Università degli Studi di Padova · Université Paris-Saclay · IPhT Saclay · Université Franco Italienne / Università Italo Francese · CEA · INFN

# Analytic Gravitational Waveforms in General Relativity from Scattering Amplitudes

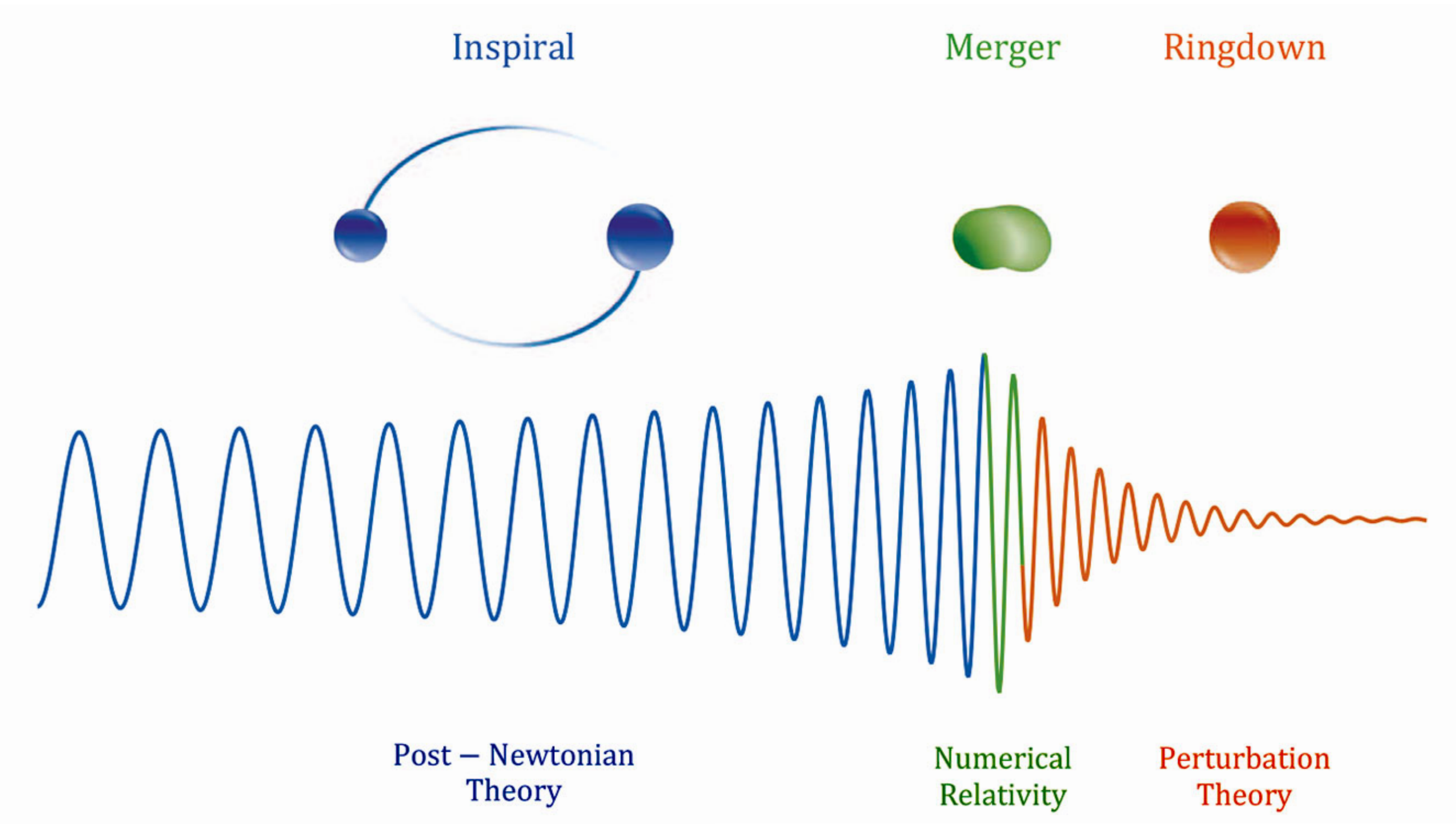
Giacomo Brunello

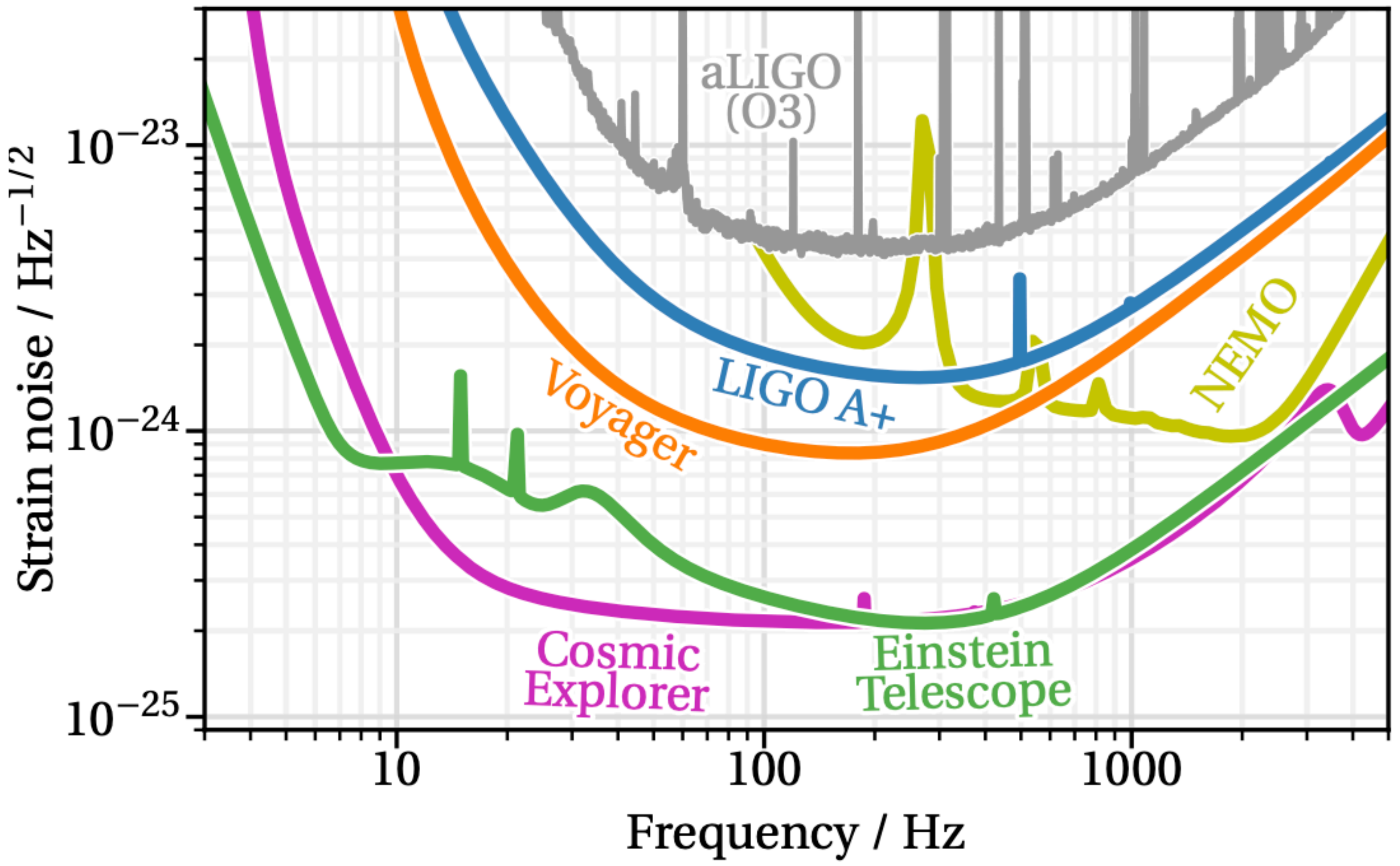
- G.B., S. De Angelis, D. A. Kosower [to appear]
- G.B., S. De Angelis [2403.08009]
- G.B., G. Crisanti, M. Giroux, P. Mastrolia, S. Smith [2311.14432]

Domoschool, Amplitools,
Domodossola, 17/07/2025

# Precision Era of Gravitational Waves Physics

Ligo-Virgo-Kagra efficiently detect GWs emitted by Coalescing Binary Systems.



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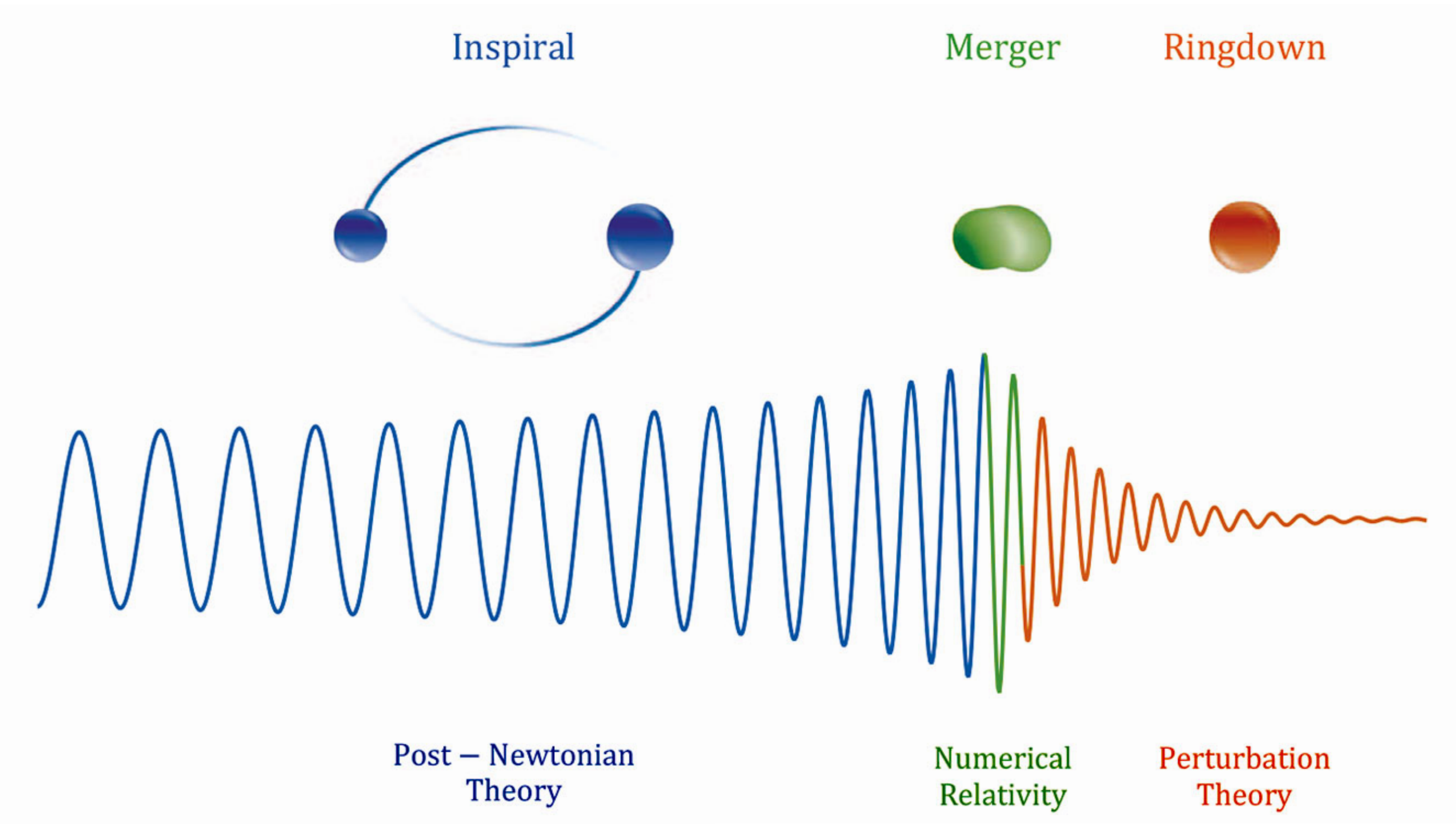


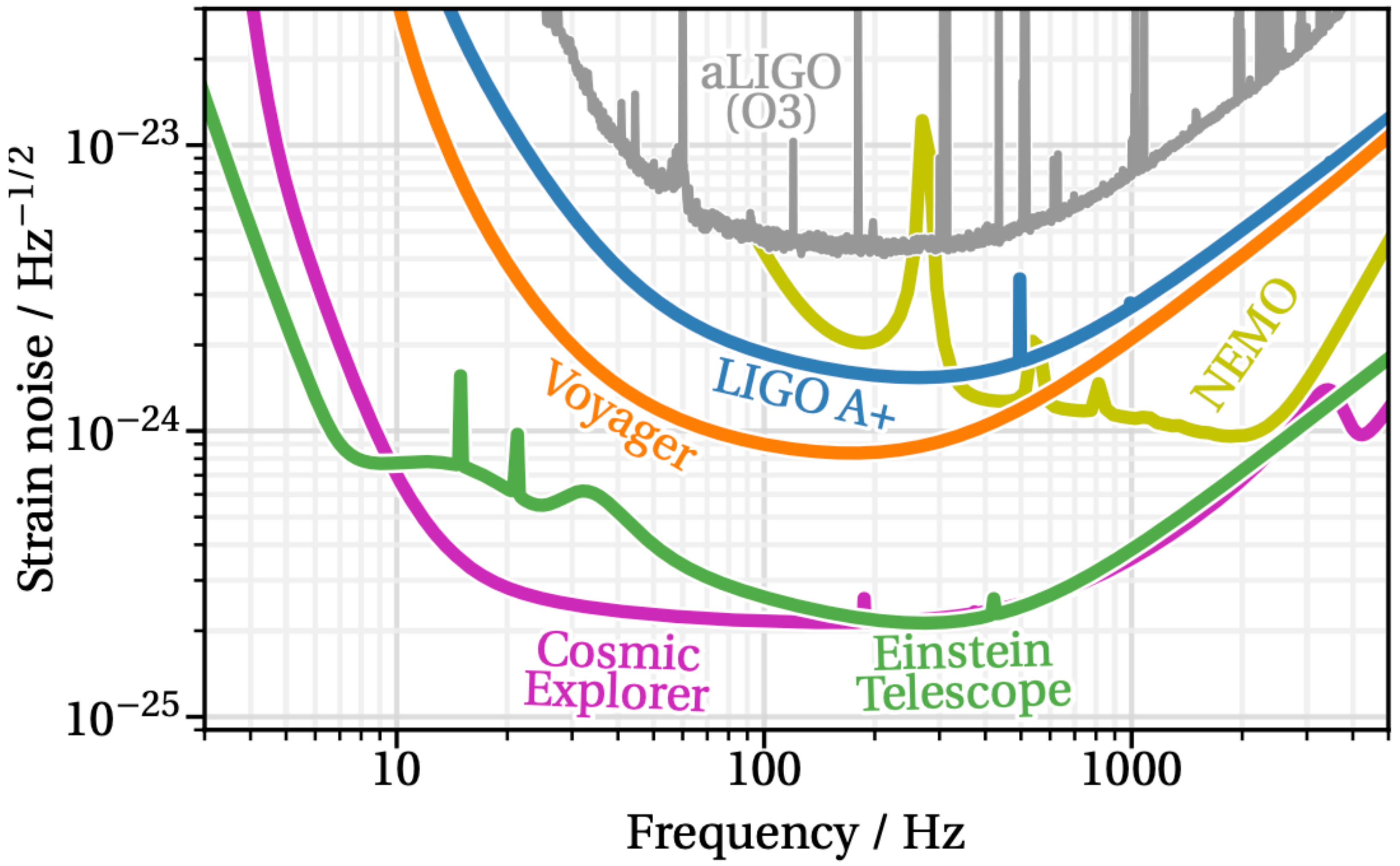


Extreme need for precise theoretical predictions for signal templates for Matched filtering analyses

# The gravitational waveform

Scattering Waveform emitted by a system of two scattering black holes

$$h_{\mu\nu}\Big|_{|x|\to\infty} = \eta_{\mu\nu} + \frac{\kappa^2 M}{8\pi|x|}\left[\frac{\kappa^2 M}{\sqrt{-b^2}}\hat{h}^{(1)}_{\mu\nu} + \left(\frac{\kappa^2 M}{\sqrt{-b^2}}\right)^2 \hat{h}^{(2)}_{\mu\nu} + \dots\right]$$

LO result:

Kovacs, Thorne
Jakobsen, Mogull, Plefka,Steinhoff
Mougiakakos, Riva, Vernizzi
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Spin:

Jakobsen, Mogull, Plefka,Steinhoff
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Template for matched filtering analyses

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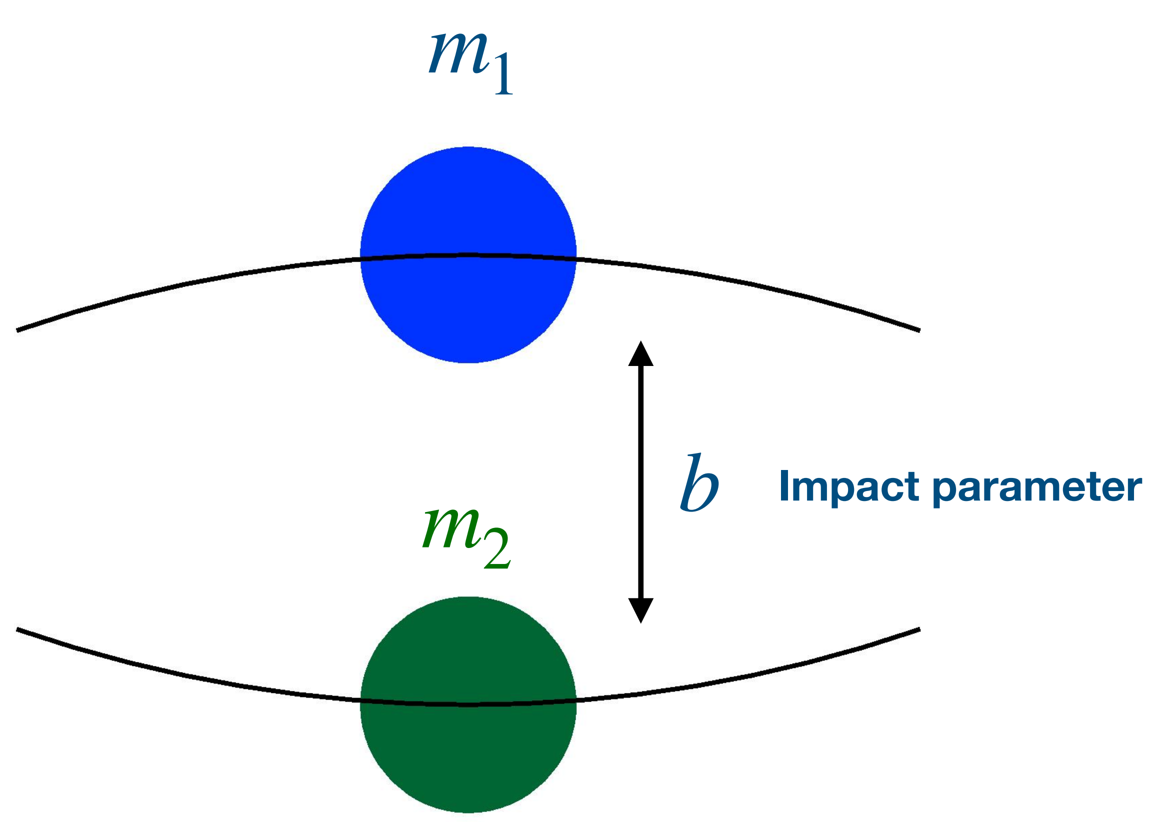
Template for matched filtering analyses

Properties of higher-points scattering amplitudes

# Classical Physics from Scattering Amplitudes



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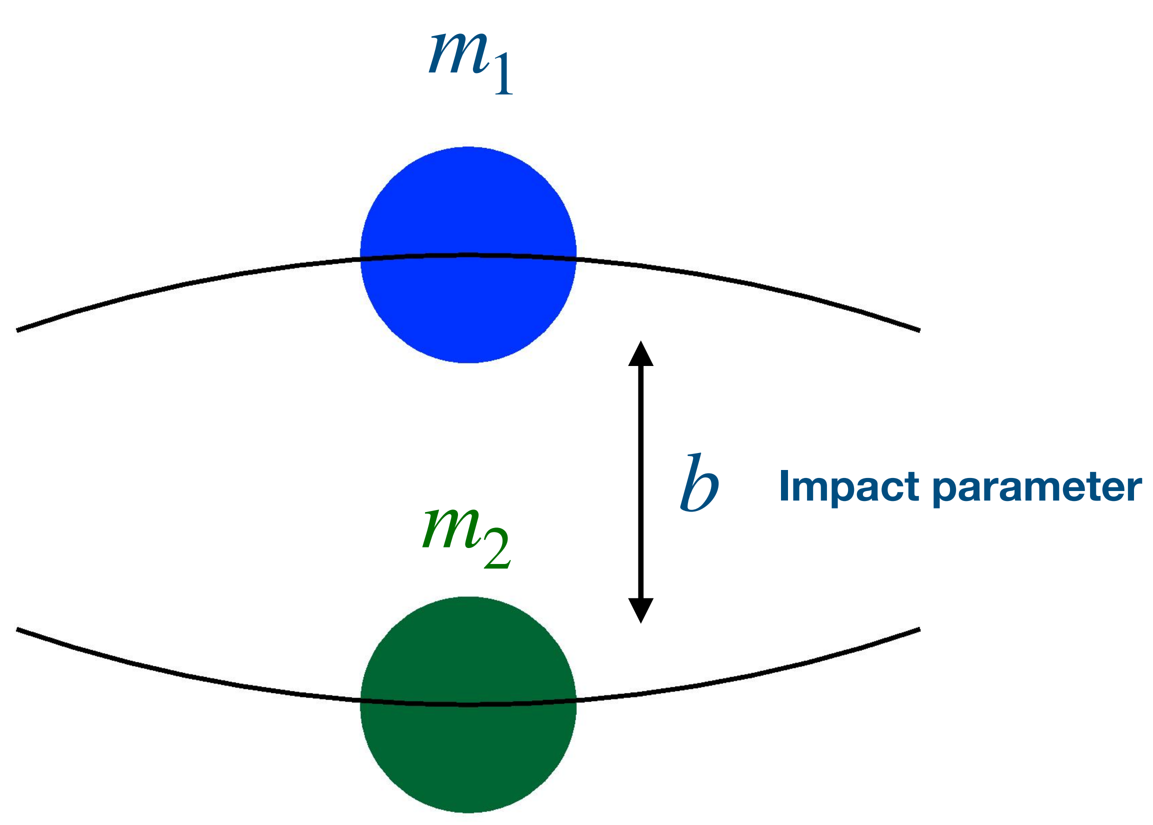


Effective Field Theory approach: Black-holes as point particles Goldberger, Rothstein

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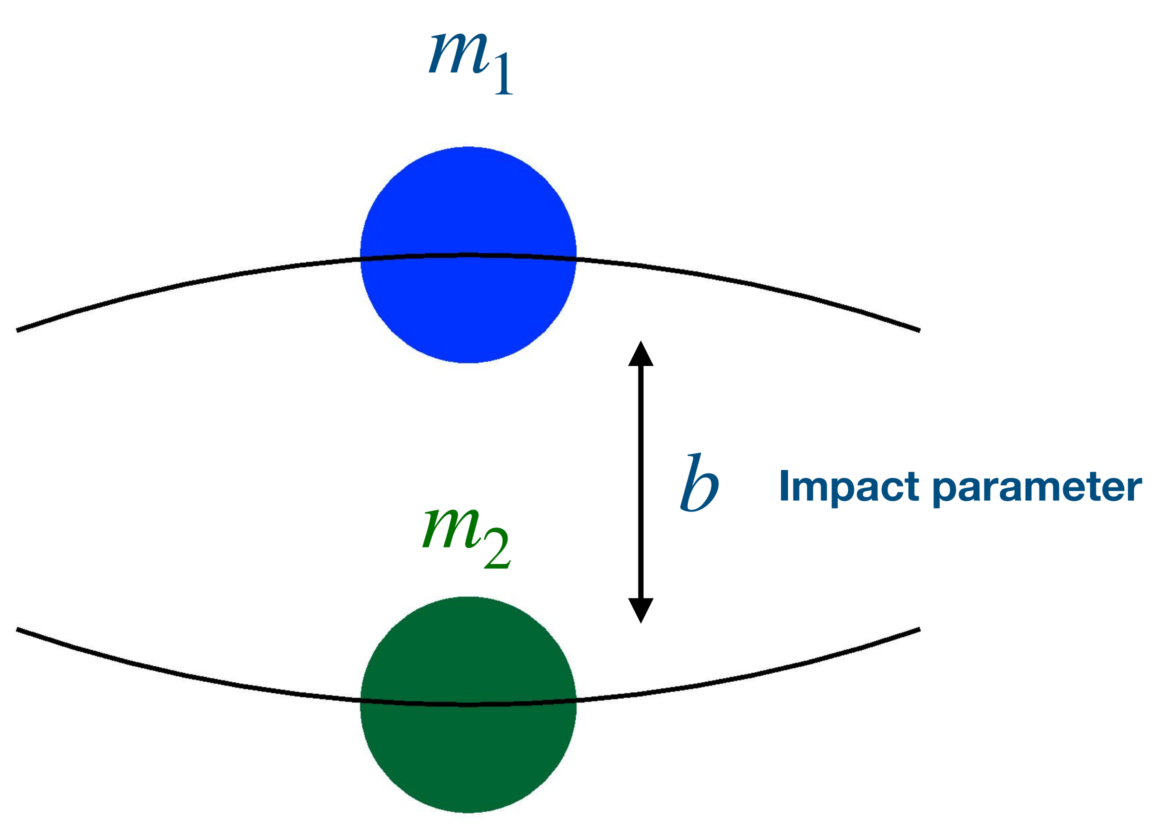


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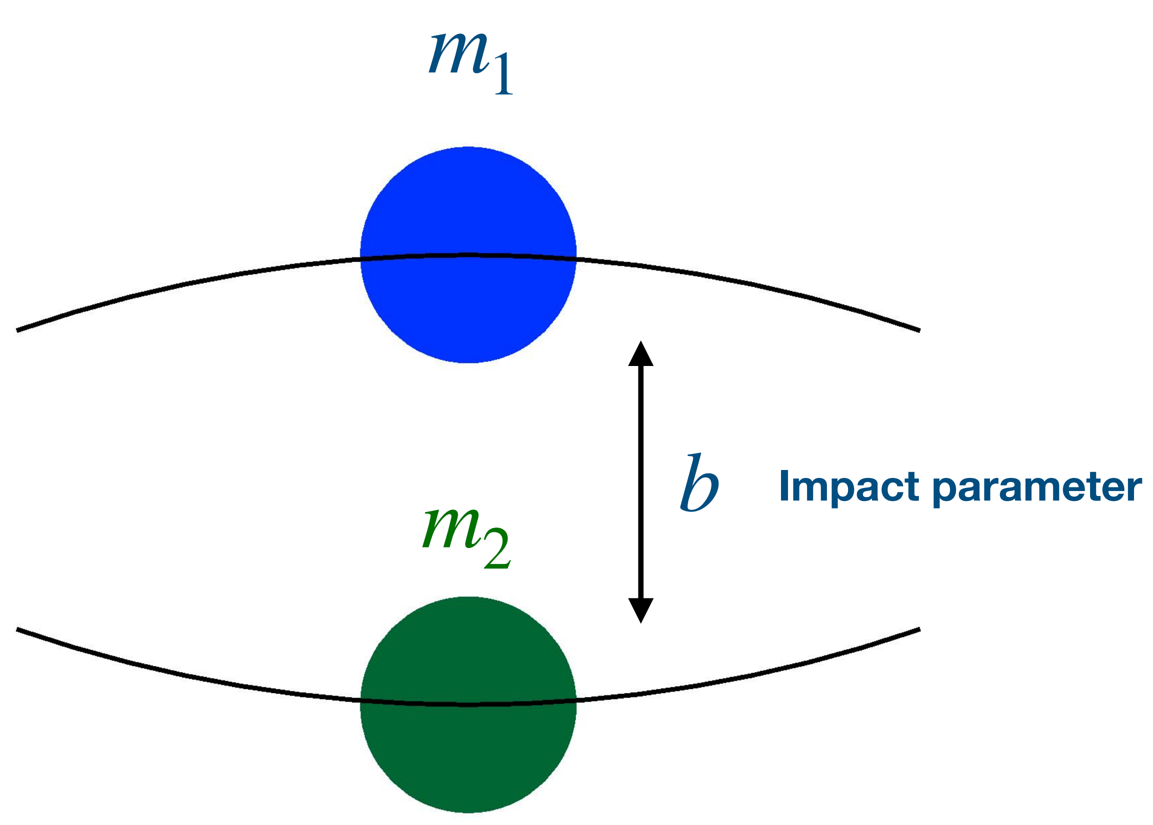


Effective Field Theory approach: Black-holes as point particles   Goldberger, Rothstein

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The scales of the problem

$$\frac{1}{m} \ll G\, m \ll |b| \ll r$$

Compton wavelength — Schwarzschild radius — Impact parameter — Distance

$$\frac{Gm}{b}$$

Post-Minkowskian expansion parameter

# Physical Observables from Scattering Amplitudes

Kosower, Maybee, O'Connell

Asymptotic states for two compact objects: wavefunctions peaked around their classical values

$$|\psi\rangle_{in} = \int d\phi(p_1)d\phi(p_2)\ \phi_1(p_1)\phi_2(p_2)\ e^{i(b_1\cdot p_1 + b_2\cdot p_2)}\ |p_1, p_2\rangle_{in} \qquad |\psi\rangle_{out} = S|\psi\rangle_{in} \qquad S = 1 + i\ T$$

On-shell phase space integral

wavefunction

Two particle momentum eigenstates

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Expectation value of a physical observable

$$\Delta\langle\mathcal{O}\rangle = {}_{out}\langle\psi|\,\mathcal{O}\,|\psi\rangle_{out} - {}_{in}\langle\psi|\,\mathcal{O}\,|\psi\rangle_{in} = {}_{in}\langle\psi|\,S^\dagger[\mathcal{O}, S]\,|\psi\rangle_{in}$$

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On-shell phase space integral — wavefunction — Two particle momentum eigenstates

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$$\mathcal{O} = \mathcal{P}_1^\mu$$

Impulse

$$\mathcal{O} = \mathcal{W}_{GR} = \epsilon_h^{\mu\nu}h_{\mu\nu} \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$$

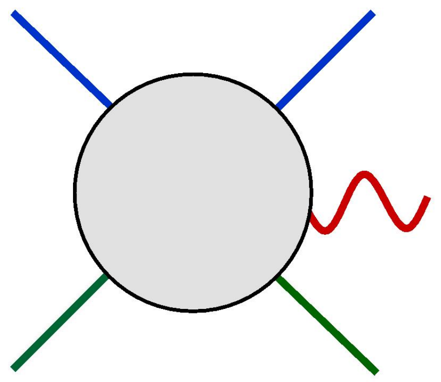
Waveform

# Scattering Waveforms from Amplitudes

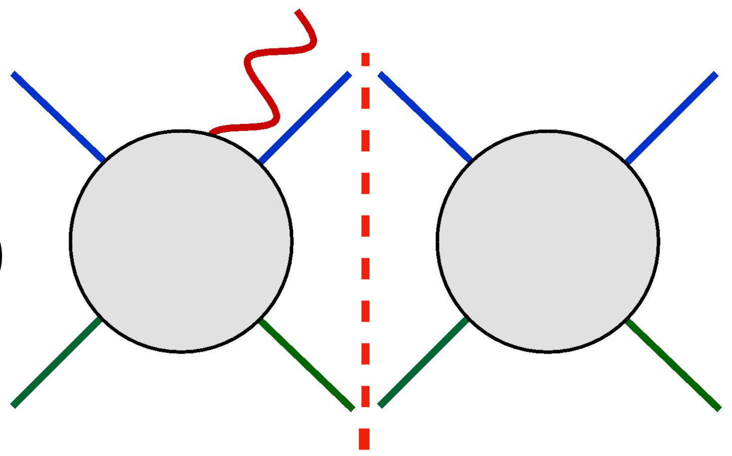
Cristofoli, Gonzo, Kosower, O'Connell

The Fourier transform of $2 \to 3$ scattering amplitudes.

$$\Delta\langle \mathscr{W}_h\rangle(u,\vec{n}) = \frac{1}{4\pi r}\int_0^\infty \hat{d}\omega \int d\mu \Big\{ \hat{\delta}^D(q_1+q_2-k)e^{-i\omega u}\Big[$$



$$- i\int d(LIPS)$$



$$\Big] + c.c.\Big\}$$

Fourier Transform

$$d\mu = \prod_{i=1}^{2} \hat{d}^D q_i \delta(2p_i \cdot q_i + q_i^2) e^{ib_i \cdot q_i}$$

Five-points amplitude

$$\mathscr{A}(p_1 p_2 \to p_1' p_2' k^h)$$

Iteration terms

$$\mathscr{A}^*(\tilde{p}_1'\tilde{p}_2' \to \tilde{X}) \otimes \mathscr{A}(p_1 p_2 \to X k^{-h})$$

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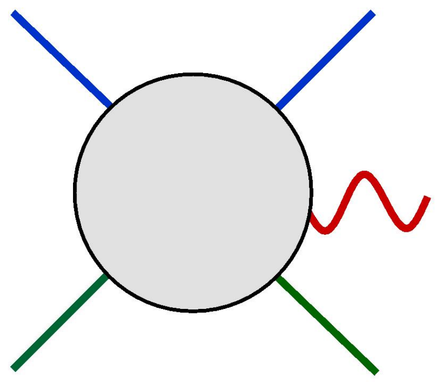

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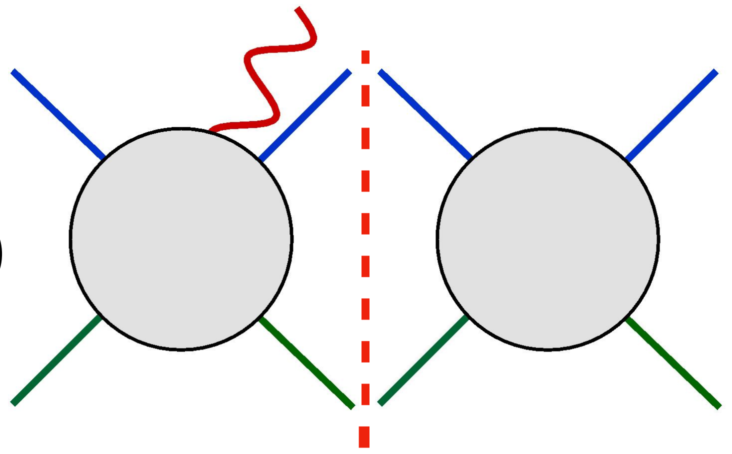

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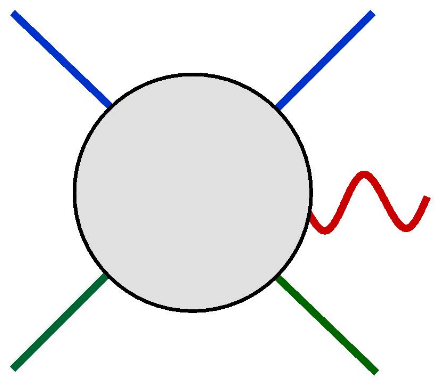

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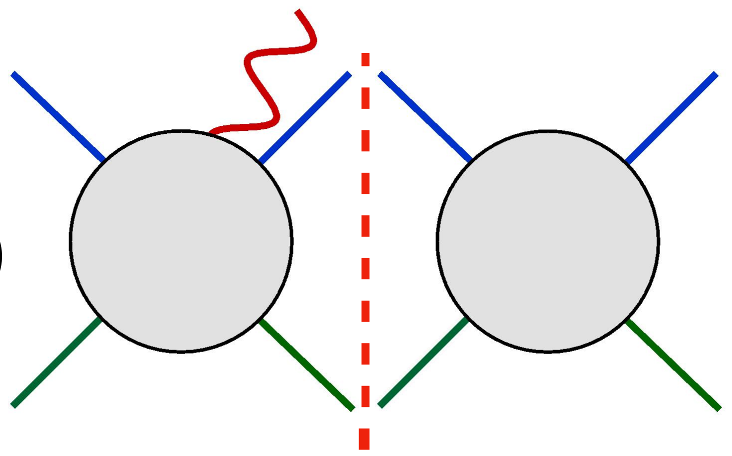

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Classically singular terms cancel between amplitude and iterations

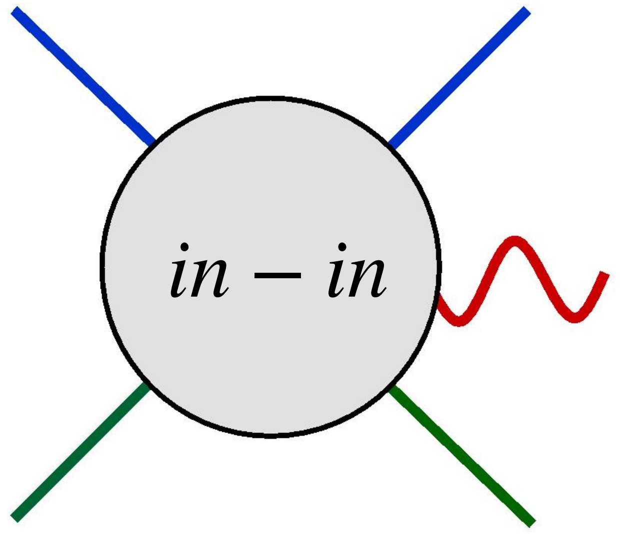
Scattering amplitudes are in-out: Iteration terms restore in-in prescription

Caron-Huot, Giroux, Hannesdottir, Mizera

# How to compute waveforms?

Loop and Fourier integration are intertwined together: loop-by-loop approach is very cumbersome

$$\Delta\langle \mathscr{W}_h\rangle(u,\vec{n}) = \int d\hat{\mu}\; e^{ib\cdot q}\Big\{$$

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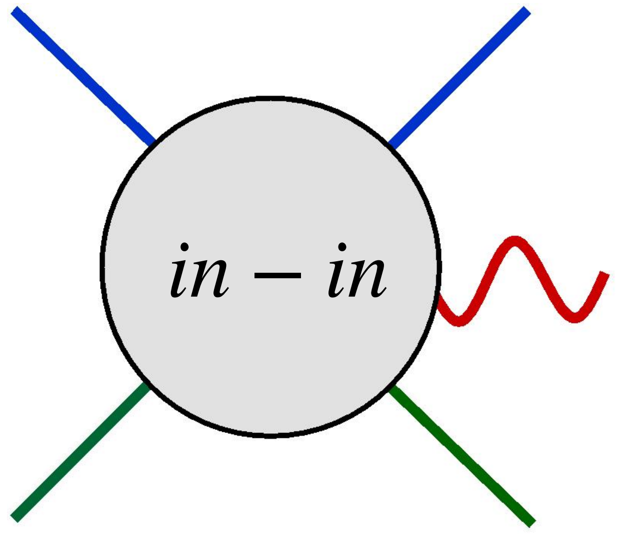


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GB, De Angelis

GB, De Angelis, Kosower

A combined Fourier-Loop approach: Scattering Amplitudes techniques in impact parameter space

**Integrand Generation**

**Integral Reduction**

**Integral Evaluation**

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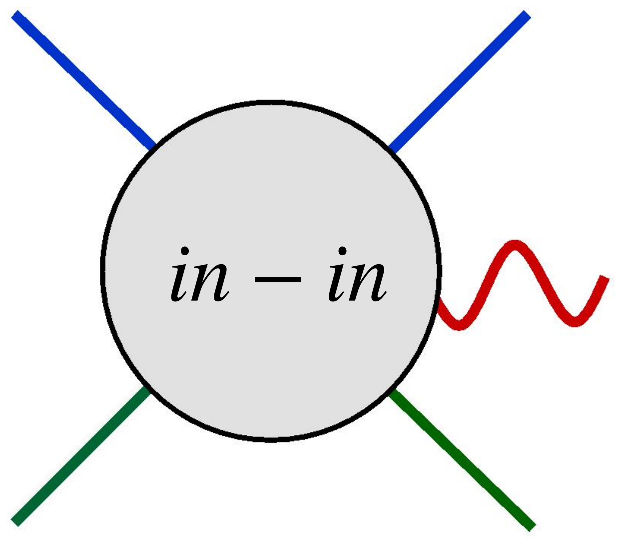


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Gravitational waveforms as a linear combination of basis functions

# Where are classical contributions?

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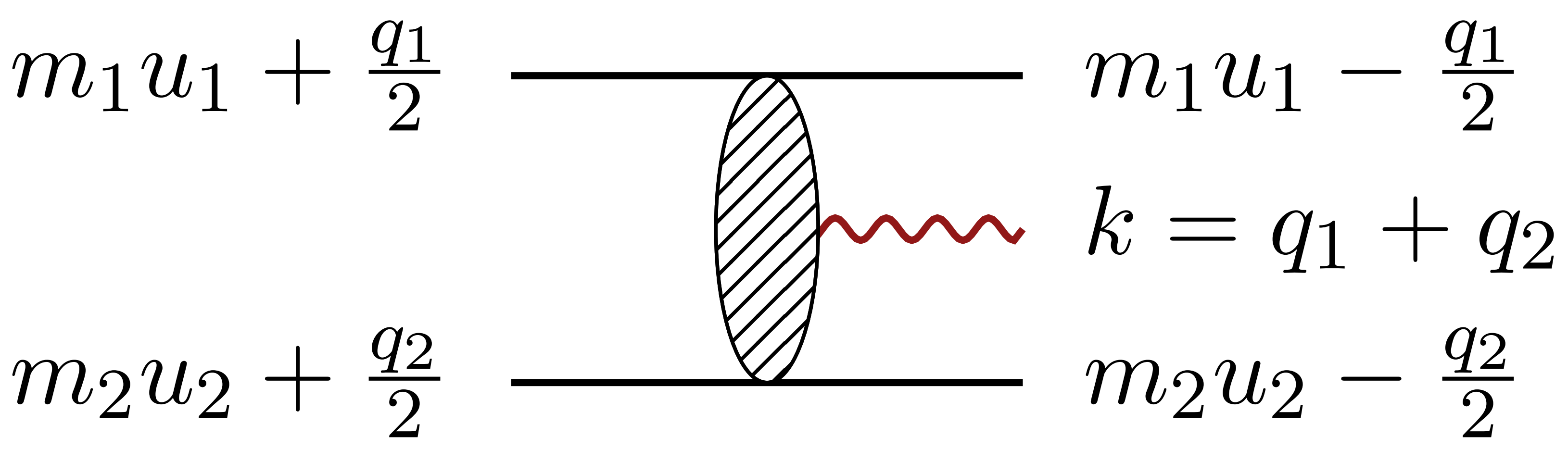
Five-points amplitudes




$$y = u_1 \cdot u_2 > 1$$

$$w_i = u_i \cdot k > 0$$

$$q_i^2 < 0$$

# Where are classical contributions?

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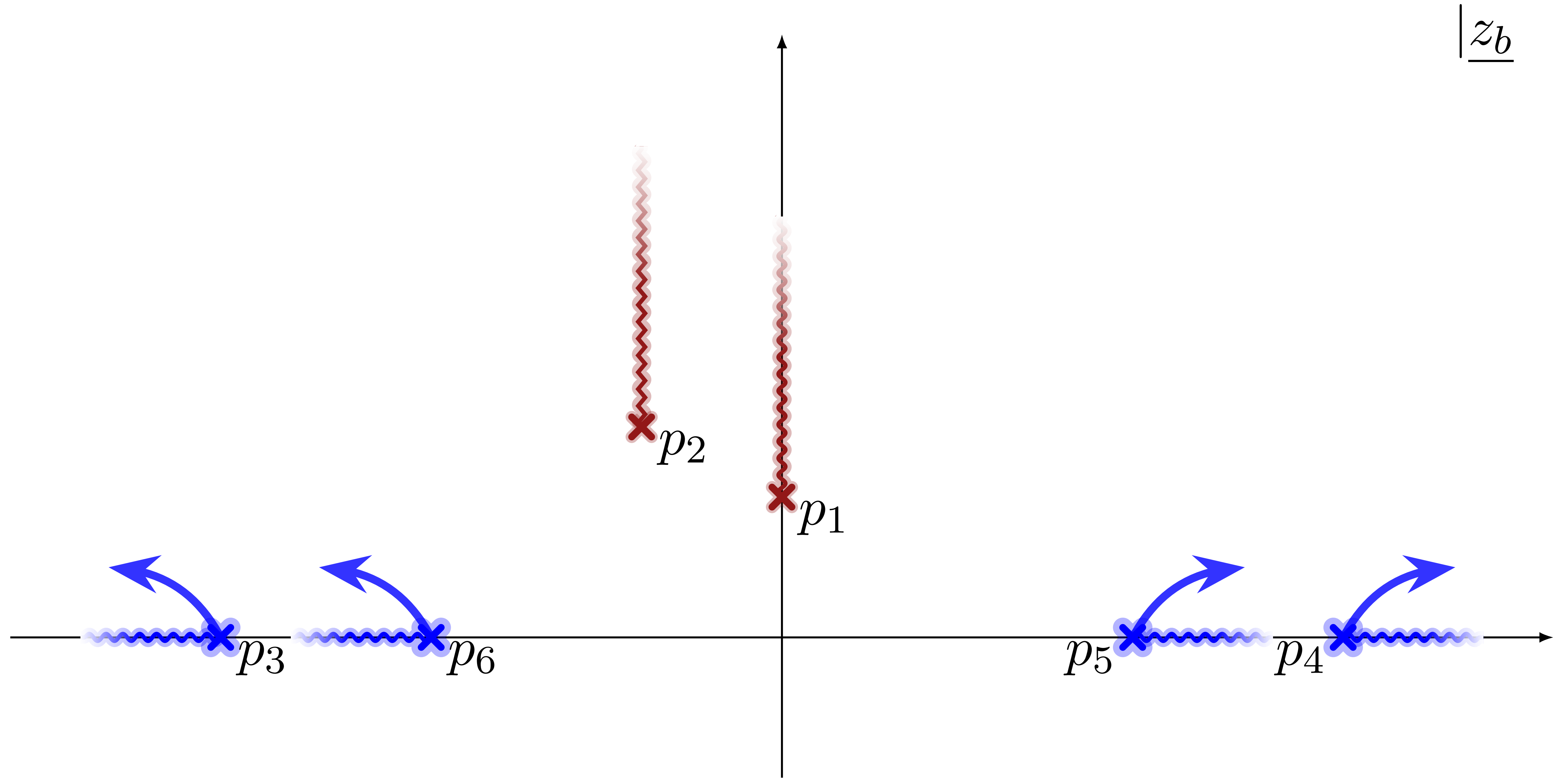
The singularities of five-points scattering amplitudes



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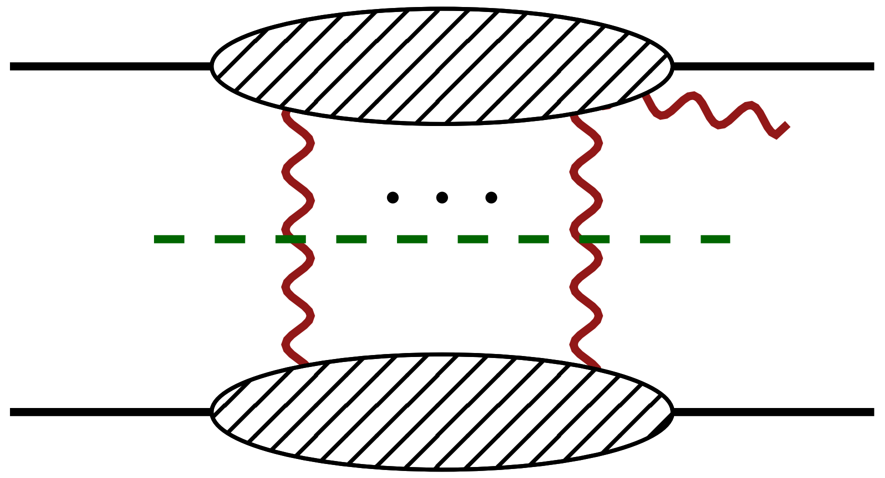
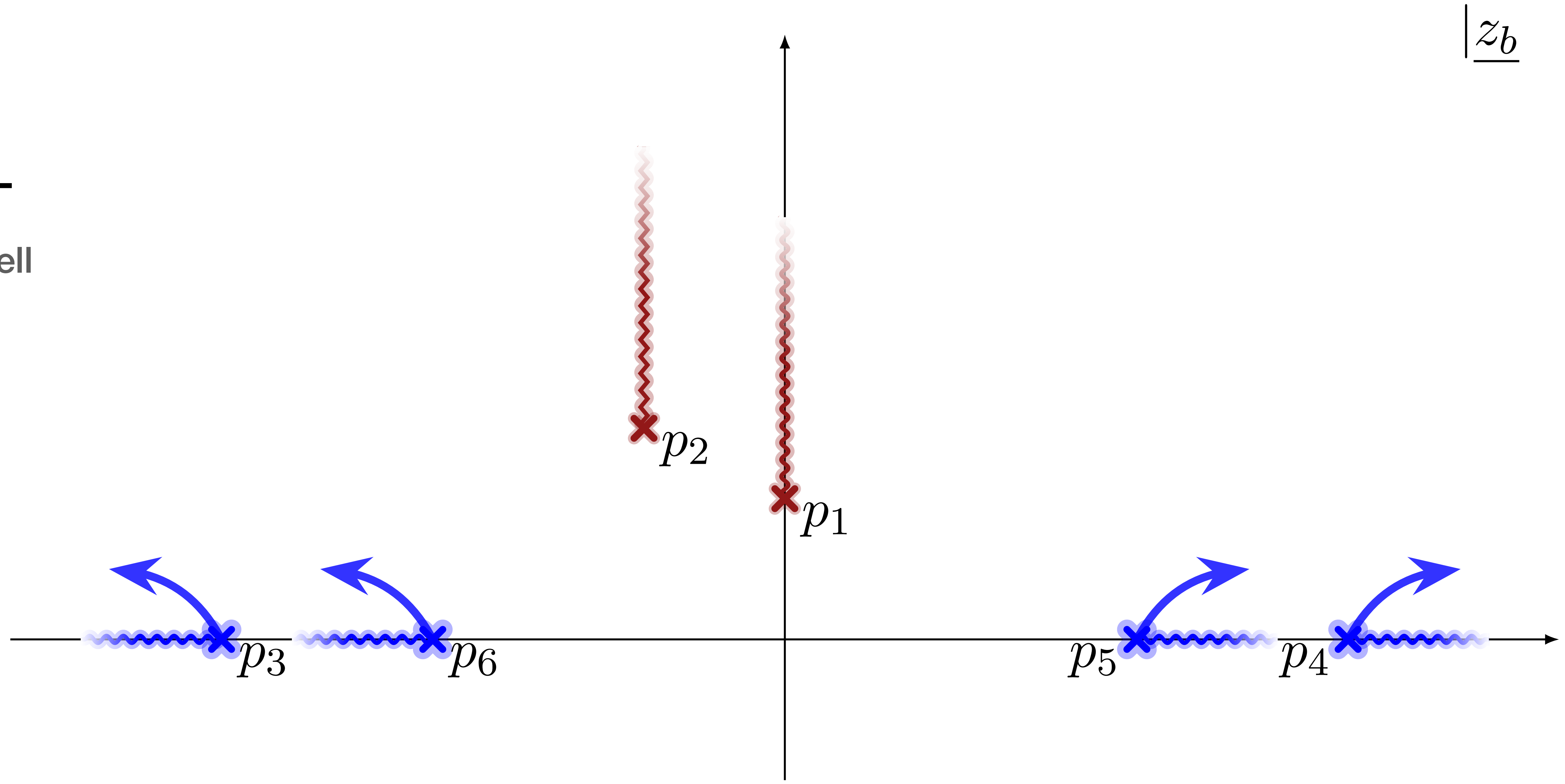


Intermediate gravitons going on-shell

$q_i^2 = 0$



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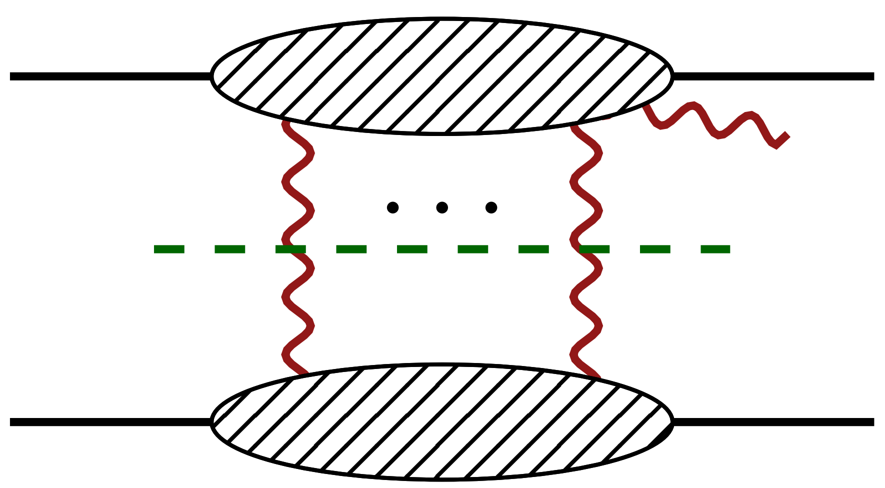

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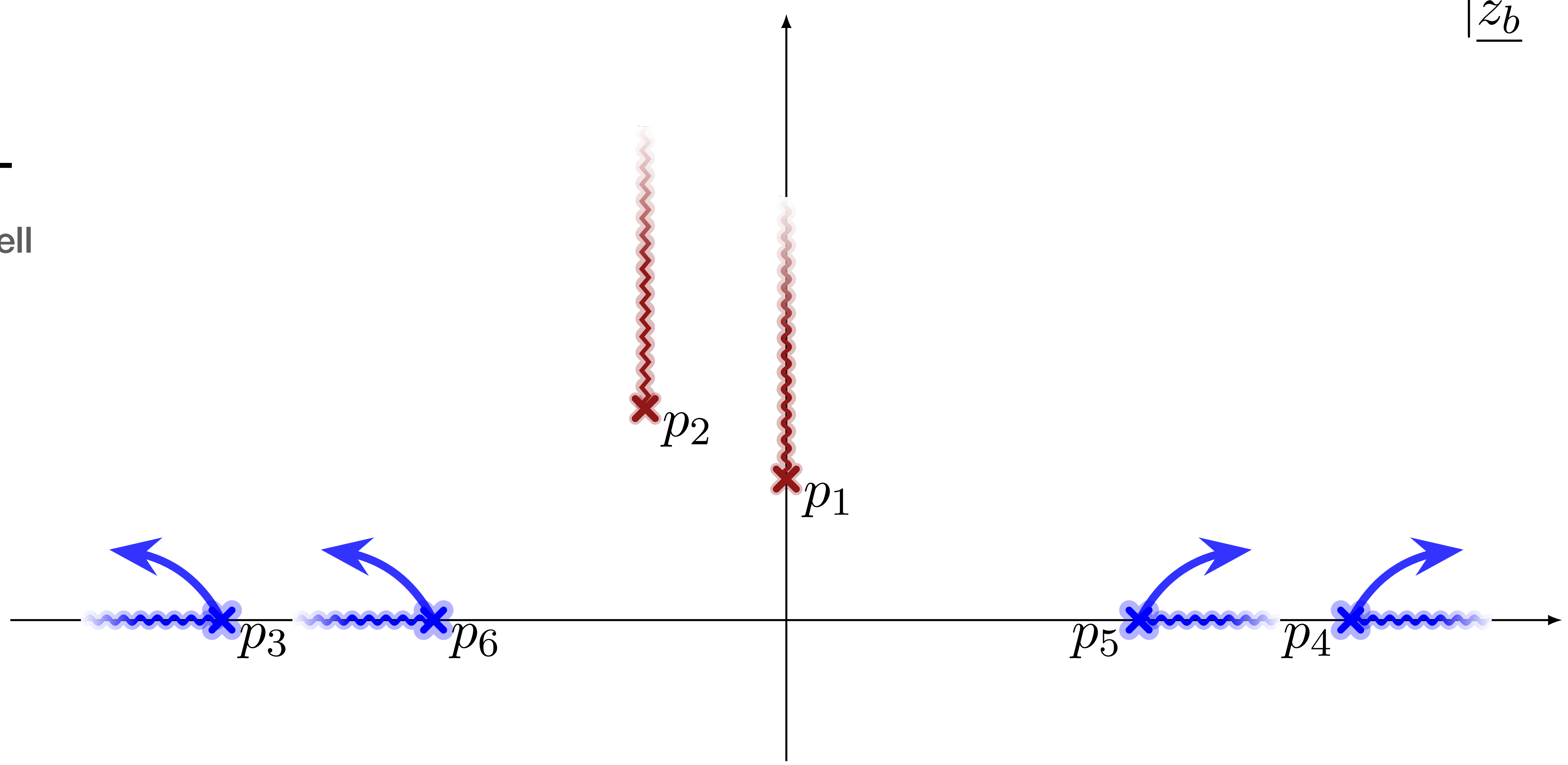


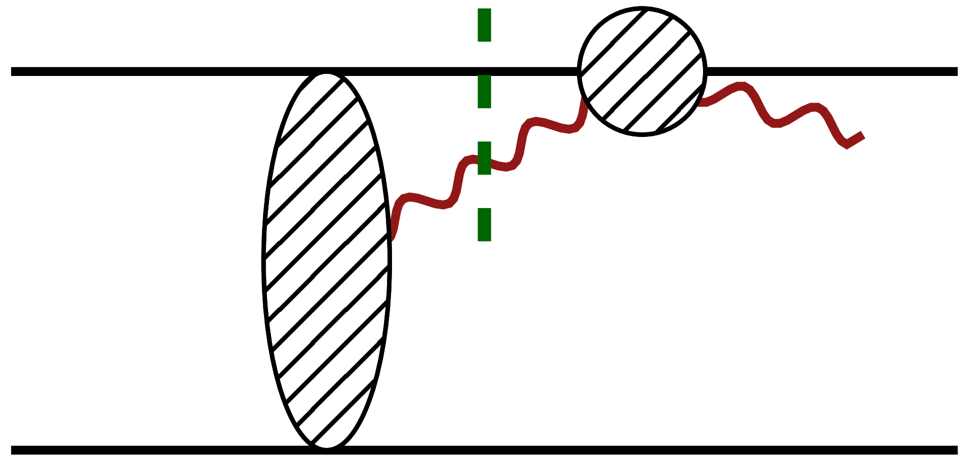

Massive (classical) particles going on-shell

$$\left(m_i u_i \pm \frac{q_i}{2} - k\right)^2 - m_i^2 = 0$$

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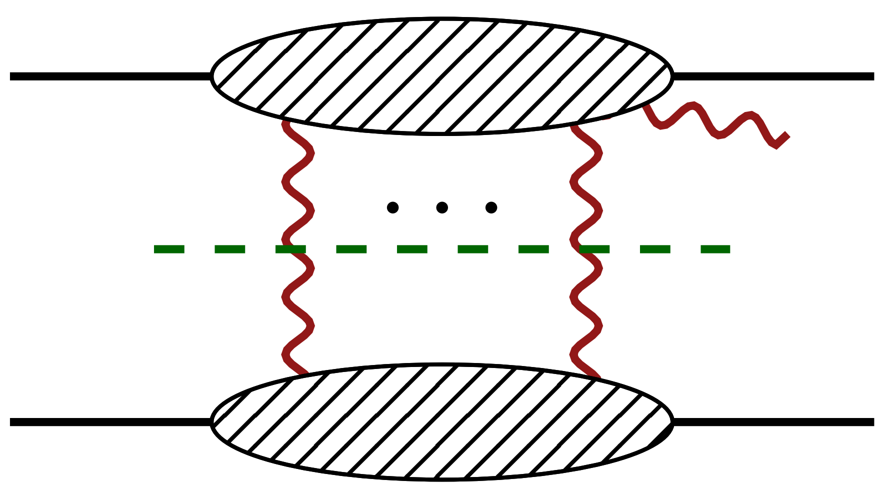

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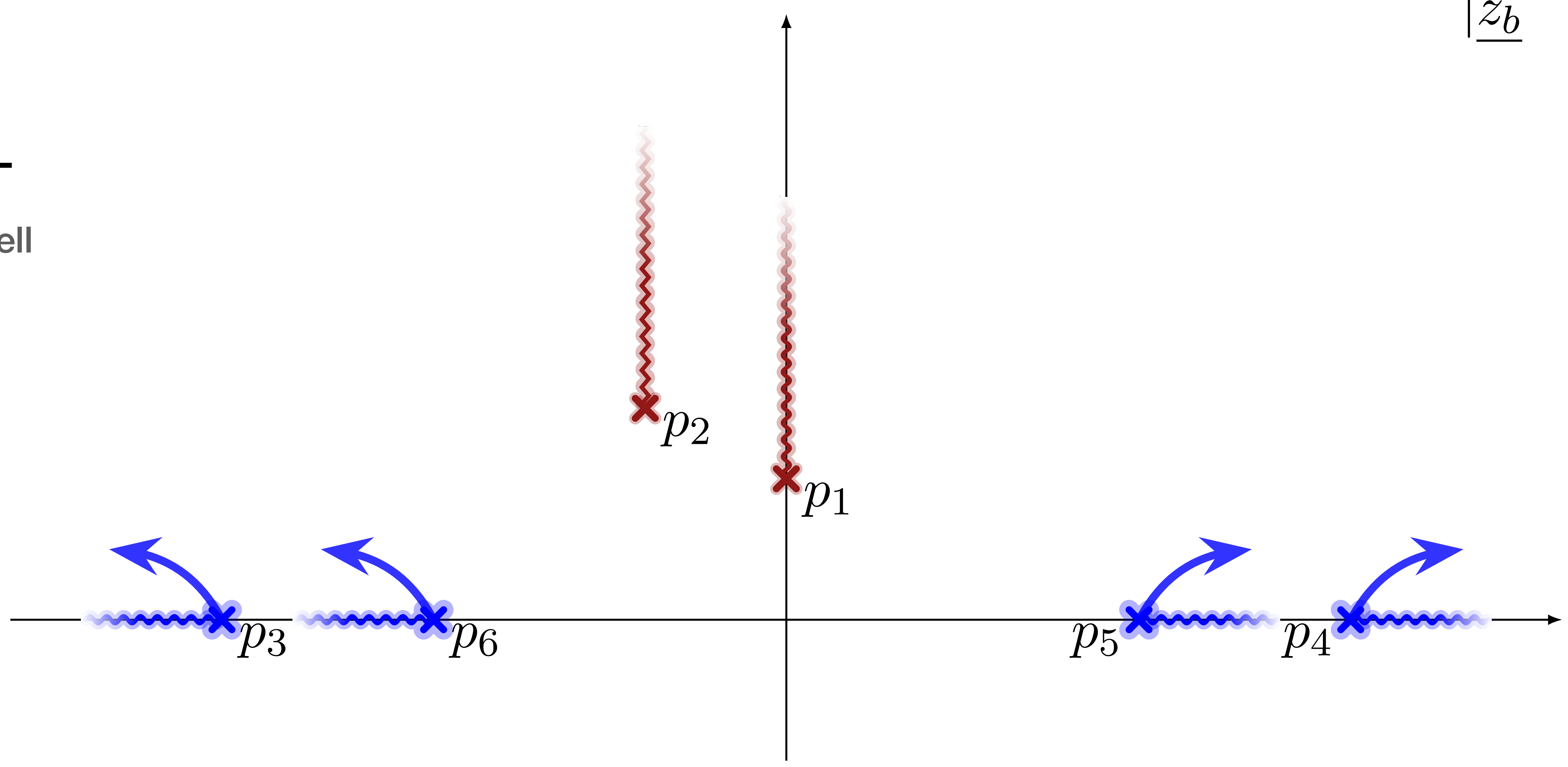


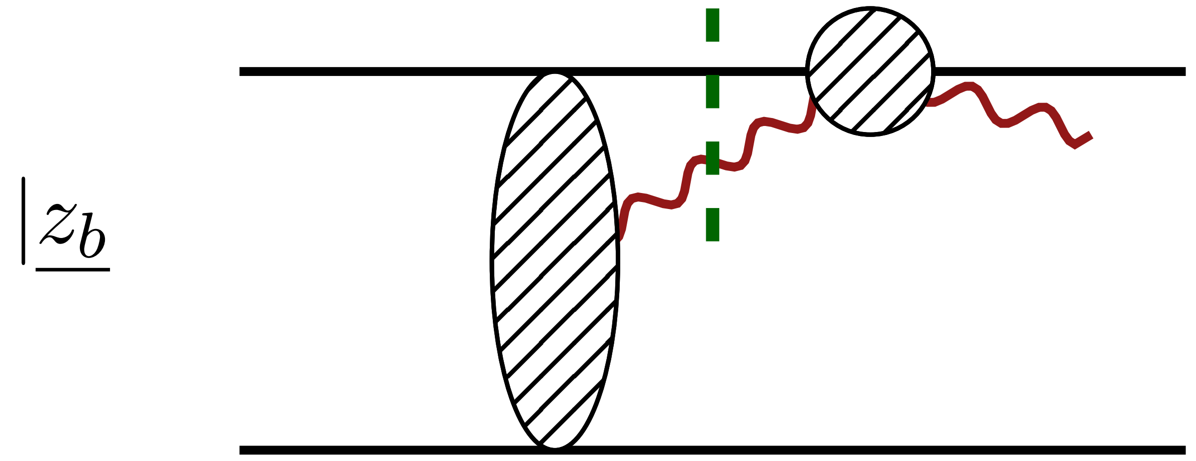

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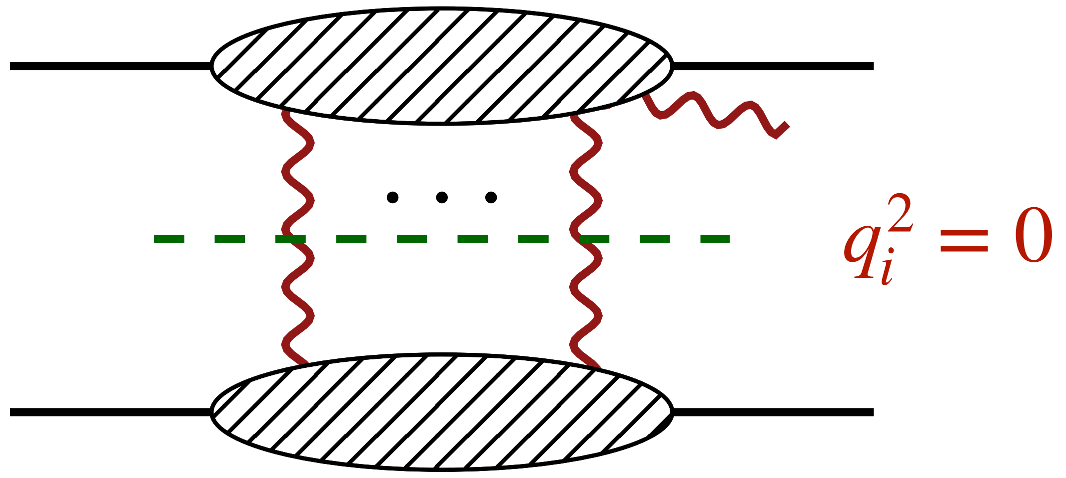
Classical limit captured by on-shell gravitons, corresponding to long-range interactions

# Integrand generation: diagrammatics

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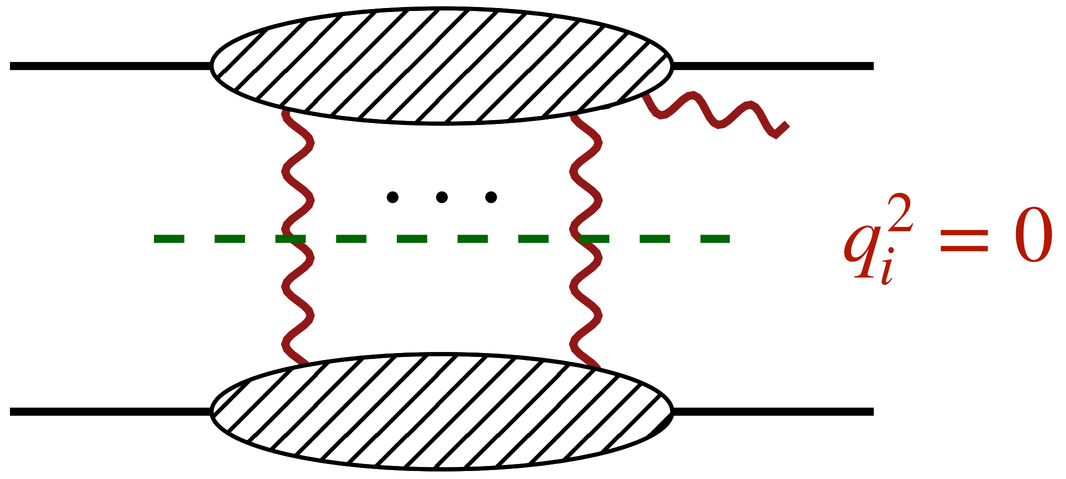


Intermediate gravitons going on-shell

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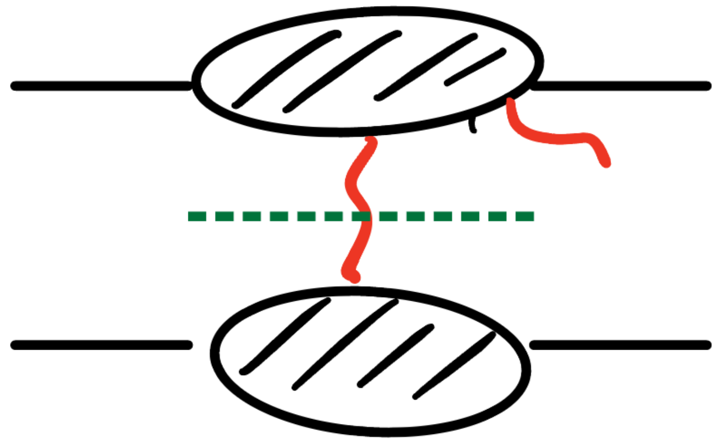

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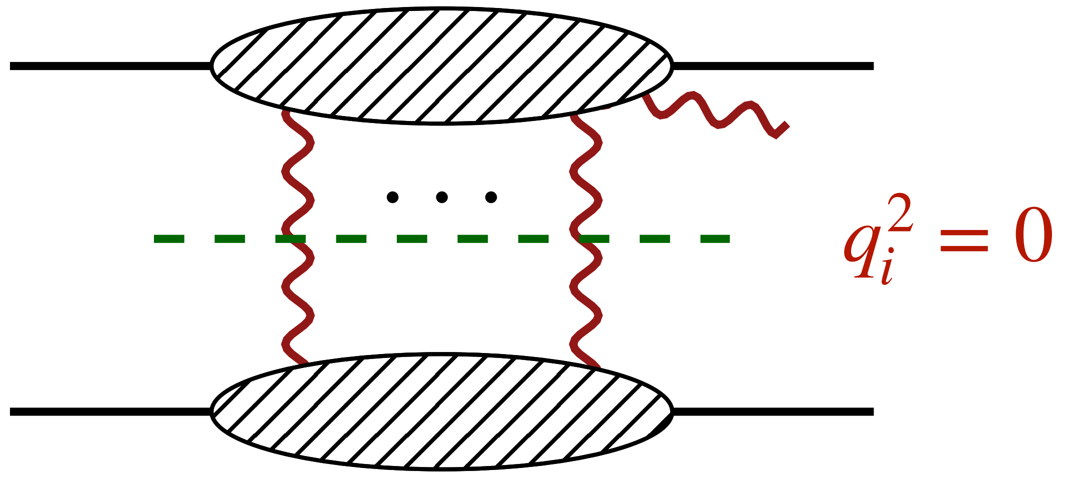


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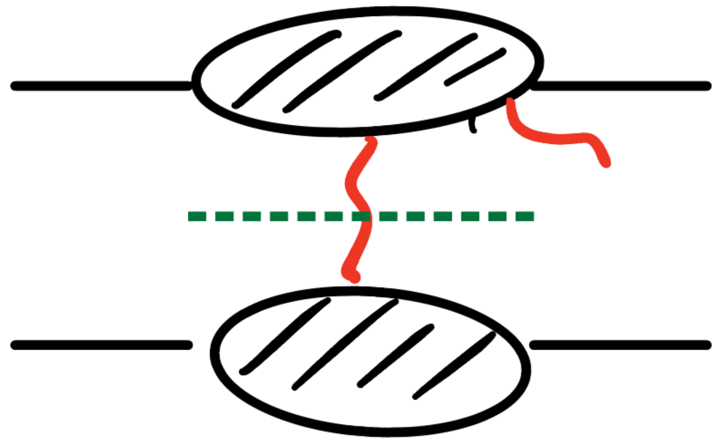

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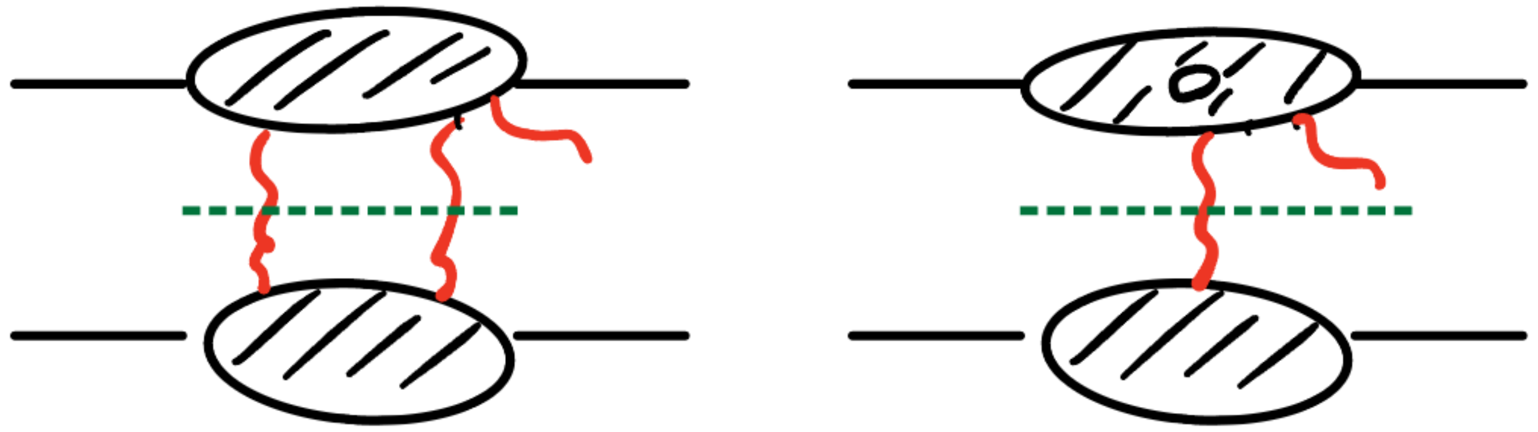

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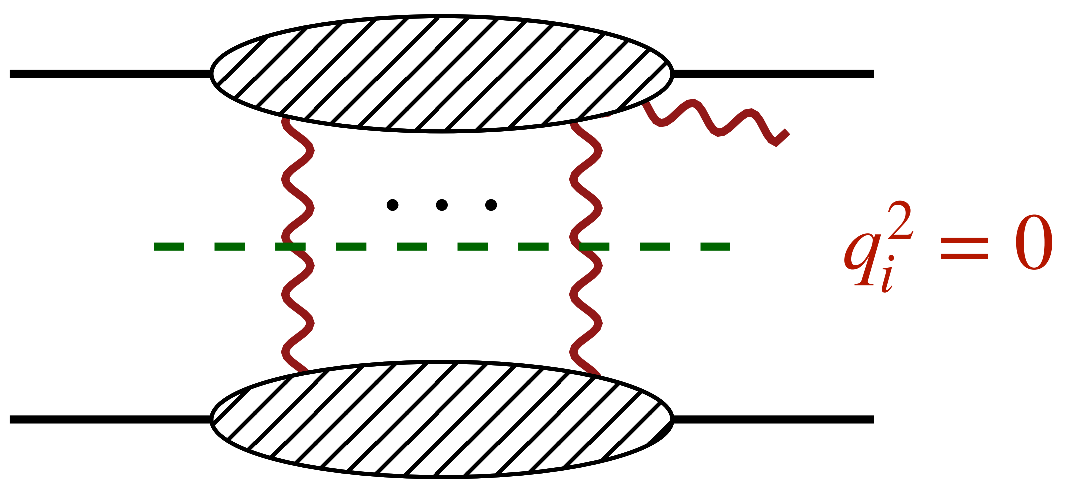


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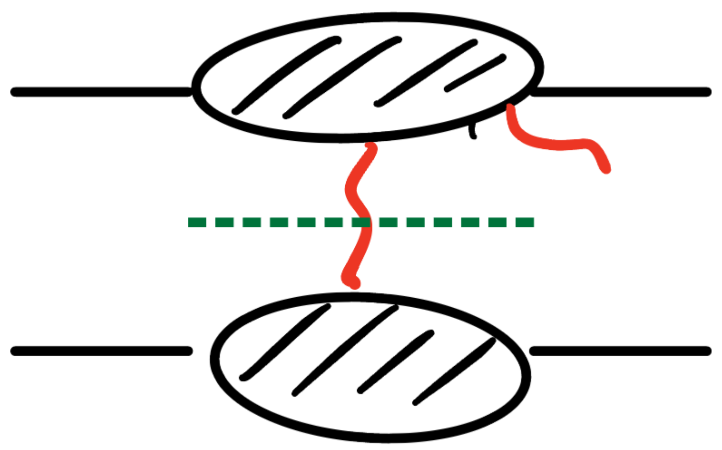

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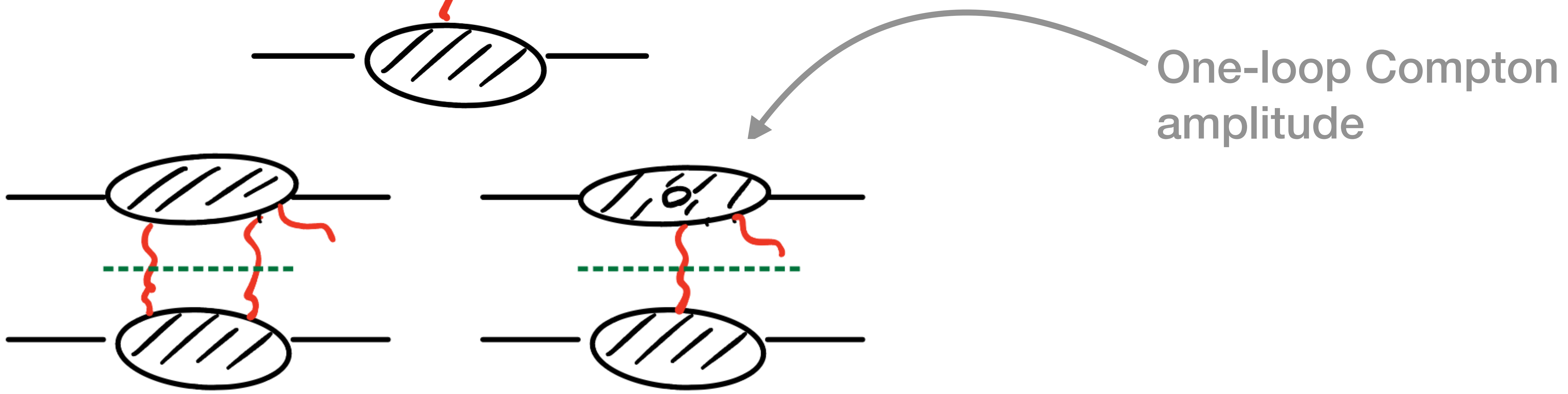

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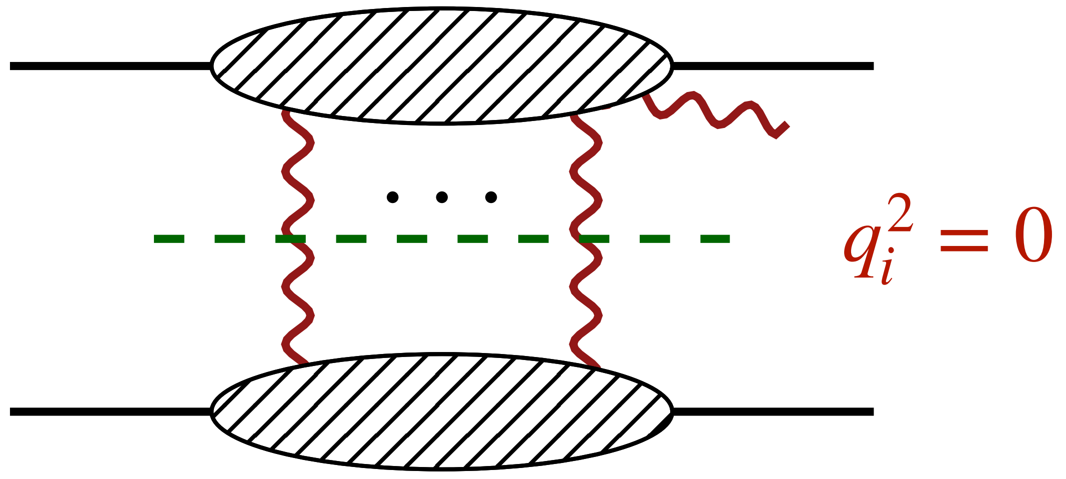


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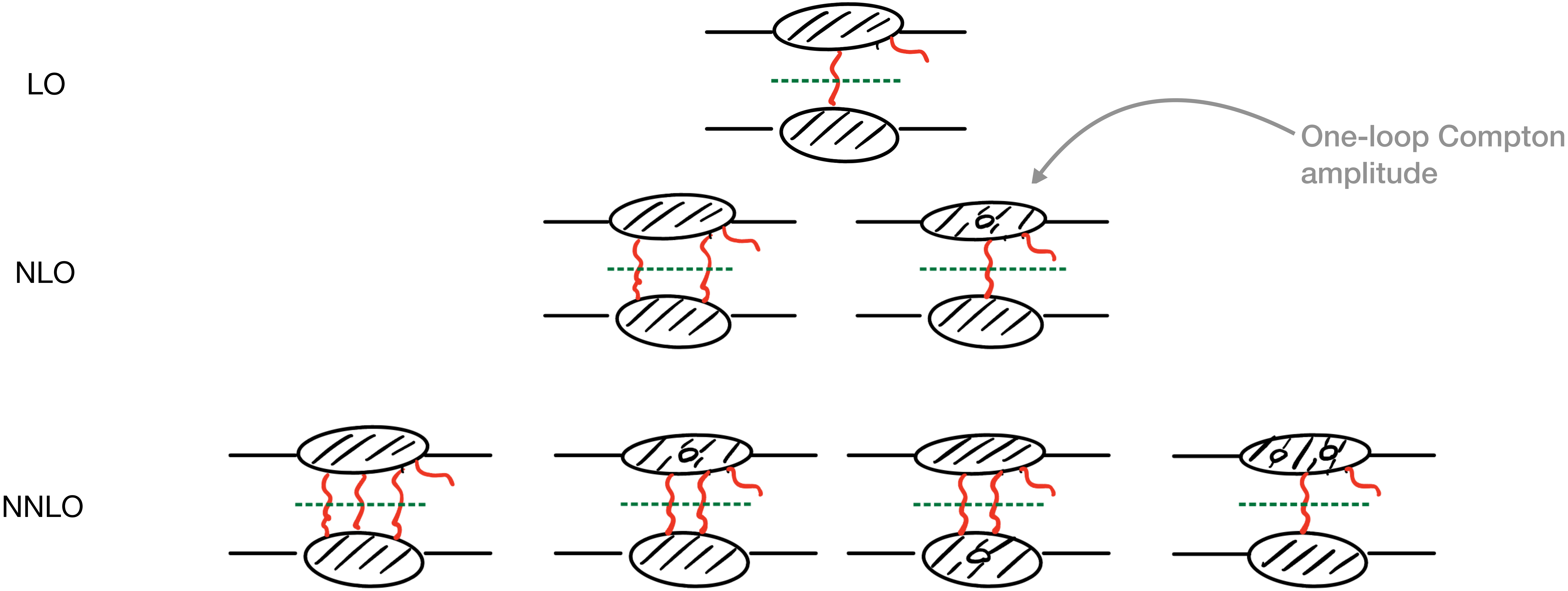

# Heavy-mass diagrammatics

Brandhuber, Chen, Travaglini, Wen
Brandhuber, Chen, Johansson, Travaglini, Wen
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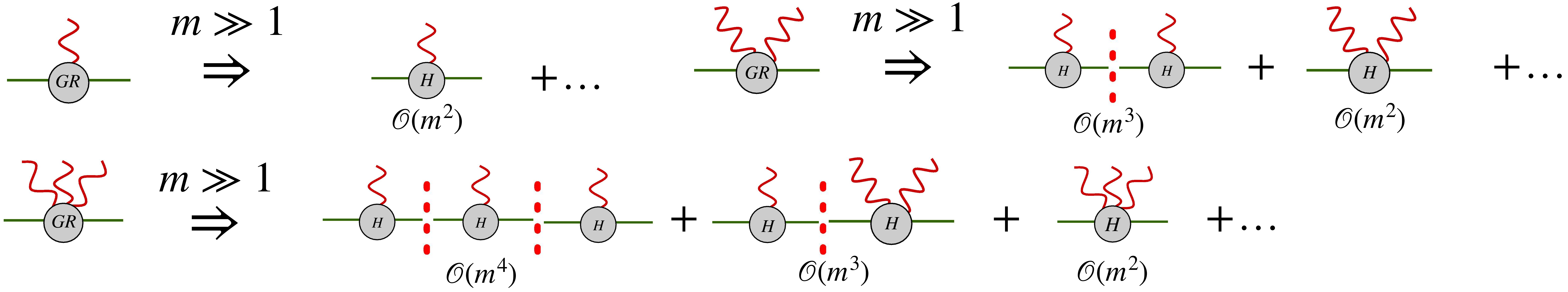

# Heavy-mass diagrammatics

Brandhuber, Chen, Travaglini, Wen
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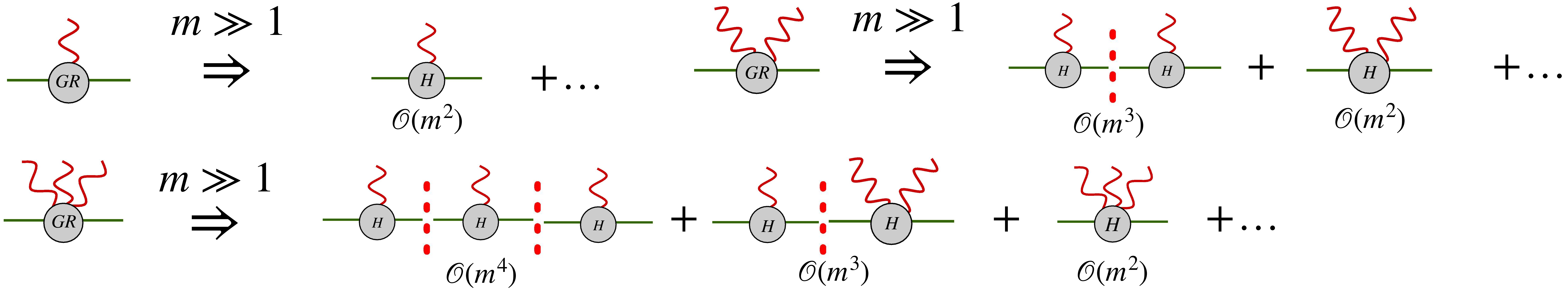


Contributing pieces from HEFT expansion $\mathcal{O}(m^{4+L})$

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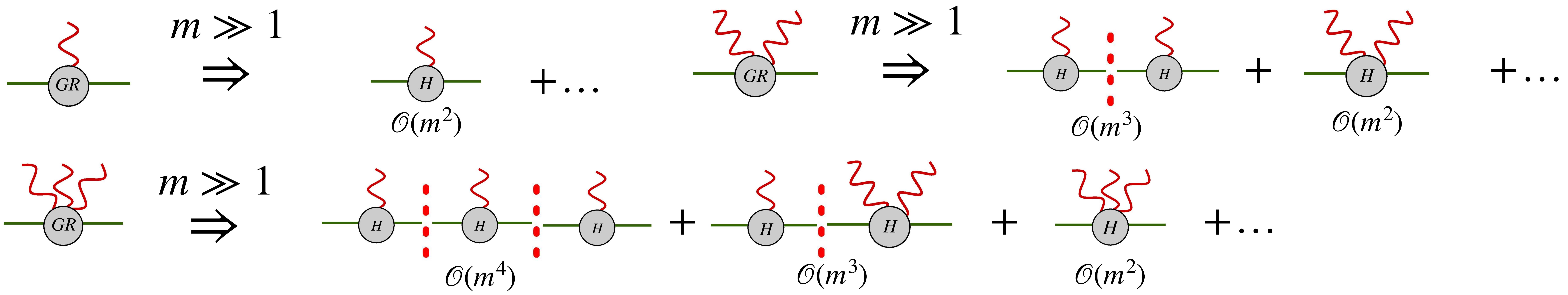
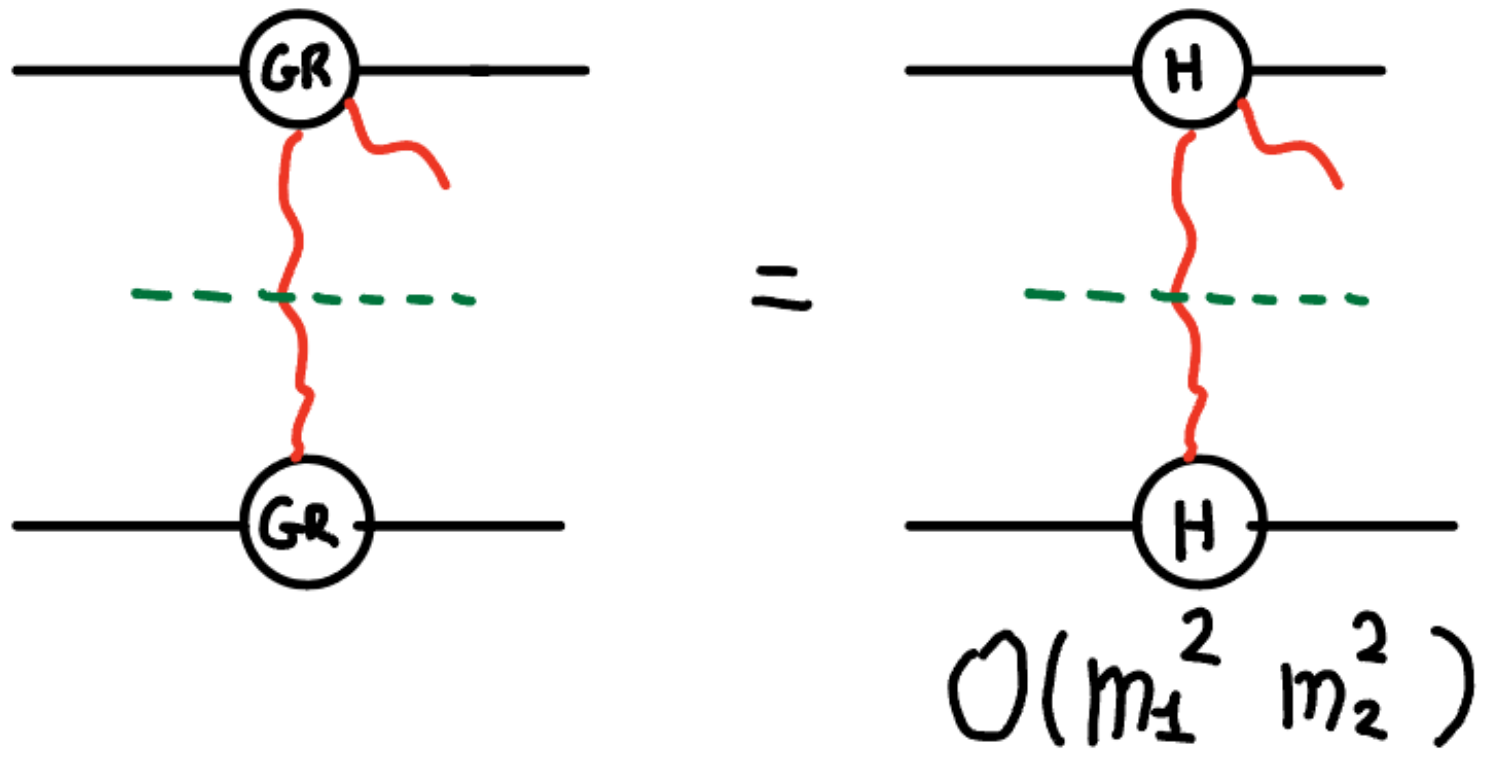



Contributing pieces from HEFT expansion $\mathcal{O}(m^{4+L})$

LO



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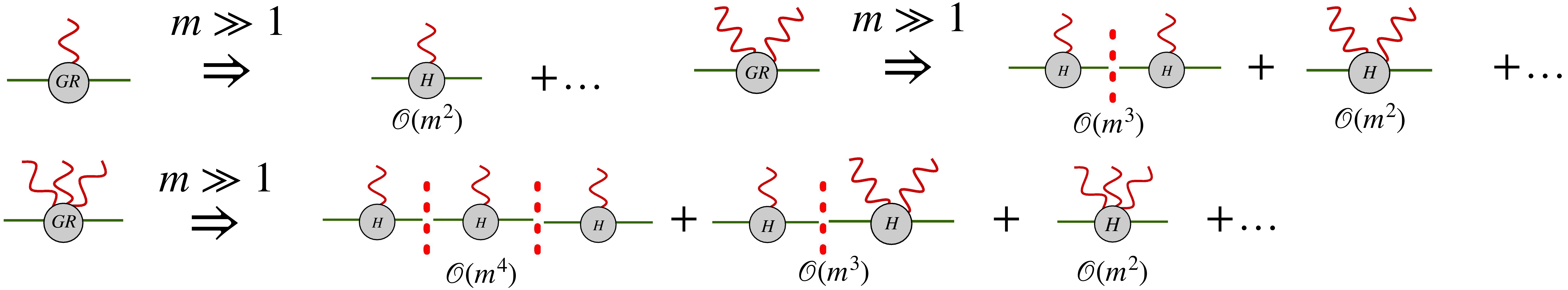


Contributing pieces from HEFT expansion $\mathcal{O}(m^{4+L})$

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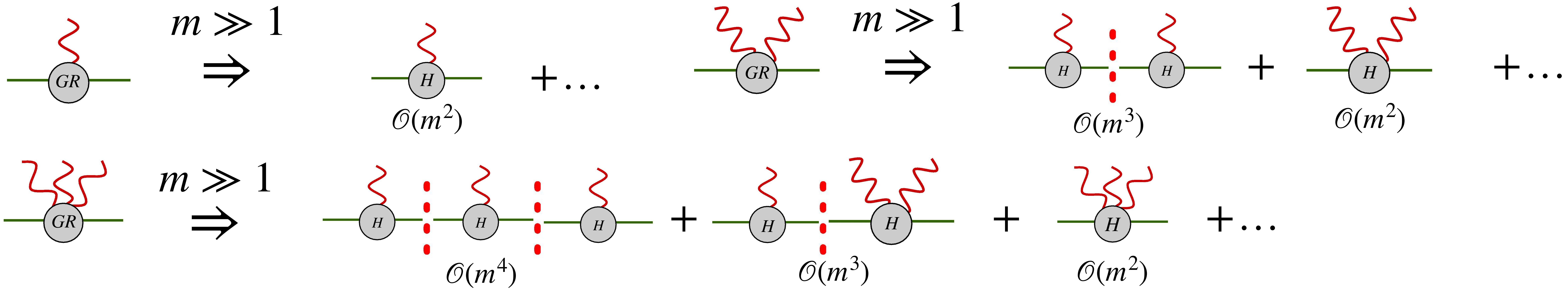
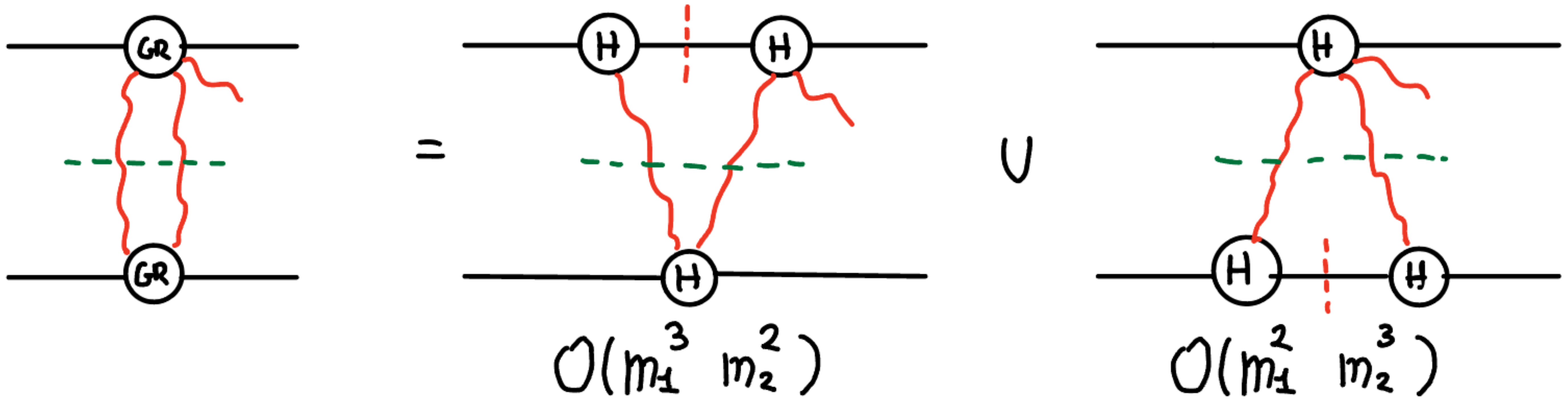



Contributing pieces from HEFT expansion $\mathcal{O}(m^{4+L})$

NLO



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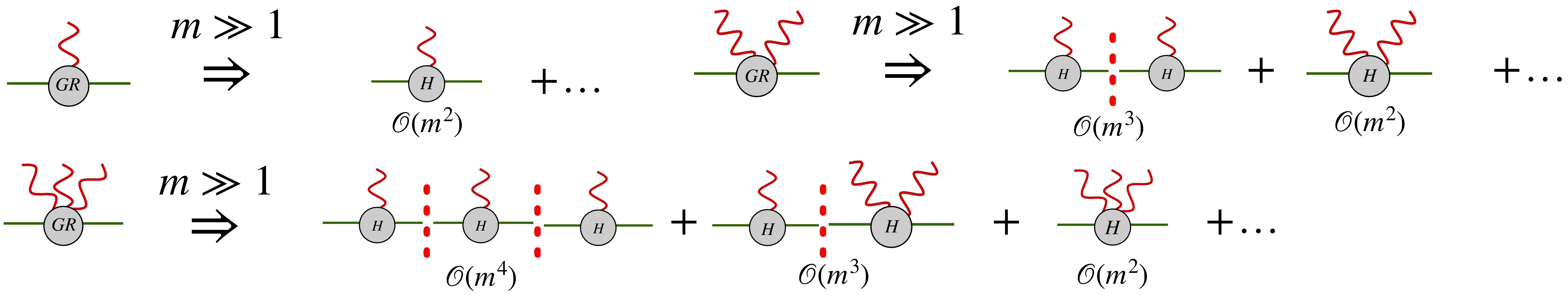


Contributing pieces from HEFT expansion $\mathcal{O}(m^{4+L})$

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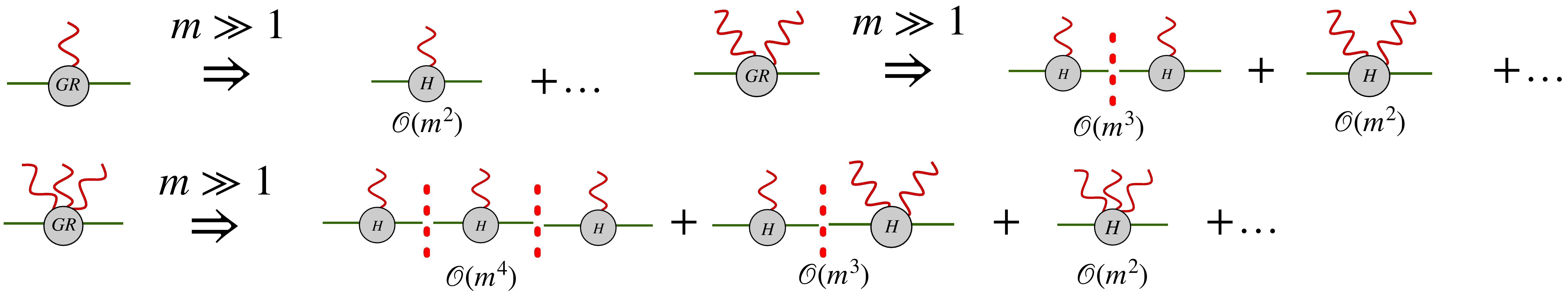
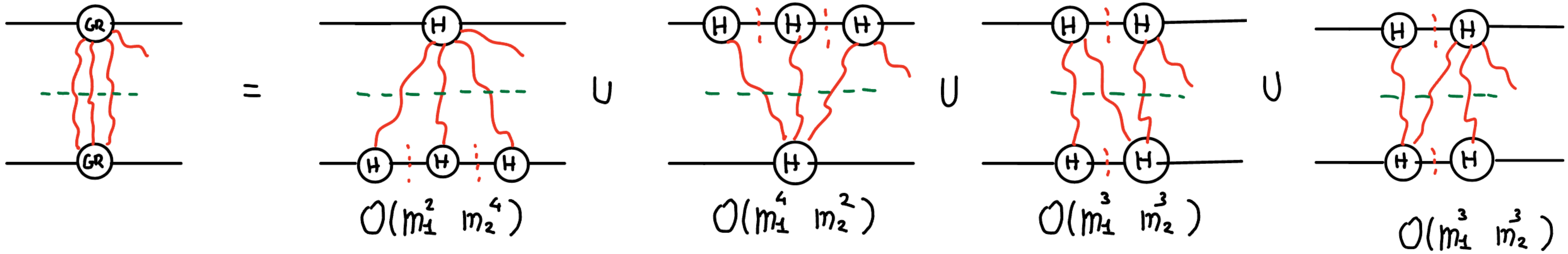



Contributing pieces from HEFT expansion $\mathcal{O}(m^{4+L})$

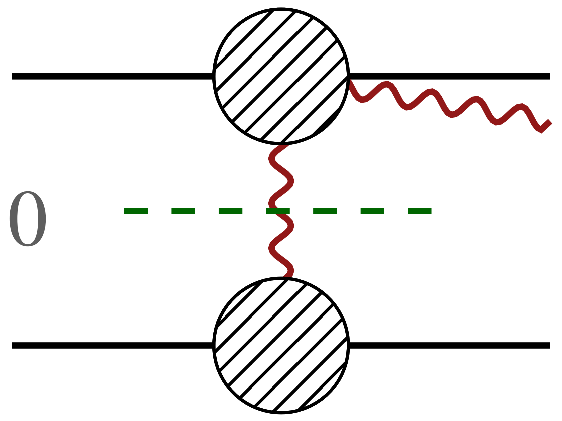
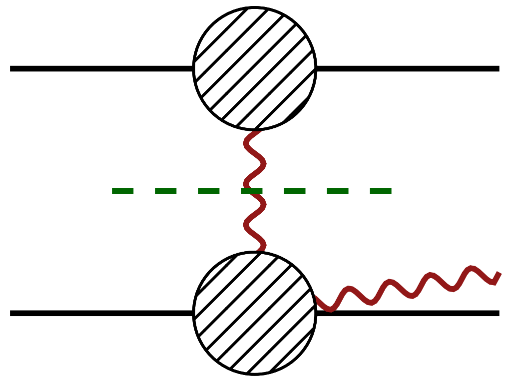
NNLO



Integrand Generation

# The leading-order waveform

Integrand generation from Generalised Unitarity

$\mathcal{I}^{(0)}_{\mathcal{W}} =$ $q_2^2 = 0$  $\cup$  $q_1^2 = 0$

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Integrand generation from Generalised Unitarity

Building blocks: heavy mass expansion

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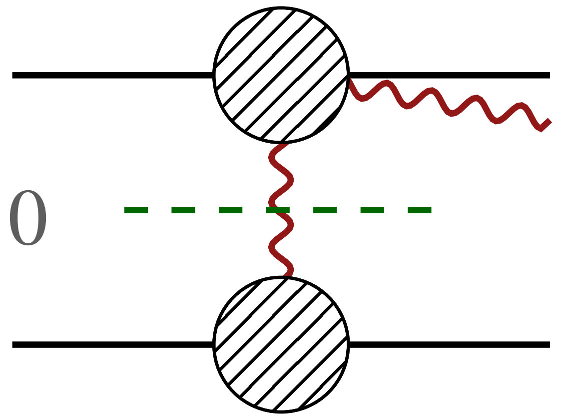

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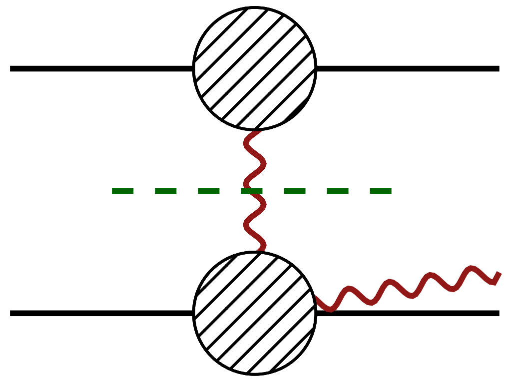

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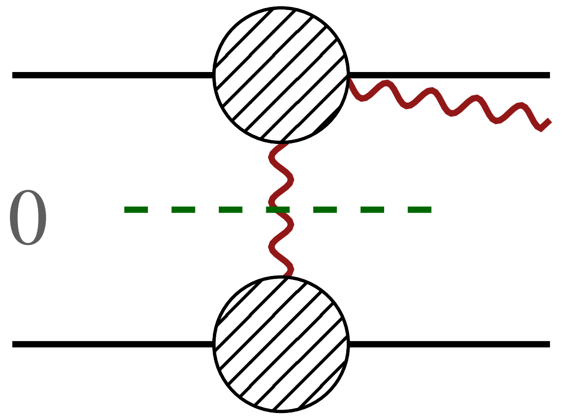
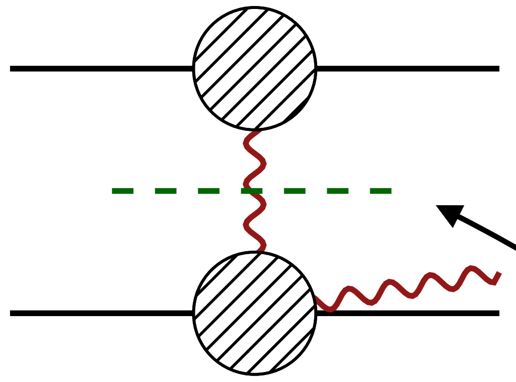


$\cup$



$q_1^2 = 0$

Polarisation sum

$$\sum_h \epsilon_{-k}^{\mu_1\nu_1}\epsilon_k^{\mu_2\nu_2} = \frac{1}{2}(P^{\mu_1\mu_2}P^{\nu_1\nu_2} + P^{\nu_1\mu_2}P^{\nu_1\mu_2}) - D_0 P^{\mu_1\nu_1}P^{\mu_2\nu_2}$$

$$D_0 = \frac{1}{2} \qquad D_0 = \frac{2}{D-2}$$

Integral Reduction

# The leading-order waveform

$$\Delta\langle \mathscr{W}_h^{(0)} \rangle = \frac{e^{i\omega\, n\cdot b_2}}{16\pi r\; m_1 m_2} \int_{\hat{q}} e^{i\, b\cdot q}\; \hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot(q-k))\; \mathcal{I}_{\mathscr{W}}^{(0)}$$

Fourier transform

Integrand

Tensor structures appearing

$\epsilon_k \cdot q$

Integral Reduction

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Decomposition in scalar Form Factors

$$u^{\mu\nu} = \sum_{i=1}^{4} \hat{v}_i^{\mu} v_i^{\nu}$$

Projector

$$v_i^{\mu} \in \{u_1^{\mu}, u_2^{\mu}, b^{\mu}, k^{\mu}\}$$

External vectors

$$v_i^{\mu}\hat{v}_{j\mu} = \delta_{ij}$$

Dual vectors

Integral Reduction

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$$\Delta\langle \mathscr{W}_h^{(0)} \rangle = \frac{e^{i\omega\, n\cdot b_2}}{16\pi r\ m_1 m_2} \int_{\hat{q}} e^{i\, b\cdot q}\ \hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot(q-k))\ \mathcal{I}_{\mathscr{W}}^{(0)}$$

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Dual vectors

Four dimensional external kinematics: factorisation of tensors

$$T^{\mu_1\cdots\mu_N} = T_{\nu_1\ldots\nu_N}\prod_{i=1}^{N} u^{\mu_i\nu_i}$$

Integral Reduction (II)

# The leading-order waveform

$$I_{a_1\,1\,1\,a_4 a_5} = \int_{\hat{q}} e^{i\,b\cdot q}\,\frac{(i\,b\cdot q)^{-a_1}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot(q-k))}{(q^2)^{a_4}((q-k)^2)^{a_5}}$$

$$I_{a_1a_2a_3a_4a_5} = \int_{\hat{q}} e^{D_1}\,\frac{1}{\prod_{i=1}^{5} D_i^{a_i}}$$

Integral family

Reverse Unitarity

$$\begin{aligned} D_1 &= i\;b\cdot q, \\ D_2 &= q^2, \\ D_3 &= (q-k)^2, \\ D_4 &= u_1\cdot q \\ D_5 &= u_2\cdot(k-q) \end{aligned}$$

Integral Reduction (II)

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$$D_1 = i\,b\cdot q,$$
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$$D_4 = u_1\cdot q$$
$$D_5 = u_2\cdot(k-q)$$

Integration-by-parts identities for Fourier integrals

$$\int_{\hat{q}} \frac{\partial}{\partial q^\mu}\left(e^{D_1}\frac{v^\mu}{\prod_{i=1}^{5} D_i^{a_i}}\right) = 0$$

$$IBP_{a_1,\ldots,a_5}[1-D_1] + IBP_{a_1-1,\ldots,a_5} = 0$$

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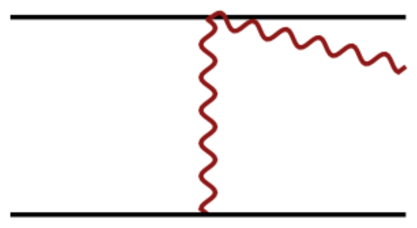
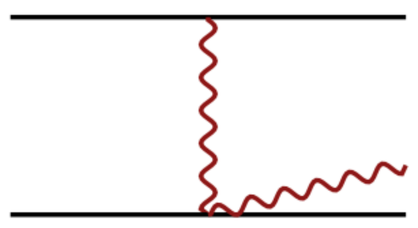
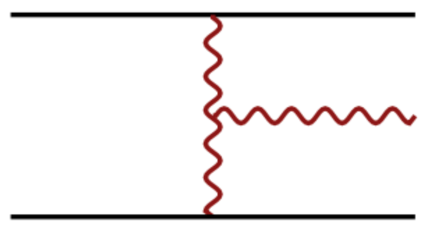
Decomposition into a basis of 6 Master Integrals

$J_{1+n} = \delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$, $\quad J_{3+n} = \delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$, $\quad J_{5+n} = \delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$ $\quad n = 0,1$

Integral Evaluation

# Evaluating Fourier Integrals

De Angelis, Gonzo, Novichkov

$$J_i \sim \int_{-\infty}^{+\infty} dz_b e^{i\, z_b} \int_{-\infty}^{\infty} dz_v f_i(z_b, z_v)$$

$$q^\mu = z_1 u_1^\mu + z_2 u_2^\mu + z_b \hat{b}^\mu + z_v \tilde{v}_\perp^\mu$$

Parametrisation

Integral Evaluation

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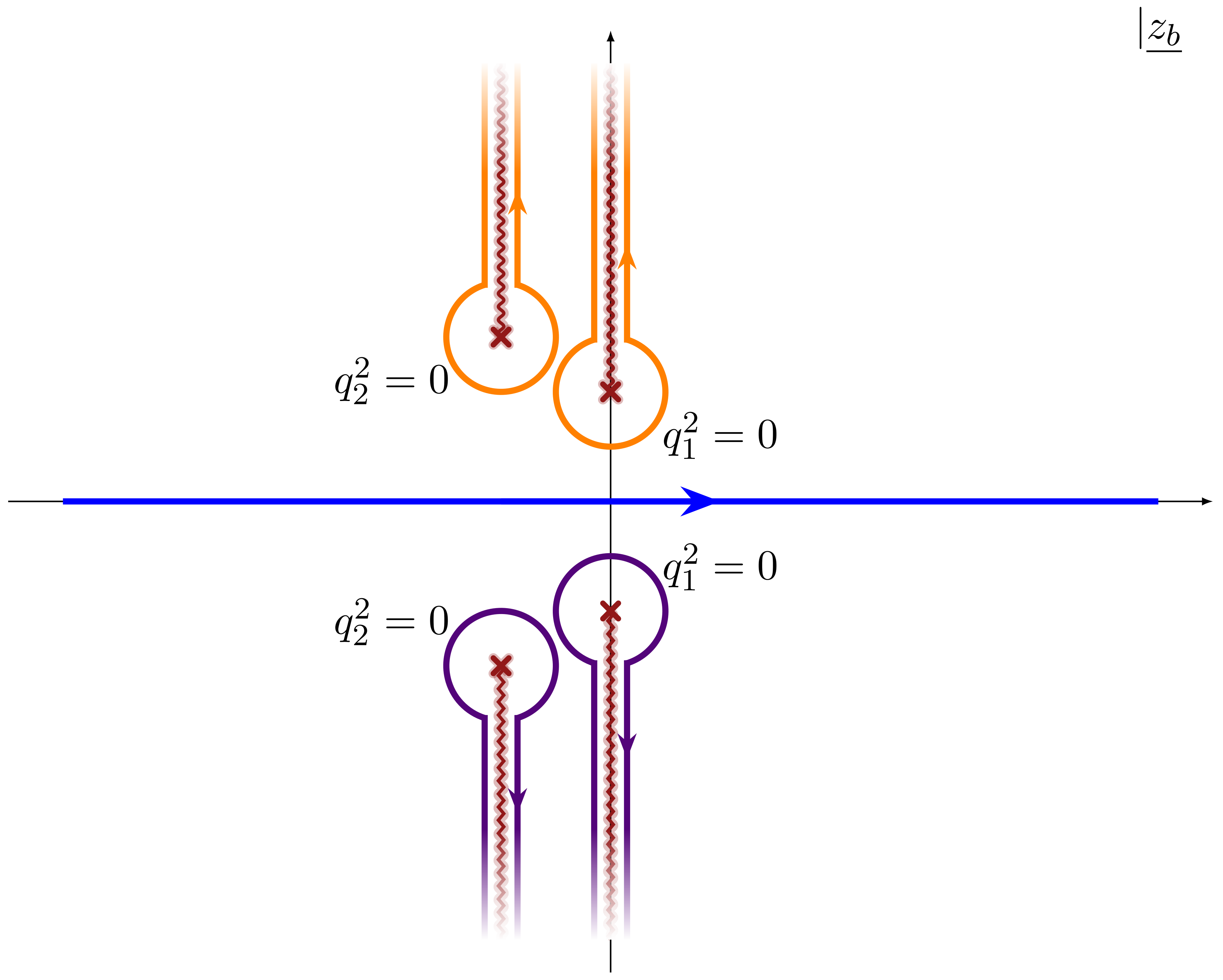

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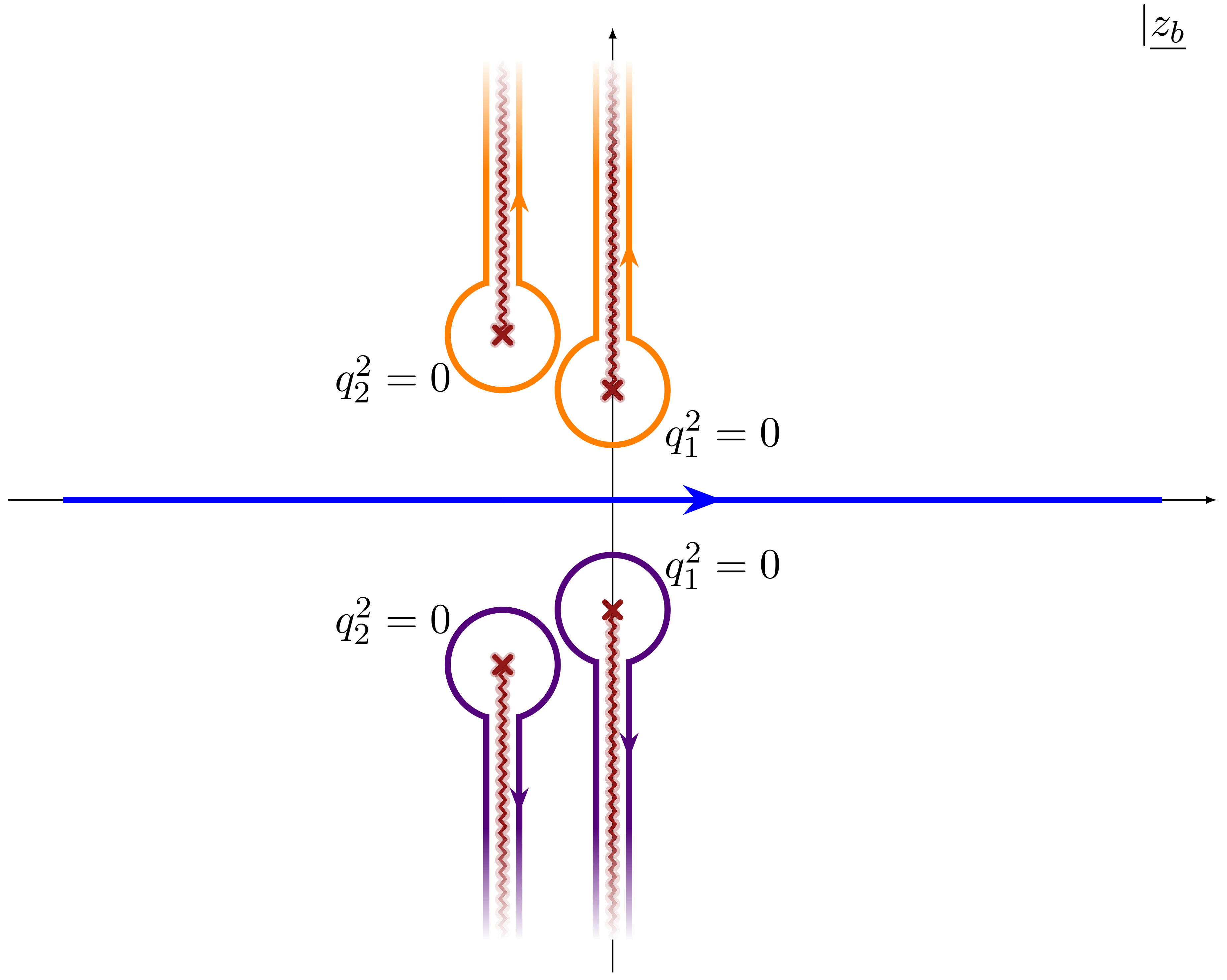


Contour deformation

$$F_i(z_b) = \sum_{j=1}^{2} \int_{\hat{z}_j \circlearrowleft \varepsilon} dz_v\ f_i(z_b, z_v) + i \int_0^\infty dx\ \underset{\hat{z}_j=0}{\text{Disc}}[f_i(z_b, z_v)]$$

Integral Evaluation

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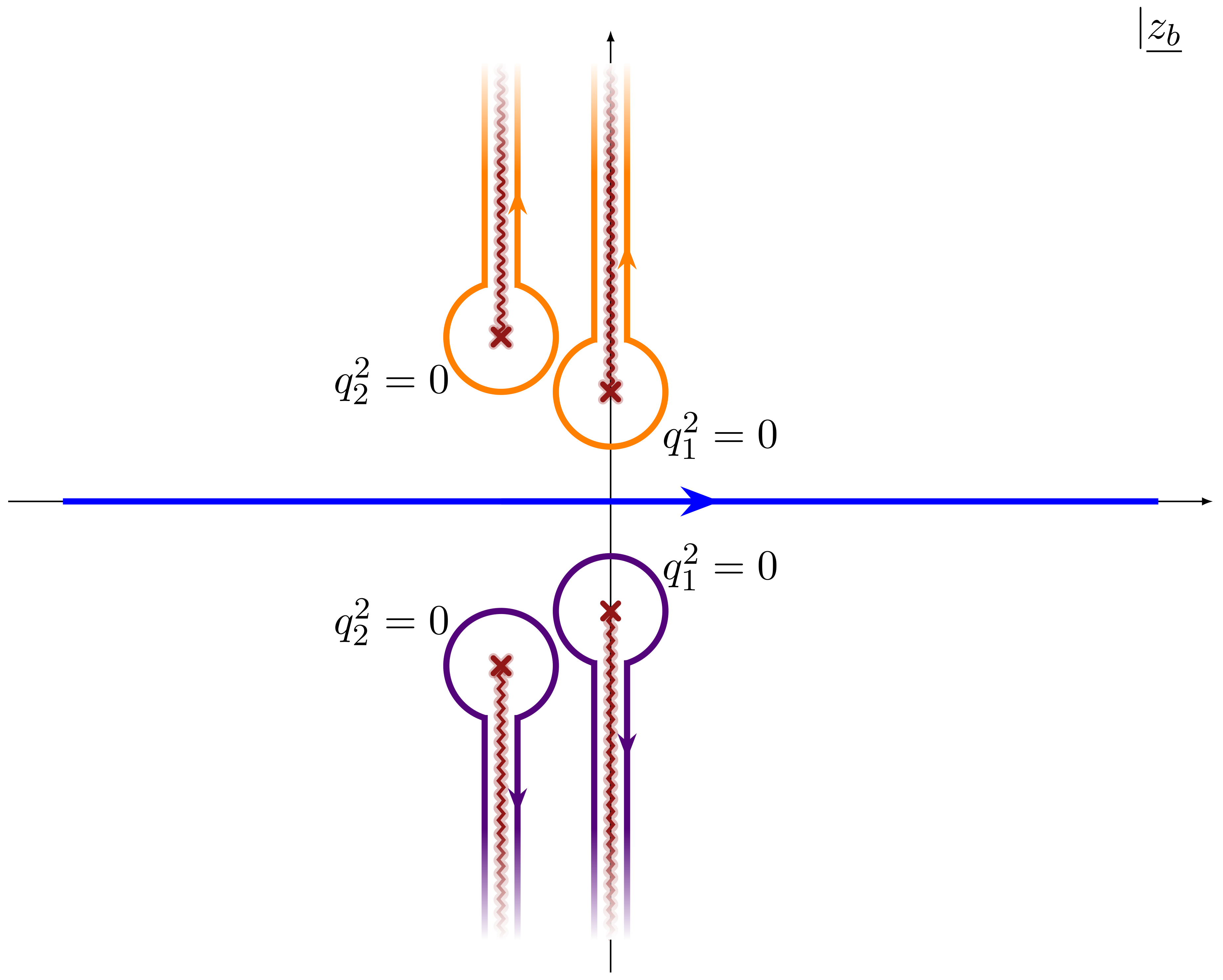


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Numerically fast-convergent one-fold integrals

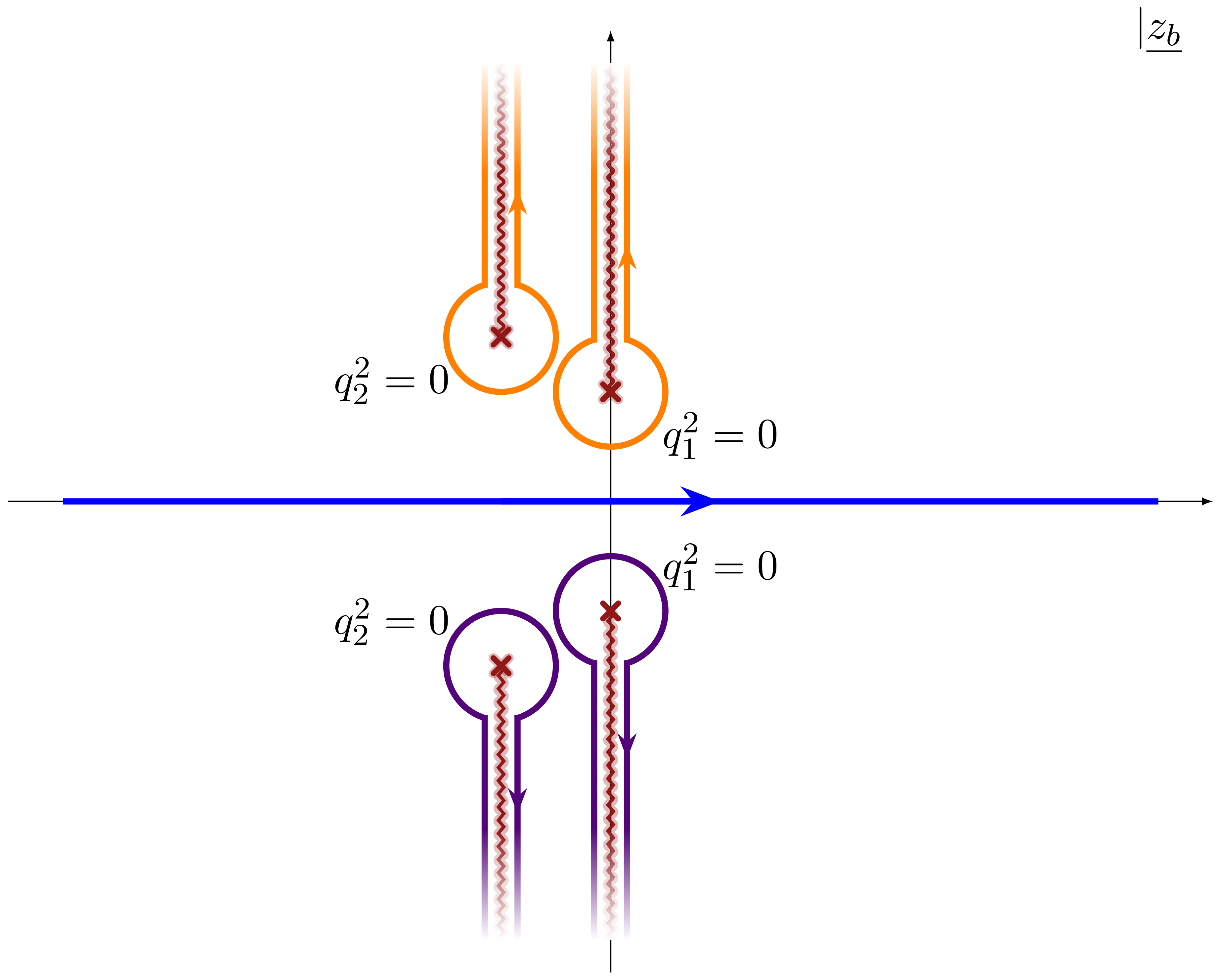
$$J_i \sim \int_{-\infty}^{+\infty} dz_b e^{i\, z_b}\ F_i(z_b)$$

Integral Evaluation

# A couple of examples

$$J_i \sim \int_{-\infty}^{+\infty} dz_b e^{i\, z_b} \int_{-\infty}^{\infty} dz_v f_i(z_b, z_v)$$

$$f_1 = \frac{\log \frac{-q_1^2}{\mu^2}}{-q_2^2} \qquad f_2 = \frac{\log \frac{-q_1^2}{\mu^2}}{-q_1^2}$$



Integral Evaluation

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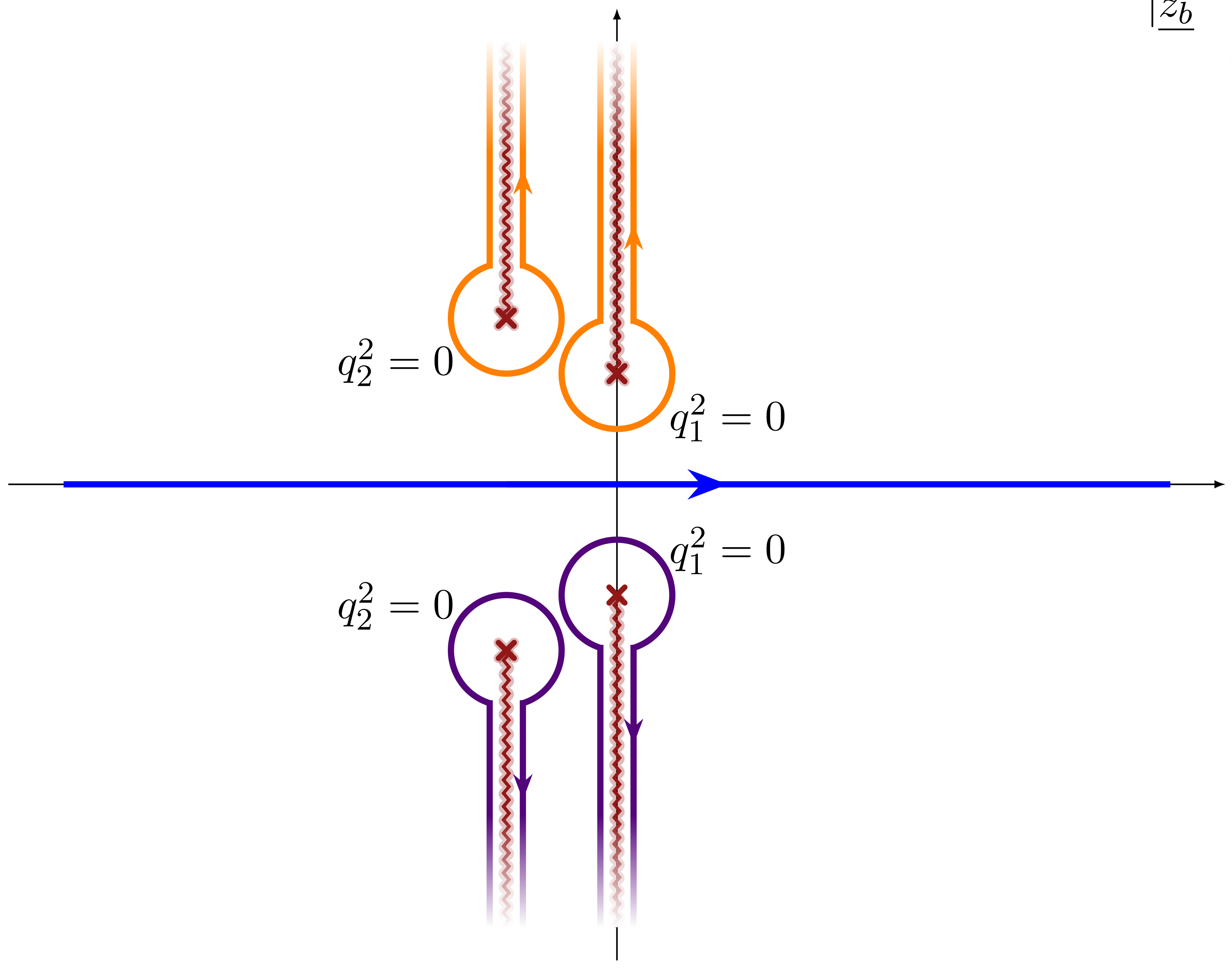


$$F_1(z_b)\Big|_{z_b} = \int_{\hat{z}_2 \circlearrowleft \varepsilon} dz_v\ f_1(z_b, z_v) + i \int_0^\infty dx\ \underset{\hat{z}_1=0}{\mathrm{Disc}}[f_1(z_b, z_v)]$$

$$= \frac{\pi}{\sqrt{(\hat{b}\cdot k + z_b)^2 + w_1^2}} \log\left[\left(\sqrt{(\hat{b}\cdot k + z_b)^2 + \hat{w}_1^2} + \sqrt{\hat{w}_2^2 + z_b^2}\right)^2 + \hat{b}\cdot k^2\right]$$

**Integral Evaluation**

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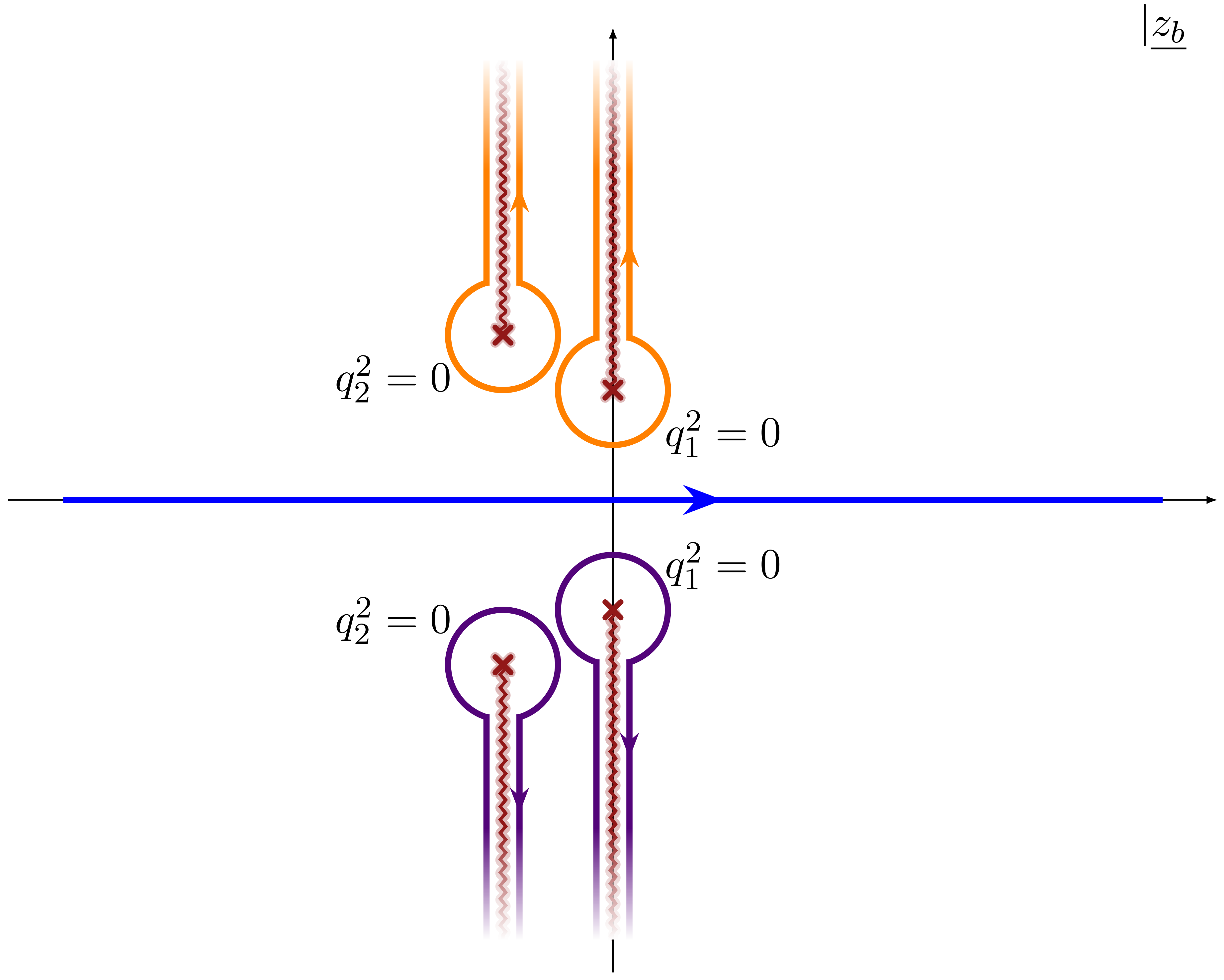


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$$F_2(z_b) = \int_{\hat{z}_1 \circlearrowleft \varepsilon} dz_v\ f_2(z_b, z_v) + i \int_0^\infty dx\ \underset{\hat{z}_1 = 0}{\mathrm{Disc}}[f_2(z_b, z_v)]$$

$$= \frac{2\pi \log\left(2\sqrt{\hat{w}_2^2 + z_b^2}\right)}{\sqrt{\hat{w}_2^2 + z_b^2}}$$

**Overlapping pole and branch cut!**

# Results at LO

$$\Delta\langle \mathscr{W}_h^{(0)} \rangle = \sum_{i=1}^{6} c_i \, J_i$$

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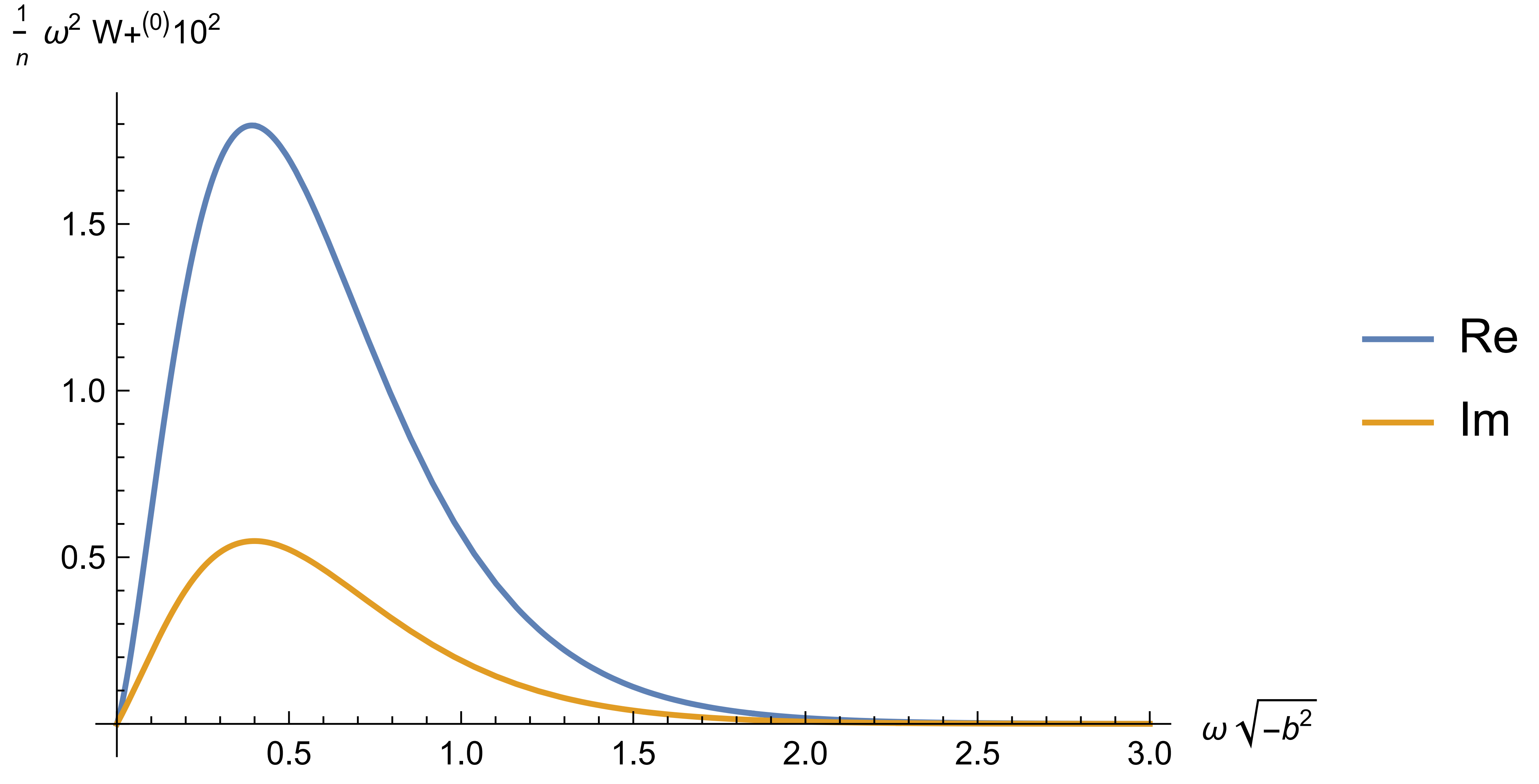
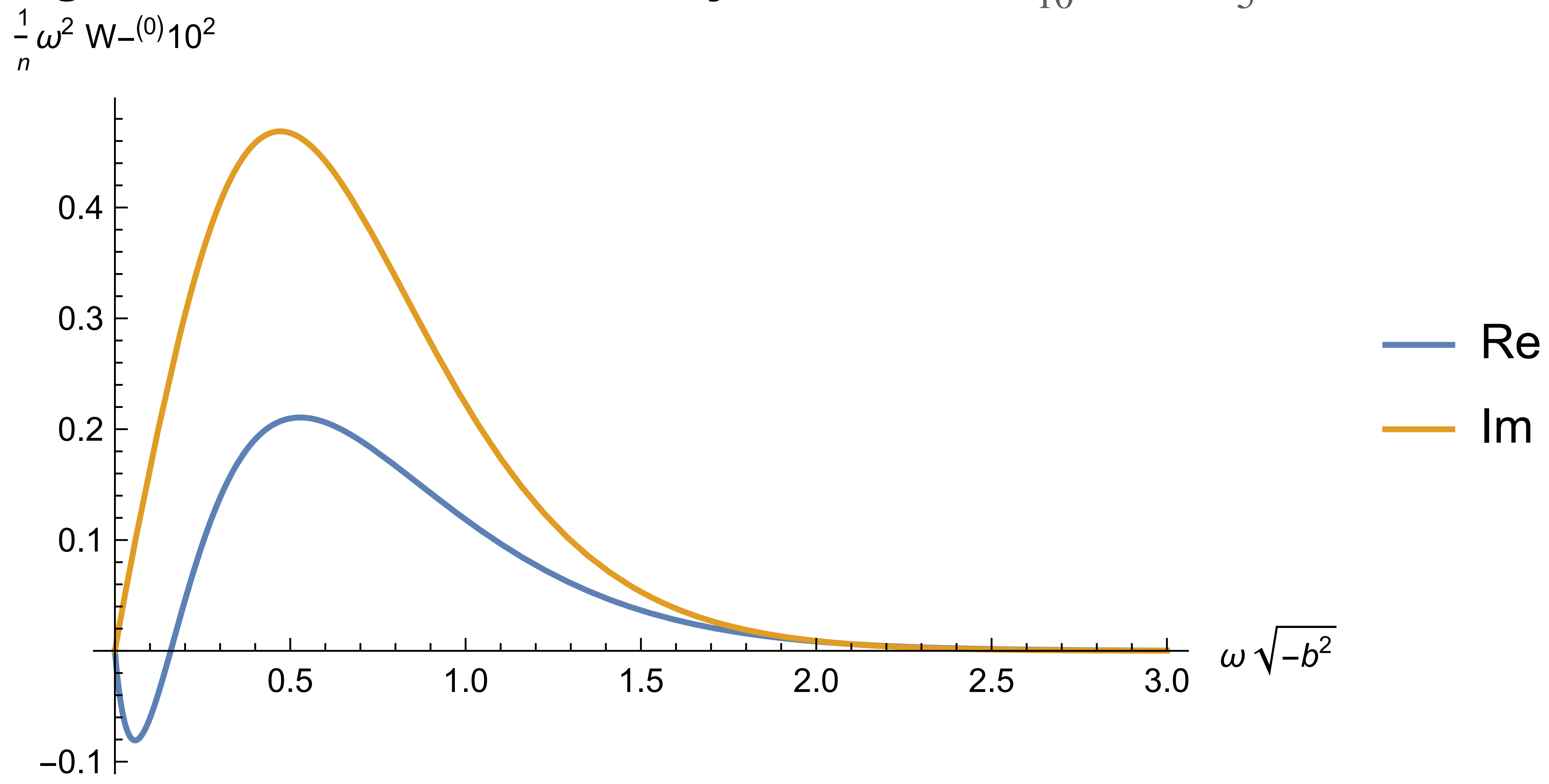
**Generation of templates for gravitational waves analysis**

$\phi = \frac{7\pi}{10}, \ \theta = \frac{7\pi}{5}, \ m_1 = m_2, \ b = 1$



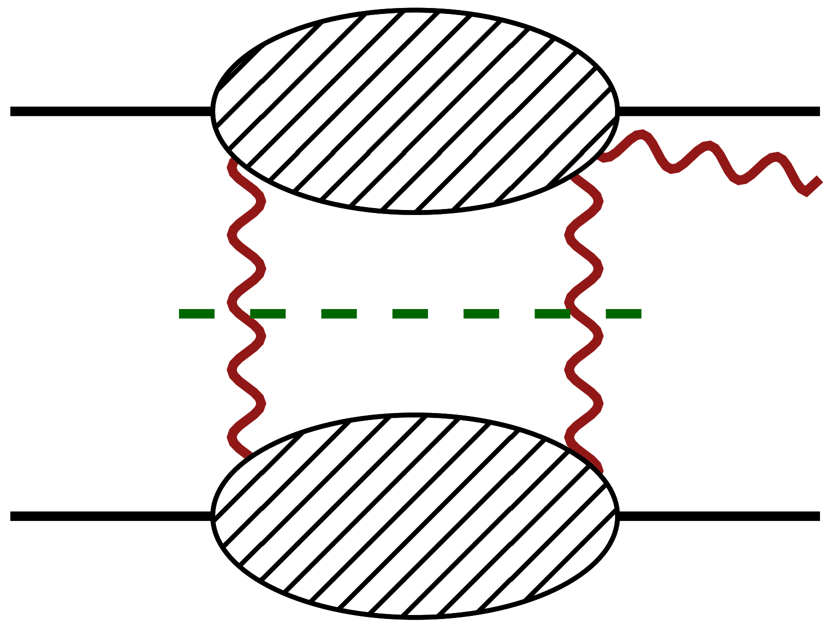



# NLO Waveform

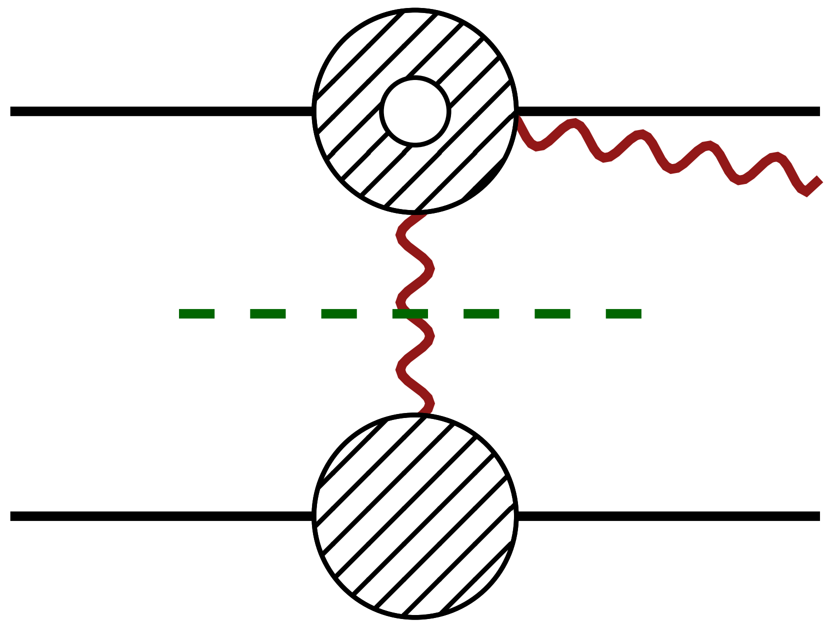
G.B., S. De Angelis, D. Kosower
In progress

Integrand generation from double and single graviton exchange

$$\mathcal{I}^{(1)}_{\mathcal{W}} =$$



$\cup$



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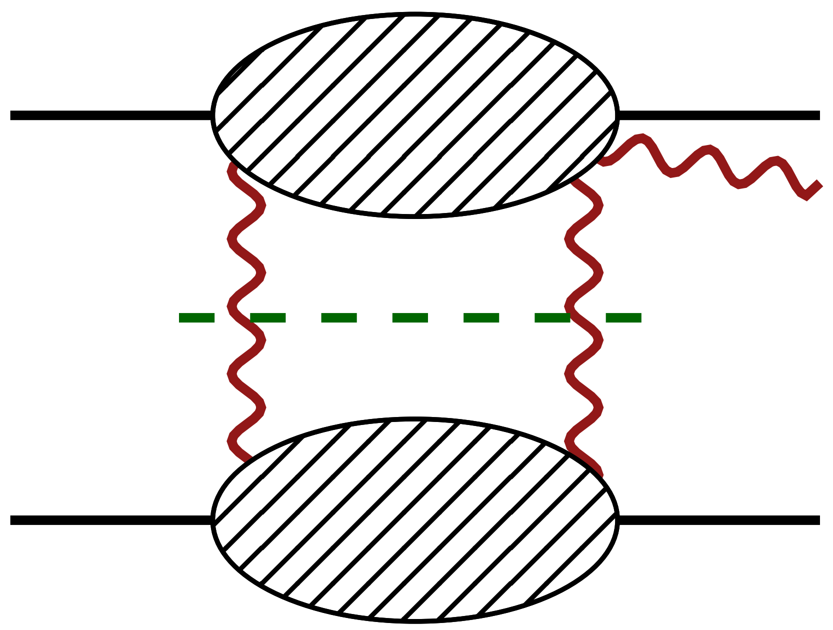
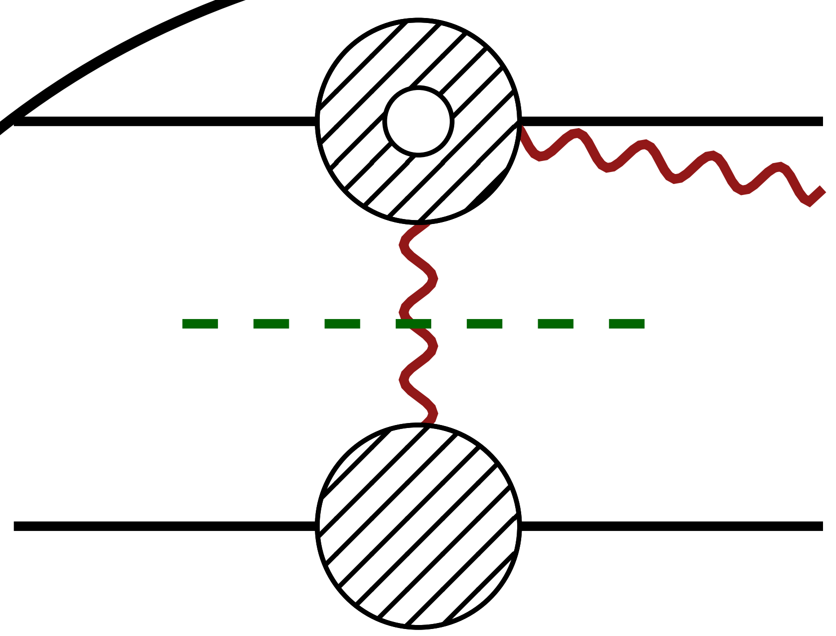
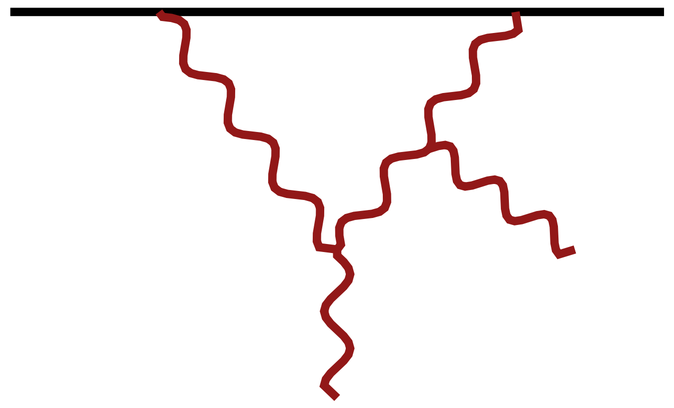
$\mathcal{I}_{\mathcal{W}}^{(1)} =$  $\cup$ 

- Cut merging



Integrand Generation

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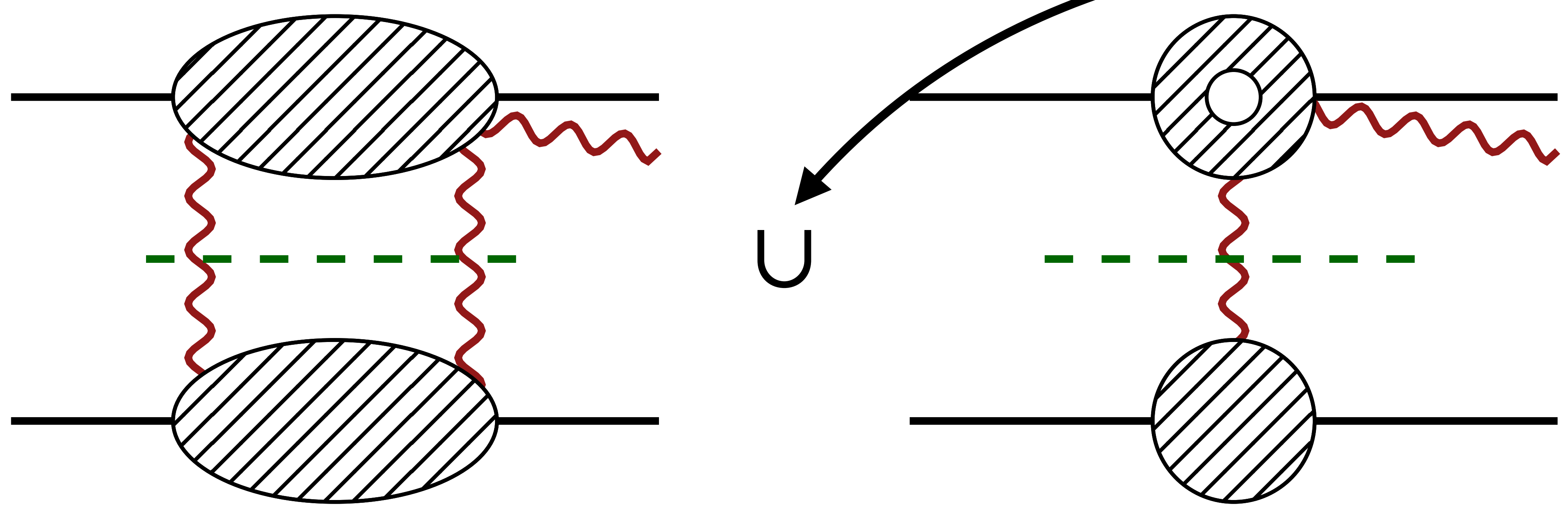
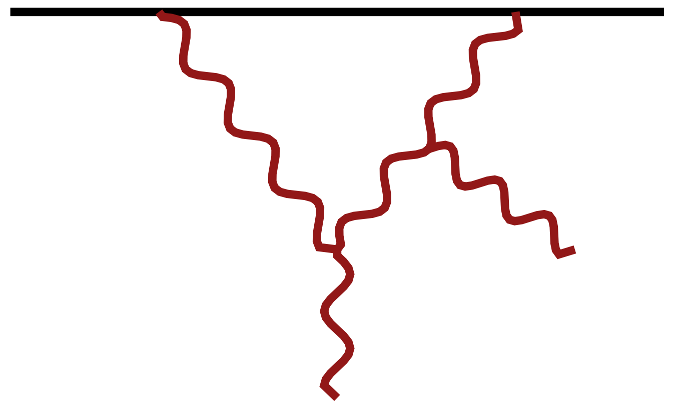
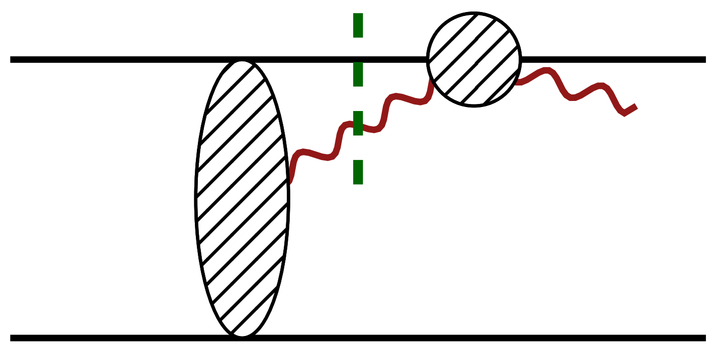


- Cut merging



- No snail diagrams



Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini
Herderschee, Roiban, Teng

Integral Reduction

# NLO Waveform

$$J^{(1)}_{a_1 a_2 a_3 111 a_7 a_8 a_9 a_{10} a_{11}} = \int_{\hat{q},\hat{\ell}} \frac{e^{D_1} D_1^{-a_1} D_{11}^{-a_{11}} \hat{\delta}(D_4)\hat{\delta}(D_5)\hat{\delta}(D_6)}{D_2^{a_2} D_3^{a_3} D_7^{a_7} D_8^{a_8} D_9^{a_9} D_{10}^{a_{10}}}$$

$D_1 = i\, b \cdot q,\ D_2 = q^2,\ D_3 = (q-k)^2,$

$D_4 = u_1 \cdot q,\ D_5 = u_2 \cdot (k-q)$

$D_6 = u_1 \cdot \ell\ ,\ \ D_7 = u_2 \cdot \ell,\ \ D_8 = \ell^2,$

$D_9 = (\ell - q_2)^2,\ \ D_{10} = (\ell + q_1)^2,\ D_{11} = i\, b \cdot \ell$

Integral Reduction

# NLO Waveform

$$J^{(1)}_{a_1a_2a_3111a_7a_8a_9a_{10}a_{11}} = \int_{\hat{q},\hat{\ell}} \frac{e^{D_1}D_1^{-a_1}D_{11}^{-a_{11}}\hat{\delta}(D_4)\hat{\delta}(D_5)\hat{\delta}(D_6)}{D_2^{a_2}D_3^{a_3}D_7^{a_7}D_8^{a_8}D_9^{a_9}D_{10}^{a_{10}}}$$

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Integration-by-parts identities for Fourier-Loop integrals

$$\int_{\hat{q},\hat{\ell}} \frac{\partial}{\partial s^\mu}\left(e^{D_1}\frac{v^\mu}{\prod_{i=1}^{11}D_i^{a_i}}\right) = 0 \qquad s^\mu \in \{q^\mu, \ell^\mu\} \qquad IBP_{a_1,\ldots,a_{11}}[1-D_1] + IBP_{a_1-1,\ldots,a_{11}} = 0$$

Integral Reduction

# NLO Waveform

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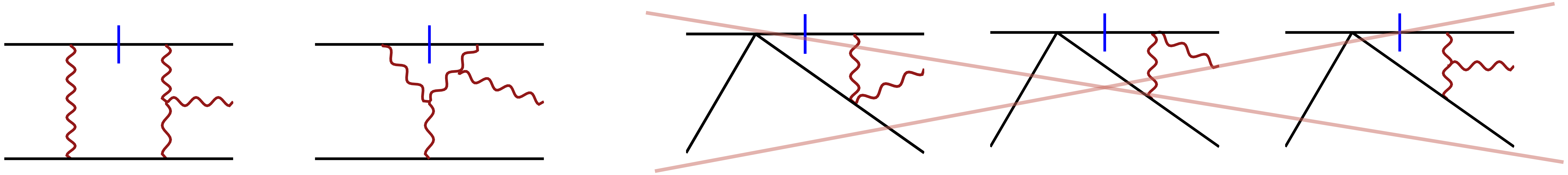
Master integrals belonging to two top topologies, no snail diagrams!



Integral Evaluation

# NLO Waveform

Loop-by-loop approach for Master Integrals Evaluation

$$J_i = \int_{\hat{q}} e^{ib\cdot q}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot(q-k)) \int_{\hat{\ell}} \mathcal{I}_i(q,\ell)$$

**Step2: Complex analysis**

**Step1: Differential Equations**

Integral Evaluation

# NLO Waveform

Loop-by-loop approach for Master Integrals Evaluation

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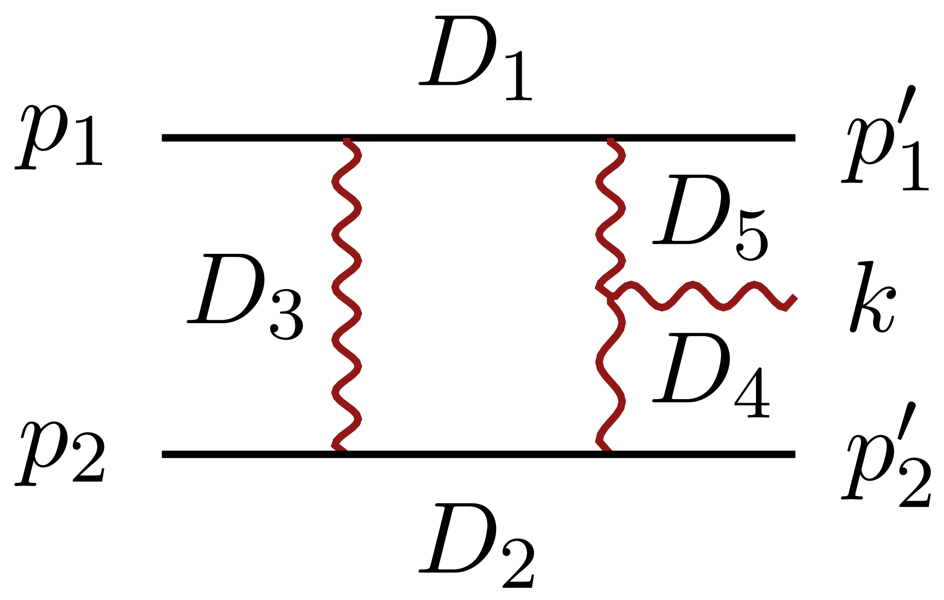
**Step2: Complex analysis** **Step1: Differential Equations**

Caron-Huot, Giroux, Hannesdottir, Mizera
Bohnenblust, Ita, Kraus, Schlenk
G.B., S. De Angelis

Loop integrals via differential equations




$$d\,\mathbf{J} = d\mathbb{A}\,\mathbf{J} \qquad d\mathbb{A} = \sum_{i=1}^{n} dz_i \mathbb{A}_{z_i} \qquad \mathbf{J} = \mathbb{P}\exp\left(\int_\gamma d\mathbb{A}\right)\mathbf{J}\Big|_{\partial\gamma}$$

Path order exponential Boundary vector

Integral Evaluation

# NLO Waveform

Loop-by-loop approach for Master Integrals Evaluation

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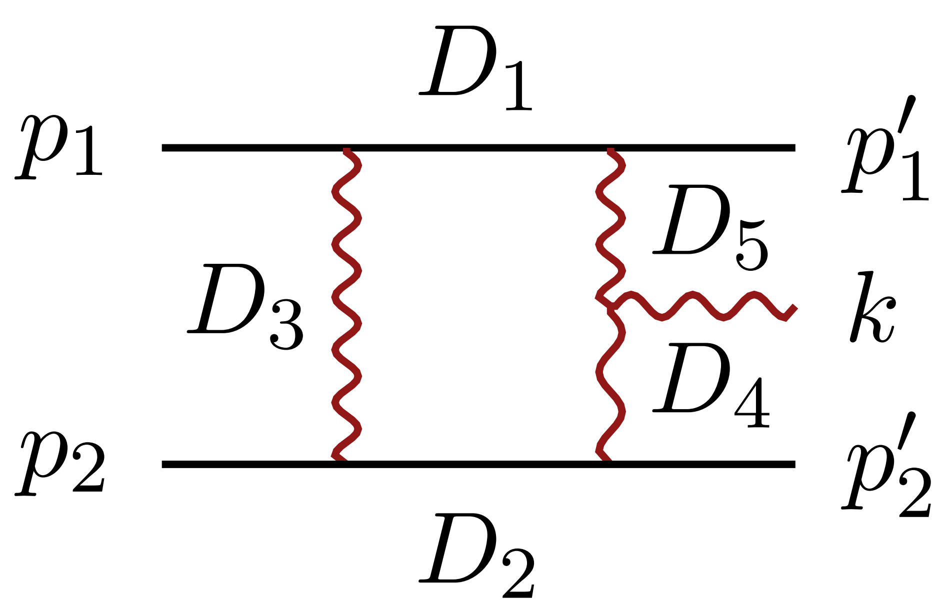


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Path order exponential Boundary vector

Canonical differential equations, retarded boundary conditions

Henn

$$d = 4 - 2\epsilon \qquad d\mathcal{G} = \epsilon\, d\hat{\mathbb{A}}\,\mathcal{G} \qquad \rightarrow \qquad \mathcal{G} = \left(\mathbb{I} + \epsilon\int_\gamma d\hat{\mathbb{A}} + \epsilon^2\int_\gamma d\hat{\mathbb{A}}\int_\gamma d\hat{\mathbb{A}} + \dots\right)\mathcal{G}\Big|_{\partial\gamma}$$

Dimensional regularisation Factorized $\epsilon$ dependence Iterated integrals

Integral Evaluation

# NLO Waveform

Loop-by-loop approach for Master Integrals Evaluation

$$J_i = \int_{\hat{q}} e^{ib\cdot q}\hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot (q-k)) \int_{\hat{\ell}} \mathcal{I}_i(q,\ell)$$

**Step2: Complex analysis**

**Step1: Differential Equations**

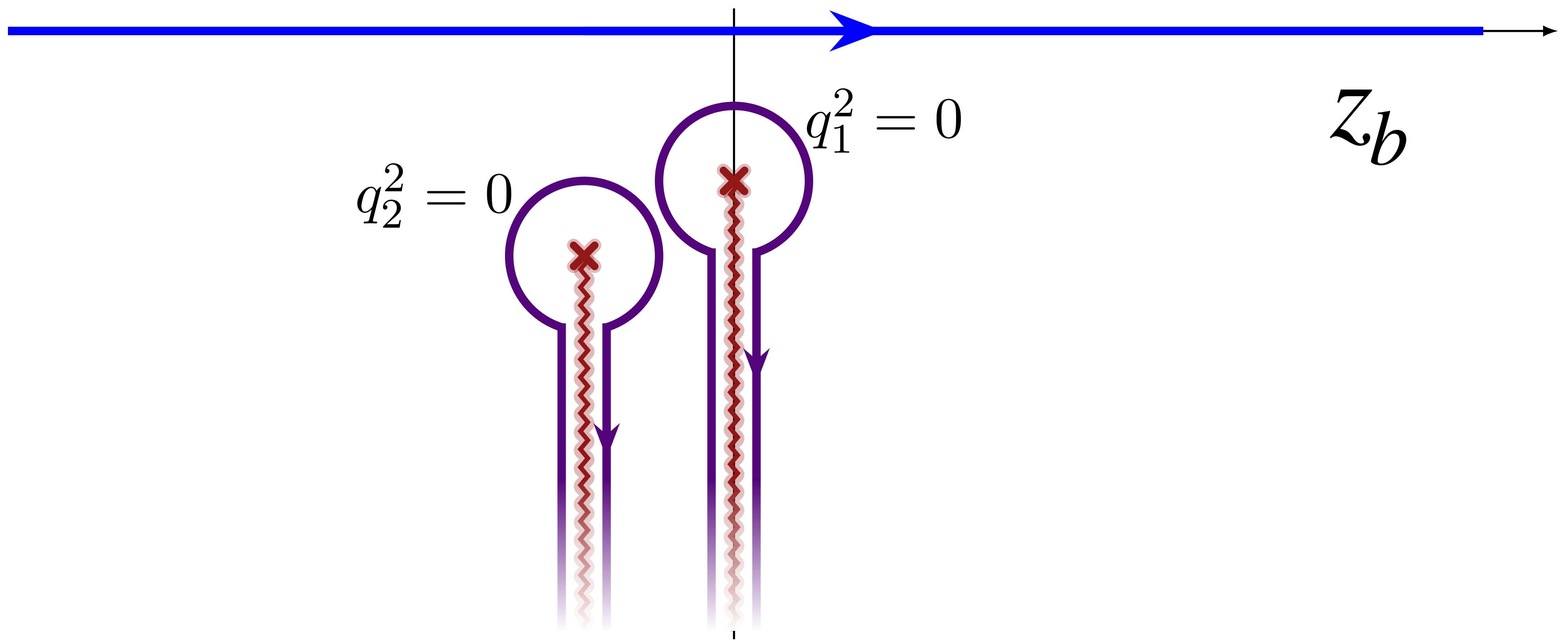
Fourier integration via contour deformation

$$J_i = \frac{1}{\epsilon}\mathscr{F}\left[J_i^{(-1)}\right] + \mathscr{F}\left[J_i^{(0)}\right]$$

# Results

$$\Delta\langle\mathscr{W}_h\rangle(u,\vec{n}) = \sum_{i=1}^{28} c_i^{(1)} J_i^{(1)} + \sum_{i=1}^{28} c_i^{(2)} J_i^{(2)}$$

$$J_i = \int_{-\infty}^{\infty} dz_b\, e^{-iz_b\sqrt{-b^2}} f(z_b)$$



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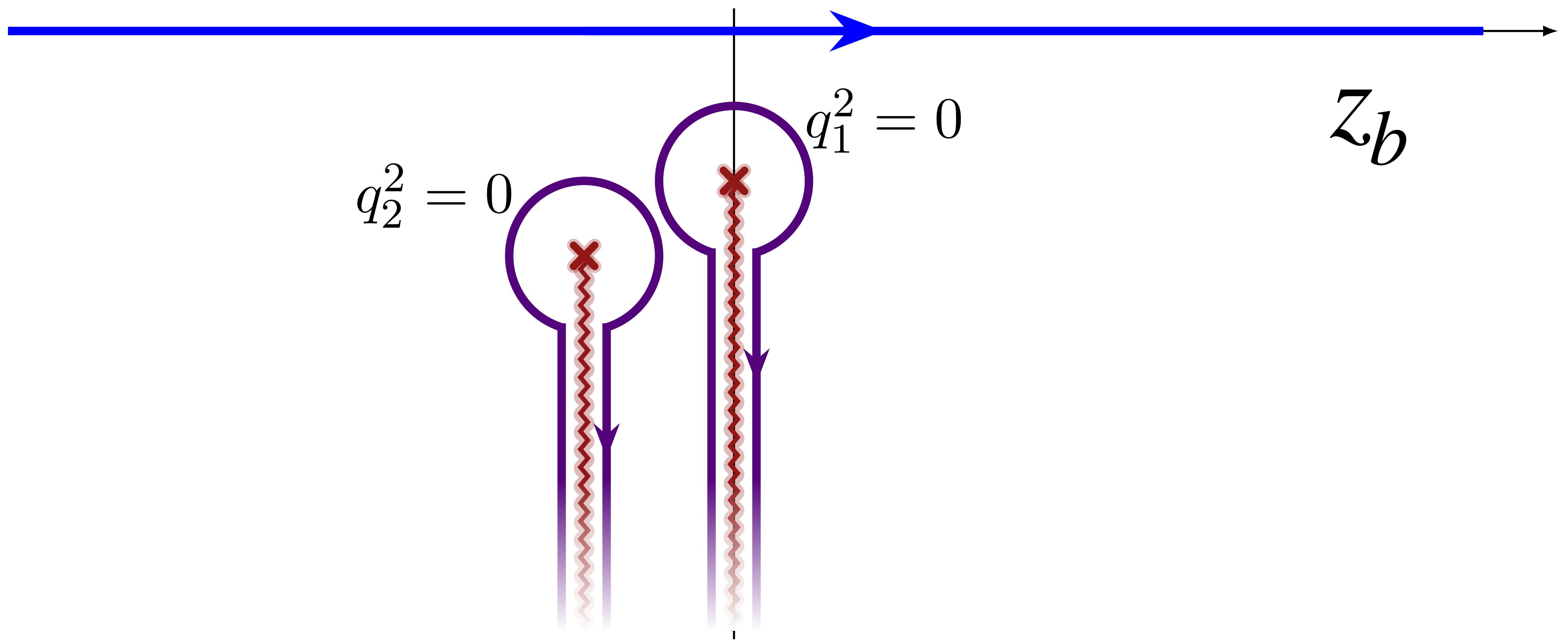


Generation of templates for gravitational waves analysis

# Results

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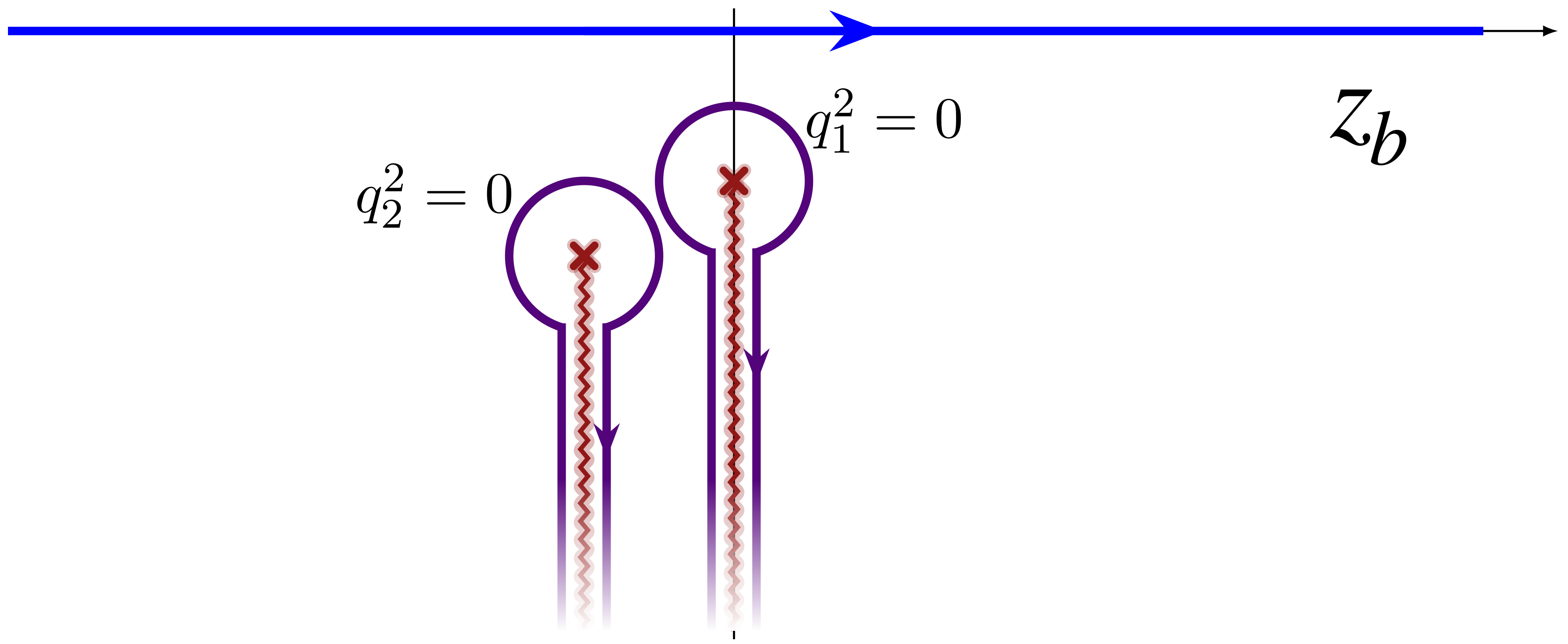


Generation of templates for gravitational waves analysis

Gravitational energy spectrum

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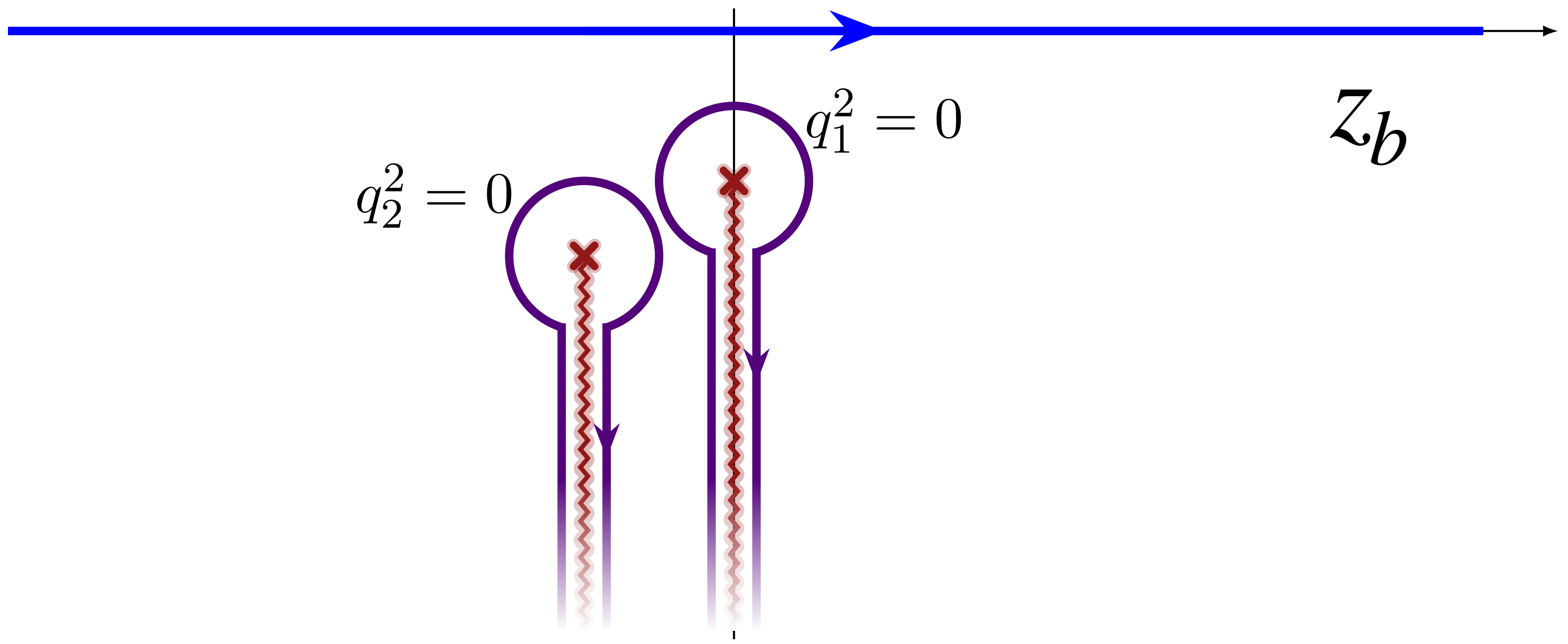


Generation of templates for gravitational waves analysis

Gravitational energy spectrum

Analytic result at NLO in Electrodynamics GB, De Angelis

# Outlooks

Gravitational Waveforms from scattering amplitudes

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Gravitational Waveforms from scattering amplitudes

Combined approach: Scattering amplitudes techniques in frequency space

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One-loop gravitational waveform expressed in a minimal basis of functions

# Outlooks

Gravitational Waveforms from scattering amplitudes

Combined approach: Scattering amplitudes techniques in frequency space

One-loop gravitational waveform expressed in a minimal basis of functions

Two-loop gravitational waveforms? Differential equations in frequency domain?

# Thank you!