Constraints on the appearance of a maximum in astrophysical S -factor

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Key fusion reactions

In a hydrogen-rich environment the light elements such as Li, Be, and B have all positive (p, α) Q values. These nuclei are easily destroyed, before fusion reactions start to play a role.

The stable carbon isotopes $^{12,13}C$ are the first nuclei with sufficiently negative (p, α) Q values. This makes $^{12}C + {}^{12}C$ the first fusion reaction that needs to be considered in nuclear astrophysics.

- ✓ In non-explosive scenarios carbon fusion takes place in the center of massive stars toward the end of their lifetime during the carbon-burning phase at temperatures of about 0.6–1 GK.
- ✓ In stellar explosions carbon fusion plays a role during the ignition phase of supernovae (1–10 GK).

Fusion reactions involving ^{16}O, plays a role only at higher temperatures.

- ✓ In explosive environments.

Key fusion reactions

The study of fusion of light heavy nuclei (C, O, Si…) at extreme sub- barrier energies (e.g. Gamow energies for C+C reaction is 1-3 MeV) is a key to understand and predict different scenarios of evolution of the stars.

Main problem:

Gamow energies are in a range where no reliable cross-section measurements can be made with present technologies.

Possible solution:

- Extrapolation of the existing experimental data to lower energies starting with data at higher energies.
- Theoretical calculations.

S-factor

More convenient representation of the fusion cross section at low energies

$$S(E_{c.m.}) = E_{c.m.}\sigma(E_{c.m.})\exp(2\pi\eta)$$

Gamow factor

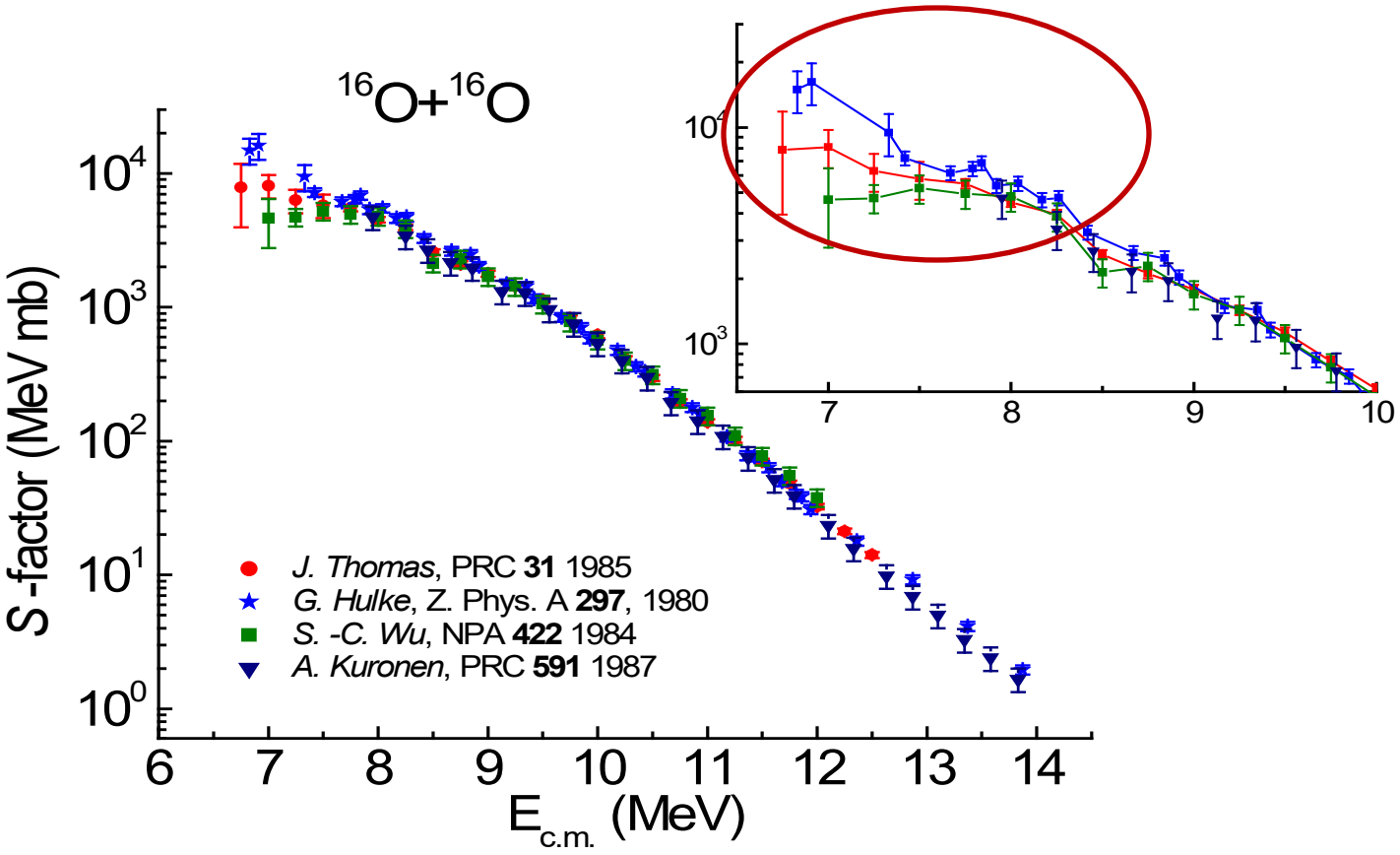
$\eta = Z_1 Z_2 e^2 / \hbar v$ --- Sommerfeld parameter

v --- beam velocity

Some models predict, that the Gamow factor accounts for the main part of the strong energy dependence of the fusion cross section, such that the *S*-factor exhibits only a weak energy dependence far below the Coulomb barrier.

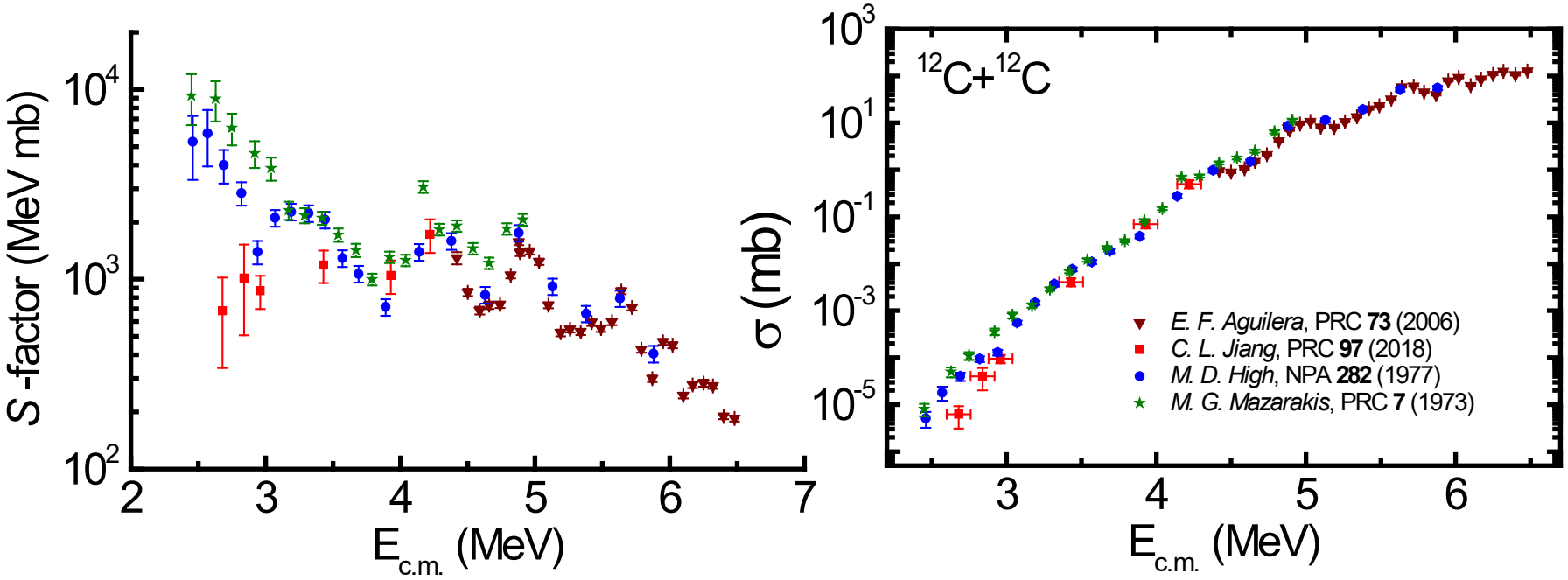
This feature is often used to extrapolate the cross section into the region of the Gamow window, in order to predict the astrophysical reaction rates.

S-factor for $^{16}O+^{16}O$ reaction



- Existence of a maximum in *S*- factor ?
- The maximum was observed in $^{64}Ni+^{64}Ni$ and $^{58}Ni+^{58}Ni$ reaction *Jiang*, PRL **93** (2004)
- The structure of *S* - factor is critical for the extrapolation of data!

S-factor for $^{12}C+^{12}C$ reaction



- More dramatic situation in $^{12}C+^{12}C$ reaction !
- Resonant behavior at low energies does not allow to make a clear conclusion about the maximum.

The capture cross section

For the light and medium-heavy nuclei the fusion is governed by the penetrability of the nuclei through the Coulomb and orbital angular momentum barriers (the so called capture).

The capture cross-section is a sum of partial capture cross-sections

$$\sigma(E_{\text{c.m.}}) = \sum_J \sigma_c(E_{\text{c.m.}}, J) = \pi \lambda\!\!\!^{-2} \sum_J (2J+1) P_{\text{cap}}(E_{\text{c.m.}}, J)$$

The partial capture probability at fixed energy and angular momentum

$$P_{\text{cap}}(E_{\text{c.m.}}, J)$$

Different approaches and models offer different methods of calculation of capture probability

The assumptions of the QD approach

The quantum diffusion approach based on the following assumptions:

1. The capture (fusion) can be treated on the one collective variable: the relative distance between the colliding nuclei: R.

2. Collective coordinate is coupled to the internal excitations.

3. The internal excitations (for example, low-lying collective modes such as dynamical quadropole and octupole excitations of the target and projectile, single particle excitations etc.) can be presented as an environment.

Sargsyan, EPJ A **45**, 125 (2010)
Sargsyan, EPJ A **47**, 38 (2011)
Sargsyan, PRC **84**, 064614 (2011)

The formalism of QD approach

The full Hamiltonian of the system:

$$H_{tot} = H_{coll}(R,P) + H_{inter}\left(b_\nu^+, b_\nu\right) + V_{coupling}\left(R, b_\nu^+ + b_\nu\right)$$

$$H_{coll} = \frac{1}{2\mu}P^2 - \frac{\mu\omega^2}{2}R^2$$

- **The collective subsystem**
  (inverted harmonic oscillator)

$$H_{inter} = \sum_\nu \hbar\omega_\nu b_\nu^+ b_\nu$$

- **The internal subsystem**

$$V_{coupling} = R\sum_\nu \Gamma_\nu\left(b_\nu^+ + b_\nu\right)$$

- **Coupling between the subsystems**
  (linear coupling)

The analytical expressions for the first and second moments

$$\overline{R(t)} = A_t R_0 + B_t P_0$$

$$\Sigma_{RR}(t) = \frac{2\hbar^2 \lambda \gamma^2}{\pi} \int_0^t d\tau' B_{\tau'} \int_0^t d\tau'' B_{\tau''} \int_0^\infty d\Omega \frac{\Omega}{\Omega^2 + \gamma^2} \coth\left[\frac{\hbar\Omega}{2T}\right] \cos[\Omega(\tau' - \tau'')]$$

$$A_t = \sum_{i=1}^{3} \beta_i \left[s_i(s_i + \gamma) + \hbar\lambda\gamma/\mu\right] e^{s_i t}$$

$$B_t = \frac{1}{\mu} \sum_{i=1}^{3} \beta_i (s_i + \gamma) e^{s_i t}$$

Functions determine the dynamic of the first and second moments

$$\beta_1 = \left[(s_1 - s_2)(s_1 - s_3)\right]^{-1}$$

$$\beta_2 = \left[(s_2 - s_1)(s_2 - s_3)\right]^{-1}$$

$$\beta_3 = \left[(s_3 - s_1)(s_3 - s_2)\right]^{-1}$$

s_i **--- are the roots of the following equation**

$$(s + \gamma)(s^2 - \omega^2) + \hbar\lambda\gamma s / \mu = 0$$

Sargsyan, PRC **77**, 024607 (2008)

Nucleus-nucleus interaction potential:

$$V(R, Z_i, A_i, J) = V_{Nucl} + V_{Coul} + \frac{\hbar^2 J(J+1)}{2\mu R^2}$$

Double-folding formalism used for nuclear part:

$$V_{Nucl} = \int d\vec{r}_1 d\vec{r}_2 \rho_1(\vec{r}_1) F(\vec{r}_1 - \vec{r}_2) \rho_2(R - \vec{r}_2)$$

Nucleus-nucleus potential:

1. **density - dependent effective nucleon-nucleon interaction**
2. **Woods-Saxon parameterization for nucleus density**

Adamian et al., Int. J. Mod. Phys E 5, 191 (1996).

Determination of the " effective frequency" of the replaced oscillator:

1. The same height and the position of the barrier of the real potential with the height and the position of inverted oscillator.
2. The condition of equality of the energy under the barrier: $V_b - E_{c.m.} = -\frac{\mu\omega^2 R_{ext}^2}{2}$

The capture probability in quantum diffusion approach

Capture probability depends on the ratio of mean value and the variance of the collective coordinate:

$$P_{\text{cap}} = \lim_{t\to\infty} \frac{1}{2} \operatorname{erfc}\left[\frac{\overline{R(t)}}{\sqrt{\Sigma_{RR}(t)}} \right]$$

- The coupling to internal excitations leads to fluctuation of collective coordinate.
- Equations for the mean value and variance contain friction and diffusion.
- The friction and diffusion are obtained in a self-consistent way.

Our approach takes into account the fluctuation and dissipation effects in the collisions of heavy ions which model the coupling with various channels.

The capture probability

$$P_{\mathrm{cap}} = \frac{1}{2}\,\mathrm{erfc}\left[\left(\frac{\pi s_1(\gamma - s_1)}{2\mu(\omega^2 - s_1^{\,2})}\right)^{1/2} \frac{\mu\omega^2 R_{ext}}{s_1\left[\gamma \ln(\gamma / s_1)\right]^{1/2}}\right]$$

Constant friction coefficient was used. $\lambda = -(s_1 + s_2)$

The frequency of the approximated oscillator.

$$\omega^2 = -\varepsilon^2 \frac{(s_1 + \gamma)(s_2 + \gamma)}{(s_1 + \gamma)(s_2 + \gamma) - 2\tilde{\lambda}\gamma\varepsilon}$$

The internal excitation width --- γ is responsible for non-Markovian effects.

The QD approach was successfully applied to describe the heavy ion fusion reactions at near- and below the Coulomb barrier energies.

To consider reactions at extremely low energies one need to extend the model!!

- The friction is proportional to the square of the nuclear force:

$$\lambda(R) = \lambda_0 (\nabla V_N(R))^2$$

- This form of friction takes into account the comparatively larger overlap of the nuclear surfaces is case of two heavier nuclei.

Gross, Phys. Rep. **45**, 175 (1978)
Weidenmuller, Prog. Part. Nucl. Phys. **3**, 49 (1980)

- Determination of the excitation width:

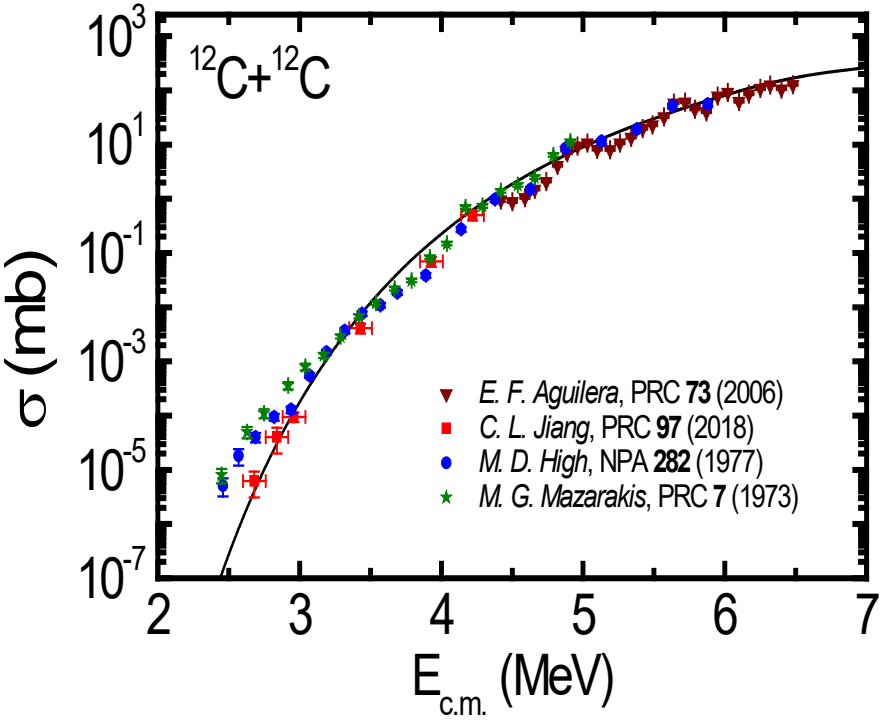
$$\gamma(R) = \gamma_0 \exp\left[k \frac{\omega(R)}{\lambda(R)} \right]$$

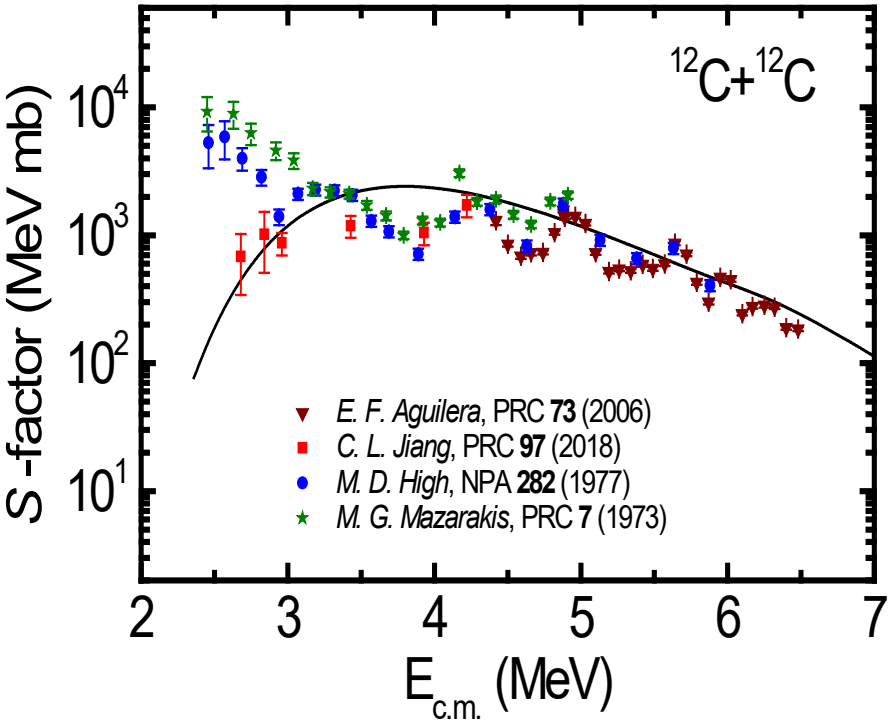
- Phenomenological coefficient!!

$$k = \sqrt{\frac{2V_b}{\mu\omega_b^2 (R_1 + R_2)^2 - 2V_b}}$$

Sargsyan, EPJ A **56**, 19 (2020)
Sargsyan, Phys. Lett. B **824**, 136792 (2022)

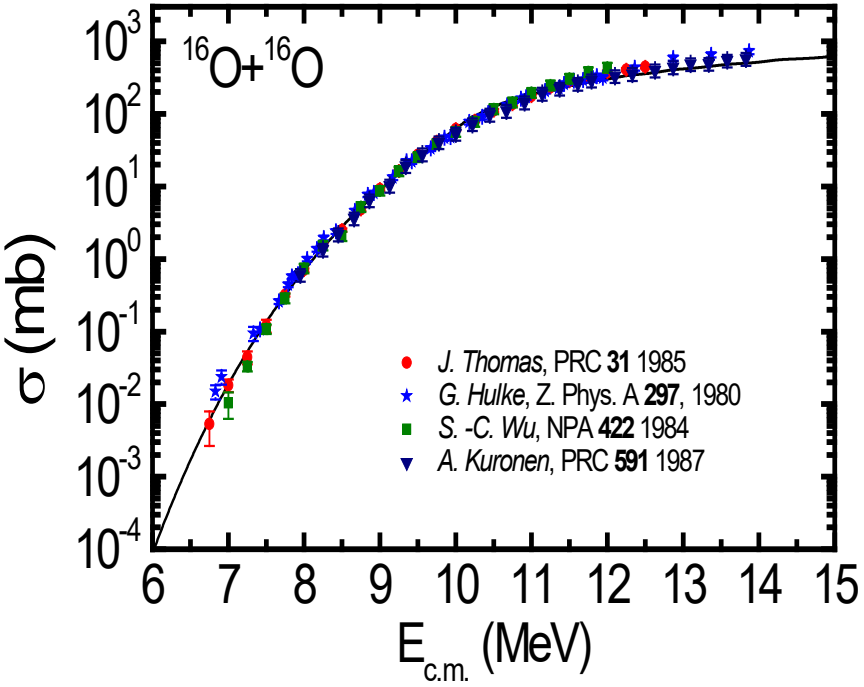
Results of calculations

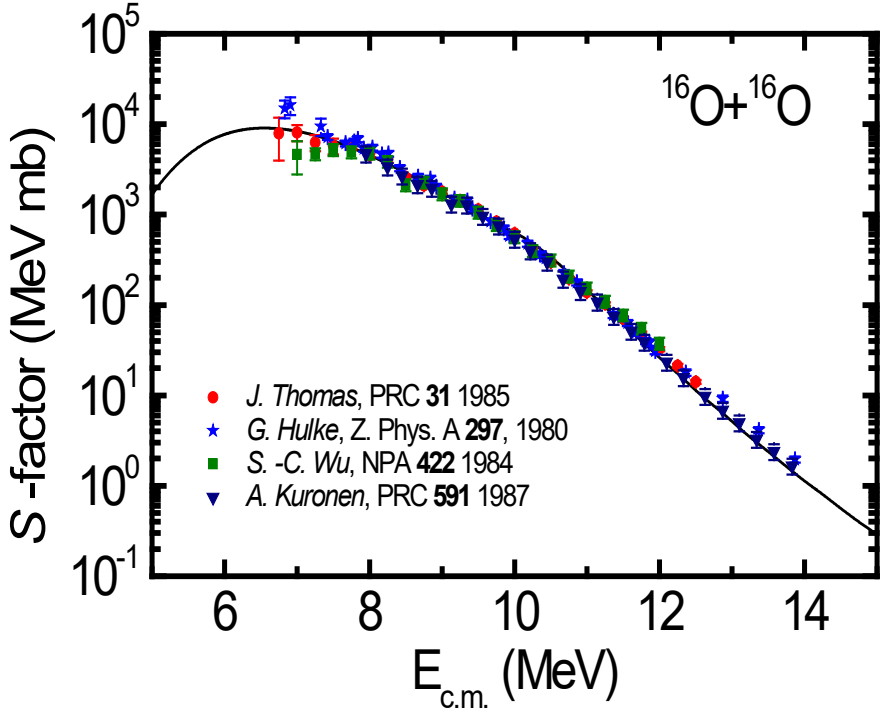




- The main trend of new experimental data is reproduced.
- We clearly identify a maximum in *S*- factor.

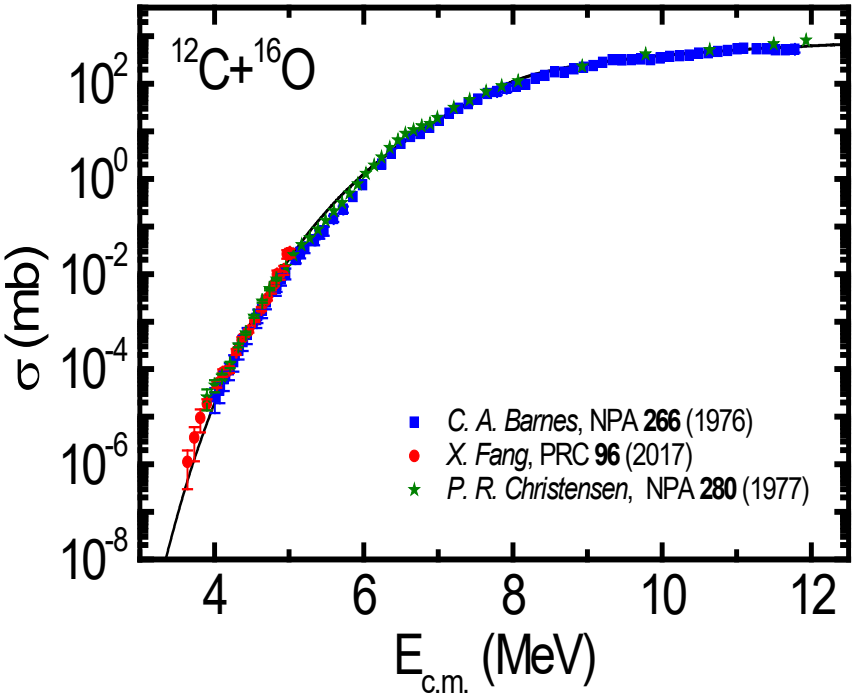
Results of calculations

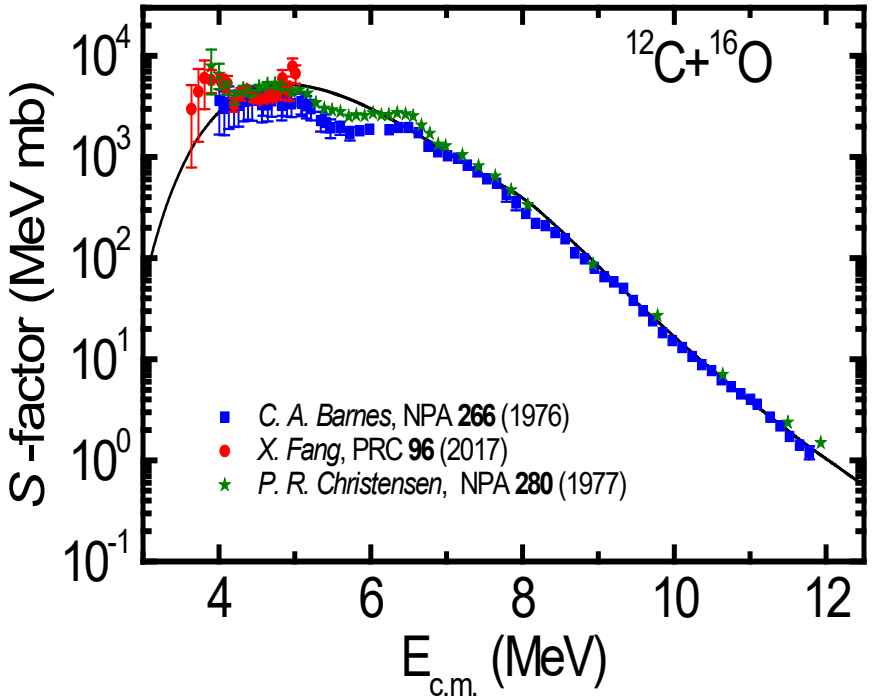




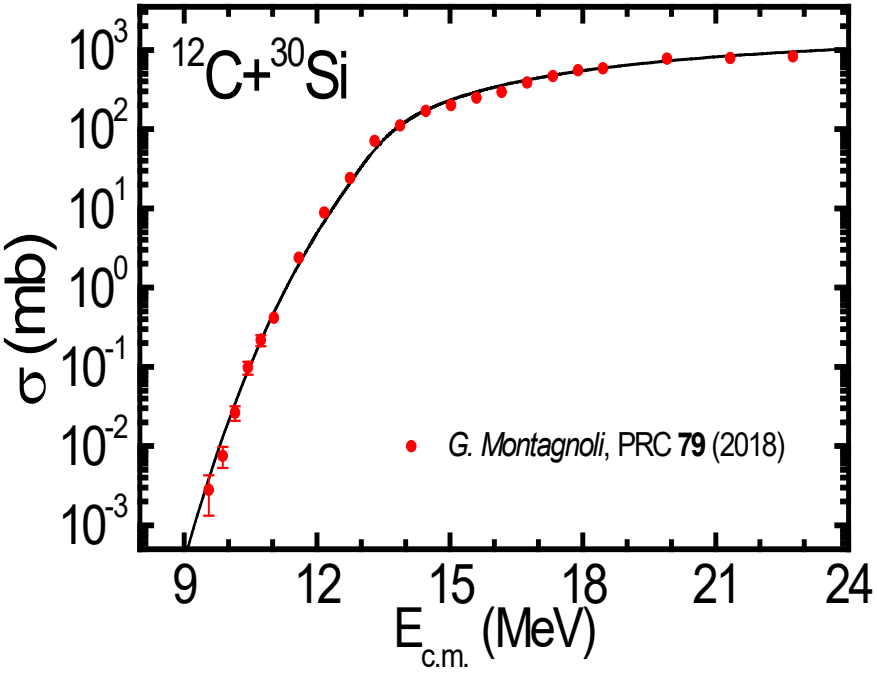
- The calculation is in good agreement with the experimental data
- The lowest data are very close to the position of the maximum of *S* factor

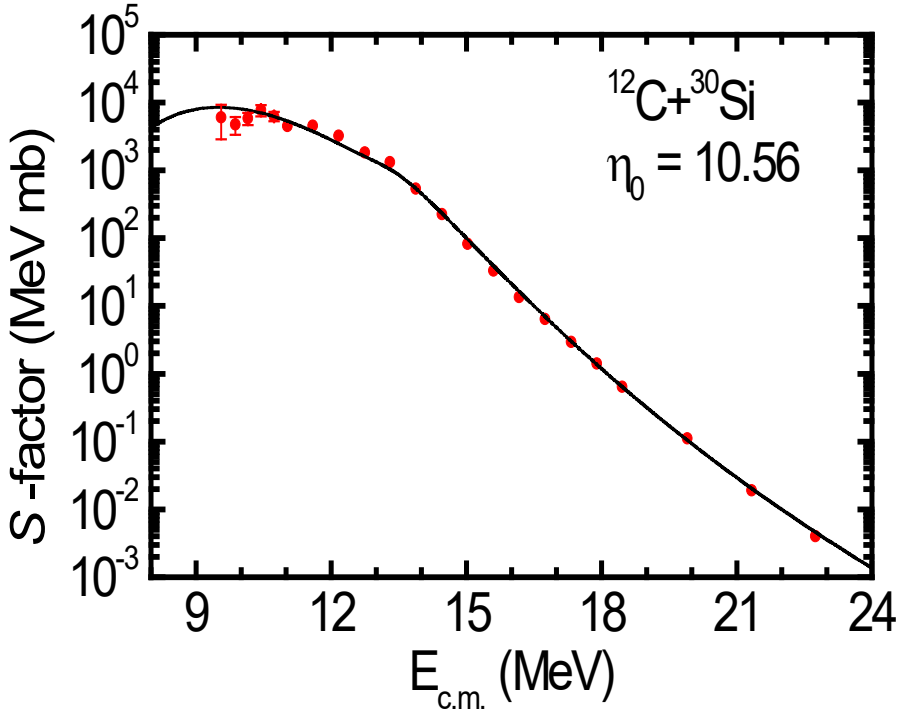
Results of calculations



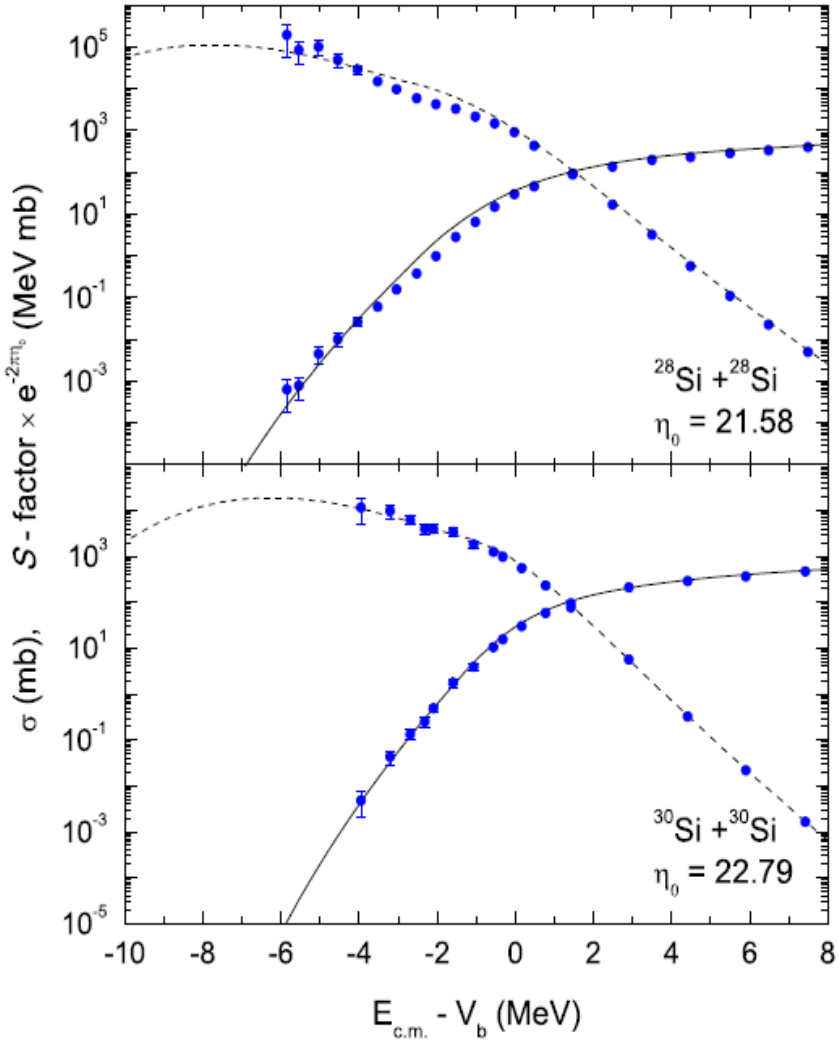


Results of calculations

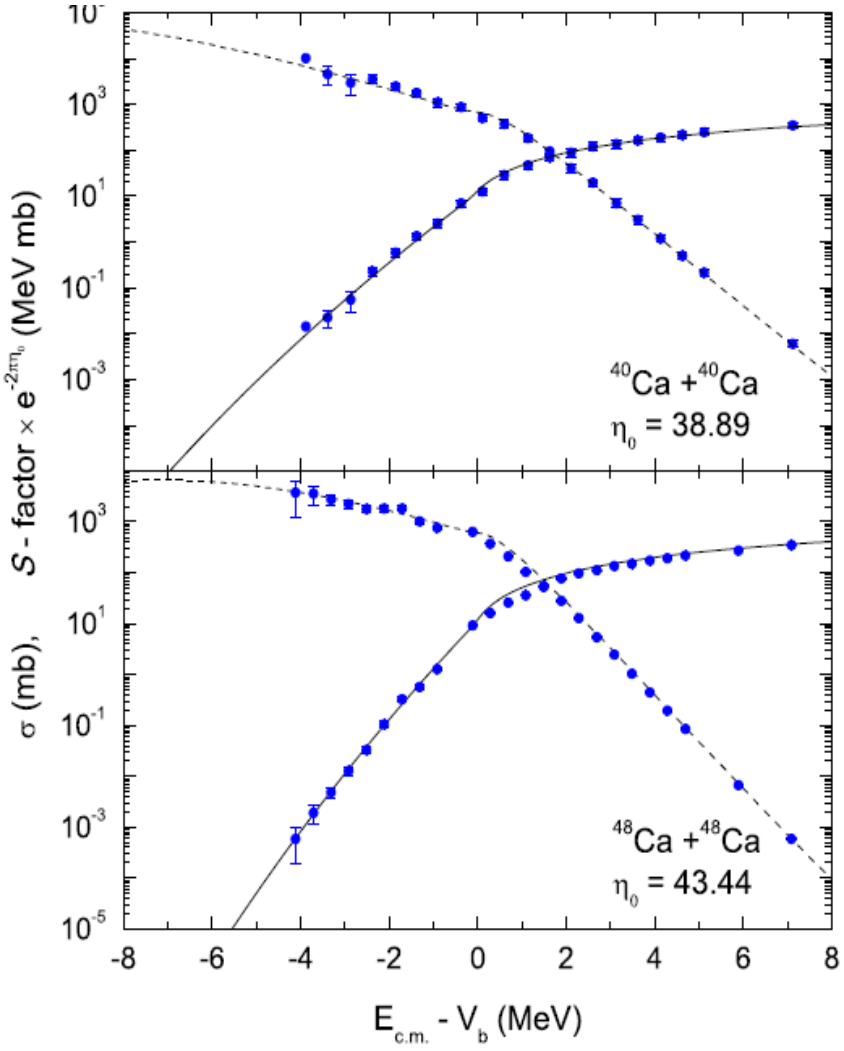




Results of calculations

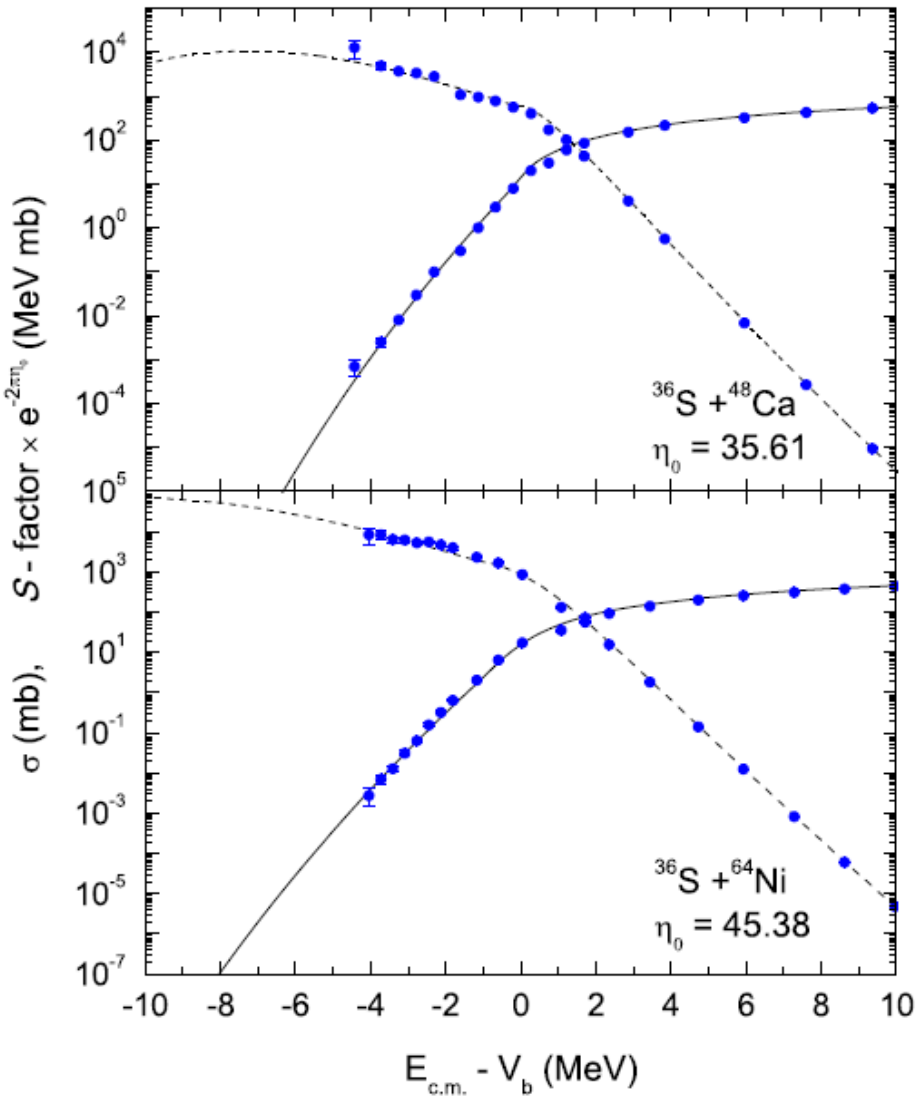


Montagnoli, PRC **90**, 044608 (2014)
Colucci, PRC **97**, 044613 (2018)



Montagnoli, PRC **85**, 024607 (2012)
Stefanini, PLB **679**, 95 (2009)

Results of calculations



Stefanini, PRC **78**, 044607 (2008)
Montagnoli, PRC **82**, 064609 (2010)

Probability at extreme sub-barrier energies

At low energies (weak friction limit) one obtains simple expression for the capture probability !

$$P_{\text{cap}} = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{\pi\,(V_b - E_{c.m.})}{k\,\omega(E_{c.m.})}}\right]$$

$$k = \sqrt{\frac{2V_b}{\mu\omega_b^2\left(R_1 + R_2\right)^2 - 2V_b}} \qquad V_b - E_{c.m.} = -\frac{\mu\omega^2 R_{ext}^2}{2}$$

- ✓ For very light nuclei the formulas are applicable already at energies 1-2 MeV below the barrier.
- ✓ The formulas could be used to obtain analytical expression for the *S*- factor position and reaction rates.

The position E_s of the S- factor maximum

Only small angular momenta contribute to the fusion!

$$\left.\frac{dS_{l=0}(E_{c.m.})}{dE_{c.m.}}\right|_{E_{c.m.}=E_s} = 0$$

The equation for the position E_s of the *S*- factor maximum:

$$\sqrt{\frac{V_b - E_s}{E_s}} + \frac{1}{2}\left(\frac{E_c - E_s}{E_c}\right)\sqrt{\frac{E_s}{V_b - E_s}} = k$$

$$E_c = \frac{Z_1 Z_2 e^2}{R_b}$$

For the considered reactions one always get a solution!

After expanding in power of $1/E_c$ and making a replacement V_b -> E_c one obtain the simple approximated expression for the position of S- factor maximum:

$$E_s \approx E_c\left(\sqrt{1+\left(k^2/2\right)^2} - k^2/2\right)$$

Sargsyan, Phys. Lett. B **824**, 136792 (2022)

The calculated exact E_s and approximated E_s^a positions of the S-factor maximum. The values of V_b, R_b, $\omega_b = \sqrt{\frac{1}{\mu}\frac{d^2V(R)}{dR^2}|_{R=R_b}}$, and κ for the spherical interacting nuclei are calculated at zero angular momentum.

Reaction	R_b (fm)	V_b (MeV)	ω_b (MeV)	κ	E_s^a (MeV)	E_s (MeV)
$^{12}C + {}^{12}C$	7.95	6.00	2.67	1.18	3.4	3.6
$^{12}C + {}^{16}O$	8.32	7.64	2.68	1.14	4.5	4.8
$^{12}C + {}^{30}Si$	8.71	12.81	3.00	0.92	9.2	9.4
$^{16}O + {}^{16}O$	8.69	9.75	2.67	1.10	6.0	6.2
$^{28}Si + {}^{28}Si$	9.08	28.64	3.27	0.81	22.4	21.0
$^{28}Si + {}^{30}Si$	9.28	28.07	3.16	0.80	22.2	21.2
$^{30}Si + {}^{30}Si$	9.45	27.52	3.04	0.79	21.9	21.2
$^{24}Mg + {}^{30}Si$	9.28	24.07	3.06	0.82	18.7	18.0
$^{40}Ca + {}^{40}Ca$	10.08	52.46	3.22	0.82	41.0	41.2
$^{48}Ca + {}^{48}Ca$	10.58	50.46	2.95	0.70	42.7	43.4
$^{36}S + {}^{48}Ca$	10.34	41.20	2.90	0.76	33.5	33.9
$^{36}S + {}^{64}Ni$	10.70	55.68	3.03	0.75	45.7	45.4

The universal ratio

$$E_s \approx E_c\left(\sqrt{1+\left(k^2/2\right)^2}-k^2/2\right)$$

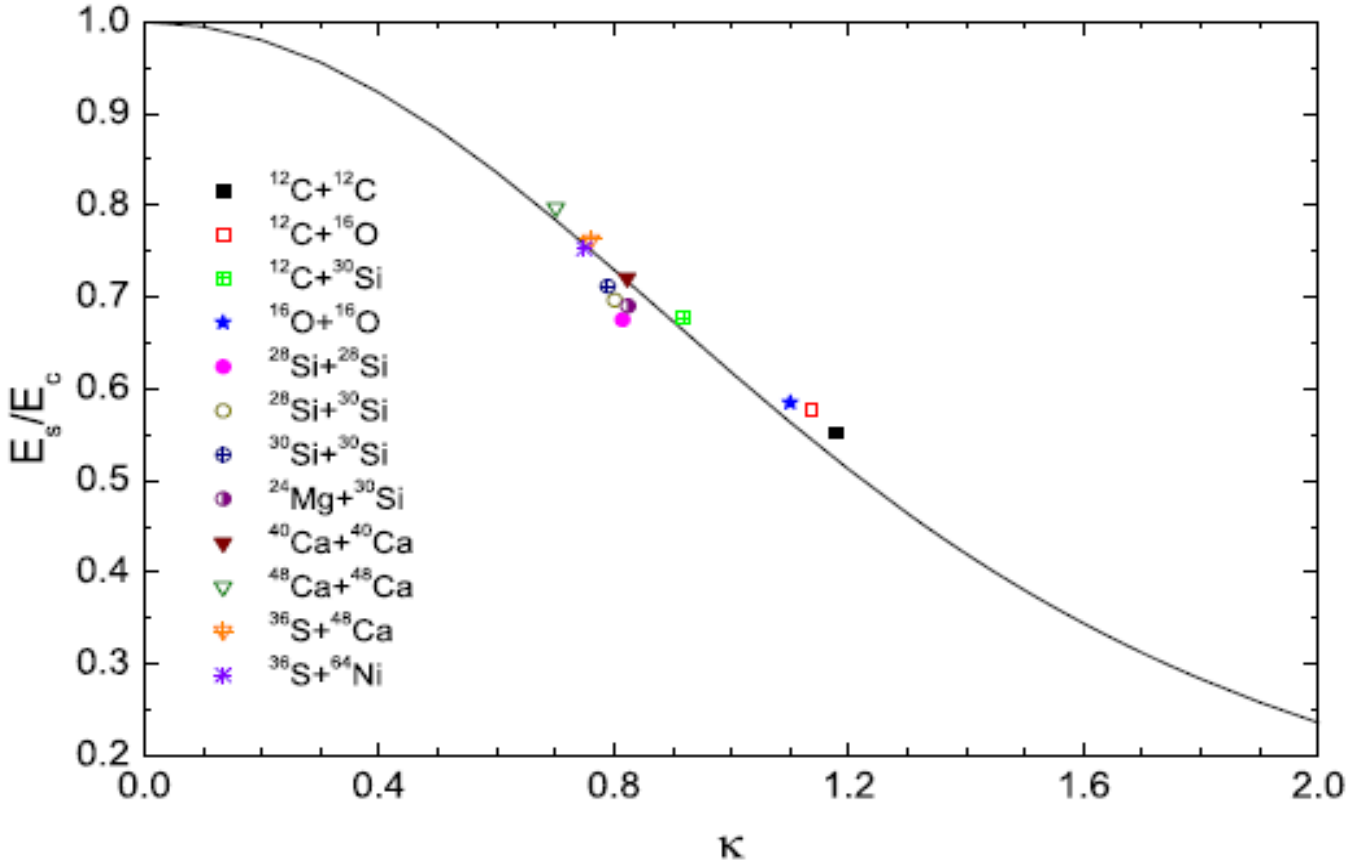


Fig. 7. The ratio E_s/E_c for the indicated reactions and the function $f(\kappa) = \sqrt{1+(\kappa^2/2)^2}-\kappa^2/2$ vs κ (line).

The origin of the *S*-factor maximum

Employing the well-known asymptotic parametrization $V_N(R) \sim e^{-R/a_0}$ we deduce in leading order the simple expression:

$$k^2/2 \sim a_0/(R_1 + R_2)$$

from which we conclude that the position of the *S*-factor maximum is determined by the thickness of the potential surface layer given by a_0 to the contact distance (R_1+R_2)between two nuclei.

- ✓ In addition to the nuclear interaction, there are fluctuation and dissipation effects in the model due to the coupling of relative motion to the excitations of various channels.
- ✓ As a result, the energy dependence of the fusion probability differs from the energy dependence of the Gamow tunneling probability, $P_G \sim e^{-2\pi\eta}$, through the simple Coulomb interaction potential.
- ✓ The origin of the maximum of *S*-factor is the nuclear interaction and dissipative effects which effectively widen the potential barrier at $E_{c.m.} < E_s$.

Summary

- The structure of astrophysical *S*-factor for various reactions is analyzed within the extended quantum diffusion approach .
- We compared the calculated fusion cross sections with the available experimental data. In all cases we obtained a good description of the experiments .
- For the considered reactions, the *S*-factor shows a clear maximum in the sub-barrier energy range $E_s \sim (0.60 - 0.86)V_b$.
- We propose an analytical expression, predicting very reliably, the dependence of the *S*-factor maximum on the ion mass and charge numbers, which may be used not only in stellar burning studies but also as a guidance for future experiments .
- Another interesting behavior of the obtained *S*-factor is its strong dependence on $E_{\mathrm{c.m.}}$ at the collision energies below the maximum which will reduce considerably the stellar burning rates and its temperature dependence.

Thank you!