# Polynomial Division and Elimination Theory Over Finite Fields

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Domodossola, 15/07/25

Based on upcoming work with

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# Introduction and Motivation (1/1)

## Motivating Example

Consider

$$f(x) = x^3 + ax^2 - (5 + 2a)x + 1 \qquad\qquad p(x) = x^2 - 2x - 1$$

For $x^*$ s.t. $p(x^*) = 0$ what is $f(x^*)$ ?

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Normal Approach: $x^* = 1 \pm \sqrt{2}$ $\longrightarrow$

$$\begin{aligned} f(x^*) &= (1 \pm \sqrt{2})^3 + a(1 \pm \sqrt{2})^2 - (5+2a)(1 \pm \sqrt{2}) + 1 \\ &= 7 \pm 5\sqrt{2} + a(3 \pm 2\sqrt{2}) - (5+2a)(1 \pm \sqrt{2}) + 1 \\ &= 3 + a \end{aligned}$$

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→ Rational Expression!

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$$f(x) = 3 + a \mod p(x)$$

Philosophy: Polynomial division can often solve problems without *explicitly* needing to solve polynomial systems

# Review of Polynomial Division (1/2)

## Quick Summary

Polynomial division allows us to decompose functions as

$$f(x) = q(x)p(x) + r(x)$$

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$$\begin{aligned}
f(x) &= x\left(p(x) + 2x + 1\right) + a\left(p(x) + 2x + 1\right) - (5 + 2a)x + 1 \\
&= p(x)(x + a) + 2x^2 + x + 2ax + a - 5x - 2ax + 1 \\
&= p(x)(x + a) + 2x^2 + a - 4x + 1 \\
&= p(x)(x + a) + 2(p(x) + 2x + 1) + a - 4x + 1 \\
&= p(x)(x + a + 2) + 4x + 2 + a - 4x + 1 \\
&= p(x)(x + a + 2) + a + 3
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no root cancellations needed

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## Can also apply the same techniques to rational functions

Define inverses as: $\frac{1}{g(x)} := g_{\text{inv}}(x) \mod p(x) \longleftrightarrow g(x)g_{\text{inv}}(x) = 1 \mod p(x)$

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# Towards Finite Fields

## Polynomial division as row reduction

If all we care about is the remainder, we can work modulo $p(x)$ from the beginning

$$x^2 = p(x) + 2x + 1 \longrightarrow x^2 = 2x + 1 \mod p(x)$$

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Useful because there exist very quick ways to do row reduction: Sample over finite fields and reconstruct output

# Finite Field Reconstruction

## Operations on Matrices

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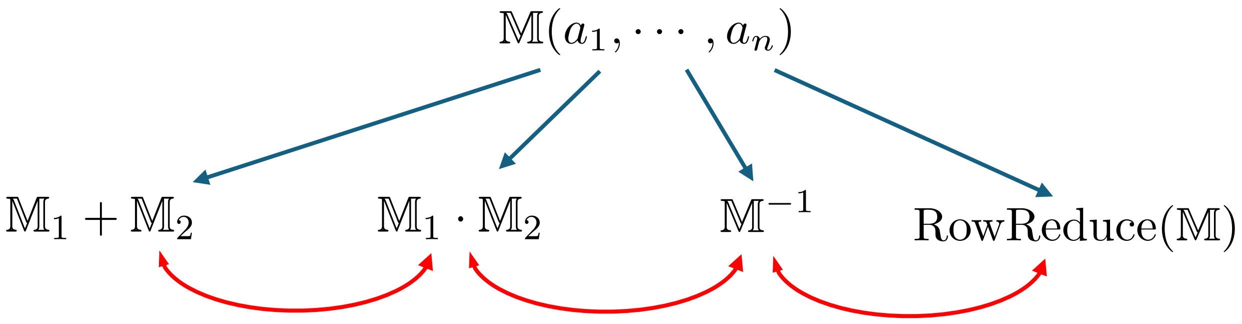


Algebraic post processing simplification — can become very intensive!

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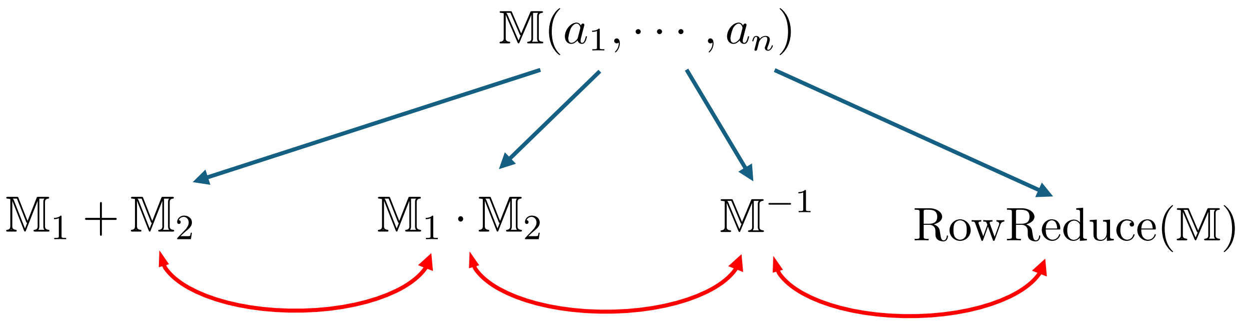


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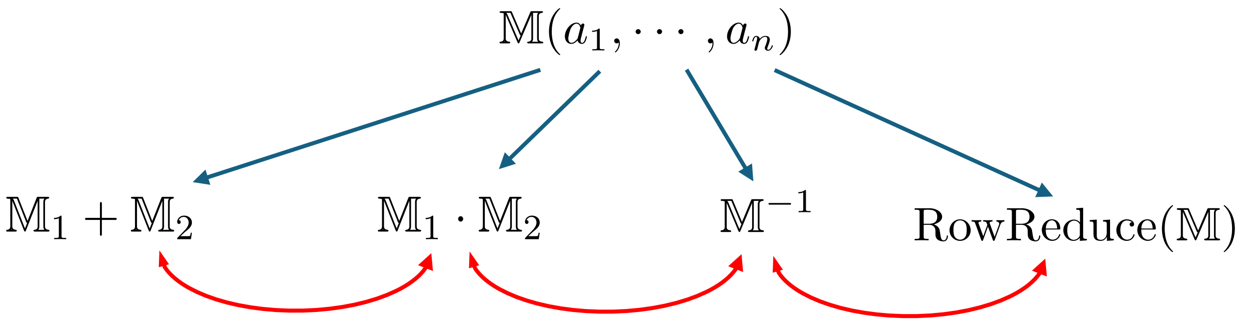


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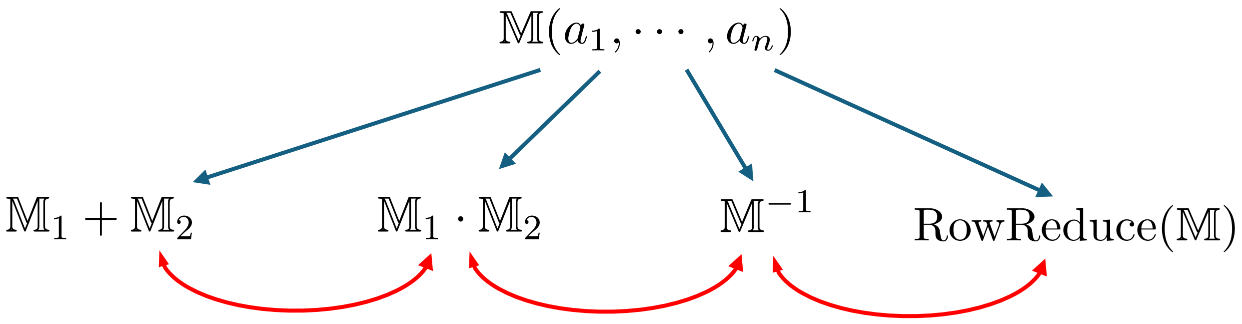


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parameters

variables

$$R = \mathbb{Q}[a][x]$$

Only need to reconstruct parameters!

# Multivariate Polynomial Division (1/3)

## Monomial Orderings

For one variable, sorting the monomials from “worst” to “best” is unambiguous

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Unfortunately, this is not enough to uniquely determine a multivariate polynomial division

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$xy = x \mod I$ $\qquad$ $xy = y + 1 \mod I$

Problem normally fixed by introducing Groebner Bases

[Buchberger, 1965]

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$$I = \langle xy - x, xy - y - 1 \rangle \longrightarrow G = \langle y^2 - 1, x - y - 1 \rangle \quad \text{(Lexicograhpic)}$$

Any possible combination of the elements of $G$ will result in the same polynomial remainder

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$$xy \overset{2}{=} (y+1)y = y + y^2 \overset{1}{=} y + 1 \mod G$$

# Multivariate Polynomial Division (2/3)

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A Groebner basis $G$ is a set of polynomials obtained from $I$ that has many nice properties

For this talk: Roots of $G = 0$ are the same as $I = 0$, and polynomial division ambiguities fixed

$$I = \langle xy - x, xy - y - 1 \rangle \longrightarrow G = \langle y^2 - 1, x - y - 1 \rangle$$ (Lexicograhpic)

Any possible combination of the elements of $G$ will result in the same polynomial remainder

$$xy \overset{2}{=} (y+1)y = y + y^2 \overset{1}{=} y + 1 \mod G$$

$$\begin{aligned} xy^2 &\overset{1}{=} x \overset{2}{=} y + 1 \\ xy^2 &\overset{2}{=} (y+1)y^2 \overset{1}{=} y + 1 \end{aligned} \mod G$$

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## Avoiding Groebner Bases

Claim: We can explicitly avoid computing a Groebner basis, and still obtain the correct result from polynomial division, using row reduction [Faugére, 1999] [Buchberger, 1985]

Allows us to compute polynomial divisions without needing to reconstruct the “intermediate” Groebner Basis

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$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f(x) \\ x^2y^2 \\ x^2y \\ x^2 \\ xy^2 \\ xy \\ x \\ y^2 \\ y \\ 1 \end{bmatrix}$$

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Irreducible monomials

Read off from top row: $f(x,y) = y + 1 \mod I$ No explicit Groebner Basis required!

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$$
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0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\
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Read off from top row: $f(x,y) = y + 1 \mod I$ No explicit Groebner Basis required!

Process can again be implemented in finite fields, with a reconstruction step at the end.

# Project Summary

## Program Overview

A (Mathematica) package that performs polynomial division over finite fields

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Finite Field Reconstruction

$$f(x_1, \cdots, x_n) = r(x_1, \cdots, x_n) \mod I$$

# Examples (1/5)

## Elimination Theory

Consider the following setup:

$$I = \langle p_1(x_1, \cdots, x_n), \cdots, p_n(x_1, \cdots, x_n) \rangle \qquad x_n > \cdots > x_1$$ (+ lexicograhpic ordering)

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Combining: $f(x_1) = x_1^m - r(x_1)$

$$\downarrow$$

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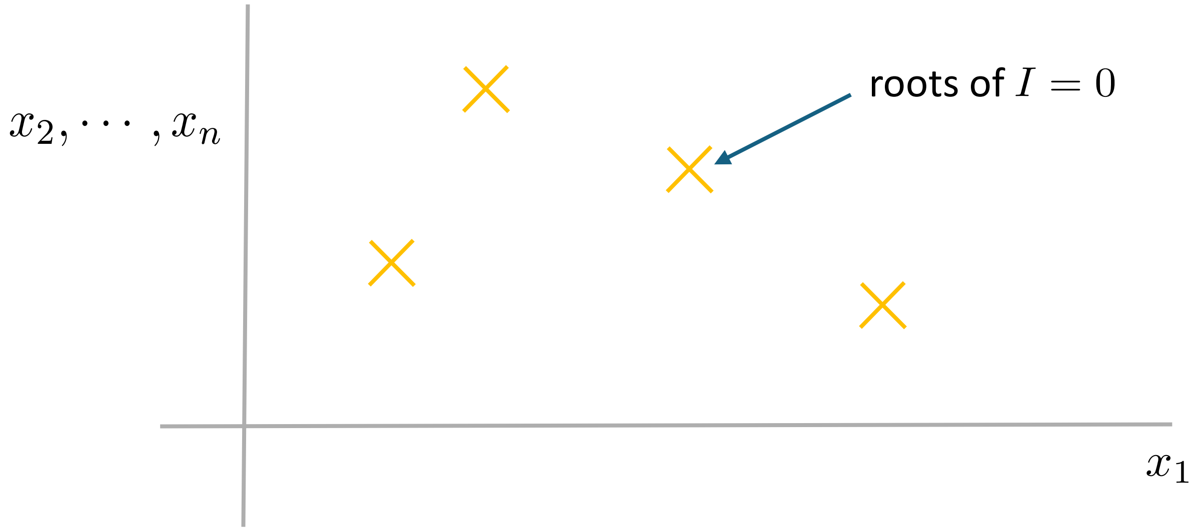

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equation for the roots in $x_1$ only

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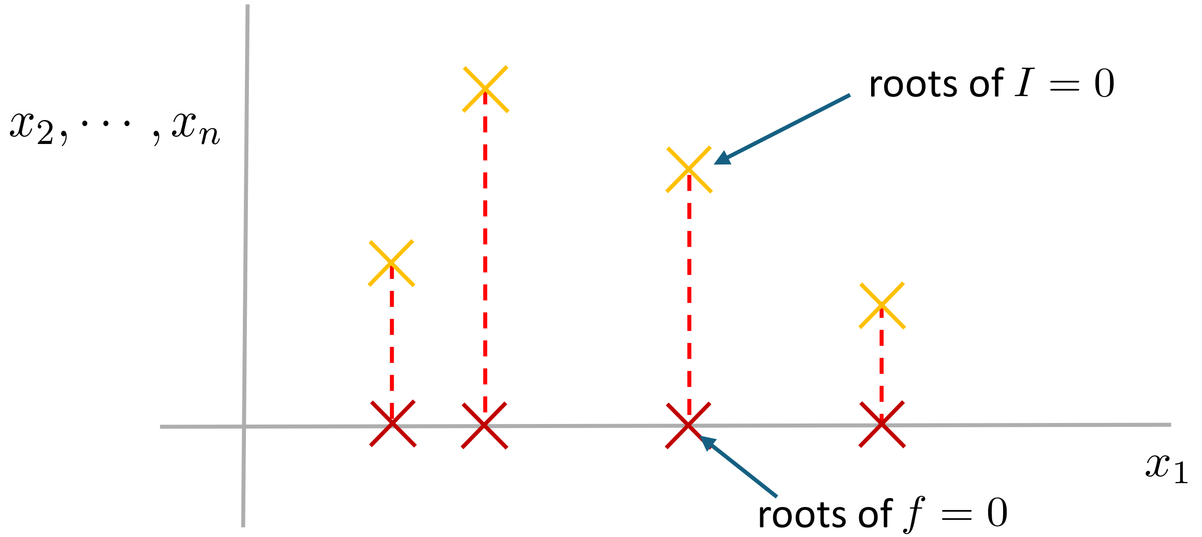

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## Elimination Benchmarking (Preliminary)

$$R = \mathbb{Q}[a, b, c, d][x, y, z]$$

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Task: Eliminate $\{x, y, z\}$ $\longrightarrow$ $\mathcal{R}(a, b, c, d)$

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Task: Eliminate $\{x,y,z\}$ ⟶ $\mathcal{R}(a,b,c,d)$

Finite Fields System Generation: Seed up to weight 17 ⟶ 2000 equations ⟶ 850 equations $\approx 1s$

| Resultant | Singular Time | Mathematica Time | Finite Fields Time (10 core) | Finite Fields Sample Points |
|---|---|---|---|---|
| $\mathcal{R}(3,5,7,d)$ | | | | |
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| $\mathcal{R}(a, b, c, d)$ | | | | |

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The Finite Fields approach is solving a larger set of equations, but isn't slowed down by intermediate cancellations

# Examples (3/5)

## Landau Singularities of Feynman (Euler) Integrals

A Feynman integral can be defined as an Euler/Twisted Period Integral

$$I(s_{ij}, m) = \int_0^\infty \mathcal{G}(x, s_{ij}, m)^{-d/2} \, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

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Parameters (Mandelstam variables)

Integration variables

[Lee, 2013]

# Examples (3/5)

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Twist: polynomial raised to a non integer (generic) power

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Parameters (Mandelstam variables)

Integration variables

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# Examples (3/5)

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For what values of $\{s_{ij}, m\}$ does $I$ diverge? ⟶ Landau Analysis [Landau, 1960] [Cutkosky, 1960]

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The Landau Variety can be defined as the values of $\{s_{ij}, m\}$ for which the Euler characteristic drops in value

[Chestnov, Matsubara-Heo, Munch, Takayama, 2023] [Mizera, Fevola, Telen, 2023/24]

# Examples (4/5)

## Computing Euler Characteristics for Landau Analysis

Computing $\chi$ : $\omega = \mathrm{d}\log\left(G(z)^{-d/2}\right)$ $\chi = \#\text{ solutions to }\omega = 0$ $\omega = -\frac{d}{2}\left(\frac{\partial_1 G}{G}dx_1 + \cdots + \frac{\partial_n G}{G}dx_n\right)$

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# Examples (4/5)

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system of polynomial equations!

# Examples (4/5)

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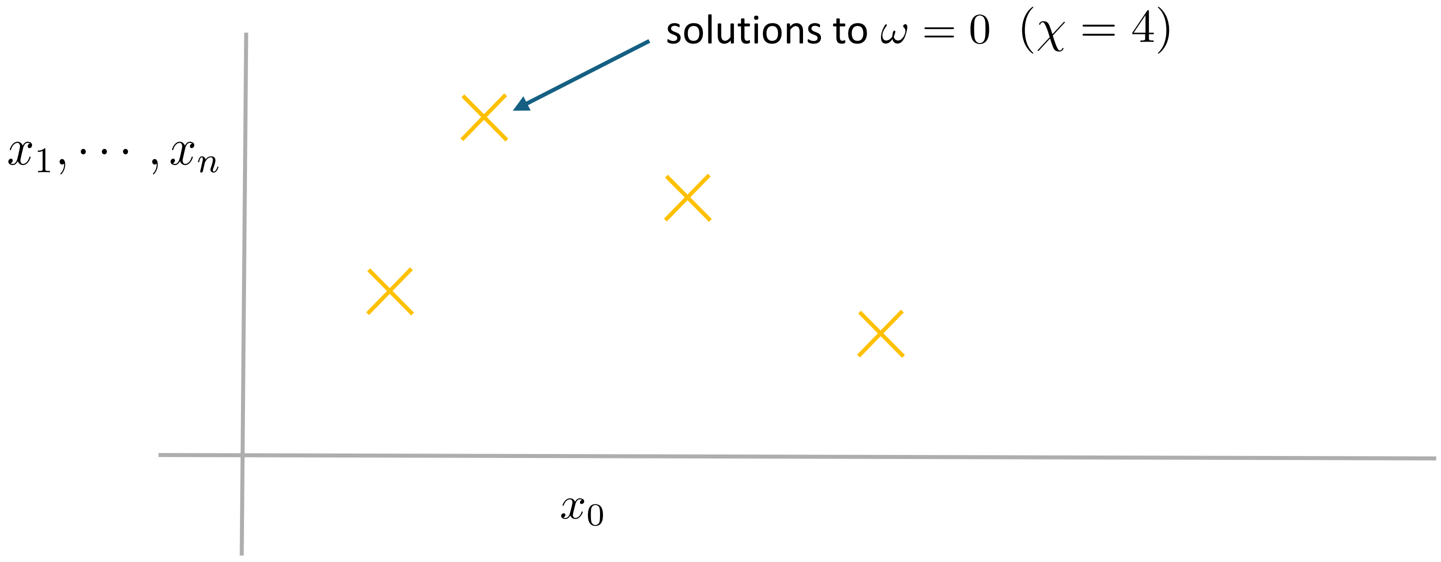
system of polynomial equations!



# Examples (4/5)

## Computing Euler Characteristics for Landau Analysis

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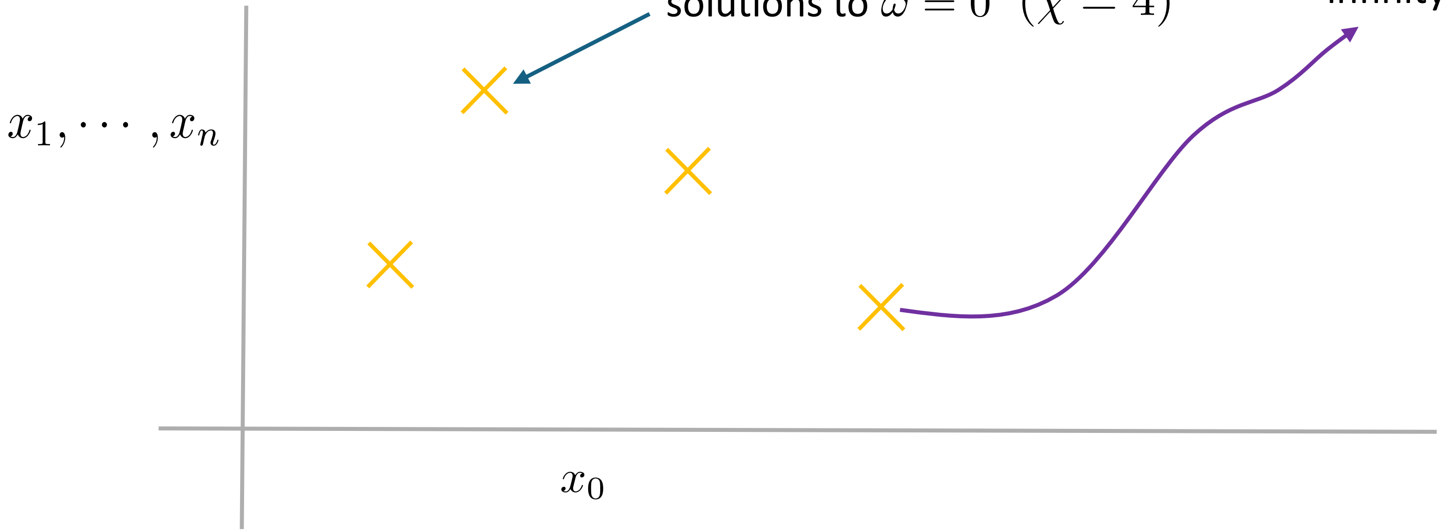
system of polynomial equations!

as the degenerate configuration is approached some solutions fly off to infinity

solutions to $\omega = 0 \quad (\chi = 4)$



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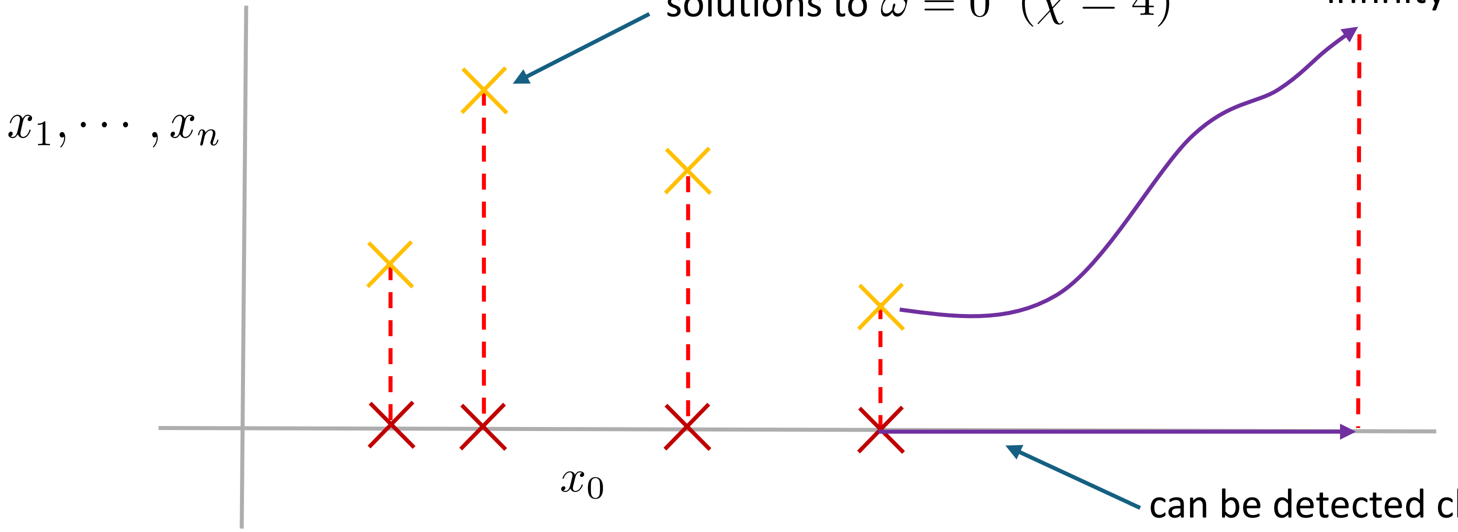

# Examples (5/5)

Computing Euler Characteristics for Landau Analysis: Three loop envelope (preliminary)

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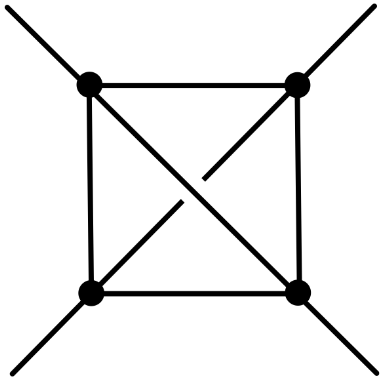

[Correia, Sever, Zhibodeev, 2021]

Horrendous integral: $\chi = 60(!)$ in the top (max cut) sector alone

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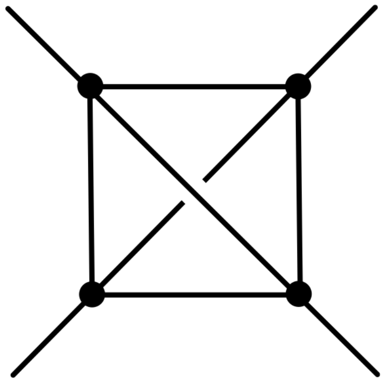

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SOFIA/PLD most complicated letter found: $27(m^2)^3 + 4s^2t + 4st^2$

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# Examples (5/5)

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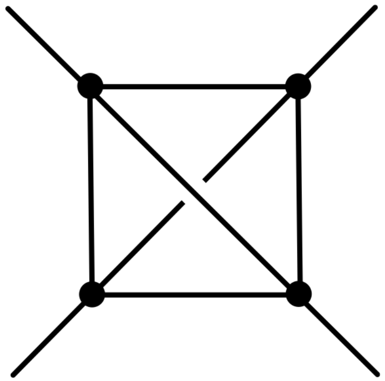

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## Euler characteristic strategy

Two new simple letters: $\{s^2 + st + t^2, m^2s^2 + m^2st + s^2t + m^2t^2 + st^2\}$

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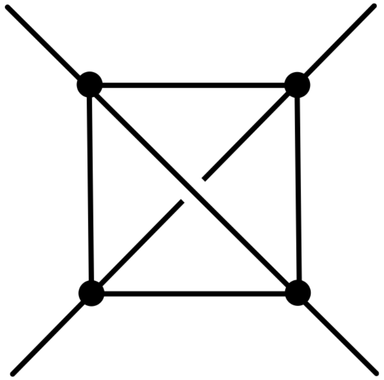

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Two new simple letters: $\{s^2 + st + t^2, m^2s^2 + m^2st + s^2t + m^2t^2 + st^2\}$

Four new complicated letters:

$$\{27\, m2^4 s^2 + 108\, m2^4 s\, t + 162\, m2^3 s^2 t + 54\, m2^2 s^3 t + 4\, m2\, s^4 t + 108\, m2^4 t^2 + 162\, m2^3 s\, t^2 + 45\, m2^2 s^2 t^2 -$$
$$6\, m2\, s^3 t^2 - s^4 t^2 - 18\, m2^2 s\, t^3 - 20\, m2\, s^2 t^3 - 2\, s^3 t^3 - 9\, m2^2 t^4 - 10\, m2\, s\, t^4 - s^2 t^4,\ 108\, m2^4 s^2 - 9\, m2^2 s^4 + 108\, m2^4 s\, t + 162\, m2^3 s^2 t -$$
$$18\, m2^2 s^3 t - 10\, m2\, s^4 t + 27\, m2^4 t^2 + 162\, m2^3 s\, t^2 + 45\, m2^2 s^2 t^2 - 20\, m2\, s^3 t^2 - s^4 t^2 + 54\, m2^2 s\, t^3 - 6\, m2\, s^2 t^3 - 2\, s^3 t^3 + 4\, m2\, s\, t^4 - s^2 t^4,$$
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$$65\,536\, m2^{12} + 270\,336\, m2^{10} s^2 + 33\,024\, m2^8 s^4 + 1024\, m2^6 s^6 + 270\,336\, m2^{10} s\, t - 458\,752\, m2^9 s^2 t + 66\,048\, m2^8 s^3 t - 1\,276\,416\, m2^7 s^4 t + 3072\, m2^6 s^5 t - 137\,472\, m2^5 s^6 t -$$
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$$3072\, m2^6 s\, t^5 - 412\,416\, m2^5 s^2 t^5 + 448\,416\, m2^4 s^3 t^5 - 92\,320\, m2^3 s^4 t^5 + 7680\, m2^2 s^5 t^5 - 336\, m2\, s^6 t^5 + 4\, s^7 t^5 + 1024\, m2^6 t^6 - 137\,472\, m2^5 s\, t^6 + 149\,472\, m2^4 s^2 t^6 -$$
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Thank you for listening!