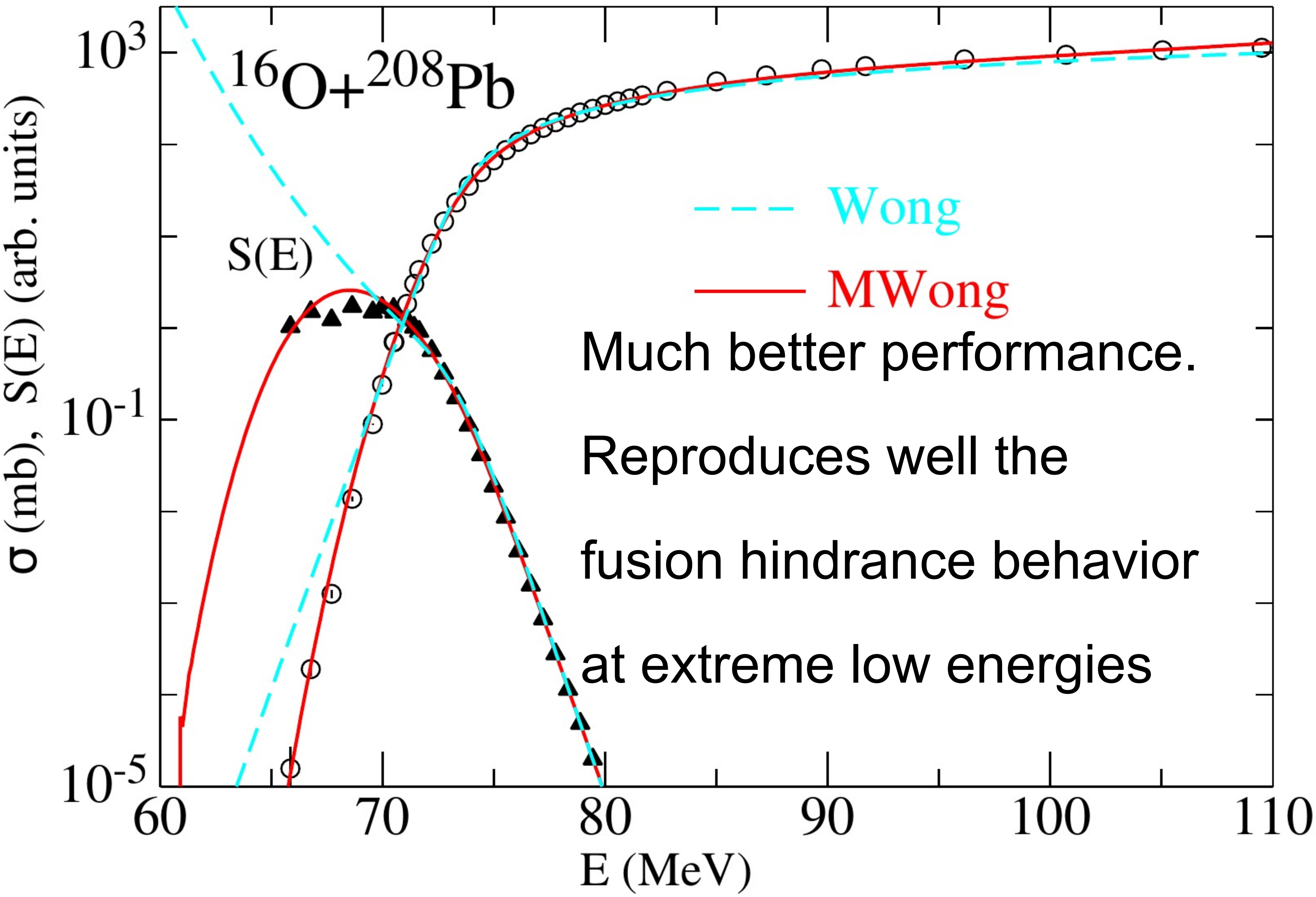
Different parameter set for each angular momentum should be used in the calculations with Wong formula

Modified-Wong formula,

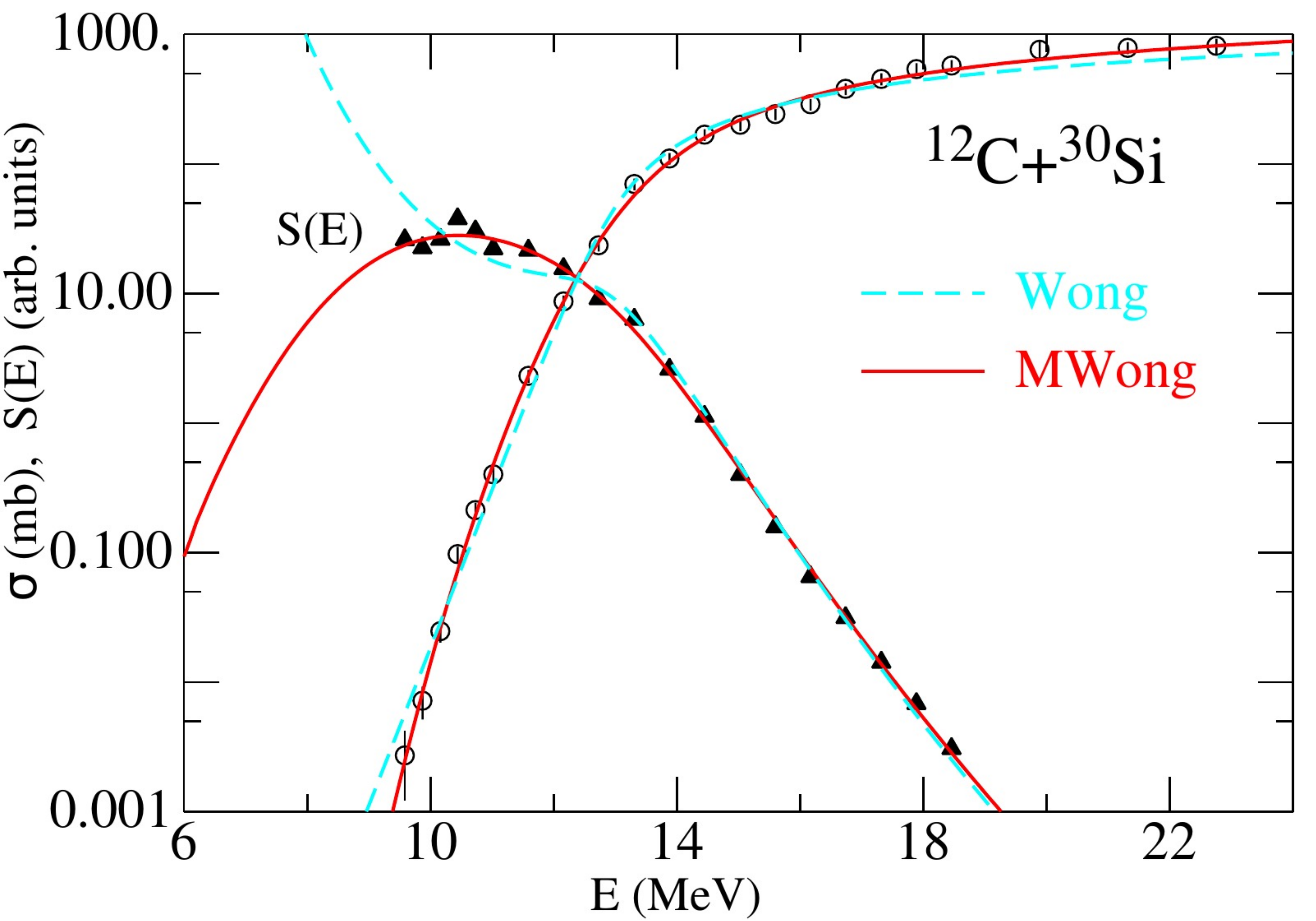
we assume:

$$\hbar\omega \rightarrow \hbar\omega \exp[\lambda \frac{E-V}{V}]$$

$$\sigma_f(E) = \frac{R^2}{2E}\hbar\omega \exp[\lambda \frac{E-V}{V}] \ln[1 + \exp((2\pi/\hbar\omega \exp[\lambda \frac{E-V}{V}])(E-V))].$$



$^{16}O+^{208}Pb$
S(E)
Wong
MWong
Much better performance.
Reproduces well the
fusion hindrance behavior
at extreme low energies
σ (mb), S(E) (arb. units)
10^3
10^{-1}
10^{-5}
60
70
80
90
100
110
E (MeV)



1000.
10.00
0.100
0.001
6
10
14
18
22
E (MeV)
σ (mb), S(E) (arb. units)
S(E)
$^{12}C+^{30}Si$
Wong
MWong

For 35 systems, whose lowest measured cross sections < 0.02 mb, all can be well reproduced, including the hindrance behaviors. (submitted to EPJ A)

system	Q	type	N	data range	χ_0^2	R	V	W	λ	E_s^f	E_s^{exp}	ref.
	MeV			mb - mb		fm	MeV	MeV		MeV	MeV	
$^{16}O+^{18}O$	24.41	EVR	21	0.006 - 224	0.12	7.13	9.67	3.02	1.18	6.54		[25]
$^{12}C+^{24}Mg$	16.30	EVR	21	.015 - 668	2.17	6.27	11.05	2.70	1.41	9.80	9.7	[26-28]
$^{12}C+^{30}Si$	14.11	EVR	22	0.0027 - 815	0.82	7.91	13.54	4.62	2.25	10.74	10.5	[29, 30]
$^{6}Li+^{198}Pt$	8.53	EVR	10	0.00017 - 348	24.5	8.22	29.53	17.91	4.32			[31]
$^{7}Li+^{198}Pt$	8.82	EVR	11	0.0002 - 1004	4.40	9.76	28.19	6.76	2.19			[32]
$^{24}Mg+^{30}Si$	17.89	EVR	20	0.0080 - 332	0.33	8.17	24.10	4.20	2.67	20.91	20.8	[33, 34]

$^{40}Ar+^{154}Sm$	-75.31	EVR+FF	15	0.0016 - 407	56.72	8.45	122.97	17.6	6.02	98.28		[53]
$^{76}Ge+^{86}Kr$	-97.91	EVR	15	0.0068 - 347	13.7	8.13	130.45	13.1	6.27	116.30		[54]
$^{58}Ni+^{124}Sn$	-112.30	EVR+FF	15	0.00046 - 570	1.07	8.41	156.86	13.3	5.49	138.80		[55, 56]
$^{64}Ni+^{124}Sn$	-117.51	EVR+FF	17	0.0008 - 605	0.82	7.66	155.28	16.8	12.9	141.9		[55, 56]

Denisov and Sedykh, EPJ A55, 153 (2019), had developed another modification of Wong formula: They gave: two simple expressions of R and V, a complex expression of $\hbar\omega$ with total 9 parameters. Fit with 1995 cross sections of 85 fusion excitation functions. Results can describe the average behaviors of these 85 systems

N. Rowley, G.R. Satchler and P.H. Stelson

Phys. Lett. B 254, 25 (1991)

On the "distribution of barrier" interpretation of heavy-ion fusion

$$D(E) = \frac{1}{\pi R^2} \frac{d^2(E\sigma(E))}{dE^2}, \int D(E) dE = 1$$

$$B_{exp}(E) = \frac{d^2(\sigma_{exp}(E)E)}{d^2E}$$

$$\sigma(E)E = \int_{E_0}^{E}\Big(\int_{E_0}^{E'} \frac{d^2(\sigma E'')}{d(E'')^2} dE''\Big) dE'$$

$$= \pi R_0^2 \int_{E_0}^{E}\Big(\int_{E_0}^{E'} D_{\text{test}}(E'') dE''\Big) dE'.$$

$$D_{\text{test}}(E) = \sum_{i=1}^{L} D_i(E) = \sum_{i=1}^{L} \frac{\omega_i}{\sqrt{2\pi} W_i} \exp\Big[-\big(\frac{E - V_i}{\sqrt{2} W_i}\big)^2\Big]$$

$$\sigma(E)E = \pi R_0^2 \sum_{i=1}^{L} \frac{\omega_i W_i}{\sqrt{2\pi}} \Big[\sqrt{\pi} Z_i \operatorname{erfc}(-Z_i) + \exp(-Z_i^2)\Big], \tag{16}$$

with $Z_i = (E - V_i)/\sqrt{2} W_i$.

A special case
Single-Gaussian barrier distribution formula

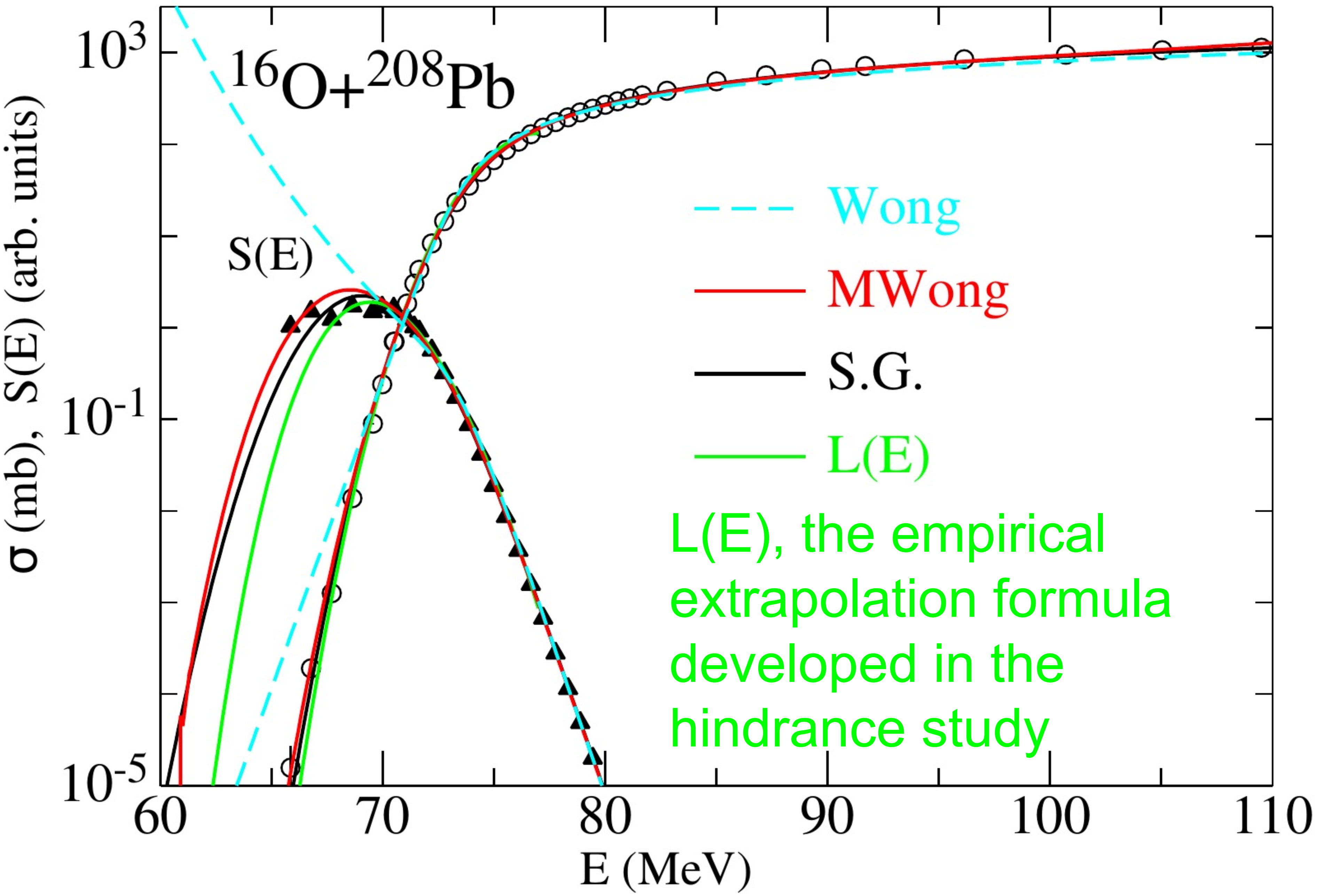
$$D_{\text{test}}(E) = \frac{1}{\sqrt{2\pi}W} \exp\left[-\left(\frac{E-V}{\sqrt{2}W}\right)^2\right]$$

V_i, W are the centroids, the standard deviations of the Gaussian function

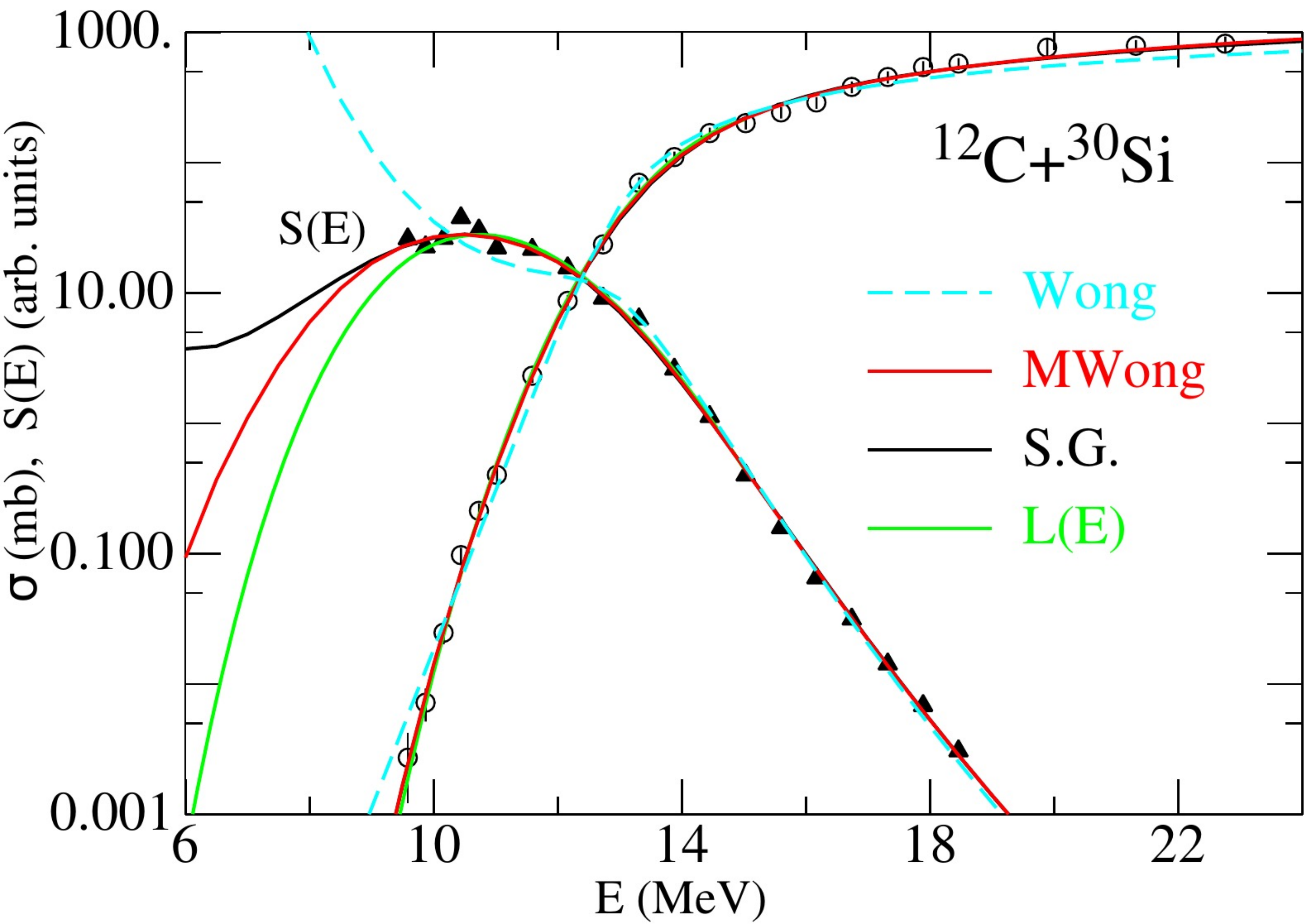
$$\sigma(E)E = \pi R^2 \frac{W}{\sqrt{2\pi}}\left[\sqrt{\pi}Z\,\mathrm{erfc}(-Z) + \exp(-Z^2)\right],$$

$Z = (E-V)/\sqrt{2}W$,
$\mathrm{erfc}(Z)$ is the complementary error function

Eur. Phys. J. A 54, 218 (2018)



$^{16}O+^{208}Pb$
S(E)
Wong
MWong
S.G.
L(E)
L(E), the empirical extrapolation formula developed in the hindrance study
σ (mb), S(E) (arb. units)
E (MeV)
10^3
10^{-1}
10^{-5}
60
70
80
90
100
110



1000.
10.00
0.100
0.001
6
10
14
18
22
E (MeV)
σ (mb), S(E) (arb. units)
S(E)
12C+30Si
Wong
MWong
S.G.
L(E)

We obtained the single-Gaussian formula from the expression:

$$\sigma(E)E = \int_{E_0}^{E}\Big(\int_{E_0}^{E'} \frac{d^2(\sigma E'')}{d(E'')^2} dE''\Big) dE'$$
$$= \pi R_0^2 \int_{E_0}^{E}\Big(\int_{E_0}^{E'} D_{\mathrm{test}}(E'') dE''\Big) dE'.$$

The same expression has been obtained by K. Siwek-Wilczynska et al., (Acta Phys. Pol. B 33, 451 (2002)) with another way, which has not been widely used and referenced in the literature

Rowley's reciple: double differentiation

$$B_{exp}(E) = \frac{d^2(\sigma_{exp}(E)E)}{d^2E}$$

Weak points for obtaining fusion barrier distribution:

Must be measured with small energy steps.

Ambiguities for B(E), since large energy

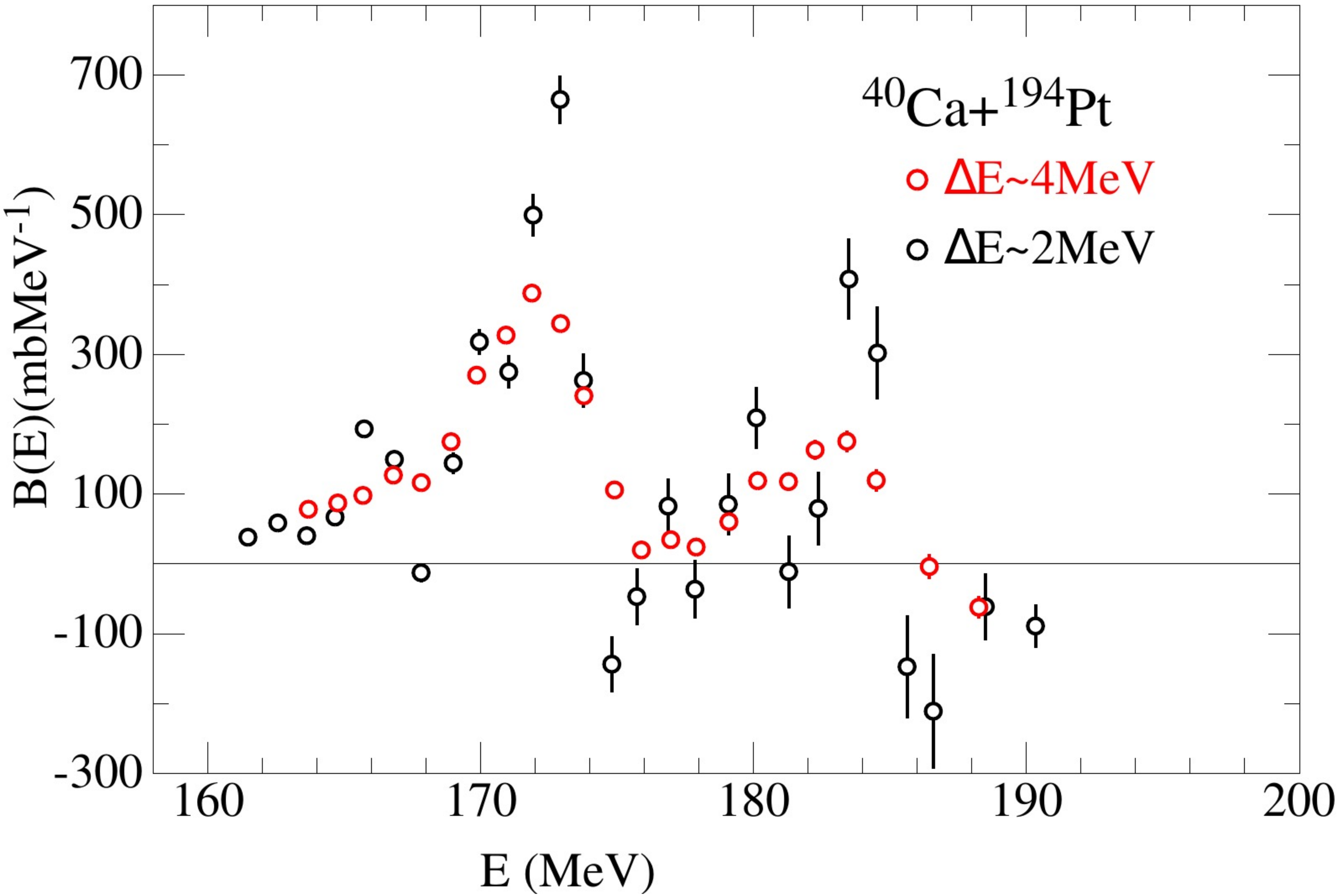
steps must be used.

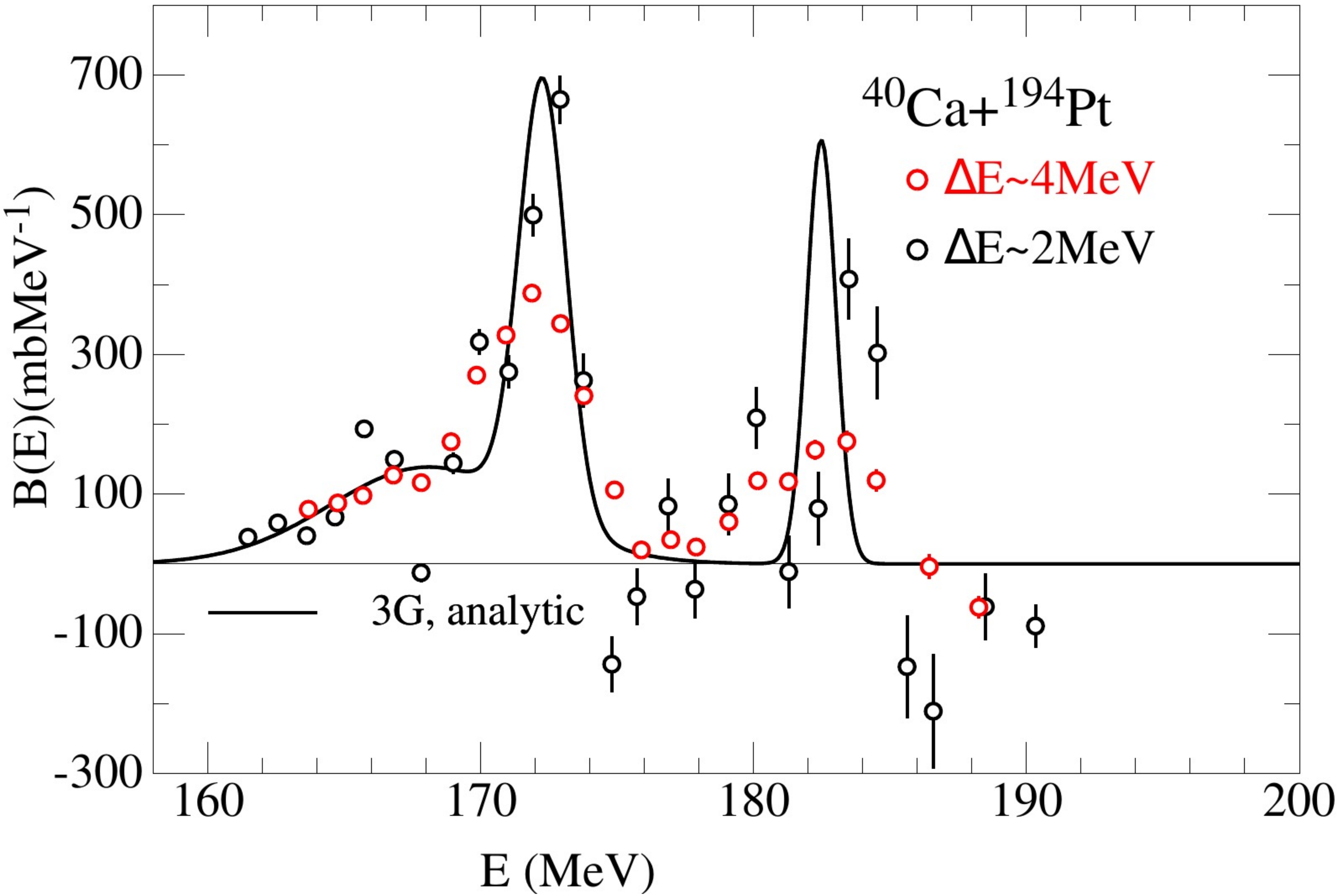
For obtaining good fit, we compare

$$\chi_0^2 = \frac{1}{N}\sum_{i=1}^{N}((\sigma_i - \sigma_{exp-i})/\Delta\sigma_{exp-i})^2,$$

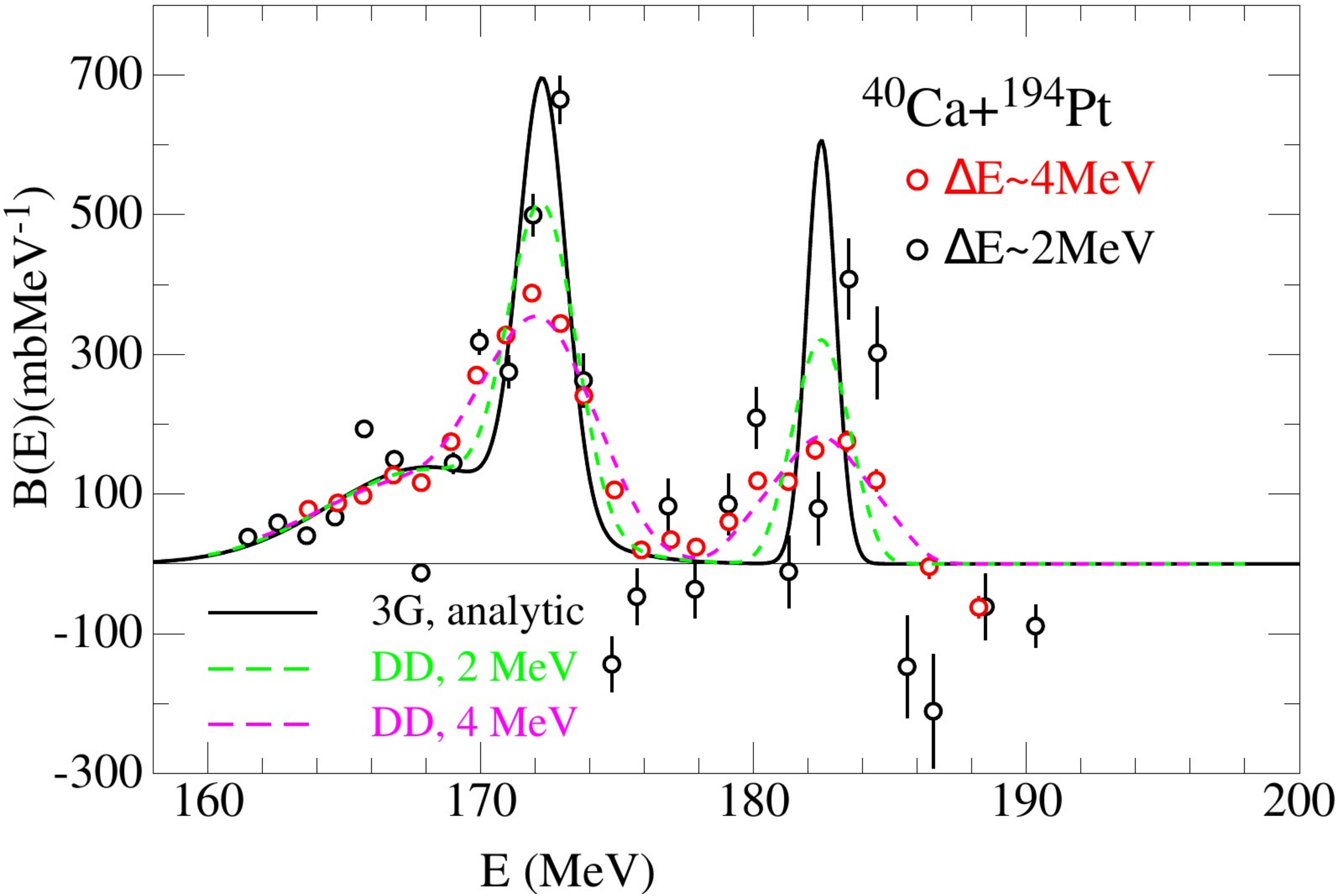
$$\chi^2 = \frac{N}{N-M}\chi_0^2$$

system	$LG,\ M$	χ_0^2	χ^2	system	$LG,\ M$	χ_0^2	χ^2
^{40}Ca+^{194}Pt	1G, 3	41.9	46.4	^{16}O+^{144}Sm	1G, 3	21.8	24.5
N=31	2G, 6	37.5	46.5	N=27	2G, 6	3.3	4.3
	3G, 9	11.0	15.5		3G, 9	3.2	4.8
^{16}O+^{154}Sm	1G, 3	13.0	14.1	^{40}Ca+^{192}Os	1G, 3	16.7	17.9
N=37	2G, 6	3.0	3.5	N=45	2G, 6	15.3	17.7
	3G, 9	2.9	3.9		3G, 9	14.8	18.5
	4G, 12	2.5	3.7		4G, 12	13.8	18.2

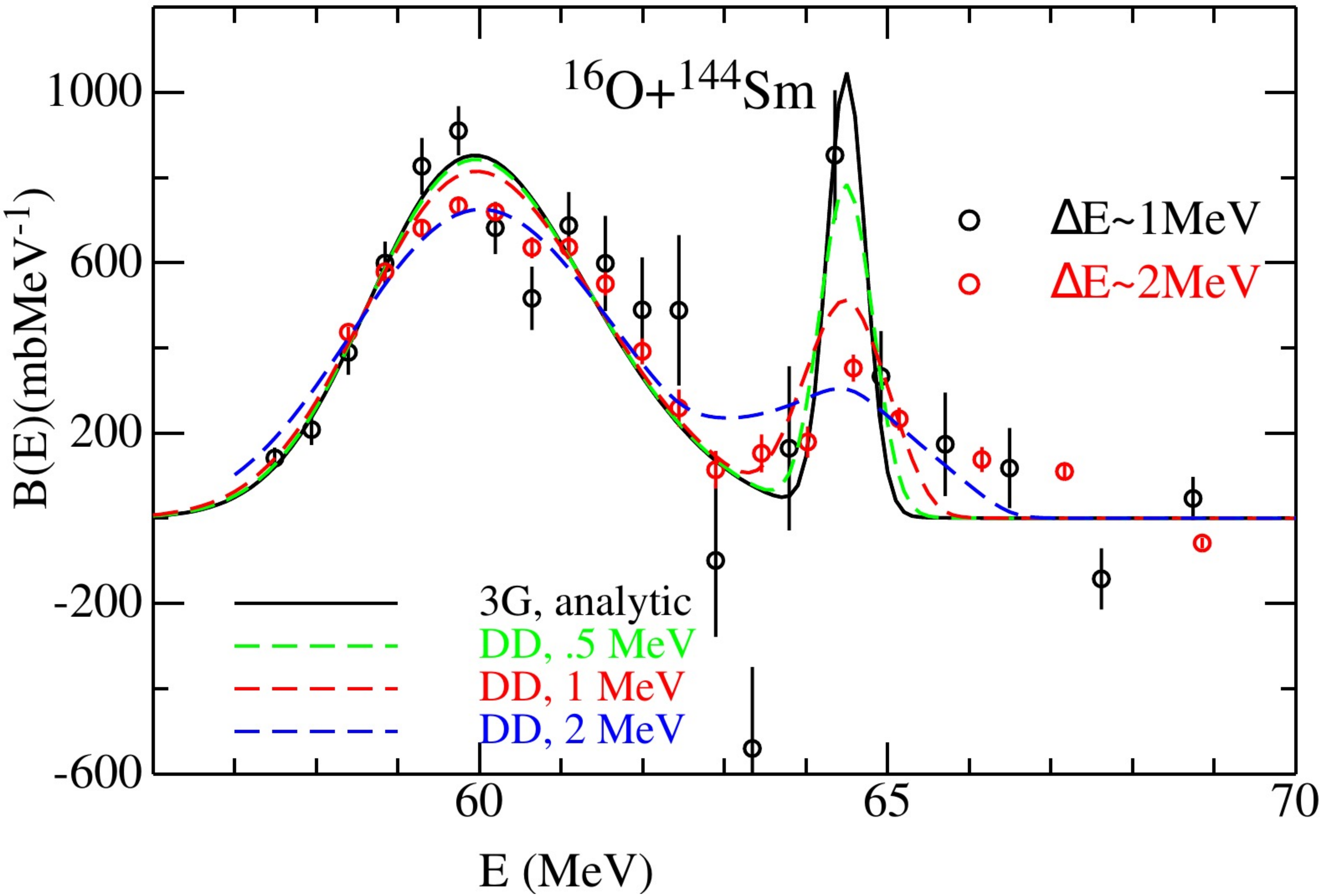

$^{40}Ca+^{194}Pt$
ΔE~4MeV
ΔE~2MeV
B(E)(mbMeV-1)
700
500
300
100
-100
-300
160
170
180
190
200
E (MeV)

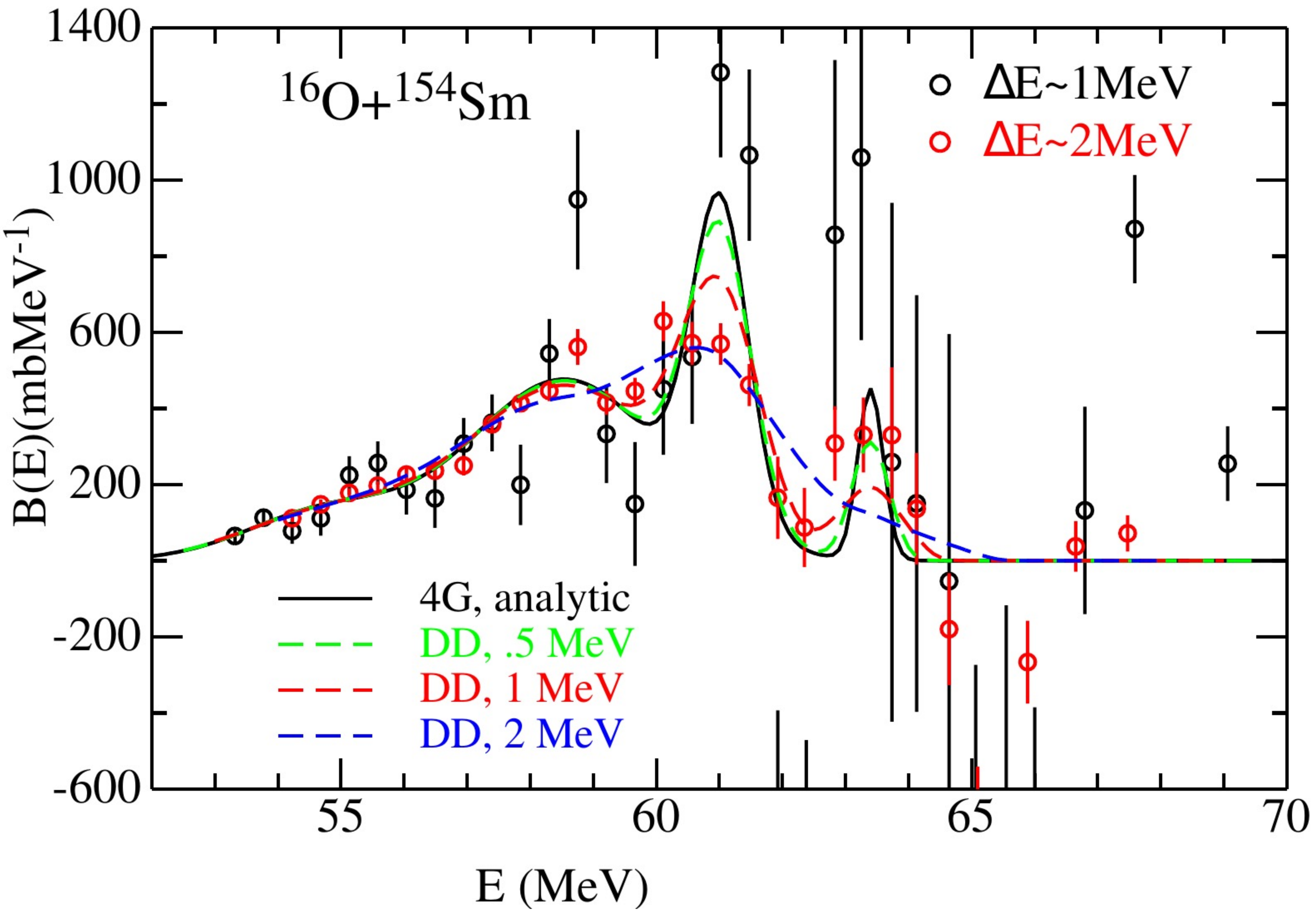


700
500
300
100
-100
-300
160
170
180
190
200
B(E)(mbMeV-1)
E (MeV)
40Ca+194Pt
ΔE~4MeV
ΔE~2MeV
3G, analytic

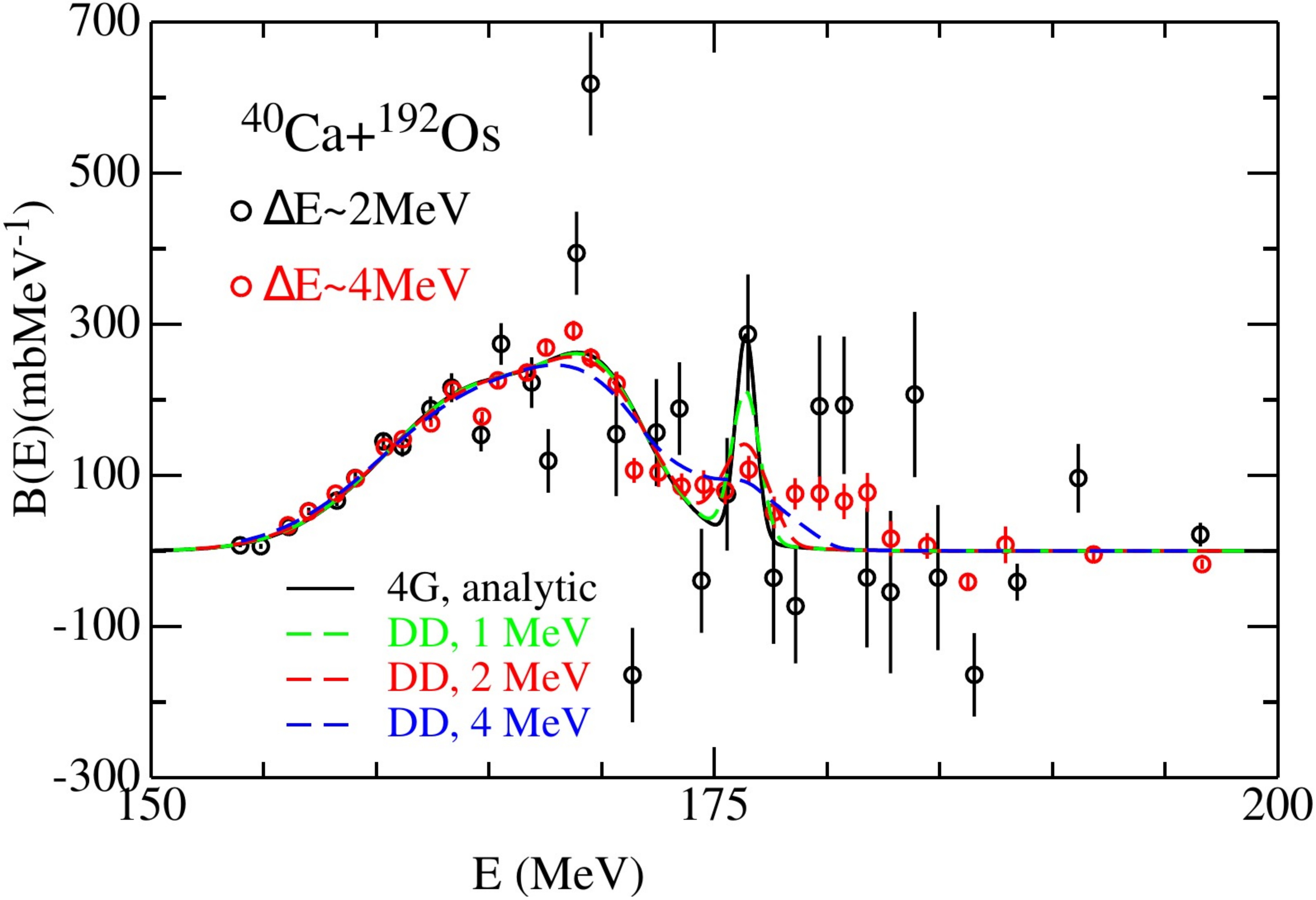


^{40}Ca+^{194}Pt
ΔE~4MeV
ΔE~2MeV
3G, analytic
DD, 2 MeV
DD, 4 MeV
B(E)(mbMeV-1)
E (MeV)
700
500
300
100
-100
-300
160
170
180
190
200


$^{16}O+^{144}Sm$
1000
600
200
-200
-600
60
65
70
B(E)(mbMeV^{-1})
E (MeV)
ΔE~1MeV
ΔE~2MeV
3G, analytic
DD, .5 MeV
DD, 1 MeV
DD, 2 MeV



$^{16}O+^{154}Sm$
ΔE~1MeV
ΔE~2MeV
4G, analytic
DD, .5 MeV
DD, 1 MeV
DD, 2 MeV
B(E)(mbMeV-1)
E (MeV)
1400
1000
600
200
-200
-600
55
60
65
70



700
500
300
100
-100
-300
150
175
200
B(E)(mbMeV-1)
E (MeV)
40Ca+192Os
ΔE~2MeV
ΔE~4MeV
4G, analytic
DD, 1 MeV
DD, 2 MeV
DD, 4 MeV

In Summary:

Two simple analytic cross section formulas:

Modified-Wong,

Single-Gaussian barrier distribution;

reproduce well the hindrance behavior

An improved method to obtain barrier distribution:

three- or four-Gaussian barrier distribution

Collaborators:

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Thank You !