Feynman Integral Reduction using Syzygy-Constrained Symbolic Reduction Rules

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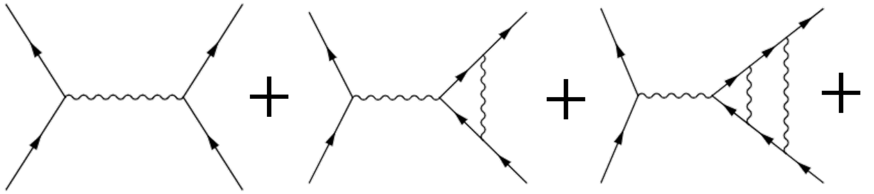
Based on 2507.XXXX with *M. Zeng*

Motivation: Amplitudes

- Observables are given by Amplitudes.
- In perturbative Quantum Field Theory, Amplitudes are given by a sum of Feynman Diagrams

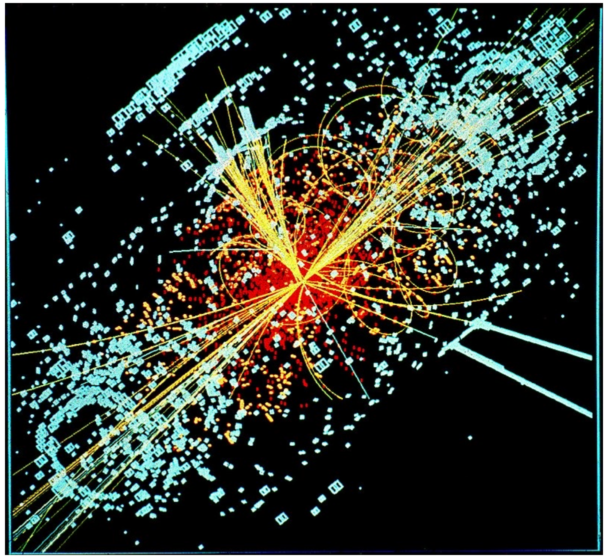
$\mathcal{A}(e^+e^- \to \mu^+\mu^-) =$

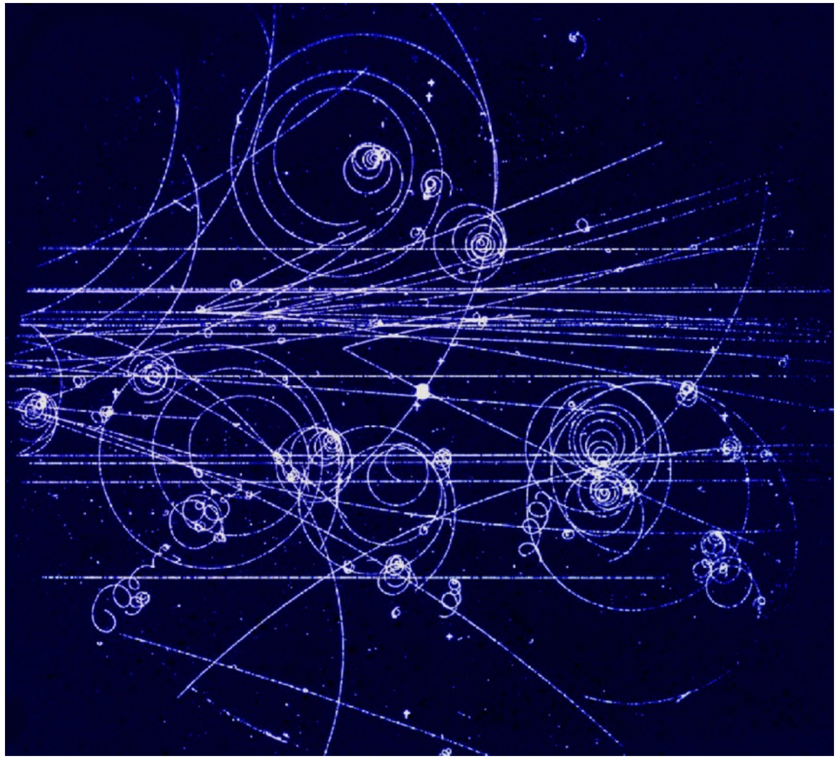


$+ \ldots$

- Each Feynman Diagram corresponds to a Feynman Integral

$$F(n_1, \ldots, n_N) = \int \prod_{a=1}^{L} d^D\ell_a \frac{1}{\rho_1^{n_1} \ldots \rho_N^{n_N}}$$



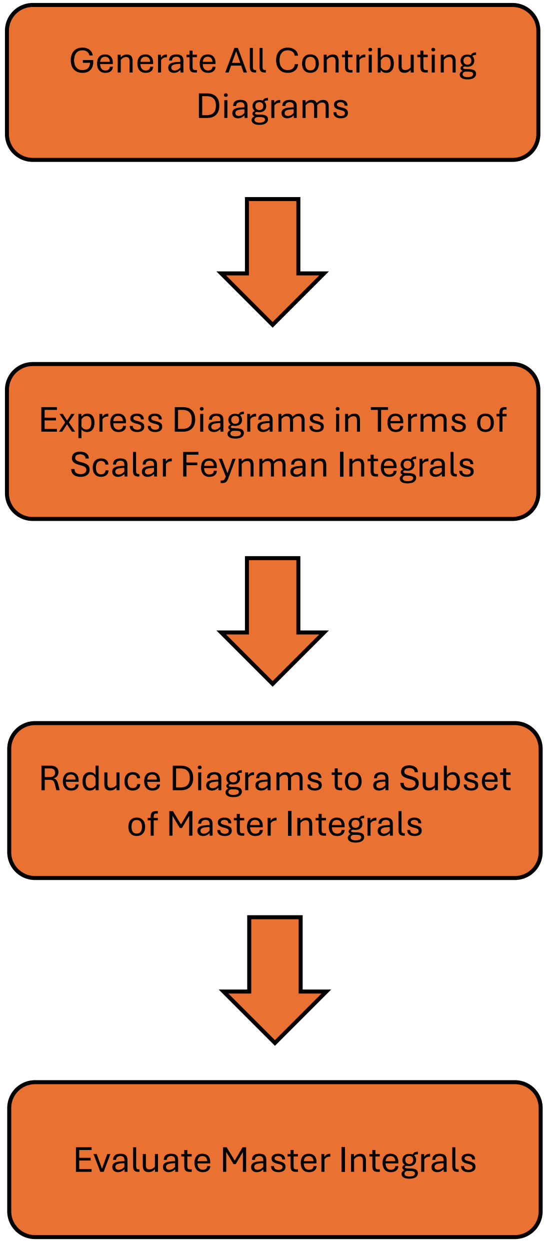


Motivation: Integration-by-parts

- Feynman Integrals $I(n_1, \ldots, n_N)$ belong to a Vector Space, the topology defines the space, and the indices n_i define the element. [Smirnov, Petukhov, 2010] [Mastrolia, Mizera, 2019]
- There exists a basis on this vector space, known as the *Master Integrals*.

$$F = \sum_i c_i J_i$$

- Integration-by-parts (IBP) Identities can be used to find the coefficients c_i [Chetyrkin, Tkachov, 1981]



Part I:
Integration-by-Parts: An Overview

[Chetyrkin, Tkachov, 1981]

[Laporta, 2000]

[LiteRed, Reduze, FIRE, Kira, NeatIBP, Blade, FiniteFlow]

IBPs: Integral Families

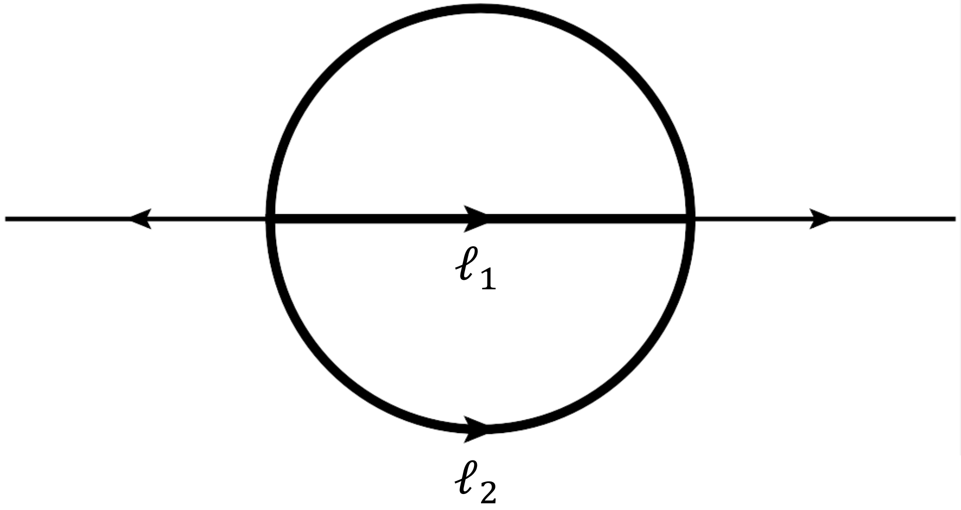
An *integral family/topology* is defined by

- A set of *loop momenta* $\ell_1, \ldots, \ell_L$.
- A set of independent *external momenta* $p_1, \ldots, p_E$.
- A set of *propagators* $\rho_1, \ldots, \rho_N$, of which some can appear as *denominators,* and some as *numerators*.

For Example:

$$\rho_1 = \ell_1^2, \qquad \rho_2 = \ell_2^2, \qquad \rho_3 = (\ell_1 + \ell_2 - p)^2$$

$$\rho_4 = \ell_1 \cdot p, \qquad \rho_5 = \ell_2 \cdot p$$



IBPs: The Laporta Algorithm

$$0 = \int \prod_{a=1}^{L} d^D \ell_a \frac{\partial}{\partial \ell_b^\mu} \frac{q_\alpha^\mu}{\rho_1^{n_1} \cdots \rho_N^{n_N}}$$

$$q_\alpha : \{\ell_1, \dots, \ell_L, p_1, \dots, p_E\}$$

- IBP Identities give us relations between integrals in a given family.

$$0 = \sum_i (\alpha_i + \vec{\beta}_i \cdot \vec{n}) F[\vec{n} + \vec{\gamma}_i]$$

- We can input values of the initial vector $\vec{n}$ into these identities to generate a system of equations, this is known as seeding.

$$\Downarrow \textit{Seeding } \vec{n}\textit{-values}$$

$$\begin{pmatrix} \# & \cdots & \# \\ \vdots & \ddots & \vdots \\ \# & \cdots & \# \end{pmatrix} \begin{pmatrix} I(\dots) \\ \vdots \\ I(\dots) \end{pmatrix} = 0$$

- The aim is to row-reduce this matrix until there exists an equation of the form

$$(Target) - \sum_i c_i J_i = 0$$

IBPs: Limitations

Current State-of-the-art:

2-Loop: 5- or 6-point

3-Loop: 4-point

[Abreu, Ita, Moriello, Page, Tschernow, Zeng, 2020]

[Chakraborty, Gambuti, 2022]

[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi, 2023]

[De Laurentis, 2024]

[Bercini, 2024]

[Henn, Matijašić, Miczajka, Peraro, Xu, Zhang, 2024]

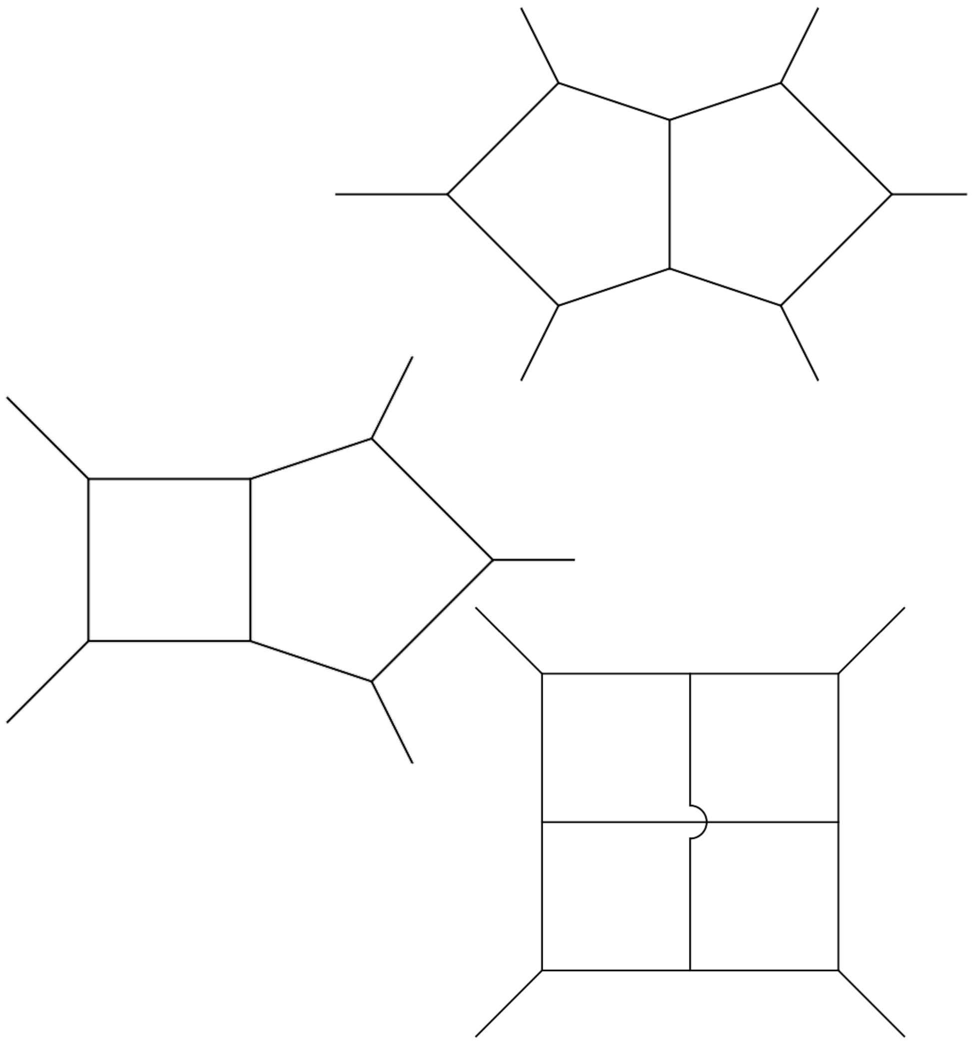
[Gehrmann, Henn, Jakubčík, Lim, Mella, 2024]

[Henn, Torres Bobadilla, Lim, 2023/24]

Multi-loop: Increased number of Equations and Variables, larger System of Equations.

Multi-scale: More External Legs and Masses mean more parameters to keep track of when performing row reduction.

High Complexity: For physical Amplitudes calculations, one has to deal with integrals with large powers of numerators and denominators



IBP Tools: Sectors & Cuts

A sector of an integral is fully described by it's denominator powers

$$F(\vec{n}) \rightarrow \vec{m} = \vec{n}\Big|_{-ve\rightarrow 0}$$

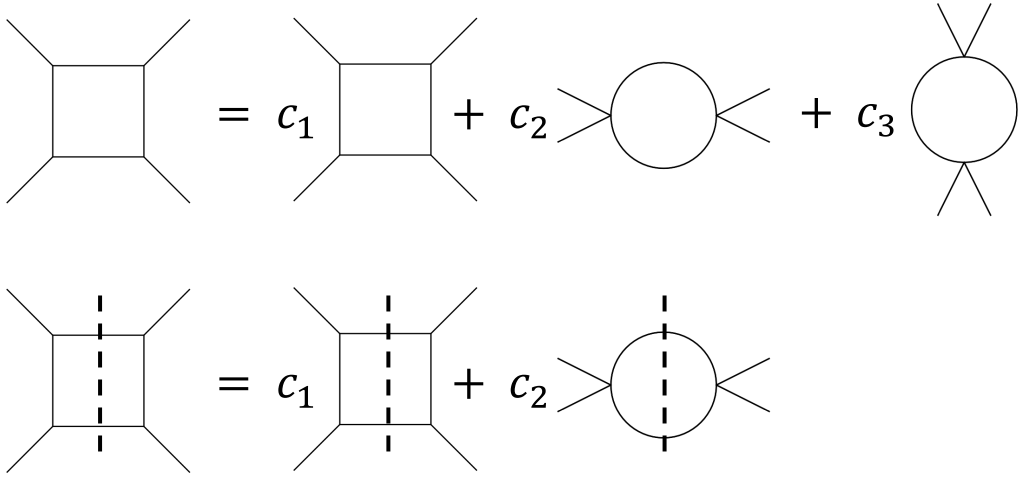
Loosely speaking, cutting a propagator means enforcing that this propagator is on-shell

$$\frac{1}{\rho_i} \rightarrow \delta(\rho_i)$$

Cuts commute with IBPs.

$$I_C = \sum_i c_i J_{C,i}$$

$$F(2,1,1,-1,-3) \rightarrow (2,1,1,0,0)$$



IBP Tools: Syzygies [Singular] [Gluza, Kajda, 2011]

$$0 = \int \prod_{b=1}^{L} d^D \ell_b \frac{\partial}{\partial \ell_a^\mu} \frac{P_{a\alpha}(\rho) q_\alpha^\mu}{\rho_1^{n_1} \cdots \rho_N^{n_N}}, \qquad P_{a\alpha}(\rho) q_\alpha^\mu \frac{\partial}{\partial \ell_a^\mu} \rho_i = f_i(\rho) \rho_i, \qquad \forall\, i \in \sigma$$

$$G_i \in \left\{ q_1^\mu \frac{\partial}{\partial \ell_1^\mu}, \dots, q_{L+E}^\mu \frac{\partial}{\partial \ell_L^\mu} \right\}, \qquad P_i(\rho) \in \{ P_{11}(\rho), \dots, P_{L,L+E}(\rho) \}$$

$$\vec{c}^T M = 0, \qquad c_i = \begin{cases} P_i(\rho), & i \le L(L+E), \\ f_{i-R}(\rho), & \text{otherwise}, \end{cases} \qquad M_{ij} = \begin{cases} G_i(\rho_{\sigma(j)}), & i \le R \\ -\rho_{\sigma(j)} \delta_{i-R,j}, & \text{otherwise} \end{cases}$$

$$M_C = M \Big|_{\rho_i \to 0, i \in C}$$

IBP Tools: Seeding

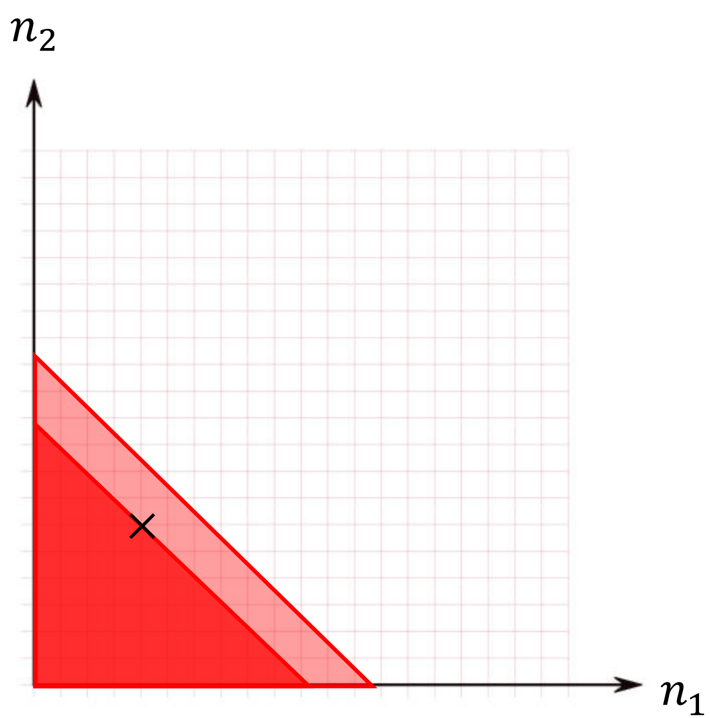
[Von Hippel, Wilhelm, 2025]

[Song, Yang, Cao, Luo, Zhu, 2025]

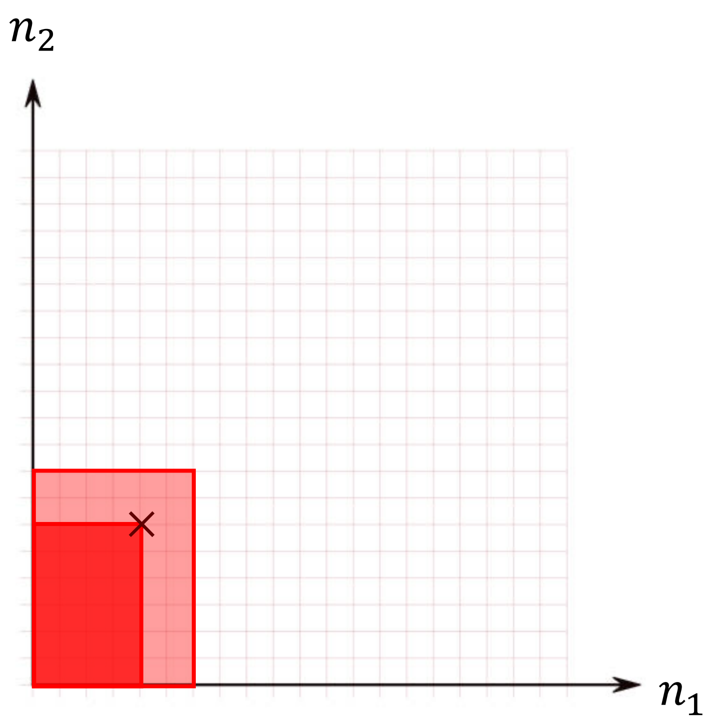
[Zeng, 2025]

[Kira]

$$I(n_1, n_2)$$



Laporta Seeding



Rectangular Seeding

IBP Tools: Seeding

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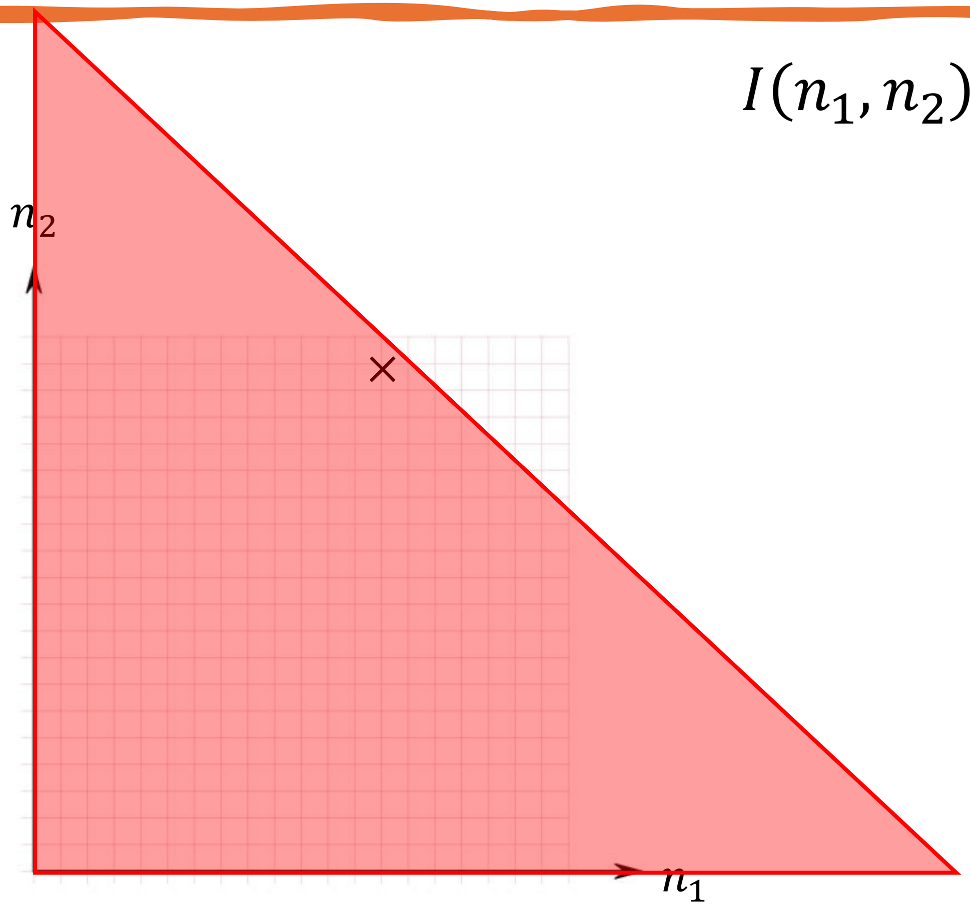
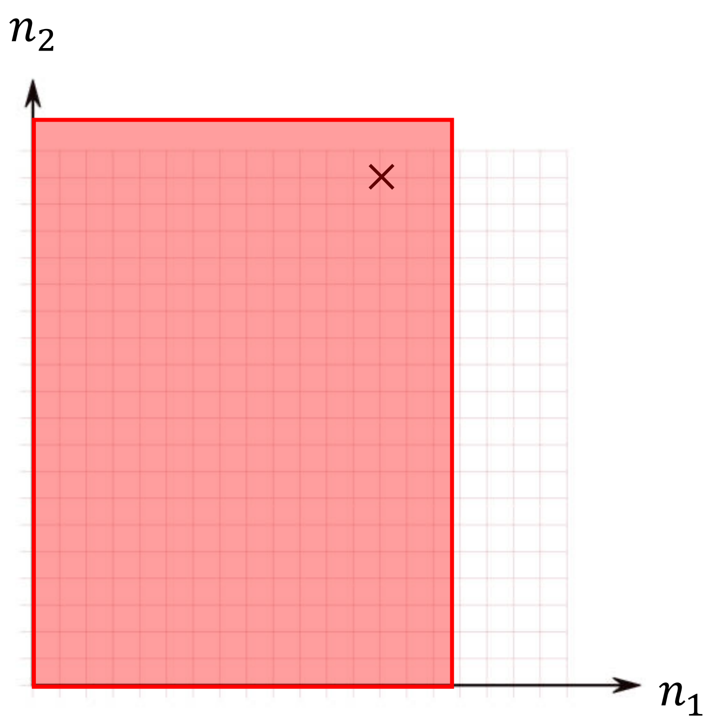


Laporta Seeding



Rectangular Seeding

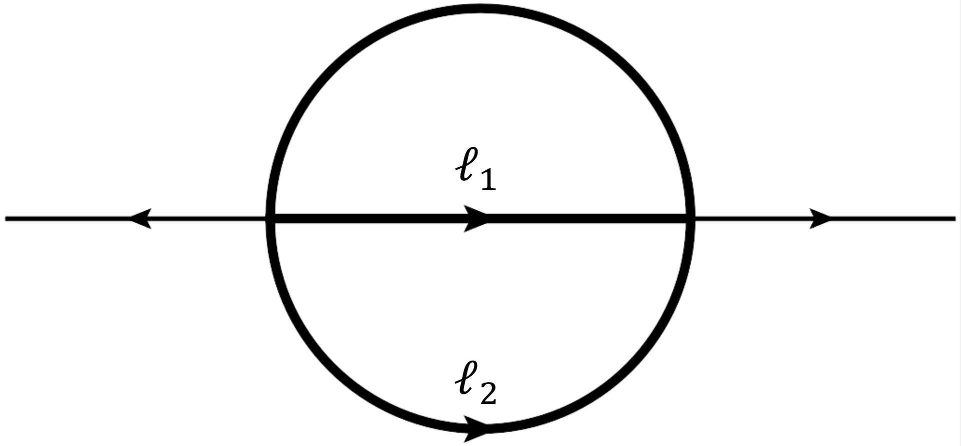
Part 2:
Algorithm for Reduction Rules

[LiteRed] [FIRE] [Kosower, 2018]

Algorithm for Reduction Rules

1. Building a complete set of reduction rules to reduce any arbitrary integral.

2. Applying these reduction rules to reduce a specific set of target integrals to master integrals



$$\rho_1 = \ell_1^2, \qquad \rho_2 = \ell_2^2, \qquad \rho_3 = (\ell_1 + \ell_2 - p)^2,$$

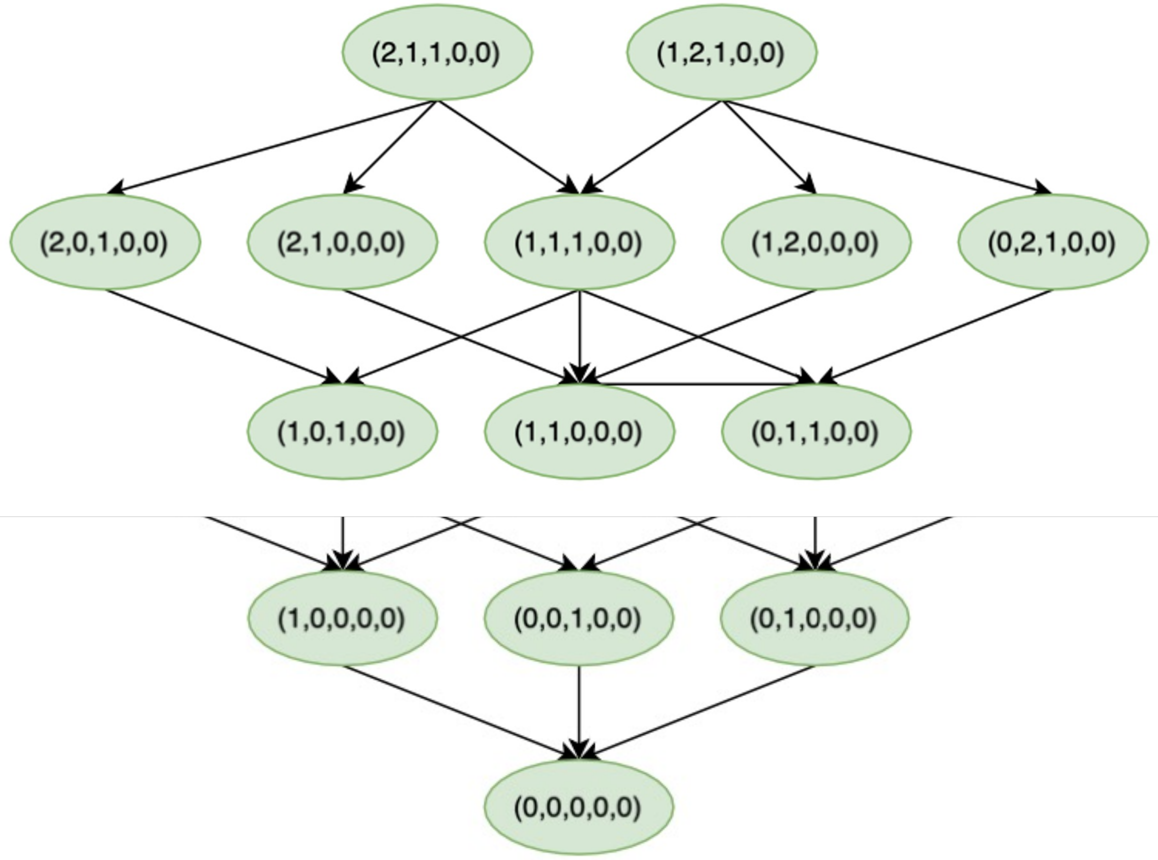
$$\rho_4 = \ell_1 \cdot p, \qquad \rho_5 = \ell_2 \cdot p$$

$$\{F(2,1,1,-4,-4), F(1,2,1,0,-7), F(1,1,1,-6,-4), F(1,1,1,-11,0)\}$$

$$F(n_1, n_2, n_3, n_4, n_5) = \int \prod_{a=1}^{2} d^D \ell_a \frac{1}{\rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5}}$$

$$\{F(2,1,1,-4,-4), F(1,2,1,0,-7), F(1,1,1,-6,-4), F(1,1,1,-11,0)\}$$

Reduction Rules: Sectors



Reduction Rules: Generating Identities

In each sector $\vec{m}$, we generate identities by solving the syzygy equations

$$P_{a\alpha}(\rho)q_\alpha^\mu \frac{\partial}{\partial \ell_a^\mu}\rho_i = f_i(\rho)\rho_i, \qquad \forall\, i \in \sigma = \{i|m_i > 0\}$$

$$\Rightarrow 0 = \sum_i (\alpha_i + \vec{\beta}_i \cdot \vec{n})F[\vec{n} + \vec{\gamma}_i]$$

We are free to fix indices for the sector we are in

$$n_i = \begin{cases} m_i, & i \in \sigma \\ \eta_i, & i \notin \sigma \end{cases}, \qquad \alpha_i \to \alpha_i + \vec{\beta}_i \cdot \vec{m}$$

Keep only identities that contain at least one $\vec{\gamma}_i$ such that $(\vec{\gamma}_i)_j = 0, \forall\, j \in \sigma$

Reduction Rules: Ordering Integrals

To write reduction rules, we need a notion of how the integrals are ordered. We therefore define a *weight function*

$$W(\vec{n}) = \left(\sum_{n_i > 0} 1 \,, \sum_{n_i > 0} (n_i - 1) \,, - \sum_{n_i < 0} n_i \,, \mathcal{O}(|\vec{n}|) \right)$$

We also need a notion of how to order shift vectors $\vec{\gamma}$ on each sector, for this we define the *sector-specific weight function*

$$w(\vec{\gamma}) = \left(\vec{\gamma} \cdot \vec{\xi}, -\vec{\gamma} \cdot \vec{\theta}, -\mathcal{O}(\vec{\gamma})\right), \qquad \vec{\xi} = \vec{m}\Big|_{+ve \to 1}, \qquad \theta_i = 1 - \xi_i$$

Reduction Rules: Ordered Identities

We then order the identities such that the highest weight shift vector comes first

$$0 = \sum_i (\alpha_i + \vec{\beta}_i \cdot \vec{\eta}) F[\vec{n} + \vec{\gamma}_i] \Rightarrow \left(\alpha_1 + \vec{\beta}_1 \cdot \vec{\eta}\right) F[\vec{n} + \vec{\gamma}_1] + \text{lower}-\text{weight integrals} = 0$$

$$\{F(2,1,1,-4,-4), F(1,2,1,0,-7), F(1,1,1,-6,-4), F(1,1,1,-11,0)\} \Rightarrow \mathcal{O}(\vec{n}) = (n_1, n_2, n_3, n_5, n_4)$$

For example, on sector $\vec{m} = (1,1,1,0,0)$, we get

$$2(2n_4 - n_5) F(1,1,1,n_4,n_5 - 1) + \text{lower}-\text{weight integrals} = 0$$

$$6s(2n_4 - n_5) F(1,1,1,n_4,n_5 - 1) + \text{lower}-\text{weight integrals} = 0$$

$$4(1 - D - n_5) F(1,1,1,n_4,n_5 - 1) + \text{lower}-\text{weight integrals} = 0$$

$$F(1,1,1,n_4,n_5)$$

$$\Downarrow$$

$$F(1,1,1,n_4,2n_4 - 1), F(1,1,1,n_4,0)$$

$$\Downarrow$$

$$F(1,1,1,0,-1), F(1,1,1,0,0), F(1,1,1,n_4,0)$$

Reduction Rules: Row Reduced Identities

Before moving on to this step, we are free to fix further indices depending on the integrals that are currently irreducible. For example, for $F(1,1,1,n_4,0)$, we set $n_5 = 0$.

We then make small perturbations around this fixed point, by inputting the following seed integrals

$$\{(1,1,1,n_4,0),(1,1,1,n_4,-1),(1,1,1,n_4-1,0)\}$$

This generates more identities, with $\vec{\eta} = (n_4)$ now

$$0 = \sum_i (\alpha_{ki} + \vec{\beta}_{ki} \cdot \vec{\eta}) F[\vec{n} + \vec{\gamma}_{ki}], \qquad k = 1, \dots, I$$

Reduction Rules: Row Reduced Identities

We then order all M shift vectors $\vec{\gamma}_i$ according to sector-specific weight, and write the identities in matrix form

$$0 = \sum_i (\alpha_i + \vec{\beta}_i \cdot \vec{\eta}) F[\vec{n} + \vec{\gamma}_i] \Rightarrow \begin{pmatrix} (\alpha_{11}, \vec{\beta}_{11}) \cdot (1, \vec{\eta}) & \cdots & (\alpha_{1M}, \vec{\beta}_{1M}) \cdot (1, \vec{\eta}) \\ \vdots & \ddots & \vdots \\ (\alpha_{I1}, \vec{\beta}_{I1}) \cdot (1, \vec{\eta}) & \cdots & (\alpha_{IM}, \vec{\beta}_{IM}) \cdot (1, \vec{\eta}) \end{pmatrix} \begin{pmatrix} F(\vec{n} + \vec{\gamma}_1) \\ \vdots \\ F(\vec{n} + \vec{\gamma}_M) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_{11} & \vec{\beta}_{11} & \cdots & \alpha_{1M} & \vec{\beta}_{1M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{11} & \vec{\beta}_{I1} & \cdots & \alpha_{IM} & \vec{\beta}_{IM} \end{pmatrix}$$

We then row reduce this matrix using FiniteFlow and reconstruct identities that are useful to resolve the irreducible integrals so far.

Reduction Rules: Direct Solution

If we are left with any irreducible integrals after the previous steps, then we move on to this final step.

Given a specific irreducible integral, we insert seeds in the vicinity of this fixed point, keeping the analytic dependence on the indices n_i in the equations.

For example, the integral $F(1,1,1,n_4,0)$ is still irreducible after the first two steps, so we input the following seeds

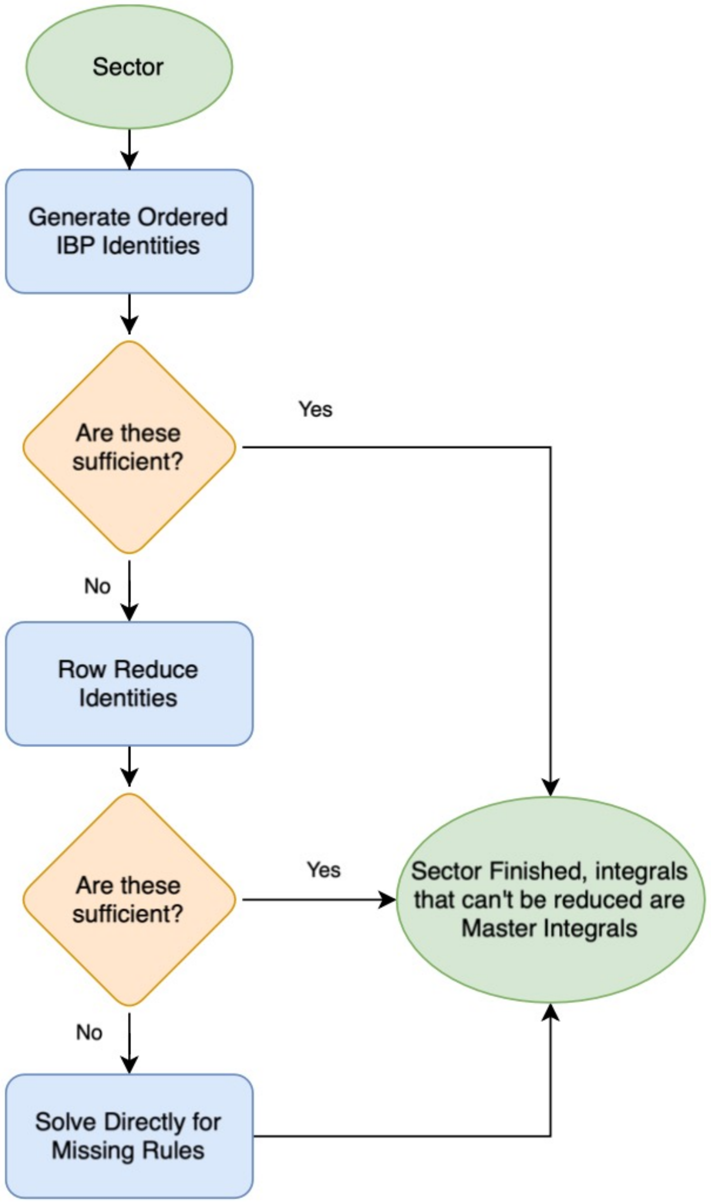
$$\{(1,1,1,n_4,0),(1,1,1,n_4-1,0)\}$$

Solving the resulting system using FiniteFlow, we are able to recover the reduction rule

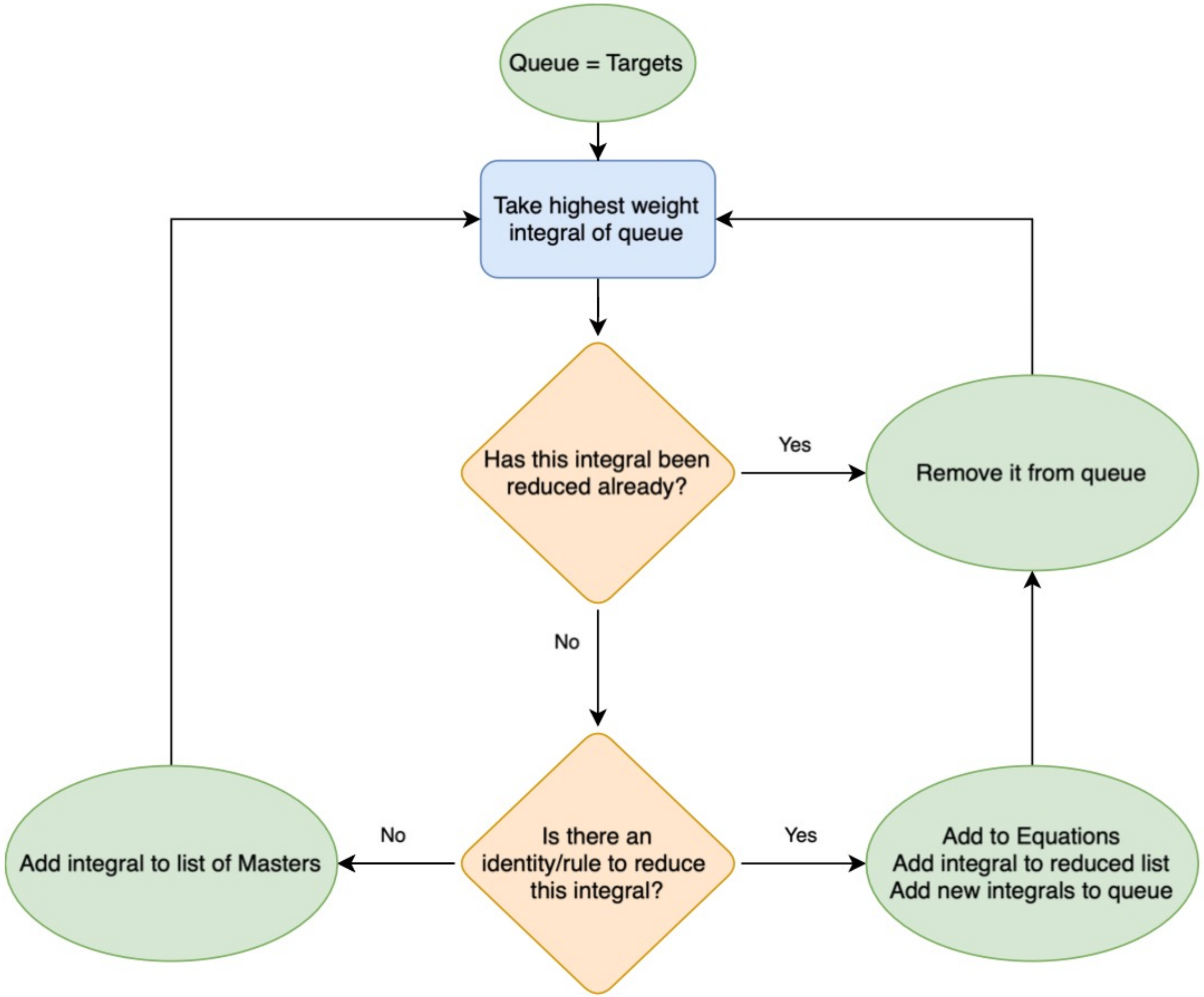
$$F(1,1,1,n_4-2) \rightarrow \text{lower}-\text{weight integrals}$$

This also works to resolve irreducible integrals with no n_i dependence, such as $F(1,1,1,-1,0)$. If an integral can not be reduced it is inferred as a master integral.

Reduction Rules: Summary



Applying Rules: Generating Equations



Applying Rules: Back Substitution

Forward-Elimination

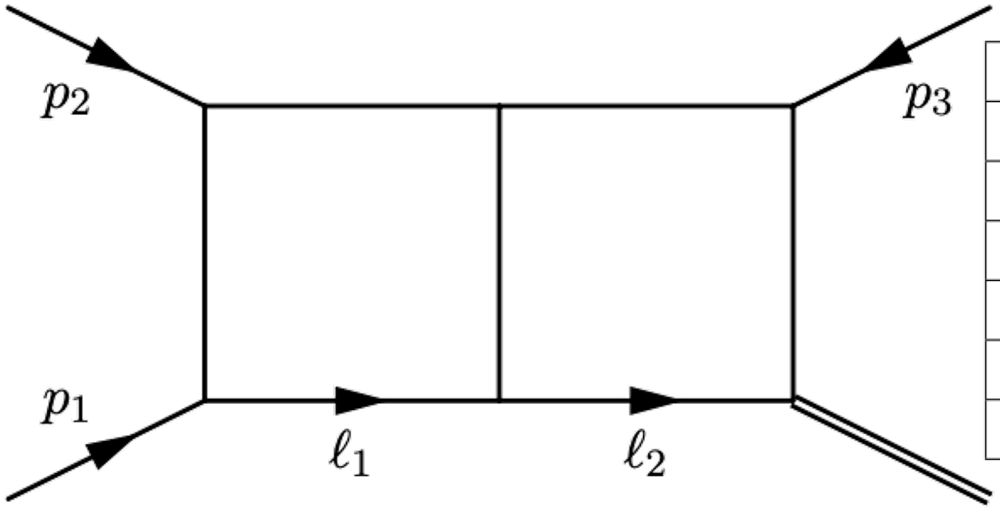
Back-Substitution

$$\begin{pmatrix} 0 & \# & \# & \# & \# \\ \# & \# & \# & 0 & \# \\ \# & 0 & 0 & \# & \# \\ \# & \# & 0 & \# & 0 \\ 0 & 0 & \# & 0 & \# \end{pmatrix} \Longrightarrow \begin{pmatrix} \# & \# & \# & \# & \# \\ 0 & \# & \# & \# & \# \\ 0 & 0 & \# & \# & \# \\ 0 & 0 & 0 & \# & \# \\ 0 & 0 & 0 & 0 & \# \end{pmatrix} \Longrightarrow \begin{pmatrix} \# & 0 & 0 & 0 & \# \\ 0 & \# & 0 & 0 & \# \\ 0 & 0 & \# & 0 & \# \\ 0 & 0 & 0 & \# & \# \\ 0 & 0 & 0 & 0 & \# \end{pmatrix}$$

Reduction Rule Equations

Part 3:
Examples

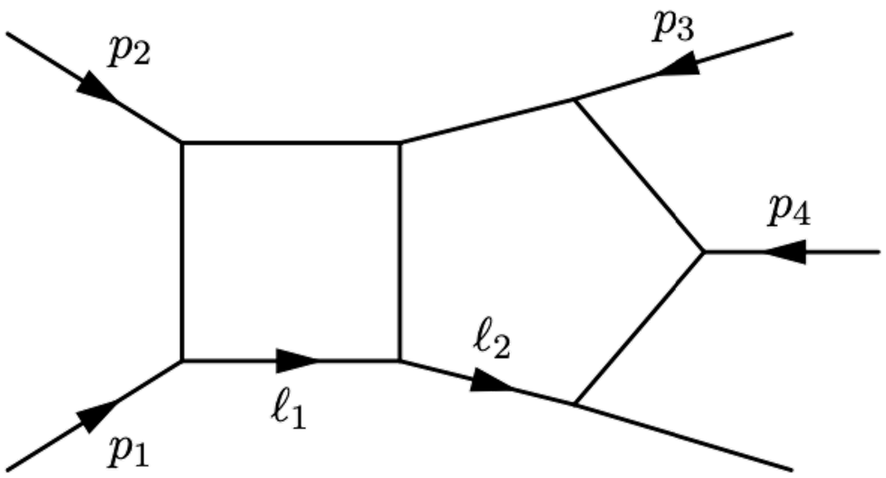
Examples: Double Box with External Mass



Cut	Time Taken	Number of Equations	Number of Masters
{5,7}	284s	18971	14
{1,4,7}	60s	8120	4
{3,6,7}	178s	12287	8
{4,6,7}	57s	2643	4
{1,3,4,6}	26s	1153	3
{1,3,5,6}	35s	2031	5

$$\{F(1,1,1,1,1,1,1,-10,-10), F(1,2,1,1,1,1,1,-6,-6), F(1,1,1,1,1,1,1,-2,-15)\}$$

Examples: Massless Pentabox

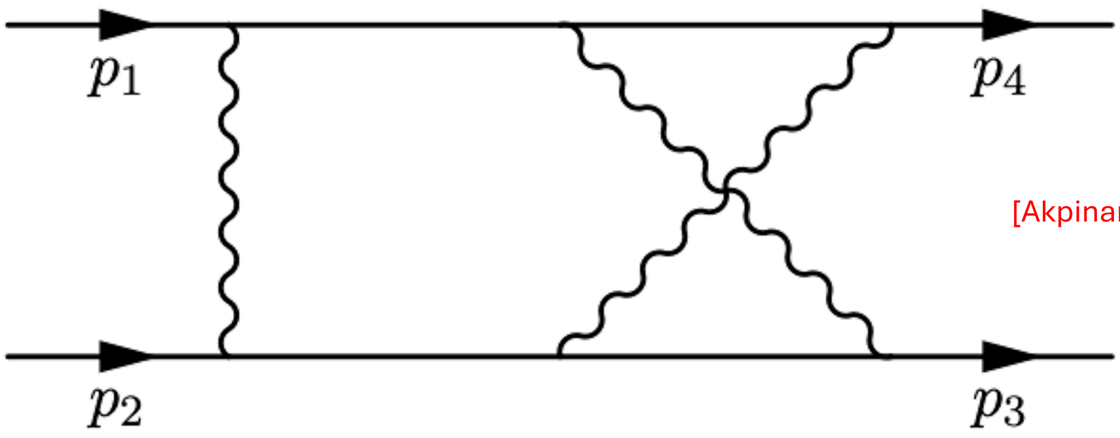


Cut	Time Taken	Number of Equations	Number of Masters
{1,4,8}	6997s	51619	21
{1,5,8}	2912s	39446	27
{2,5,8}	23721s	112188	31
{1,3,4,6}	974s	3979	13
{1,3,4,7}	1063s	4275	9
{2,4,7,8}	7064s	28338	12

$$\{F(1,1,1,1,1,1,1,1,-10,-10,0), F(1,1,1,1,1,1,1,1,-5,-6,-3)\}$$

[Brunello, Chestnov, Mastrolia, 2024]

Examples: Spinning Black Hole



[Akpinar, Cordero, Kraus, Smirnov, Zeng, 2024]

47365 Target Integrals $\{F(11,1,1,1,1,1,1,-10,0), F(3,5,1,5,1,1,1,-7,-3), \ldots\}$

10 days ⇒ 11 hours

Conclusion

- We presented a novel algorithm for reducing Feynman integrals by generating symbolic reduction rules that can be applied to an arbitrary set of Feynman integrals.

- The motivation behind this algorithm is for the reduction of Feynman integrals with high powers of numerators and denominators.

- We tested the algorithm against two highly non-trivial examples of rank-20 integrals for the double box with an external mass and the massless pentabox.

- We also presented an application of this algorithm to a physical problem, the spinning black hole

Outlook

- This algorithm can be incredibly effective for the computation of amplitudes in non-renormalizable field theories such as gravity.

- Currently, the reduction rule part of the algorithm is the bottleneck, but we foresee plenty of ways to improve the implementation of this.

- A similar approach with Laporta identities without the syzygy constraints could prove useful for more complex topologies, where Singular struggles to solve the syzygy equations.

Thank you! ☺