

# Two analytic formulas of Heavy-ion Fusion Cross sections

Cheng Lie Jiang

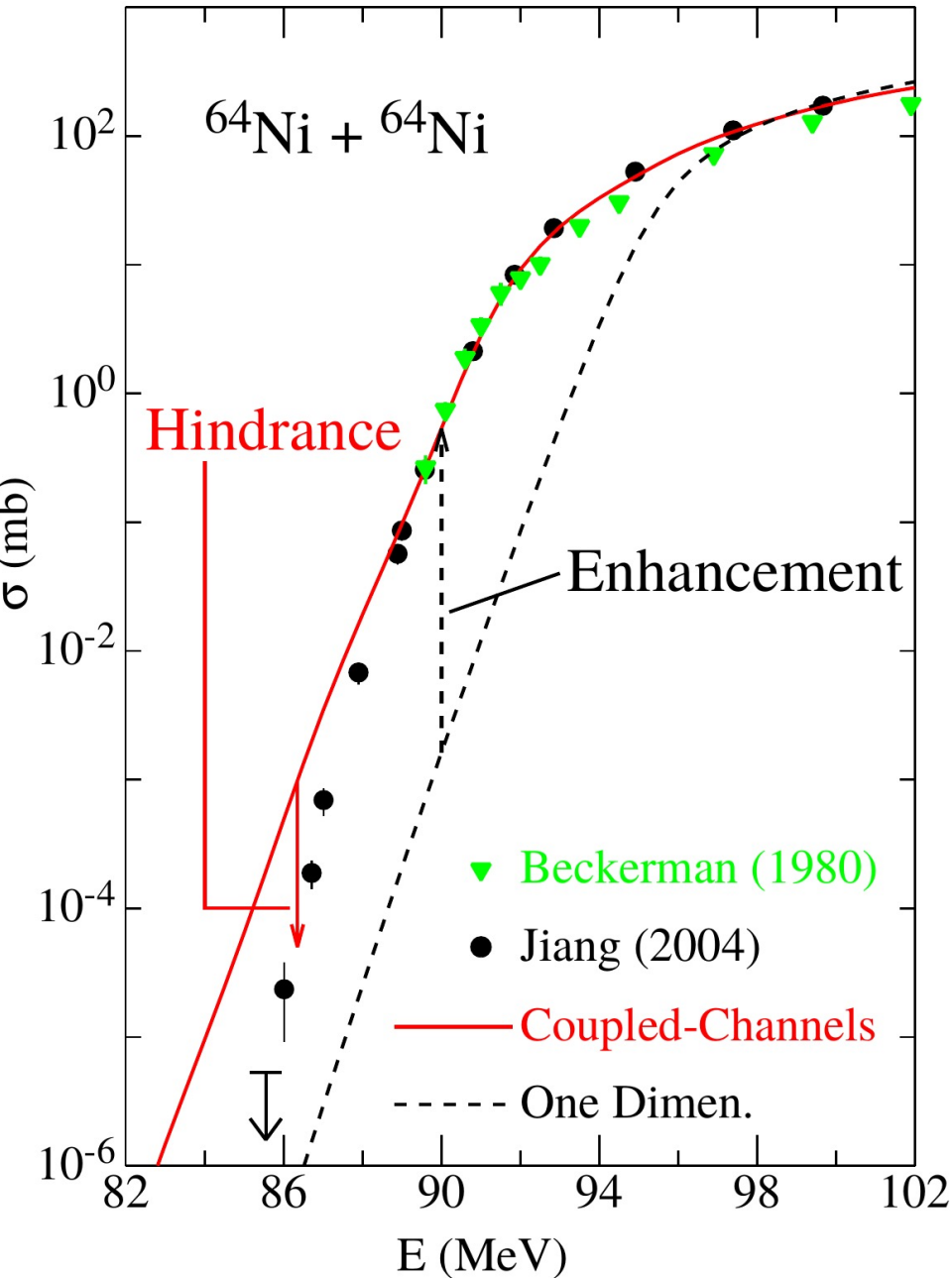
Argonne National Laboratory



Office of Science  
U.S. Department of Energy

A U.S. Department of Energy  
Office of Science Laboratory  
Operated by The University of Chicago





Fusion hindrance at extreme sub-barrier energies discovered twenty years ago.

We will demonstrate two simple, analytic cross section formulas which can reproduce the hindrance behavior very well

EPJ A57, 235 (2021)



**Wong formula:** Includes the effects of quantum mechanical tunneling through the Coulomb potential

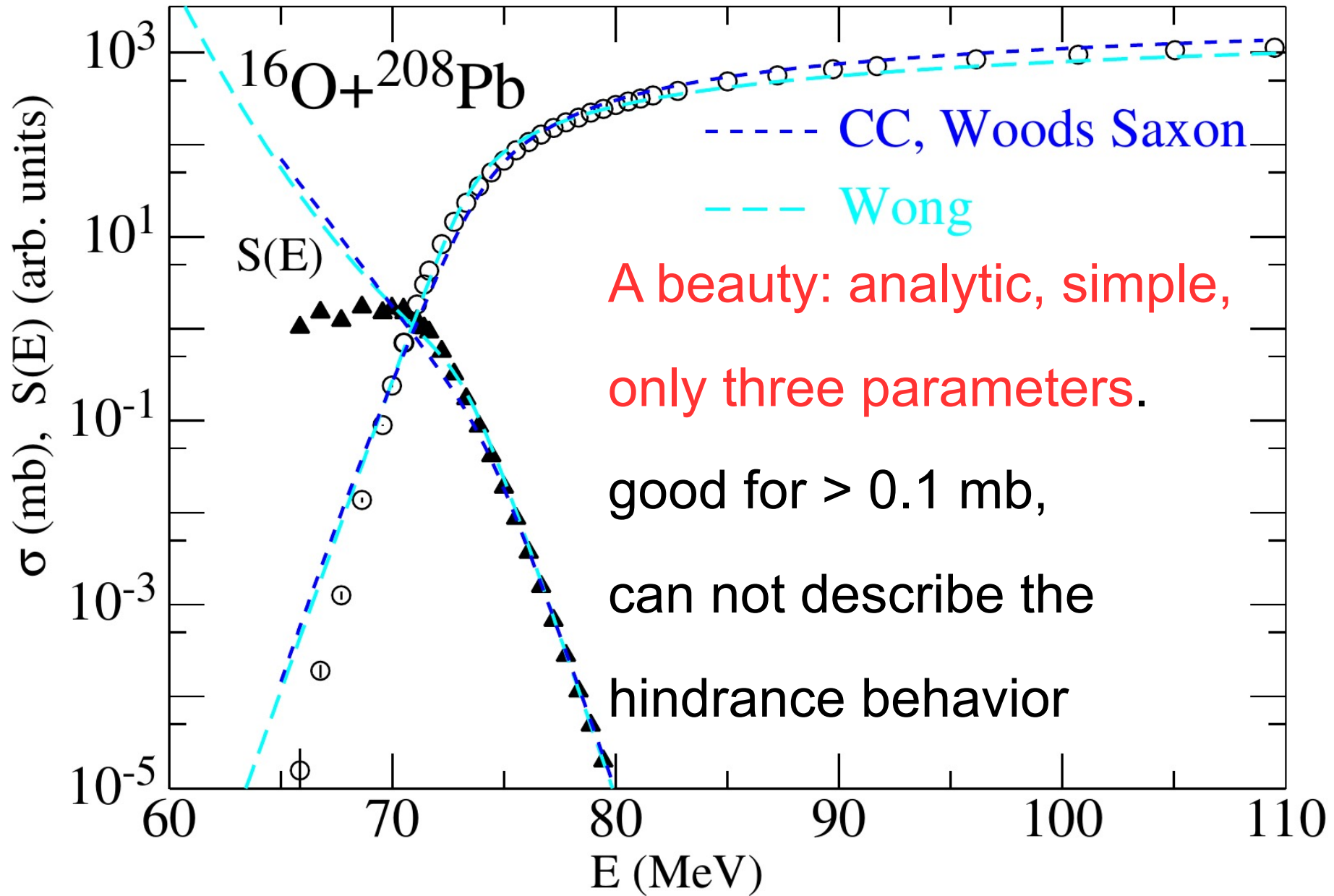
$$\sigma_W(E) = \frac{R^2}{2E} \hbar\omega \ln[1 + \exp((2\pi/\hbar\omega)(E - V))]$$

$R$  : radius

$V$  : potential barrier height

$\hbar\omega$ : curvature of the potential barrier

All three parameters, are L independent



Rowley and Hagino, PRC 91, 044617 (2015)

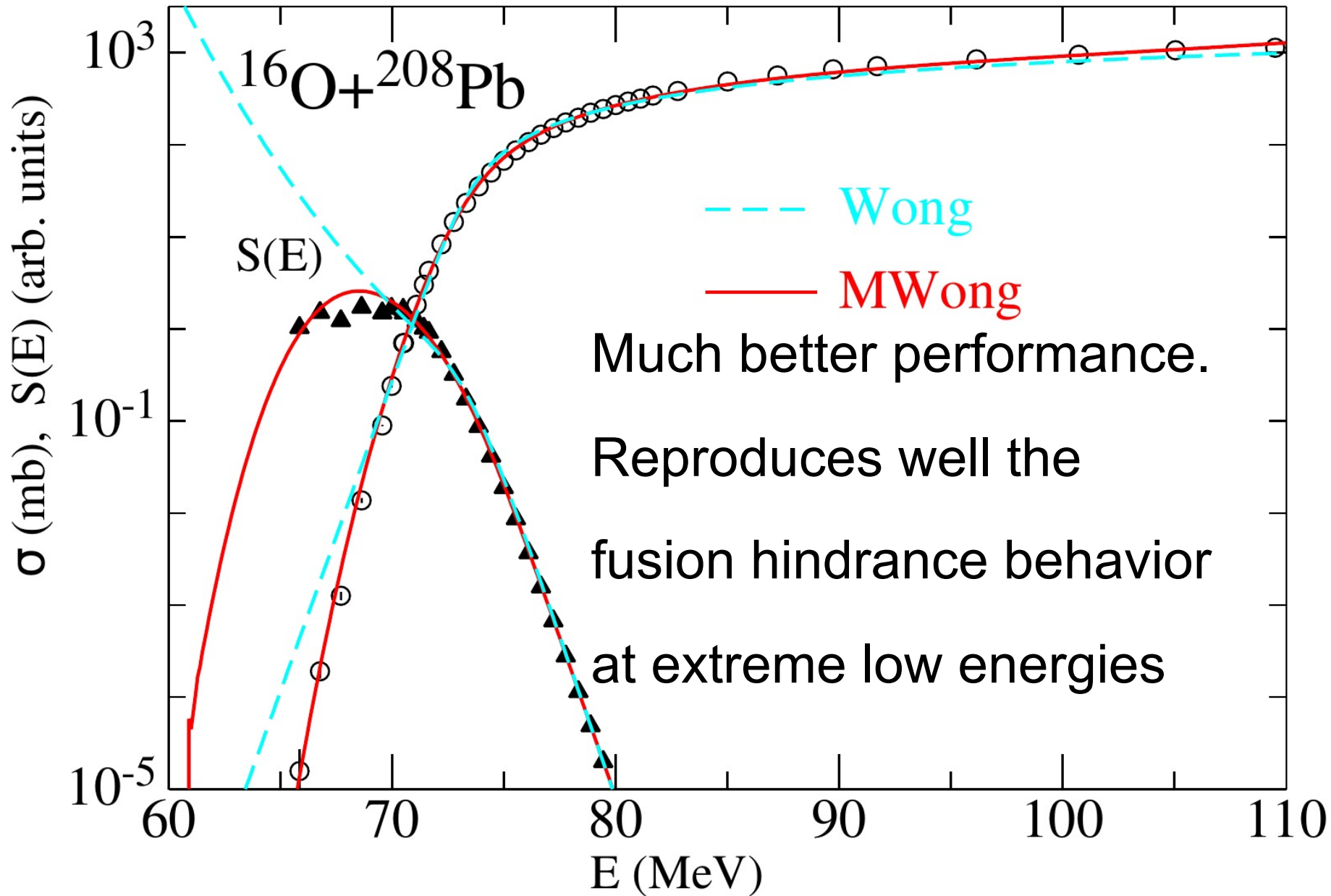
The  $R$ ,  $V$  and  $\hbar\omega$  are sensitive to angular momentum  $L$ , they suggested: Different parameter set for each angular momentum should be used in the calculations with Wong formula

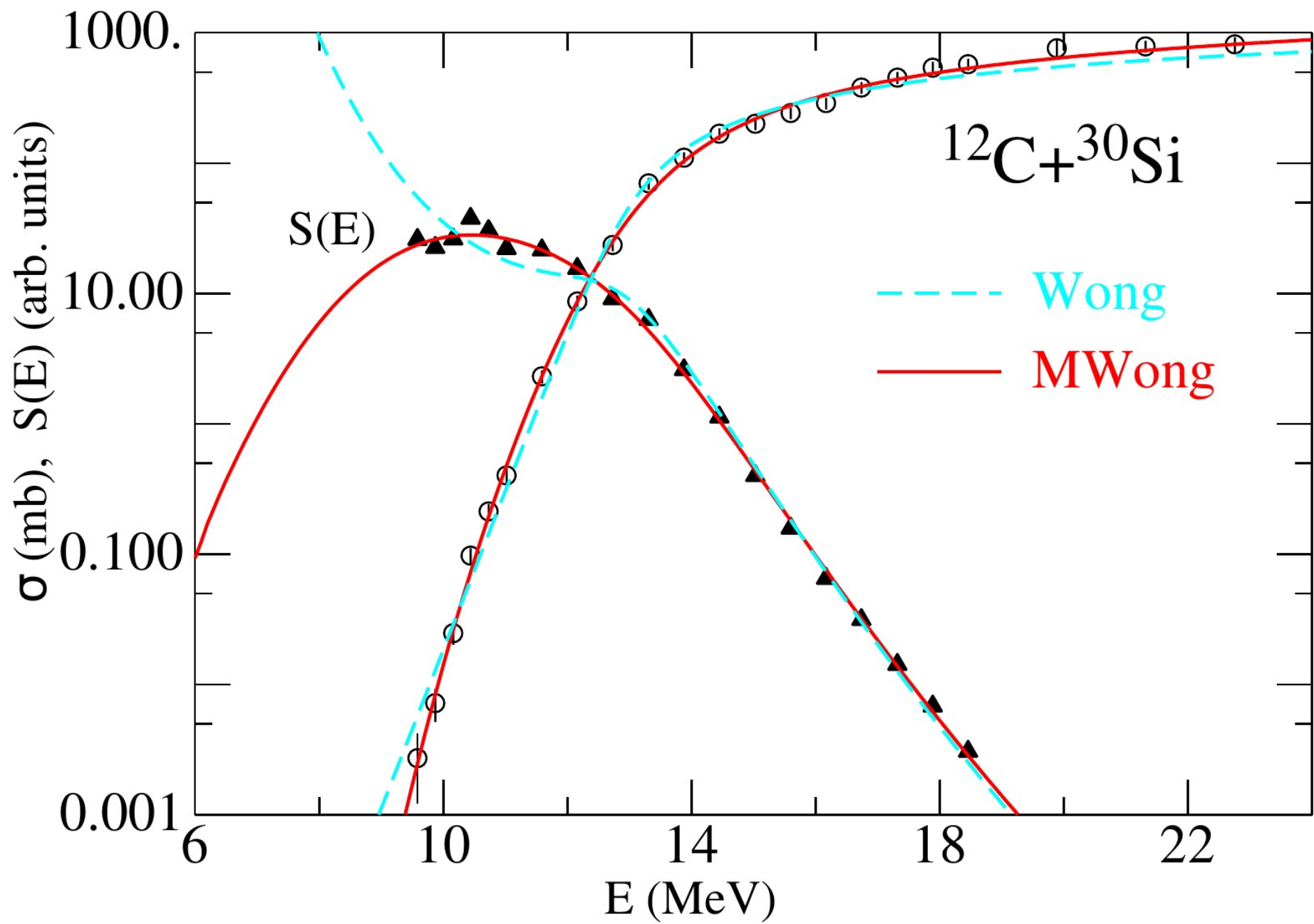
# Modified-Wong formula,

we assume:

$$\hbar\omega \rightarrow \hbar\omega \exp\left[\lambda \frac{E - V}{V}\right]$$

$$\sigma_f(E) = \frac{R^2}{2E} \hbar\omega \exp\left[\lambda \frac{E - V}{V}\right] \ln\left[1 + \exp\left(\left(2\pi / \hbar\omega \exp\left[\lambda \frac{E - V}{V}\right]\right)(E - V)\right)\right].$$







For 35 systems, whose lowest measured cross sections  $< 0.02$  mb, all can be well reproduced, including the hindrance behaviors. (submitted to EPJ A)

system	$Q$	type	$N$	data range	$\chi_0^2$	$R$	$V$	$W$	$\lambda$	$E_s^f$	$E_s^{exp}$	ref.
	MeV			mb - mb		fm	MeV	MeV		MeV	MeV	
$^{16}\text{O}+^{18}\text{O}$	24.41	EVR	21	0.006 - 224	0.12	7.13	9.67	3.02	1.18	6.54		[25]
$^{12}\text{C}+^{24}\text{Mg}$	16.30	EVR	21	.015 - 668	2.17	6.27	11.05	2.70	1.41	9.80	9.7	[26-28]
$^{12}\text{C}+^{30}\text{Si}$	14.11	EVR	22	0.0027 - 815	0.82	7.91	13.54	4.62	2.25	10.74	10.5	[29, 30]
$^6\text{Li}+^{198}\text{Pt}$	8.53	EVR	10	0.00017 - 348	24.5	8.22	29.53	17.91	4.32			[31]
$^7\text{Li}+^{198}\text{Pt}$	8.82	EVR	11	0.0002 - 1004	4.40	9.76	28.19	6.76	2.19			[32]
$^{24}\text{Mg}+^{30}\text{Si}$	17.89	EVR	20	0.0080 - 332	0.33	8.17	24.10	4.20	2.67	20.91	20.8	[33, 34]
-----												
$^{40}\text{Ar}+^{154}\text{Sm}$	-75.31	EVR+FF	15	0.0016 - 407	56.72	8.45	122.97	17.6	6.02	98.28		[53]
$^{76}\text{Ge}+^{86}\text{Kr}$	-97.91	EVR	15	0.0068 - 347	13.7	8.13	130.45	13.1	6.27	116.30		[54]
$^{58}\text{Ni}+^{124}\text{Sn}$	-112.30	EVR+FF	15	0.00046 - 570	1.07	8.41	156.86	13.3	5.49	138.80		[55, 56]
$^{64}\text{Ni}+^{124}\text{Sn}$	-117.51	EVR+FF	17	0.0008 - 605	0.82	7.66	155.28	16.8	12.9	141.9		[55, 56]

Denisov and Sedykh, EPJ A55, 153 (2019),  
had developed another modification of  
Wong formula: They gave: two simple  
expressions of  $R$  and  $V$ , a complex  
expression of  $h\omega$  with total 9 parameters.  
Fit with 1995 cross sections of 85 fusion  
excitation functions. Results can describe  
the average behaviors of these 85 systems

N. Rowley, G.R. Satchler and P.H. Stelson

Phys. Lett. B 254, 25 (1991)

On the “distribution of barrier”

interpretation of heavy-ion fusion

$$D(E) = \frac{1}{\pi R^2} \frac{d^2(E\sigma(E))}{dE^2}, \quad \int D(E) dE = 1$$

$$B_{exp}(E) = \frac{d^2(\sigma_{exp}(E)E)}{d^2E}$$

$$\begin{aligned}\sigma(E)E &= \int_{E_0}^E \left( \int_{E_0}^{E'} \frac{d^2(\sigma E'')}{d(E'')^2} dE'' \right) dE' \\ &= \pi R_0^2 \int_{E_0}^E \left( \int_{E_0}^{E'} D_{\text{test}}(E'') dE'' \right) dE' .\end{aligned}$$

$$D_{\text{test}}(E) = \sum_{i=1}^L D_i(E) = \sum_{i=1}^L \frac{\omega_i}{\sqrt{2\pi}W_i} \exp \left[ - \left( \frac{E - V_i}{\sqrt{2}W_i} \right)^2 \right]$$

$$\sigma(E)E = \pi R_0^2 \sum_{i=1}^L \frac{\omega_i W_i}{\sqrt{2\pi}} \left[ \sqrt{\pi} Z_i \operatorname{erfc}(-Z_i) + \exp(-Z_i^2) \right], \quad (16)$$

with  $Z_i = (E - V_i)/\sqrt{2}W_i$ .

## A special case

### Single-Gaussian barrier distribution formula

$$D_{\text{test}}(E) = \frac{1}{\sqrt{2\pi}W} \exp\left[-\left(\frac{E - V}{\sqrt{2}W}\right)^2\right]$$

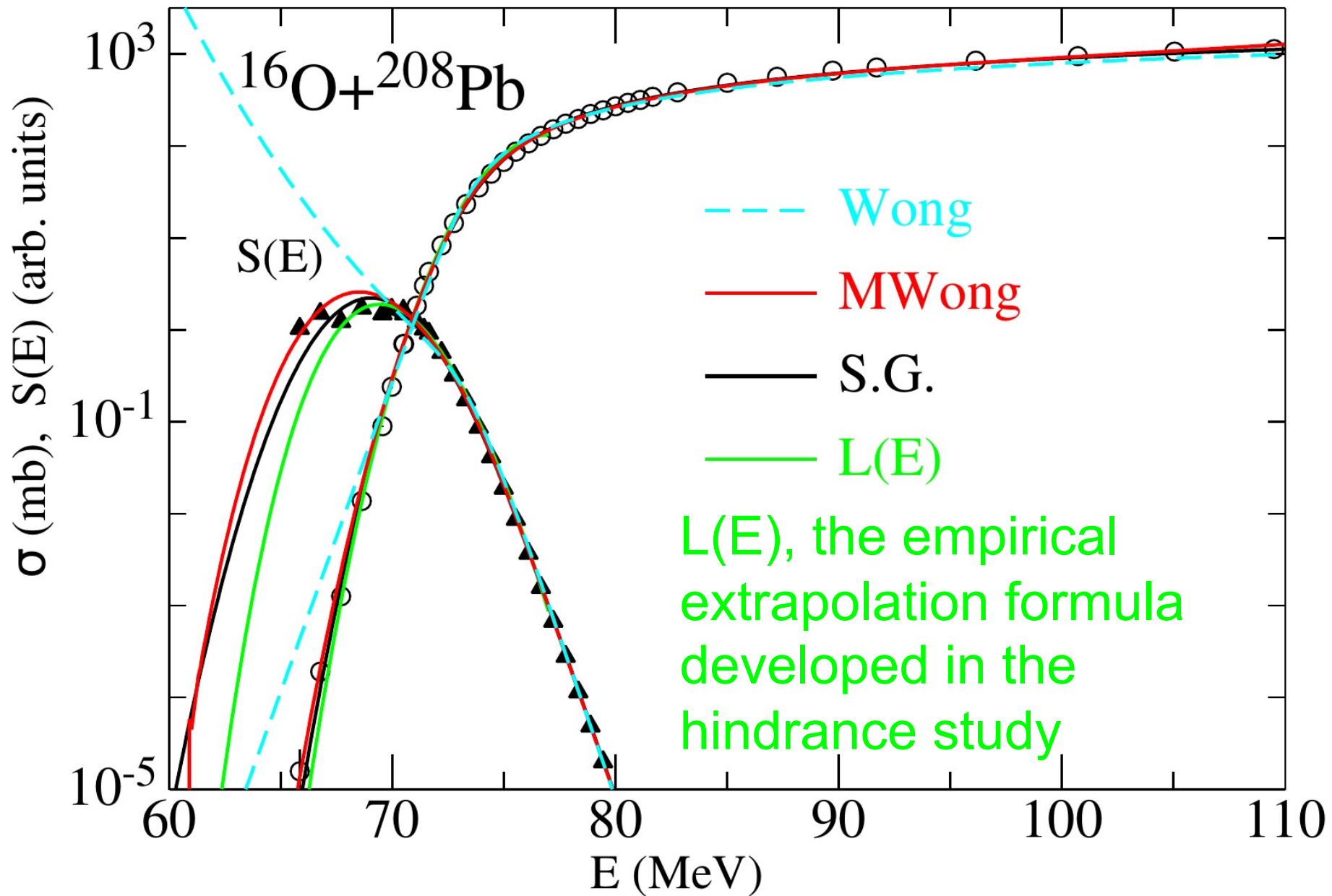
$V_i$ ,  $W$  are the centroids, the standard deviations of the Gaussian function

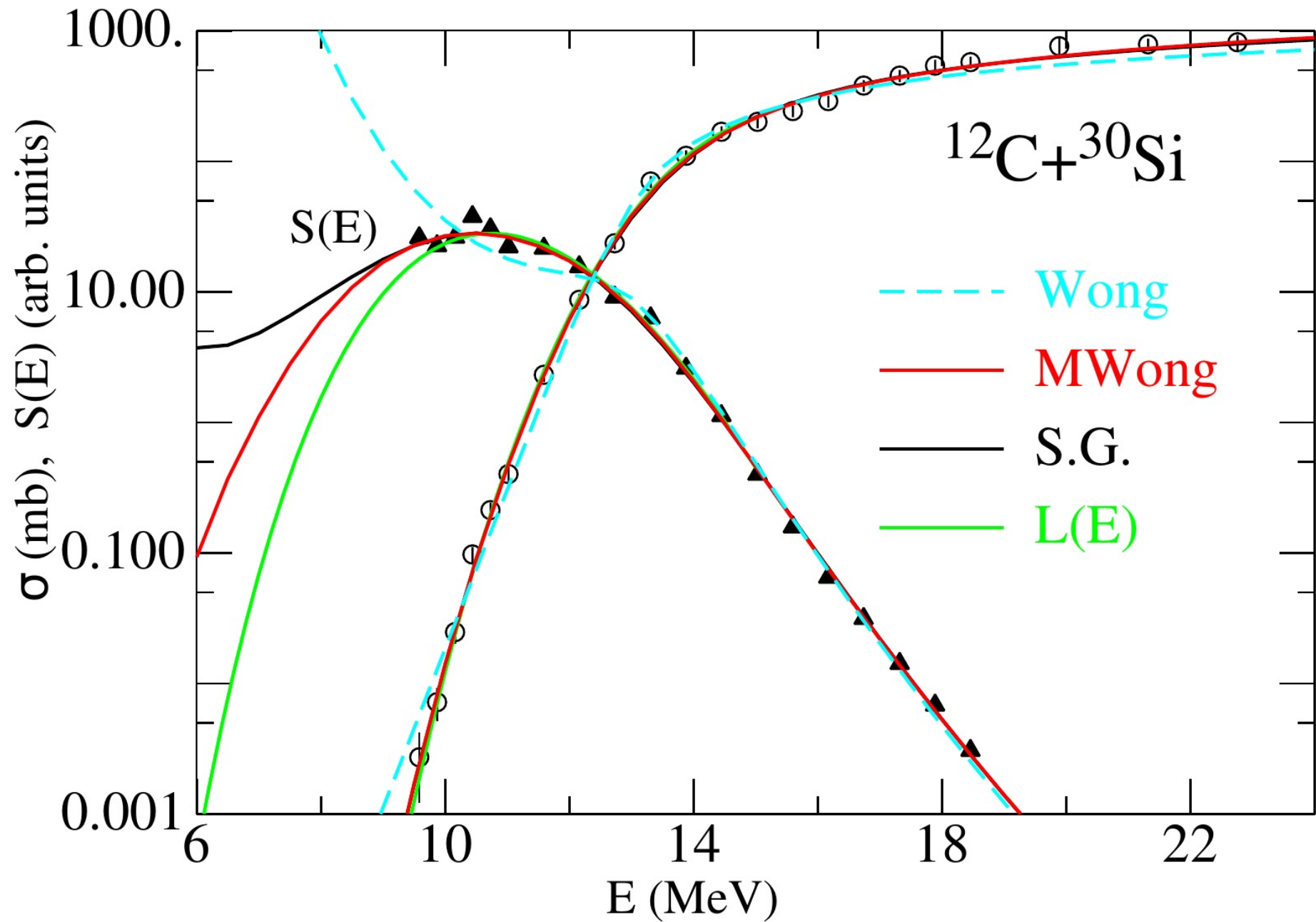
$$\sigma(E)E = \pi R^2 \frac{W}{\sqrt{2\pi}} \left[ \sqrt{\pi} Z \operatorname{erfc}(-Z) + \exp(-Z^2) \right],$$

$$Z = (E - V)/\sqrt{2}W,$$

$\operatorname{erfc}(Z)$  is the complementary error function

Eur. Phys. J. A 54, 218 (2018)





We obtained the single-Gaussian formula from the expression:

$$\begin{aligned}\sigma(E)E &= \int_{E_0}^E \left( \int_{E_0}^{E'} \frac{d^2(\sigma E'')}{d(E'')^2} dE'' \right) dE' \\ &= \pi R_0^2 \int_{E_0}^E \left( \int_{E_0}^{E'} D_{\text{test}}(E'') dE'' \right) dE' .\end{aligned}$$

The same expression has been obtained by K. Siwek-Wilczynska et al., (Acta Phys. Pol. B 33, 451 (2002)) with another way, which has not been widely used and referenced in the literature





Rowley's recipe: double differentiation

$$B_{exp}(E) = \frac{d^2(\sigma_{exp}(E)E)}{d^2E}$$

Weak points for obtaining fusion barrier distribution:

Must be measured with small energy steps.

Ambiguities for  $B(E)$ , since large energy steps must be used.

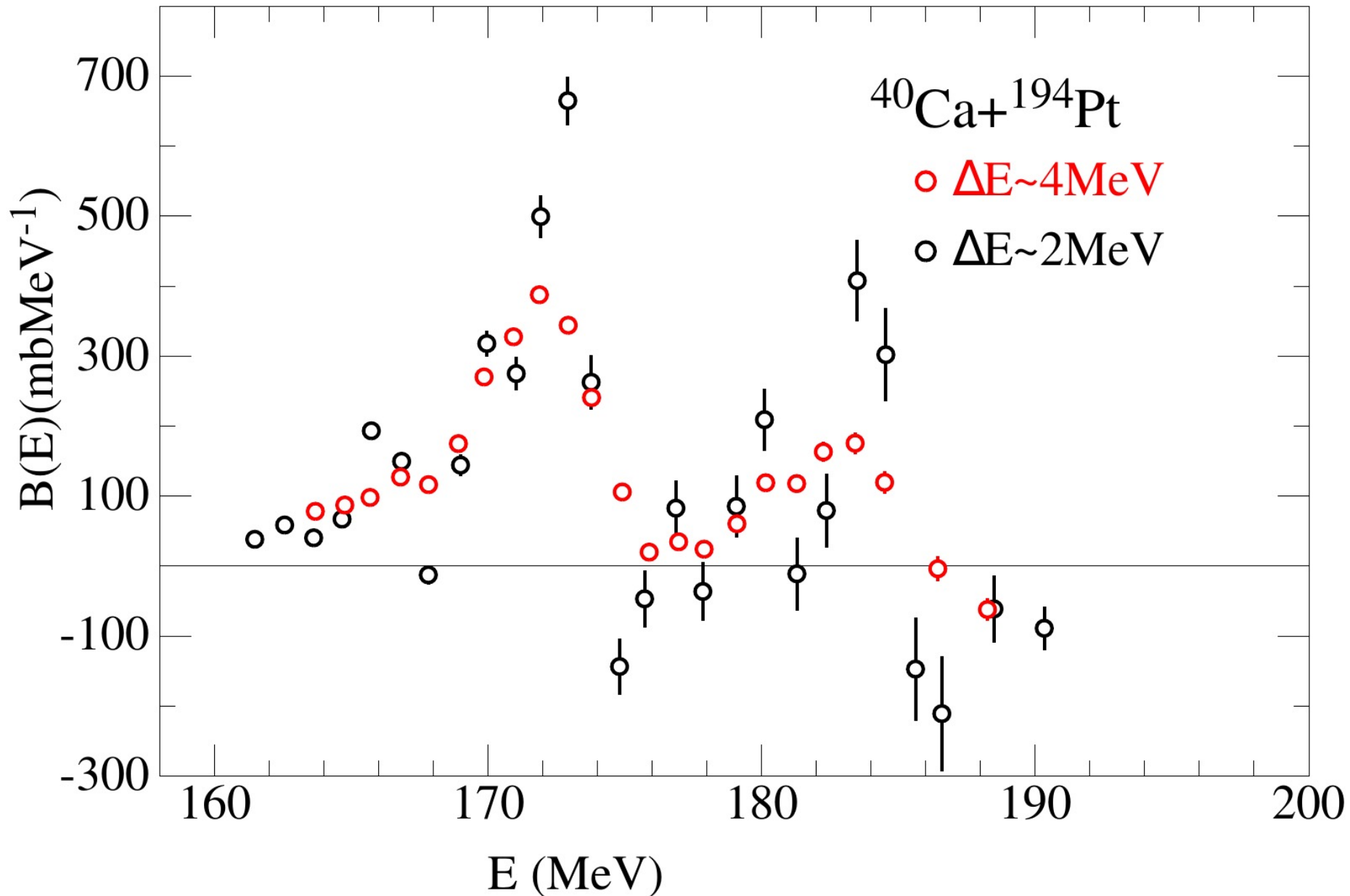
For obtaining good fit, we compare

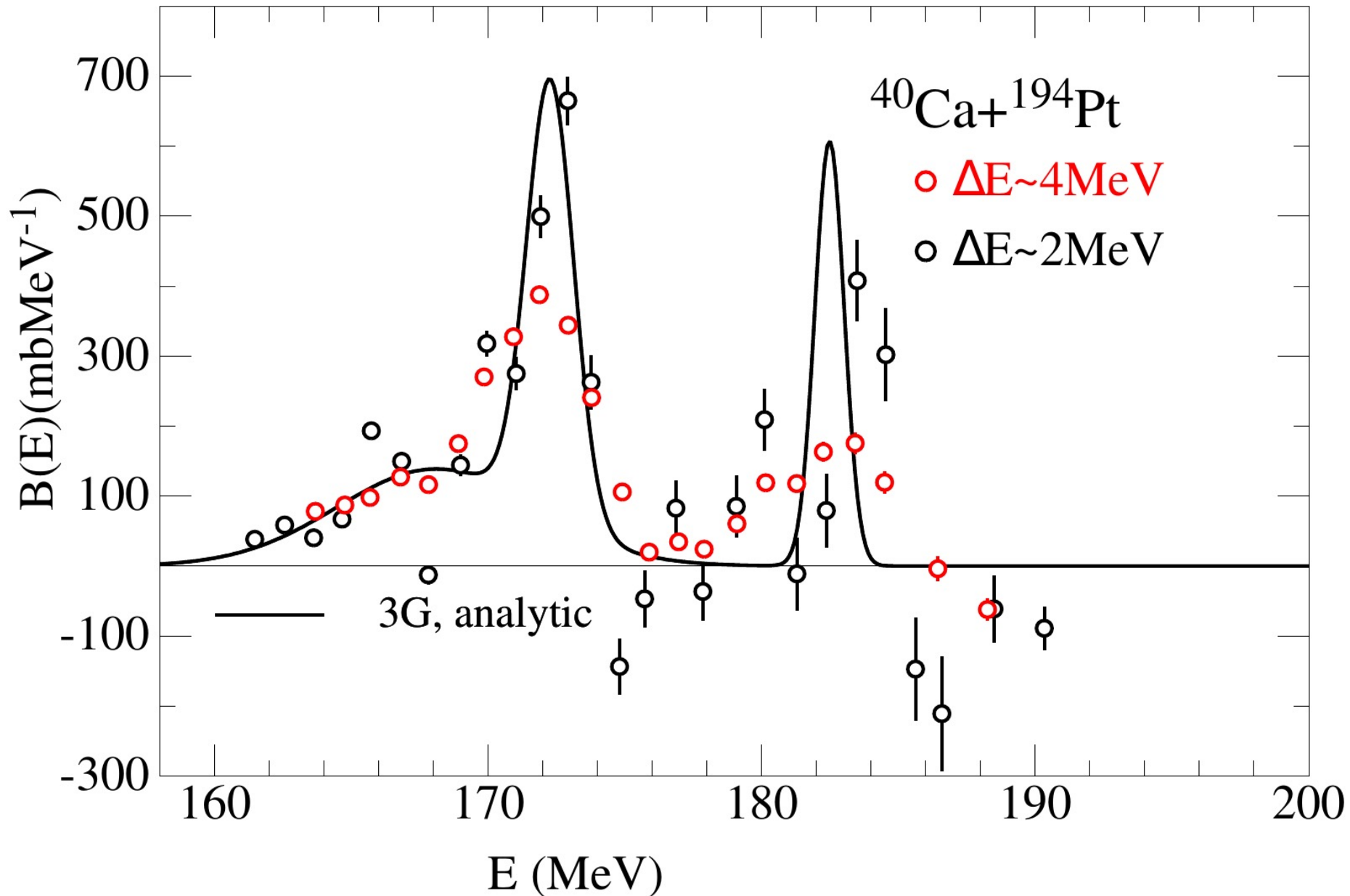
$$\chi_0^2 = \frac{1}{N} \sum_{i=1}^N ((\sigma_i - \sigma_{exp-i}) / \Delta\sigma_{exp-i})^2,$$

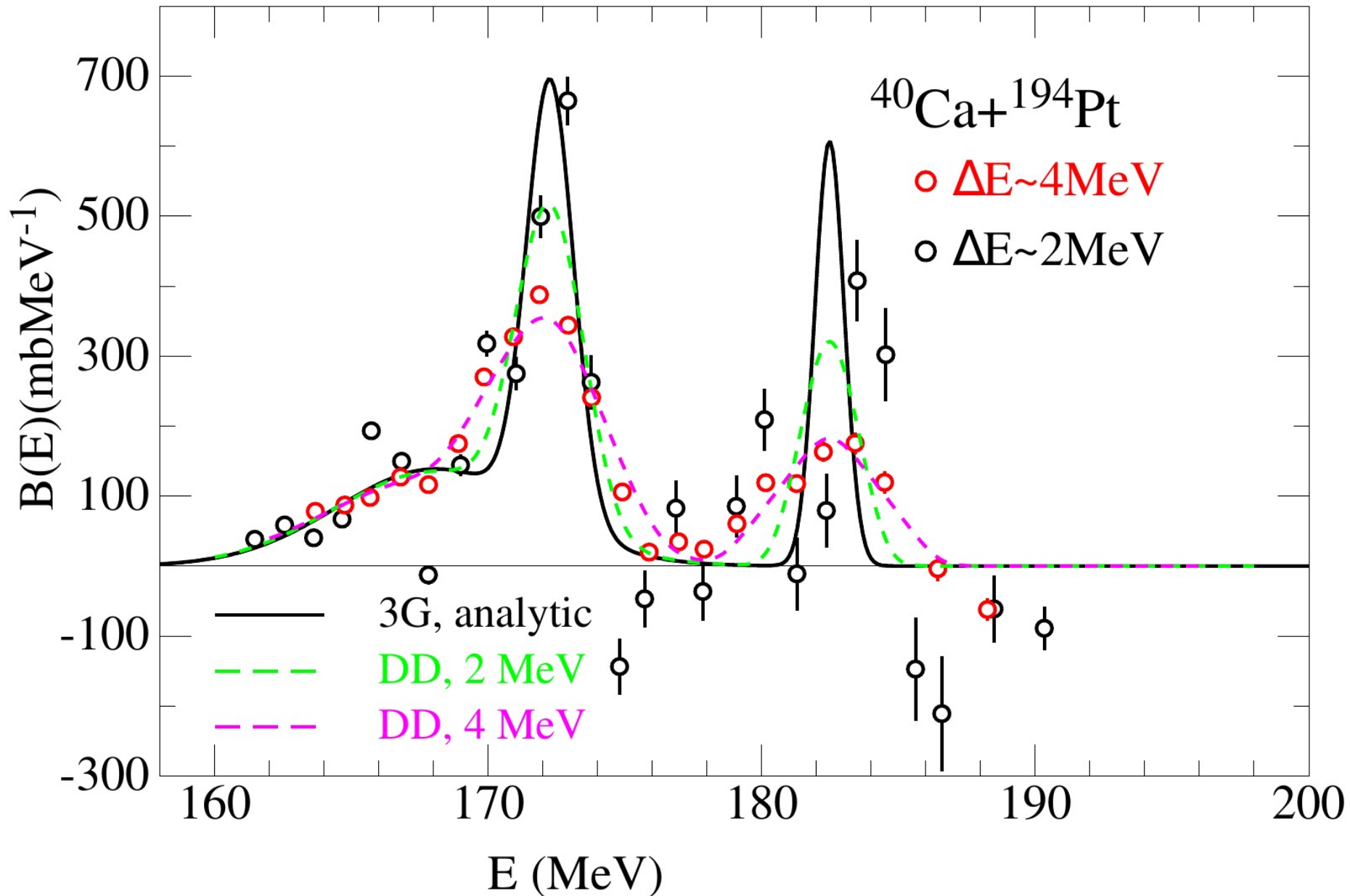
$$\chi^2 = \frac{N}{N - M} \chi_0^2$$

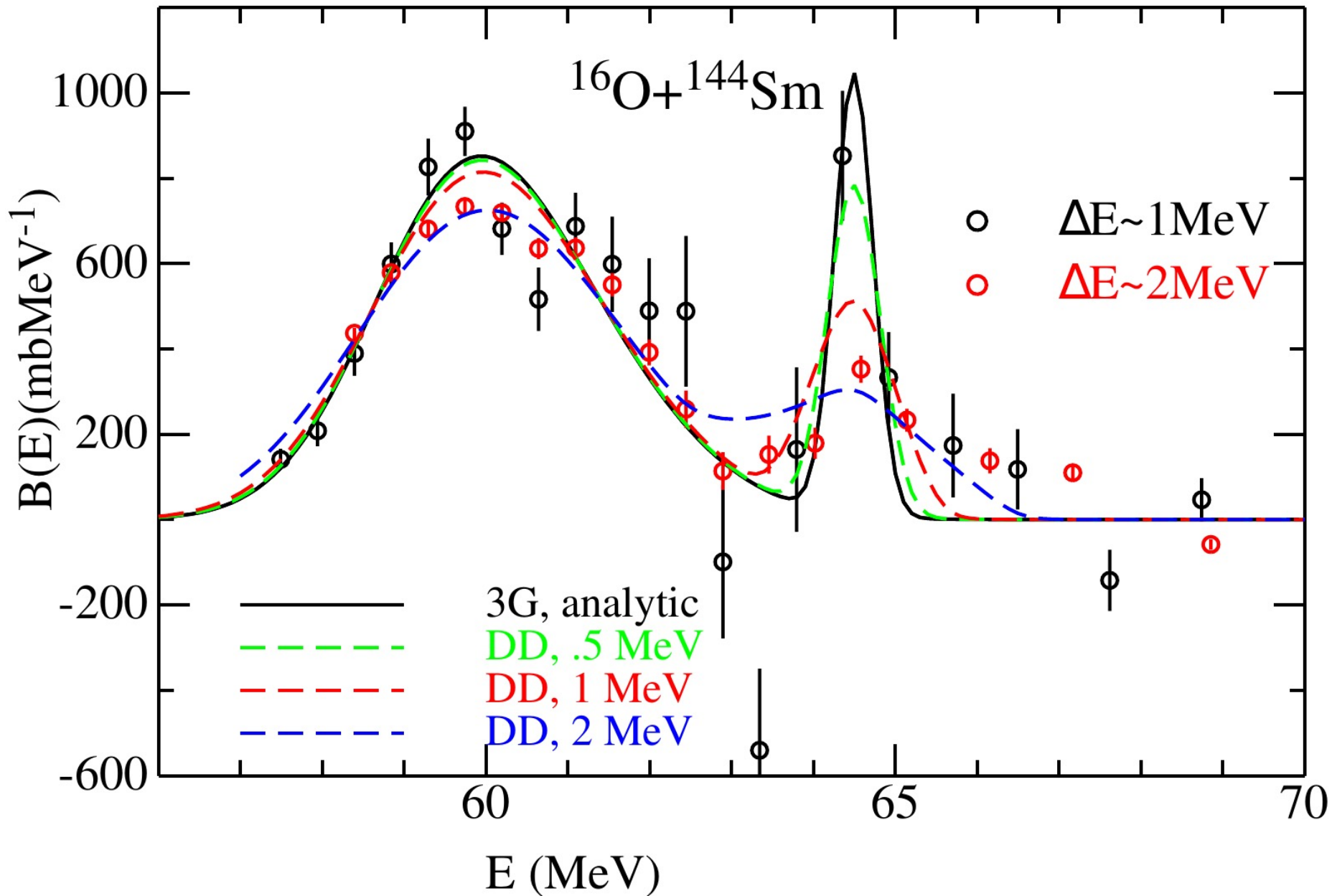
system	LG, M	$\chi_0^2$	$\chi^2$	system	LG, M	$\chi_0^2$	$\chi^2$
$^{40}\text{Ca} + ^{194}\text{Pt}$ $N=31$	1G, 3	41.9	46.4	$^{16}\text{O} + ^{144}\text{Sm}$ $N=27$	1G, 3	21.8	24.5
	2G, 6	37.5	46.5		2G, 6	3.3	4.3
	3G, 9	11.0	15.5		3G, 9	3.2	4.8
$^{16}\text{O} + ^{154}\text{Sm}$ $N=37$	1G, 3	13.0	14.1	$^{40}\text{Ca} + ^{192}\text{Os}$ $N=45$	1G, 3	16.7	17.9
	2G, 6	3.0	3.5		2G, 6	15.3	17.7
	3G, 9	2.9	3.9		3G, 9	14.8	18.5
	4G, 12	2.5	3.7		4G, 12	13.8	18.2

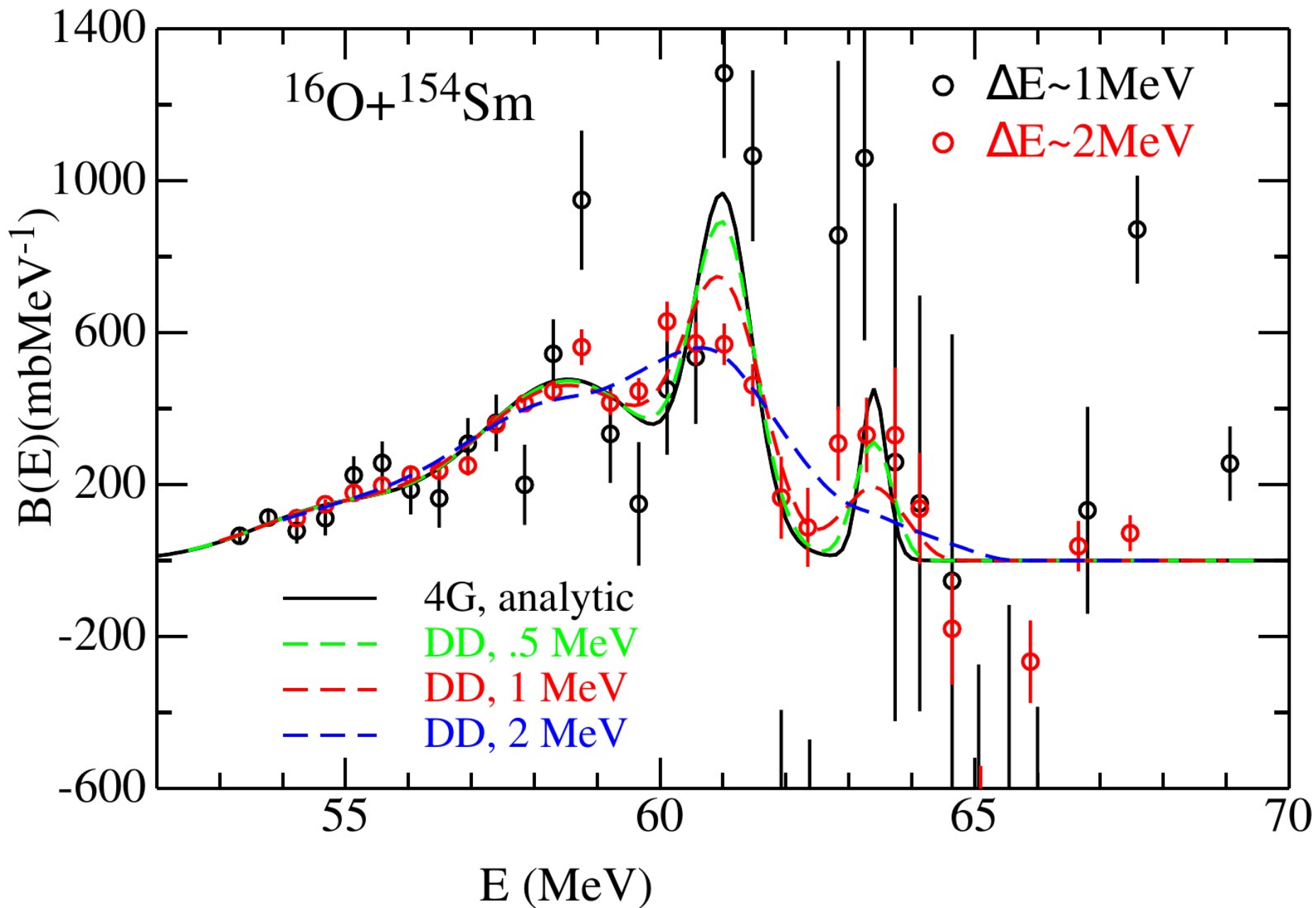


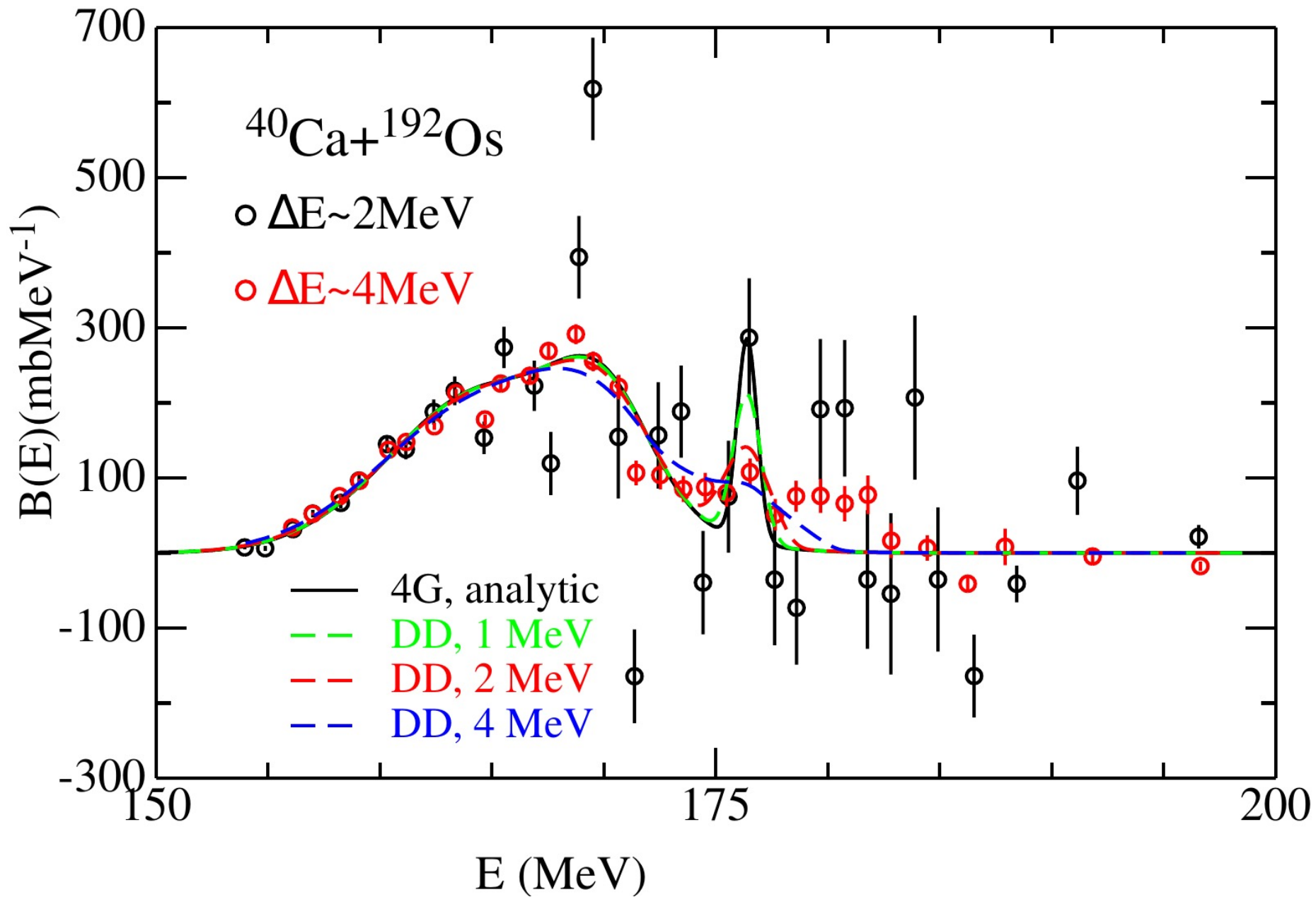














In Summary:

Two simple analytic cross section formulas:

Modified-Wong,

Single-Gaussian barrier distribution;

reproduce well the hindrance behavior

An improved method to obtain barrier

distribution:

three- or four-Gaussian barrier distribution

## Collaborators:

Ernst Rehm

Birger Back

Henning Esbensen

Kouichi Hagino

Alberto Stefanini

Giovanna Montagnoli

Benjamin Kay

Daniel Santiago

# Thank You !