# Wall crossing structure from quantum phenomena to Feynman Integrals

**Anthony Massidda**

Domodossola 14/07/2025

Based on

arXiV: 2506.03252
with Angius Roberta and Sergio L. Cacciatori

amassidda@uninsubria.it
anthony.massidda@mi.infn.it

UNIVERSITÀ DEGLI STUDI DELL'INSUBRIA

INFN Istituto Nazionale di Fisica Nucleare

# Outline of the talk

- Introduction and motivations
- Exponential integrals for holomorphic functions: Pearcey Integral
- Exponential integrals for closed forms: Legendre Family
- Conclusions and Outlooks

*Consider the exponential integral:*

$$I(\gamma) = \int_\Gamma e^{-\gamma f} \mu$$

Where:

- $f : X \to \mathbb{C}$
- $\Gamma$ *is a n-chain with* $D_0 \supset \partial\Gamma \neq 0$
- $\gamma \in \mathbb{C}^*$
- $\mu$ *is an algebraic volume n-form*

*Consider the exponential integral:*

$$I(\gamma) = \int_\Gamma e^{-\gamma f}\mu$$

*Where:*

- $f : X \to \mathbb{C}$
- $\Gamma$ *is a n-chain with* $D_0 \supset \partial\Gamma \neq 0$
- $\gamma \in \mathbb{C}^*$
- $\mu$ *is an algebraic volume n-form*

*We want to:*

- *Identify* $H^k(-)$ *and* $H_k(-)$ *such that:* $I(\gamma) : H^k(-) \otimes H_k(-) \to \mathbb{C}$

# Exponential integrals

*Consider the exponential integral:*

$$I(\gamma) = \int_\Gamma e^{-\gamma f}\mu$$

*Where:*

- $f : X \to \mathbb{C}$
- $\Gamma$ *is a n-chain with* $D_0 \supset \partial\Gamma \neq 0$
- $\gamma \in \mathbb{C}^*$
- $\mu$ *is an algebraic volume n-form*

*We want to:*

- *Identify* $H^k(-)$ *and* $H_k(-)$ *such that:* $I(\gamma) : H^k(-) \otimes H_k(-) \to \mathbb{C}$
- *Determine the dimension and a basis of such spaces:* *Master Integrals*

# Exponential integrals

*Consider the exponential integral:*

$$I(\gamma) = \int_\Gamma e^{-\gamma f} \mu$$

Where:

- $f : X \to \mathbb{C}$
- $\Gamma$ *is a n-chain with* $D_0 \supset \partial\Gamma \neq 0$
- $\gamma \in \mathbb{C}^*$
- $\mu$ *is an algebraic volume n-form*

*We want to:*

- *Identify* $H^k(-)$ *and* $H_k(-)$ *such that:* $I(\gamma) : H^k(-) \otimes H_k(-) \to \mathbb{C}$
- *Determine the dimension and a basis of such spaces: Master Integrals*
- *Compute the intersection numbers*

*Consider the exponential integral:*

$$I(\gamma) = \int_\Gamma e^{-\gamma f}\mu$$

Where:

- $f : X \to \mathbb{C}$
- $\Gamma$ is a n-chain with $D_0 \supset \partial\Gamma \neq 0$
- $\gamma \in \mathbb{C}^*$
- $\mu$ is an algebraic volume n-form

We want to:

- Identify $H^k(-)$ and $H_k(-)$ such that: $I(\gamma) : H^k(-) \otimes H_k(-) \to \mathbb{C}$
- Determine the dimension and a basis of such spaces: Master Integrals
- Compute the intersection numbers
- Analyze the dependence on $\gamma$ : wall crossing structure

*Multiloop Feynman integrals in Baikov representation*

$$I = \int_{\Delta} \mathscr{B}(x_i)^{-\gamma} \omega$$

- $\mathscr{B}(x_i)$ *is the Baikov Polynomial*
- $\omega$ *is a rational form*
- *In dimensional regularization:* $\gamma(D) \notin \mathbb{Z}$

*Multiloop Feynman integrals in Baikov representation*

$$I = \int_{\Delta} \mathscr{B}(x_i)^{-\gamma} \omega$$

- $\mathscr{B}(x_i)$ *is the Baikov Polynomial*
- $\omega$ *is a rational form*
- *In dimensional regularization:* $\gamma(D) \notin \mathbb{Z}$

↓

*We lose geometric interpretation*

*Multiloop Feynman integrals in Baikov representation*

$$I = \int_\Delta \mathscr{B}(x_i)^{-\gamma}\omega$$

$$I = \int_\Delta e^{-\gamma \log \mathscr{B}(x_i)}\omega$$

- $\mathscr{B}(x_i)$ *is the Baikov Polynomial*
- $\omega$ *is a rational form*
- *In dimensional regularization:* $\gamma(D) \notin \mathbb{Z}$

# The pairing

$$\int exp[f]\mu : H_{\bullet}^{Betti,global}(X, D_0, f) \otimes H^{\bullet}_{dR,global}(X, D_0, f) \to \mathbb{C}$$

[Kontsevich and Soibelman: arXiv:2402.07343, 2024 ]

# The pairing

$$\int exp[f]\mu : H_\bullet^{Betti,global}(X, D_0, f) \otimes H^\bullet_{dR,global}(X, D_0, f) \to \mathbb{C}$$

[Kontsevich and Soibelman: arXiv:2402.07343, 2024 ]

*Global Betti homology*

$$H^\bullet_{Betti,global}\left((X, D_0), f, \mathbb{Z}\right) \equiv H^\bullet\left((X, D_0), f^{-1}(\infty), \mathbb{Z}\right)$$

$$\int exp[f]\mu : H_\bullet^{Betti,global}(X, D_0,f) \otimes H^\bullet_{dR,global}(X, D_0,f) \to \mathbb{C}$$

[Kontsevich and Soibelman: arXiv:2402.07343, 2024 ]

*Global Betti homology*

$$H^\bullet_{Betti,global}\left((X, D_0), f, \mathbb{Z}\right) \equiv H^\bullet\left((X, D_0), f^{-1}(\infty), \mathbb{Z}\right)$$

*Global Twisted de Rham cohomology*

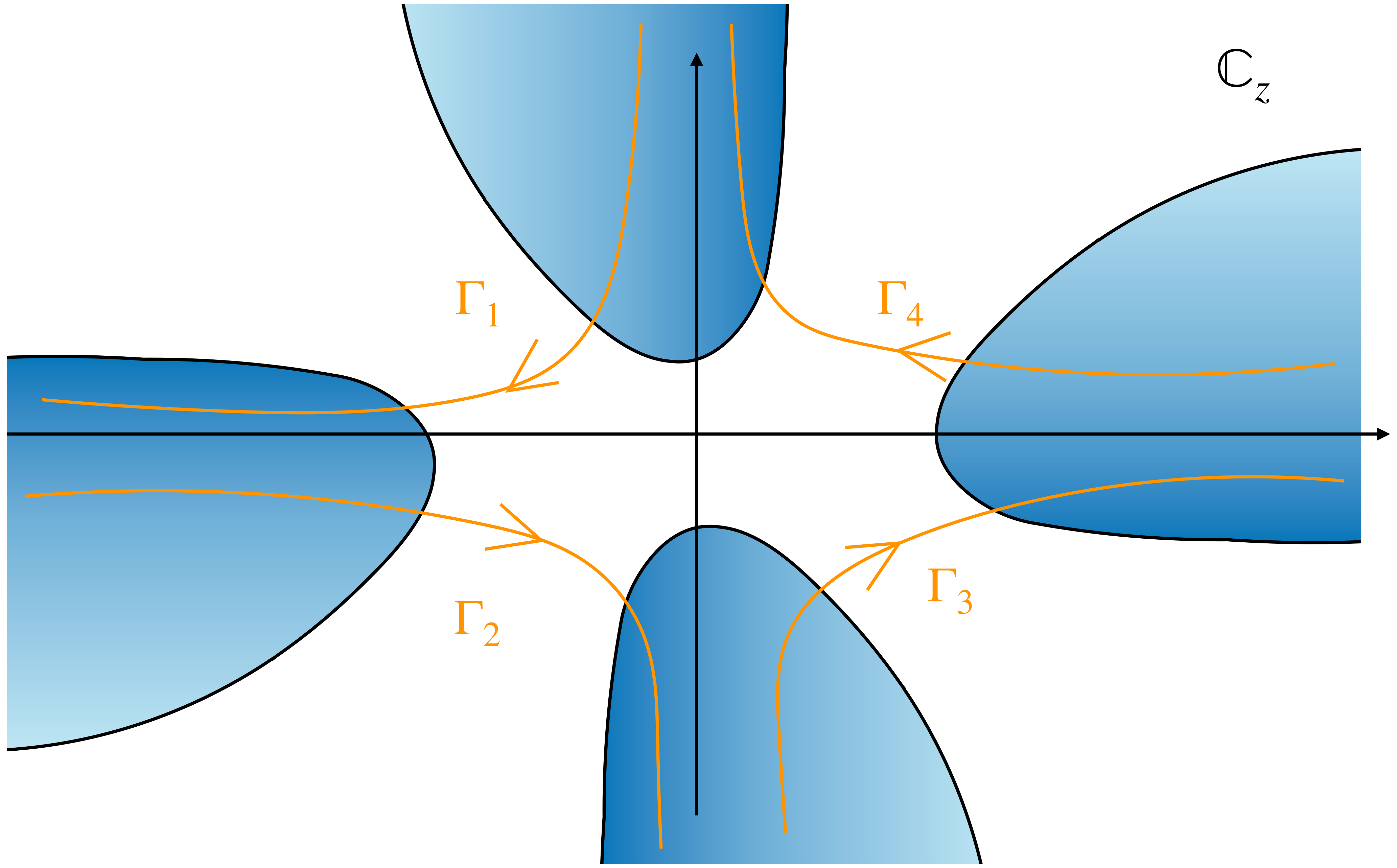
$$H^\bullet_{dR}(X, D_0, f) \cong \mathbb{H}^\bullet(X, \Omega^\bullet_{X,D_0}, \nabla_f)$$

*With*

$$(\Omega^\bullet_{X,D_0}, \nabla_f) : \ \Omega^0_{X,D_0} \xrightarrow{\nabla_f} \Omega^1_{X,D_0} \xrightarrow{\nabla_f} \dots \xrightarrow{\nabla_f} \Omega^n_{X,D_0}$$

$$\nabla_f = d - df \wedge$$

*Let* $X \cong \mathbb{C}^n$ *and consider the set:*




$$X \supset D_N = \{z \in \mathbb{C}^n \,|\, \mathrm{Re}(\gamma f(z)) \geq N\}$$

*Any reasonable cycle* $\Gamma$ *should connect two distinct regions in* $D_N$ *:*

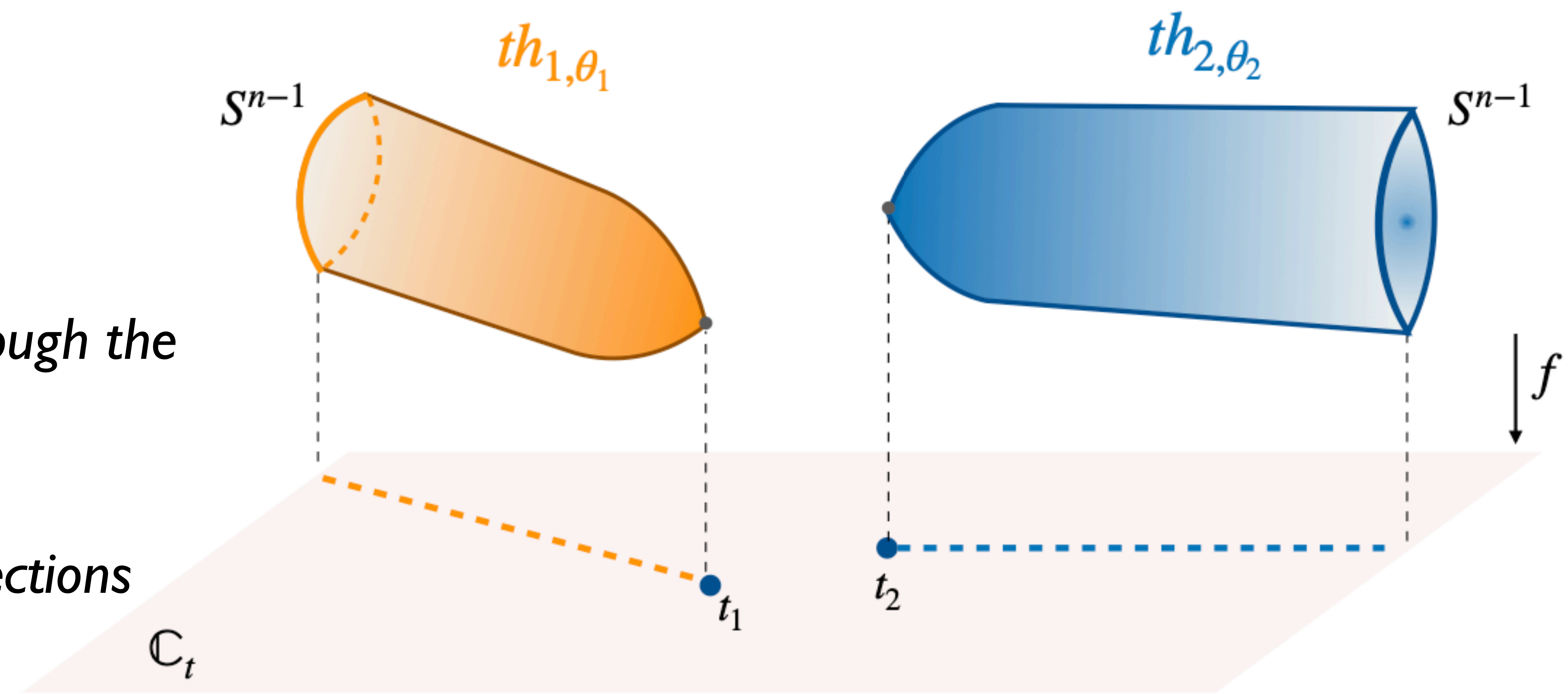
$$\Gamma \in H_n(X, D_N, \mathbb{Z})$$

*Moreover, in order to avoid oscillations* $\mathrm{Im}(\gamma f)$ *must remain constant along* $\Gamma$

# Thimbles decomposition

*A good basis for $H_n(X, D_N, \mathbb{Z})$ is provided by the so called Lefschetz thimbles:*

- *Gradient flow lines with constant phase passing through the critical points of* $h = \mathrm{Re}(\gamma f(z))$
- *Ascendent paths* $\Gamma_i^+$ *and descendent paths* $\Gamma_i^-$
- *"Traces" of the vanishing cycles along vanishing directions*

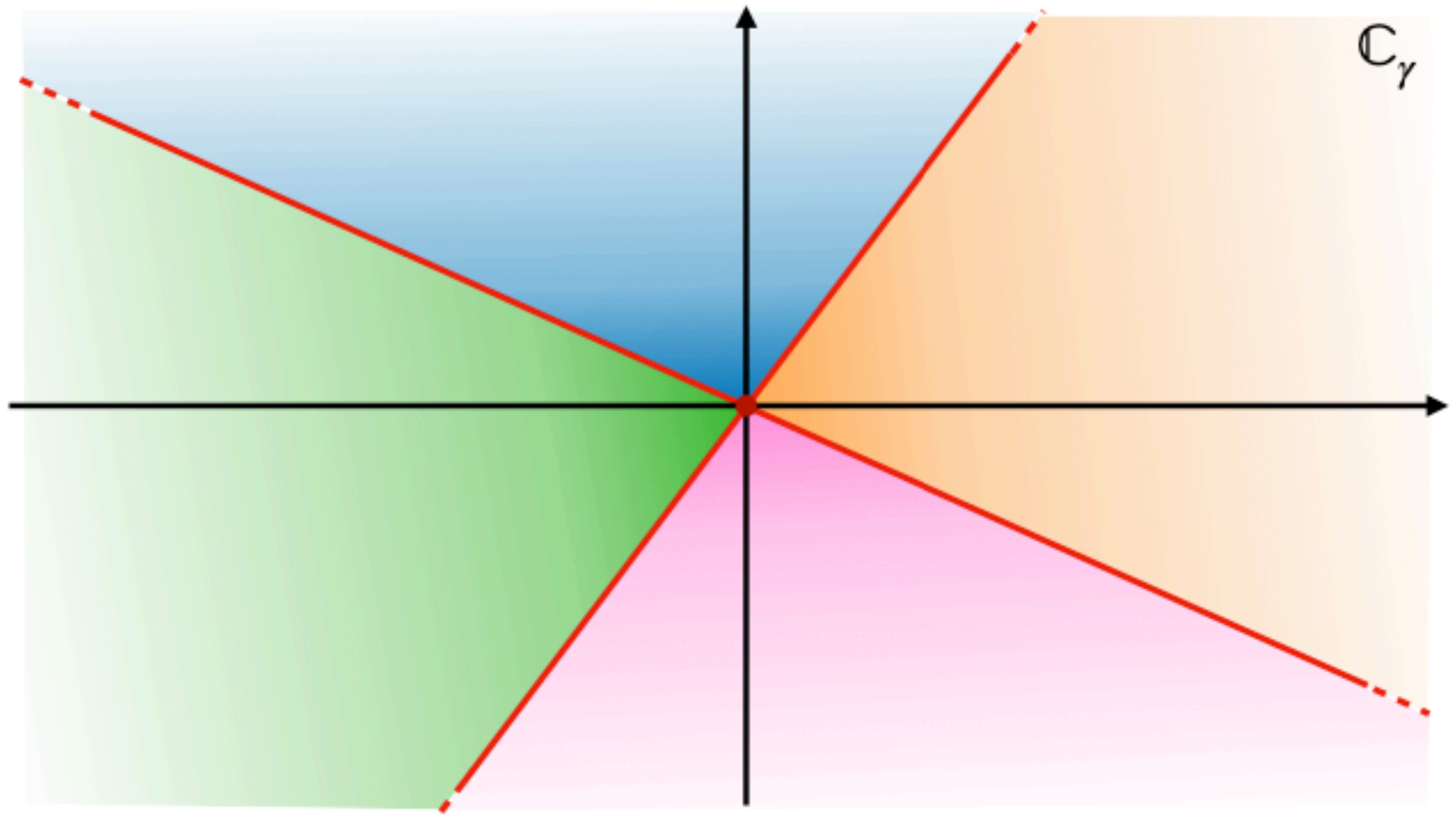



$$rank[H_k(X, D_N, \mathbb{Z})] = \begin{cases} 0, & k < n, \\ \#\Sigma, & k = n\,. \end{cases}$$

*Set of critical points*

*As $\gamma$ varies it may happens that a thimble crosses more then one singular point:*

$$l = \{\gamma \in \mathbb{C}^* \,|\, \mathrm{Im}(\gamma f(z))\,|_{\sigma_i} = \mathrm{Im}(\gamma f(z))\,|_{\sigma_j}\}$$ *Stokes' line*

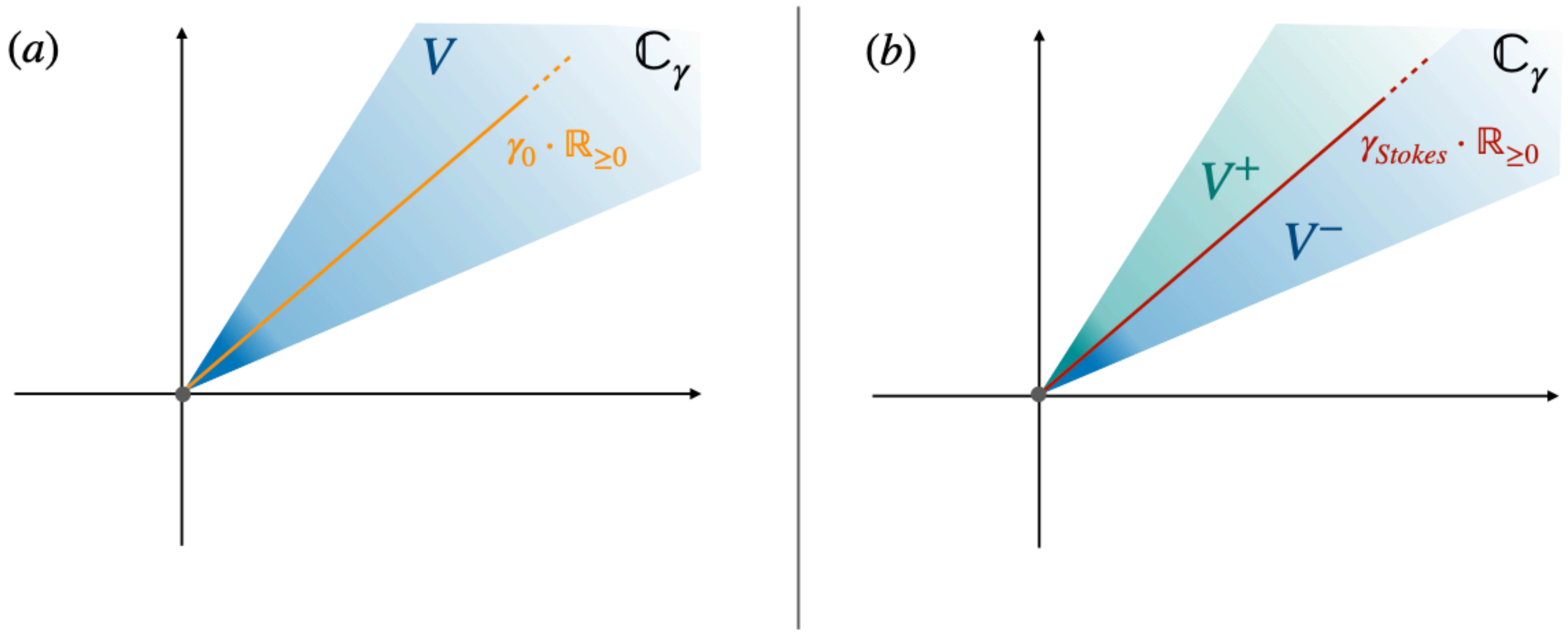



*The number of Thimbles for such values of $\gamma$ is less then for generic $\gamma$*

*The space $\mathbb{C}_\gamma$ splits into regions separated by Stokes lines*

# Wall crossing structure

*As we cross a Stokes ray, associated with a Stokes line between the critical points $\sigma_i$ and $\sigma_j$ the corresponding thimbles $\Gamma_j^+$ and $\Gamma_i^+$ undergo a discontinuous jump to the adjacent region of the form:*

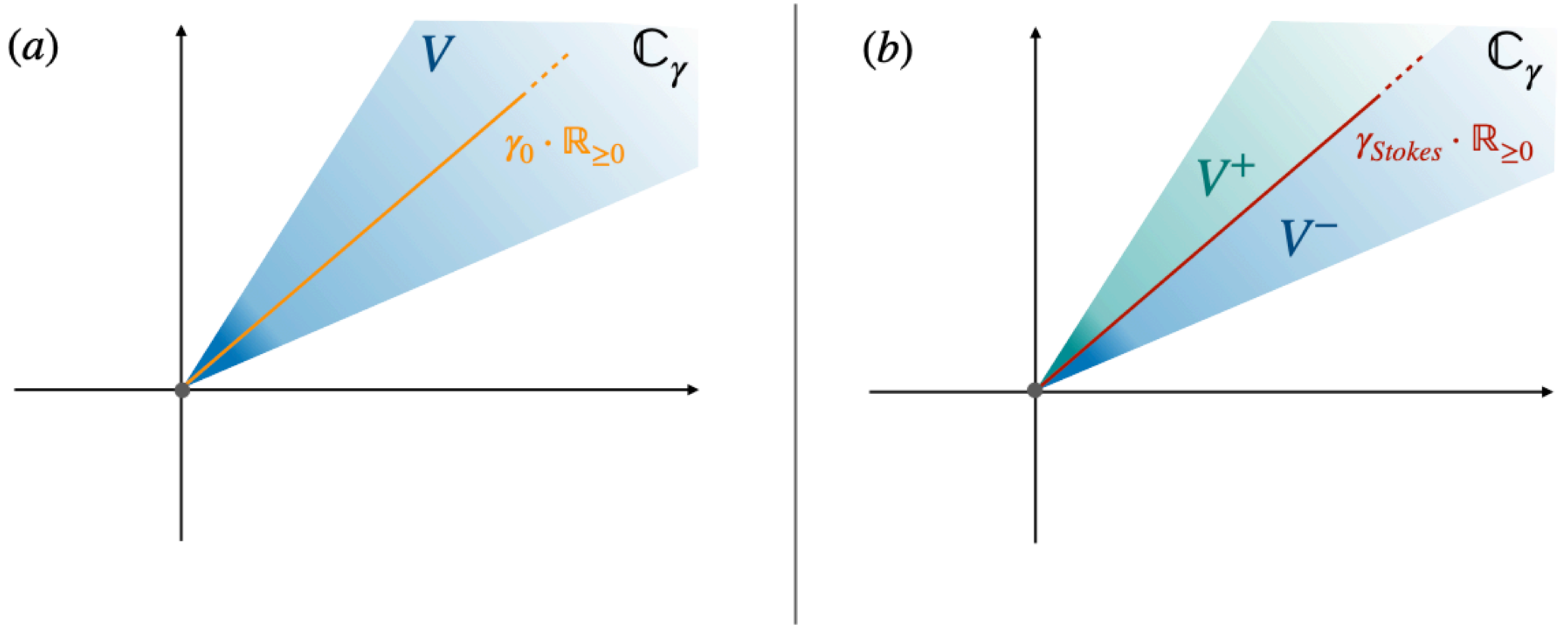



$$\begin{pmatrix} \Gamma_i^{+(1)} \\ \Gamma_j^{+(1)} \end{pmatrix} = \begin{pmatrix} 1 & \Delta_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_i^{+(0)} \\ \Gamma_j^{+(0)} \end{pmatrix} , \quad \text{for } \; h_{\sigma_i} < h_{\sigma_j}$$

Where $\Delta_{ij} = (\pm 1)\Delta_i \circ \Delta_j \, .$

# Wall crossing structure

*As we cross a Stokes ray, associated with a Stokes line between the critical points $\sigma_i$ and $\sigma_j$ the corresponding thimbles $\Gamma_j^+$ and $\Gamma_i^+$ undergo a discontinuous jump to the adjacent region of the form:*



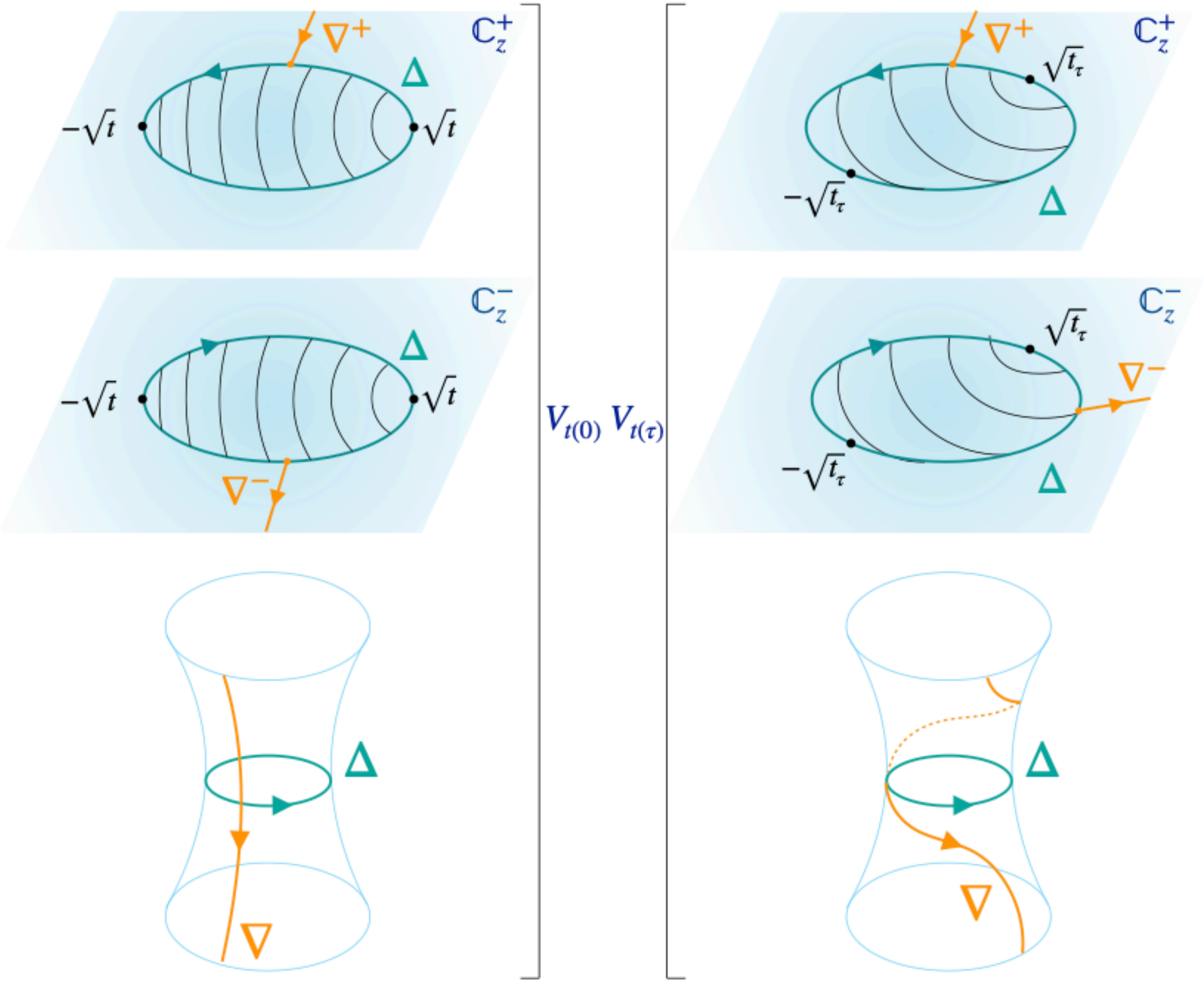

$$\begin{pmatrix} \Gamma_i^{+(1)} \\ \Gamma_j^{+(1)} \end{pmatrix} = \begin{pmatrix} 1 & \Delta_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_i^{+(0)} \\ \Gamma_j^{+(0)} \end{pmatrix}, \quad \text{for } h_{\sigma_i} < h_{\sigma_j}$$

Where $\Delta_{ij} = (\pm 1) \Delta_i \circ \Delta_j .$ → *Intersection among the corresponding vanishing cycles*

*Picard-Lefshetz theorem:*

$$h_\tau(a) = a + (-1)^{n(n+1)/2}(a \circ \Delta)\Delta$$

*Representation of the monodromy acting on vanishing cycles*



# Holomorphic exponent: Twisted de Rham cohomology

*Let consider* $f = \mathscr{P}_\ell \in \mathbb{C}[z]$ *and extend it to* $\overline{\mathscr{P}}_\ell : \mathbb{P}^1 \to \mathbb{P}^1$.
*We want to compute the Hypercohomology of the complex:*

$$(\Omega^\bullet_{\mathbb{P}^1,p}, \nabla) : 0 \to \mathcal{O}_{\mathbb{P}^1}(*p) \xrightarrow{\nabla} \Omega^1_{\mathbb{P}^1}(*p) \to 0$$

*With respect to the differential:*

$$\nabla \equiv (\gamma^{-1} d + d\mathscr{P}_\ell \wedge)$$

*Let consider* $f = \mathscr{P}_\ell \in \mathbb{C}[z]$ *and extend it to* $\overline{\mathscr{P}_\ell} : \mathbb{P}^1 \to \mathbb{P}^1$.
*We want to compute the Hypercohomology of the complex:*

$$(\Omega^\bullet_{\mathbb{P}^1,p}, \nabla) : 0 \to \mathscr{O}_{\mathbb{P}^1}(*p) \xrightarrow{\nabla} \Omega^1_{\mathbb{P}^1}(*p) \to 0$$

*With respect to the differential:*

$$\nabla \equiv (\gamma^{-1} d + d\mathscr{P}_\ell \wedge)$$

*Skipping the details, the final result is*

$$H^0_{dR}(\mathbb{C}, d\mathscr{P}_l) \cong 0$$
$$H^1_{dR}(\mathbb{C}, d\mathscr{P}_\ell) \cong \mathbb{C}^{\ell-1}$$

# Holomorphic exponent: Twisted de Rham cohomology

*Let consider* $f = \mathscr{P}_\ell \in \mathbb{C}[z]$ *and extend it to* $\overline{\mathscr{P}}_\ell : \mathbb{P}^1 \to \mathbb{P}^1$.
*We want to compute the Hypercohomology of the complex:*

$$(\Omega^\bullet_{\mathbb{P}^1,p}, \nabla) : 0 \to \mathscr{O}_{\mathbb{P}^1}(*p) \xrightarrow{\nabla} \Omega^1_{\mathbb{P}^1}(*p) \to 0$$

*With respect to the differential:*

$$\nabla \equiv (\gamma^{-1} d + d\mathscr{P}_\ell \wedge)$$

*Skipping the details, the final result is*

$$H^0_{dR}(\mathbb{C}, d\mathscr{P}_l) \cong 0$$

$$H^1_{dR}(\mathbb{C}, d\mathscr{P}_\ell) \cong \mathbb{C}^{\ell-1}$$

*This result depends on the global Jacobian ring*

$$J_{\mathscr{P}_\ell} \cong \mathbb{C}^{\ell-1}$$

*that is NOT sensitive to the coalescence of critical points! If one point as multiplicity m, by considering the middle extension of the connexion we get*

$$H^1_{dR}(\mathbb{C}, d\mathscr{P}_\ell)_{(m)} \cong \mathbb{C}^{\ell-m}$$

$$P(\gamma) = \int_\Gamma e^{-\gamma(z^4+bz^2+cz+d)}dz$$

*Describing the Grand-canonical partition function of gauge Skyrme models for nuclear matter*

[Cacciatori, Canfora, Lagos, Muscolino,Vera: JHEP 12:150, 2021 ]

*The singular locus is the set*

$$\Sigma = \{z \in \mathbb{C} \,|\, f'(z) = 0\} = \{\sigma_i(\Delta)\}$$

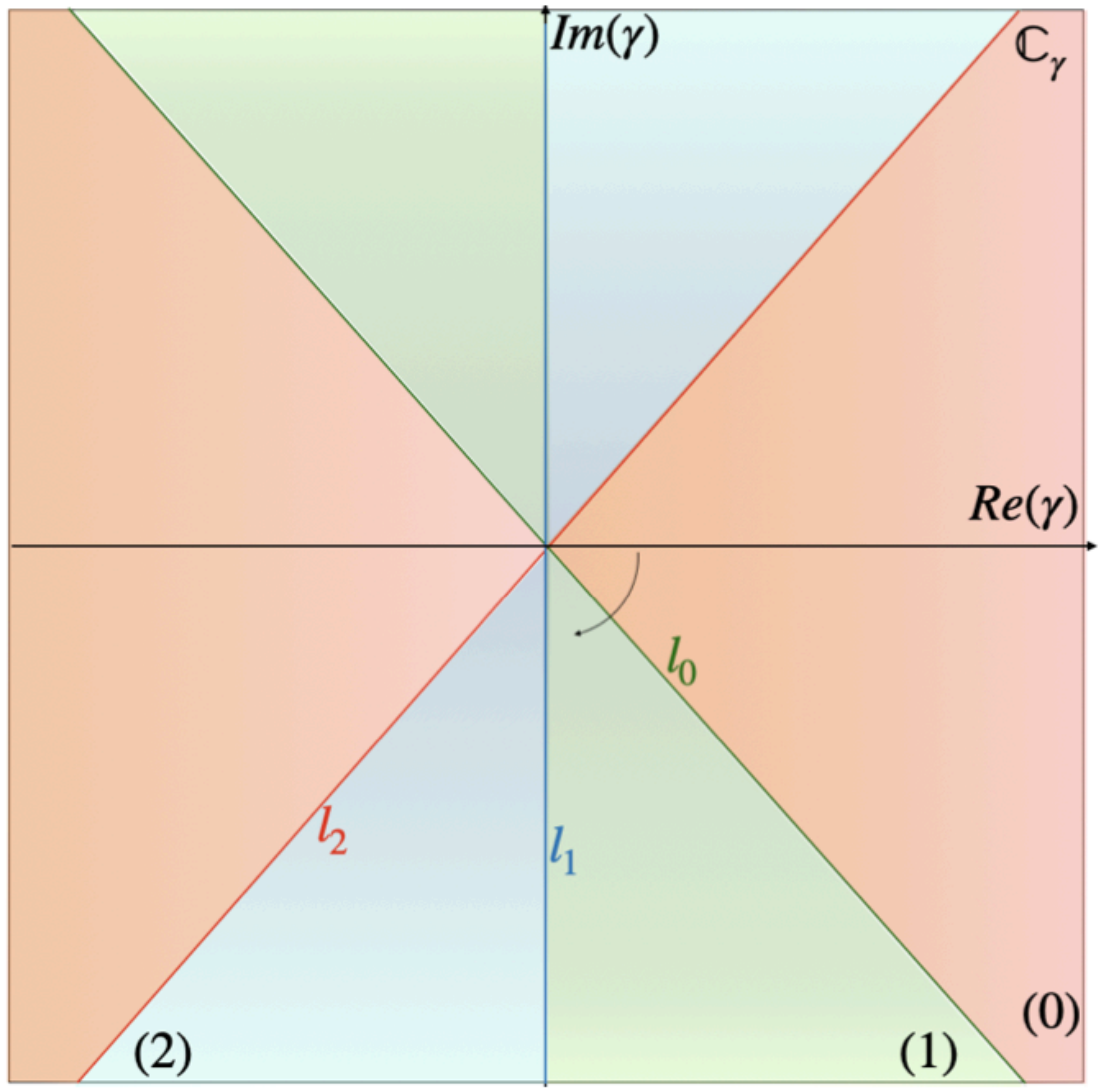
*Where* $\Delta = 8b^2 + 27c^2$

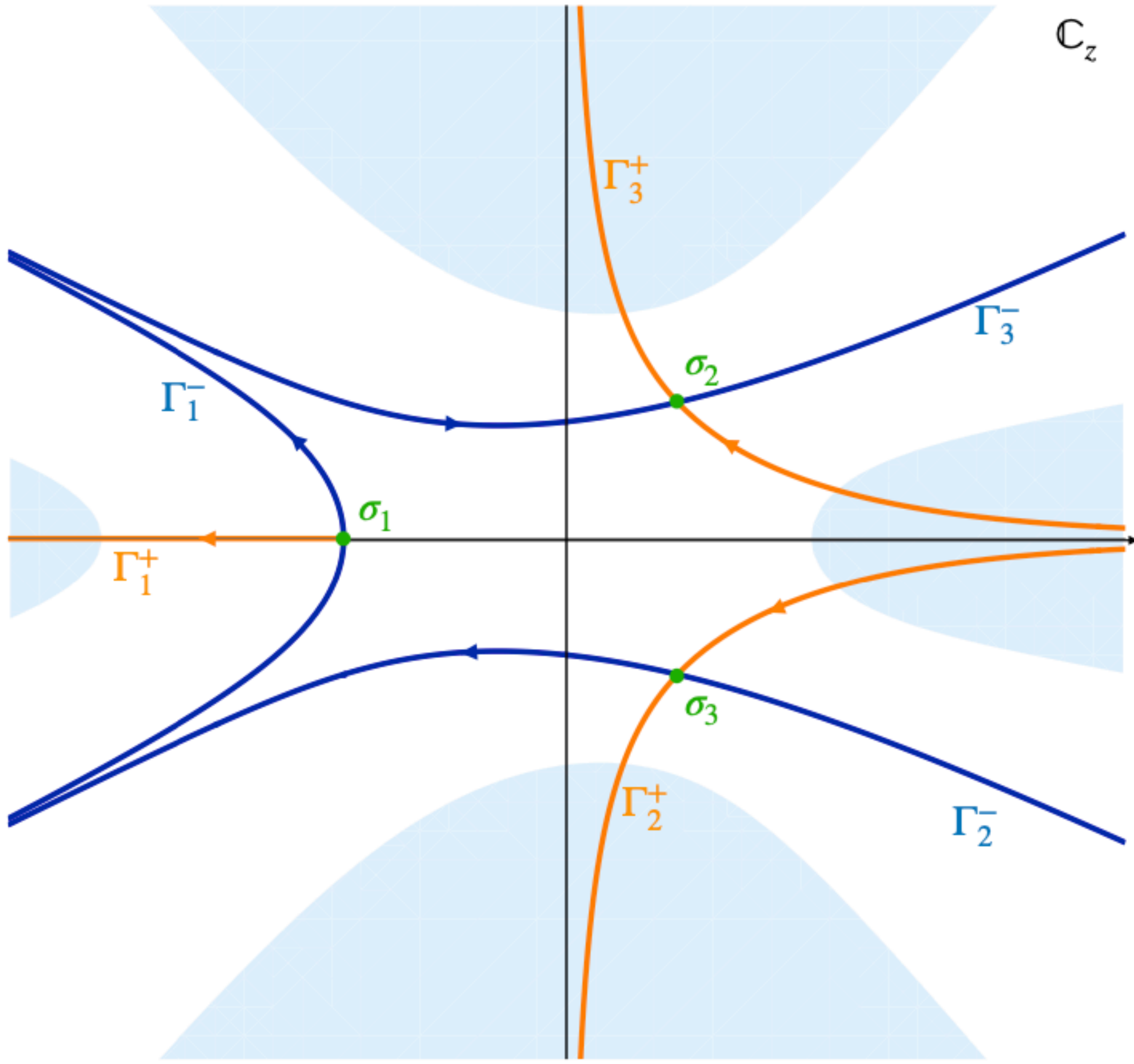
$$\Delta \equiv \begin{cases} > 0 & 1 \text{ real and } 2 \text{ complex conjugate solutions,} \\ < 0 & 3 \text{ real different solutions,} \\ = 0 & 3 \text{ real solutions with at least a multiple root.} \end{cases}$$

$$l_0 : Re(\gamma) = -\frac{11}{16}\sqrt{\frac{3}{2}}\mathrm{Im}(\gamma), \qquad \textit{where} \quad \mathrm{Im}(\gamma f(z))|_{\sigma_1} = \mathrm{Im}(\gamma f(z))|_{\sigma_2},$$

$$l_1 : Re(\gamma) = 0, \qquad \textit{where} \quad \mathrm{Im}(\gamma f(z))|_{\sigma_2} = \mathrm{Im}(\gamma f(z))|_{\sigma_3},$$

$$l_2 : Re(\gamma) = \frac{11}{16}\sqrt{\frac{3}{2}}\mathrm{Im}(\gamma), \qquad \textit{where} \quad \mathrm{Im}(\gamma f(z))|_{\sigma_1} = \mathrm{Im}(\gamma f(z))|_{\sigma_3}.$$




*Fix* $\gamma \notin l_i$

# Pearcey Integral

*Consider the preimage of the level set* $f(z) = t$

$$f^{-1}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix}$$

*Consider the preimage of the level set* $f(z) = t$

$$f^{-1}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix}$$

*Vanishing cycles*

$$\Delta_1 = \{z_3\} - \{z_4\} \quad \textit{and} \quad \Delta_2 = \Delta_3 = \{z_1\} - \{z_4\}$$

*Consider the preimage of the level set* $f(z) = t$

$$f^{-1}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix}$$

*Vanishing cycles*

$$\Delta_1 = \{z_3\} - \{z_4\} \quad \textit{and} \quad \Delta_2 = \Delta_3 = \{z_1\} - \{z_4\}$$

*Monodromy matrices*

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad M_2 = M_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

*Consider the preimage of the level set* $f(z) = t$

$$f^{-1}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix}$$

*Vanishing cycles*

$$\Delta_1 = \{z_3\} - \{z_4\} \quad \textit{and} \quad \Delta_2 = \Delta_3 = \{z_1\} - \{z_4\}$$

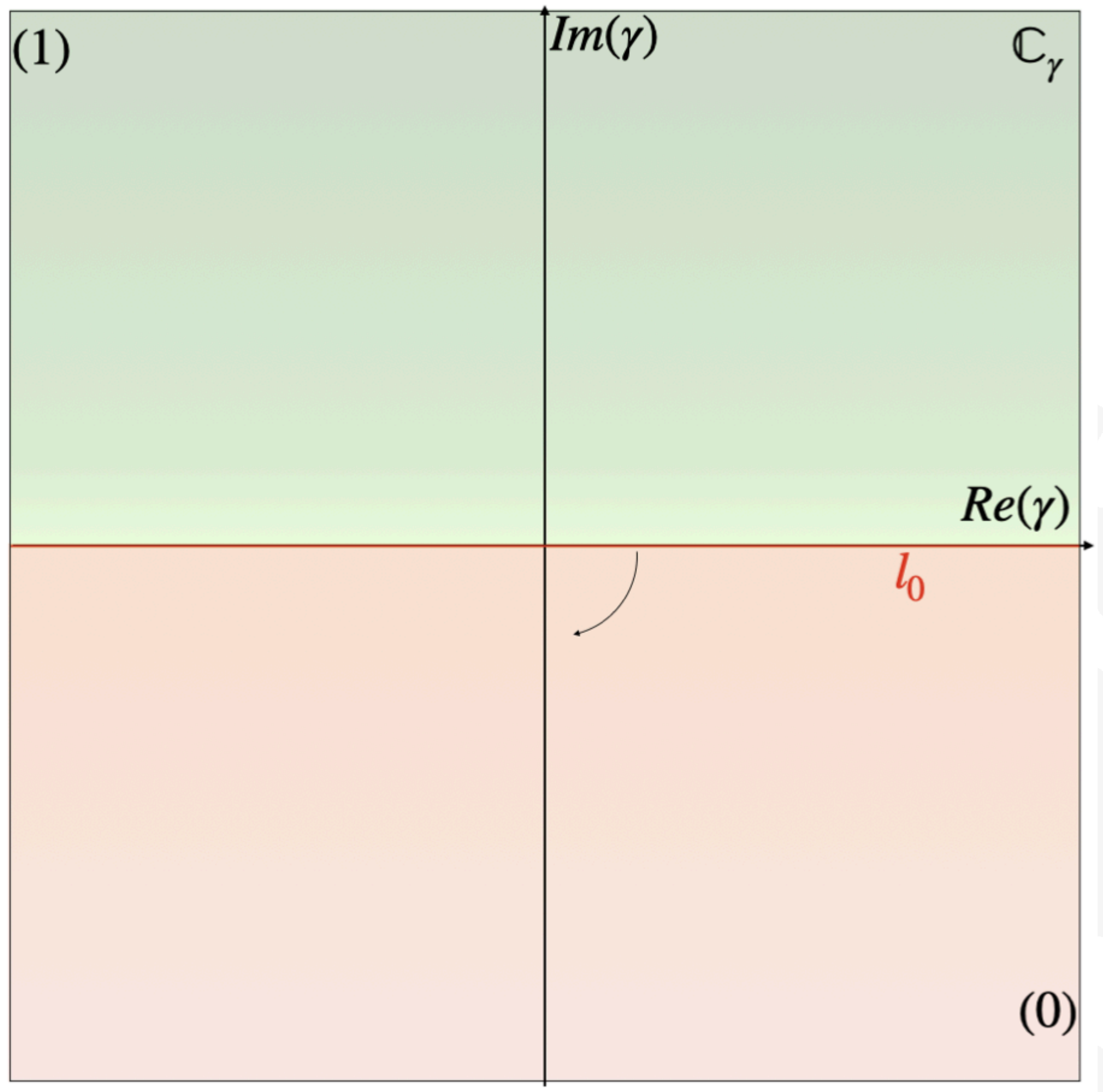
*Monodromy matrices*

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad M_2 = M_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

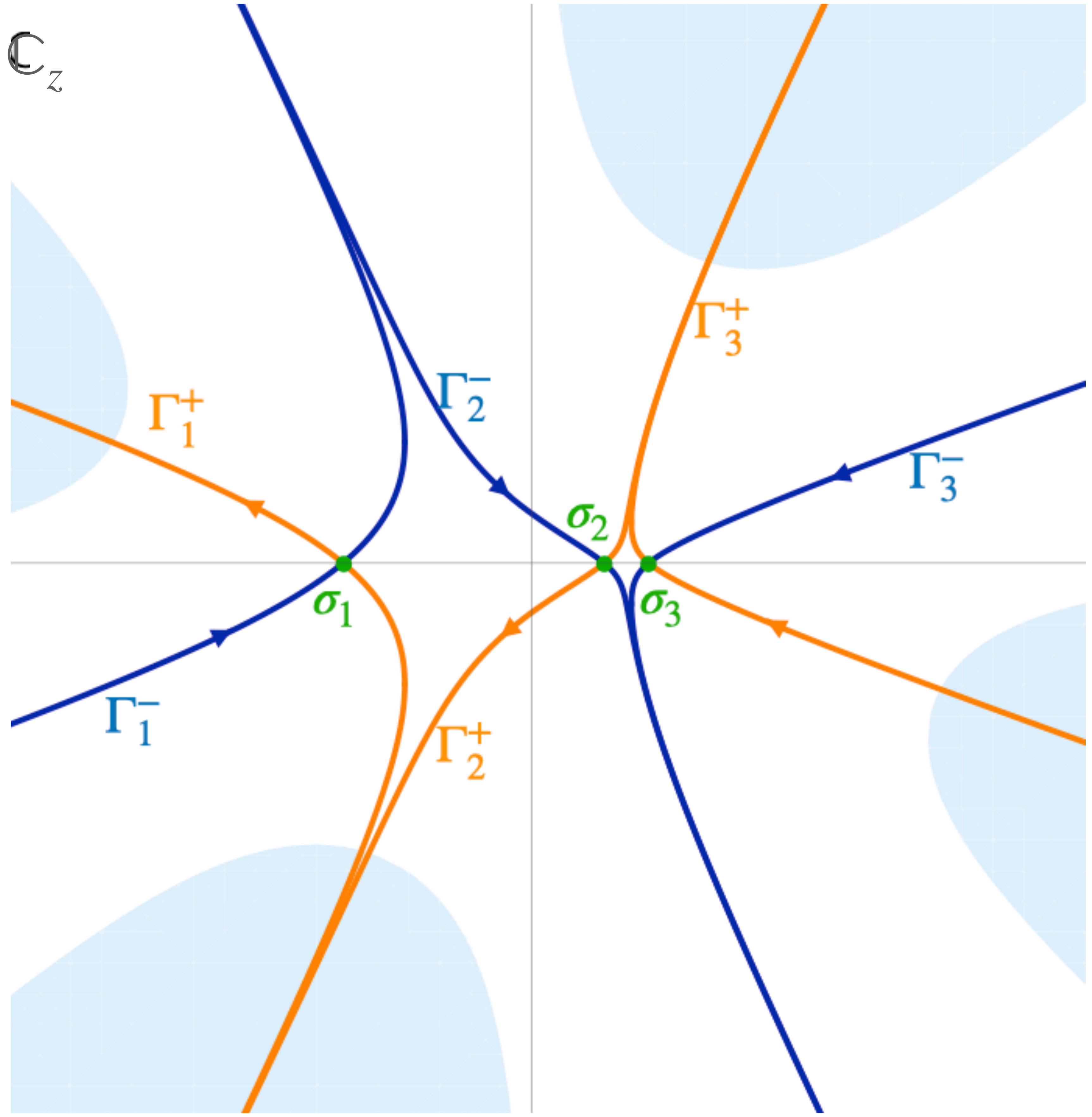
*Jump matrices*

$$T^{(0)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \qquad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

# Pearcey Integral: $\Delta < 0$




*Stoke's line:* $\text{Im}(\gamma) = 0$

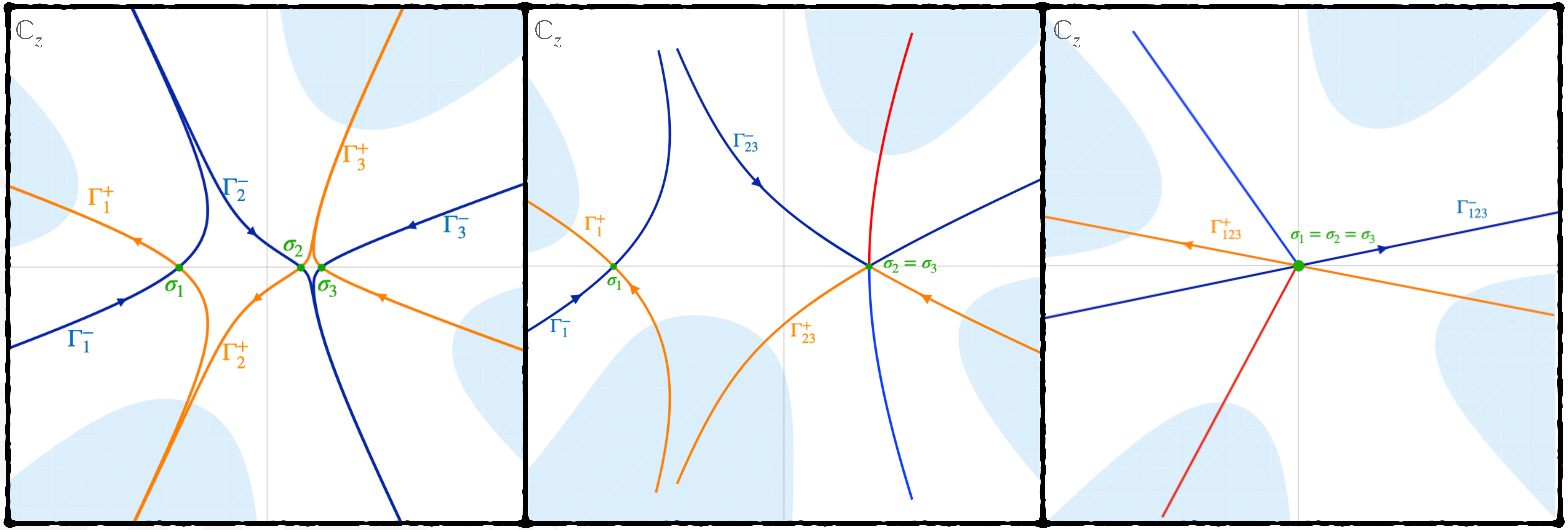



*Vanishing cycles*

$$\Delta_1 = \{z_1\} - \{z_2\}\,, \quad \Delta_2 = \{z_3\} - \{z_4\}, \ \Delta_3 = \{z_1\} - \{z_4\}$$

*Monodromy matrices*

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$




$$H_1(X, D_N, \mathbb{Z}) = span\{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+\} \cong \mathbb{Z}^3$$
$$H_1(X, D_N, \mathbb{Z})^\vee = span\{\Gamma_1^-, \Gamma_2^-, \Gamma_3^-\} \cong \mathbb{Z}^3$$

$$H_1(X, D_N, \mathbb{Z}) = span\{\Gamma_1^+, \Gamma_{23}^+\} \cong \mathbb{Z}^2$$
$$H_1(X, D_N, \mathbb{Z})^\vee = span\{\Gamma_1^-, \Gamma_{23}^-\} \cong \mathbb{Z}^2$$

$$H_1(X, D_N, \mathbb{Z}) = span\{\Gamma_{123}^+\} \cong \mathbb{Z}$$
$$H_1(X, D_N, \mathbb{Z})^\vee = span\{\Gamma_{123}^-\} \cong \mathbb{Z}$$

# Logarithmic exponent

*Let us consider now*

$$f(z_1, \ldots, z_n) = log\mathscr{B}(z_1, \ldots, z_n)$$

$f$ *is defined on* $X = \mathbb{C}^n \backslash \{\mathscr{B} = 0\}$ → $\bar{X} = \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$

*We classify the added divisor according to th behavior of* $\overline{\mathscr{B}}$*, the extension of* $\mathscr{B}$ *over* $\bar{X}$*:*

$$\bar{X} - X = D_h \cup D_v \cup D_{log}$$

# Logarithmic exponent

*Let us consider now*

$$f(z_1, \ldots, z_n) = log\mathscr{B}(z_1, \ldots, z_n)$$

*$f$ is defined on* $X = \mathbb{C}^n \backslash \{\mathscr{B} = 0\}$ → $\overline{X} = \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$

*We classify the added divisor according to th behavior of* $\overline{\mathscr{B}}$*, the extension of* $\mathscr{B}$ *over* $\overline{X}$*:*

$$\overline{X} - X = D_h \cup D_v \cup D_{log}$$

*Horizontal divisor:* $\overline{\mathscr{B}}$ *si finite*

# Logarithmic exponent

*Let us consider now*

$$f(z_1, \ldots, z_n) = log\mathscr{B}(z_1, \ldots, z_n)$$

*f is defined on* $X = \mathbb{C}^n \backslash \{\mathscr{B} = 0\}$ → $\overline{X} = \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$

*We classify the added divisor according to th behavior of* $\overline{\mathscr{B}}$*, the extension of* $\mathscr{B}$ *over* $\overline{X}$*:*

$$\overline{X} - X = D_h \cup D_v \cup D_{log}$$

*Horizontal divisor:* $\overline{\mathscr{B}}$ *si finite*

*Vertical divisor(at infinity):* $\overline{\mathscr{B}}$ *has poles*

# Logarithmic exponent

*Let us consider now*

$$f(z_1, \ldots, z_n) = log\mathscr{B}(z_1, \ldots, z_n)$$

$f$ *is defined on* $X = \mathbb{C}^n \backslash \{\mathscr{B} = 0\}$ → $\overline{X} = \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$

*We classify the added divisor according to th behavior of* $\overline{\mathscr{B}}$*, the extension of* $\mathscr{B}$ *over* $\overline{X}$*:*

$$\overline{X} - X = D_h \cup D_v \cup D_{log}$$

*Horizontal divisor:* $\overline{\mathscr{B}}$ *si finite*

*Vertical divisor(at infinity):* $\overline{\mathscr{B}}$ *has poles*

*Logarithmic divisor:* $d\overline{\mathscr{B}}$ *has logarithmic poles*

# Logarithmic exponent

*Consider the local system.* $\mathscr{L}_{\alpha,\gamma}(U) = \ker \nabla|_U, \qquad U \subset X$ *with* $\nabla = d + \gamma\alpha$

*The global Betti cohomology is defined as*

$$H^\bullet(X,\alpha) \cong H^\bullet\left(\widetilde{X}, D_v^{\mathbb{R}+} \cup D_{log}^{\mathbb{R}-}, \Pi_*(\mathscr{L}_{\alpha,\gamma})\right)$$

*We get:*

$$H^\bullet(X,\alpha) \cong \begin{cases} H^\bullet\left(\tilde{X}, D_\infty^{\mathbb{R}}, \Pi_*(\mathscr{L}_{\alpha,\gamma})\right), & Re(\gamma) > 0, \\ H^\bullet\left(\tilde{X}, D_{\mathscr{B}}^{\mathbb{R}}, \Pi_*(\mathscr{L}_{\alpha,\gamma})\right), & Re(\gamma) < 0. \end{cases}$$

# Legendre family of Elliptic curves

*Consider the integral*

$$\mathcal{I} = \int_\Gamma \frac{dx \wedge dy}{\left[y^2 + x(x-1)(x-\lambda)\right]^\gamma} = \int_\Gamma e^{-\gamma \log\left[y^2 + x(x-1)(x-\lambda)\right]} dx \wedge dy = \int_\Gamma e^{-\gamma \log \mathcal{B}(x,y;\lambda)} dx \wedge dy$$

*We extend $\mathcal{B}$ to $\mathbb{P}^2$:*

$$\overline{\mathcal{B}}(x, y, \eta; \lambda) = y^2\eta - x(x-\eta)(x-\eta\lambda)$$

*And define the close form*

$$d \, \log \overline{\mathcal{B}} = \frac{2\eta y dy + [y^2 + x^2 + x\lambda(x - 2\eta)]d\eta + [-3x^2 - \eta^2\lambda + 2x\eta(1+\lambda)]dx}{y^2\eta - x(x-\eta)(x-\eta\lambda)}$$

*Analyzing the behavior of* $\overline{\mathscr{B}}$ *we get*

$$D_h = D_v = \varnothing \qquad\qquad D_{log} = D_{\overline{\mathscr{B}}} \cup D_\infty$$

*With:*

$$D_{\overline{\mathscr{B}}} = \overline{\mathscr{E}}_\lambda = \{[x : y : \eta] \in \mathbb{P}^2 \,|\, \overline{\mathscr{B}} = 0\}$$
$$D_\infty = \mathbb{P}^1 = \{[x : y : 0] \in \mathbb{P}^2\}$$

*Intersecting at* $D_{\overline{\mathscr{B}}} \cap D_\infty = [0 : 1 : 0]$

*The computation of Betti cohomology gives:*

$$H^\bullet_{Betti,glob,\gamma}(X,\alpha)(\widetilde{X}, S^3, \Pi_*(\mathscr{L}_{\alpha,\gamma})) \cong 0 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus 0 \oplus \mathbb{C}^2$$
$$H^\bullet_{Betti,glob,\gamma}(X,\alpha)(\widetilde{X}, D^R_{\overline{\mathscr{B}}}, \Pi_*(\mathscr{L}_{\alpha,\gamma})) \cong 0 \oplus 0 \oplus \mathbb{C}^2 \oplus \mathbb{C}^4 \oplus 0$$

# Conclusions and Outlooks

- The identification of exponential integrals as periods of twisted de Rham cohomology and Betti homology over complex manifolds, allows to accomodate in the same framework a wide range of physically relevant integrals
- The analysis of the wall crossing structure allows to analytic continue the master integral decomposition in the parameter $\gamma$ and carefully avoid Stokes' phenomena for a sharp counting of the co-homology dimension

*Outlooks*

- *Concrete application of this formalism to Feynman integrals in different representation and multiple variables (working progress with Angius, Cacciatori, Mastrolia and Noja)*
- *Analysis of multi parameter dependence (Complex structure moduli of algebraic varieties = kinematic physical variables)*
- *Application to conformal correlators: string amplitudes*

# Thank you