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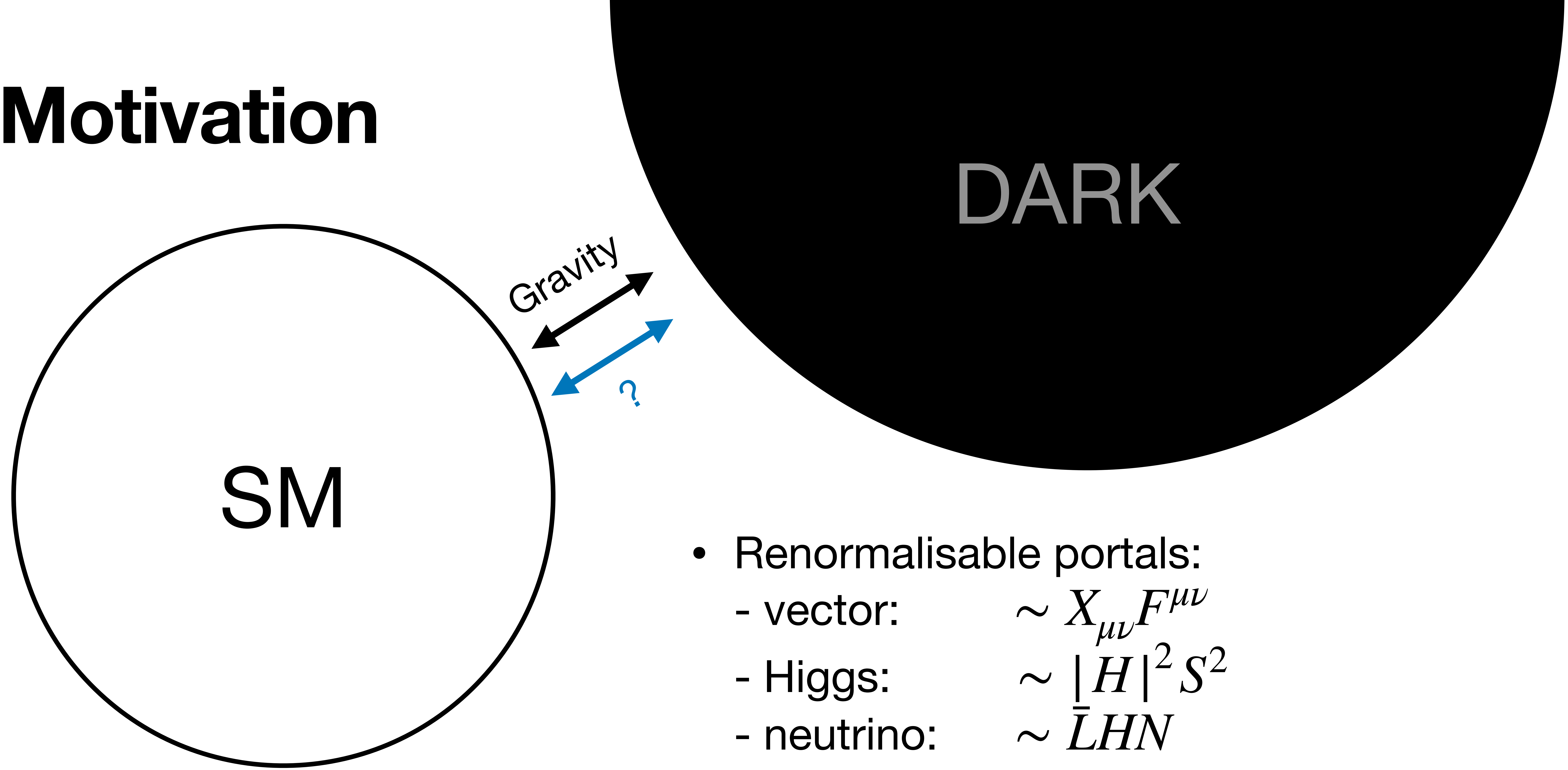
Topological Portal to the Dark Sector

Nudžeim Selimović, INFN Padova

Joe Davighi, Admir Greljo, NS, Phys.Rev.Lett. 134 (2025)

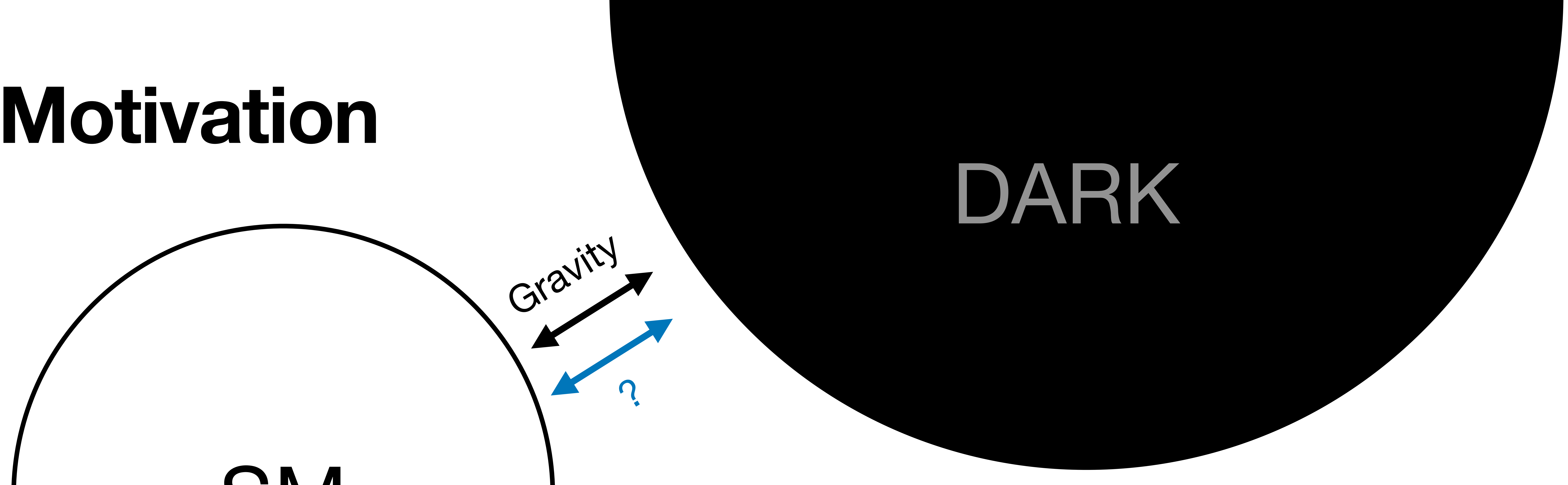
Motivation

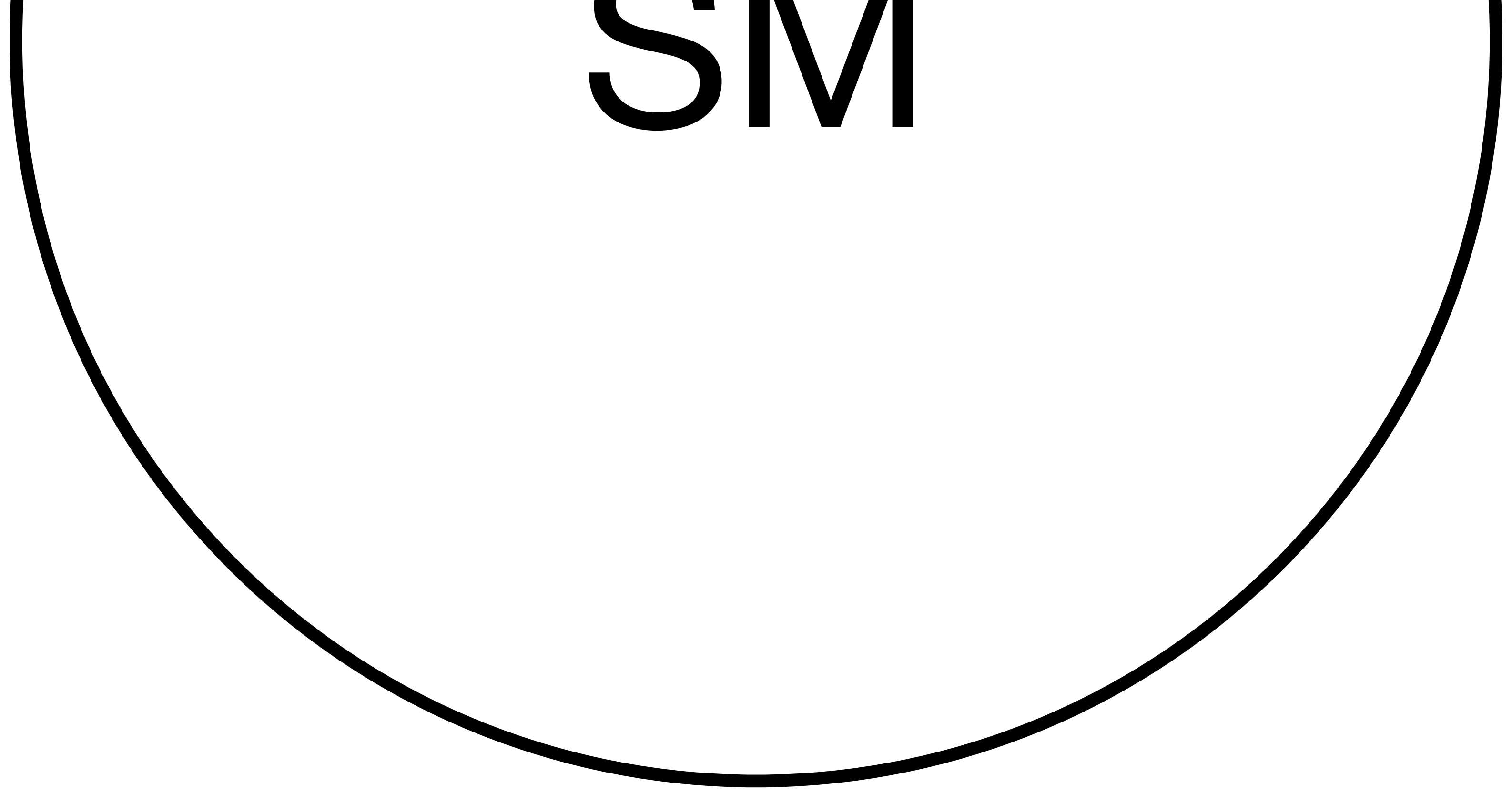


- Renormalisable portals:
  - vector: $\sim X_{\mu\nu}F^{\mu\nu}$
  - Higgs: $\sim |H|^2 S^2$
  - neutrino: $\sim \bar{L}HN$

- Non-renormalisable portals:
  - ALPs: $\sim a\tilde{F}_{\mu\nu}F^{\mu\nu}$ ...

Motivation





- This work:

  A portal that is

  - topological
  - unique*
  - non-renormalisable

  connecting 3 QCD pions to 2 dark pions.

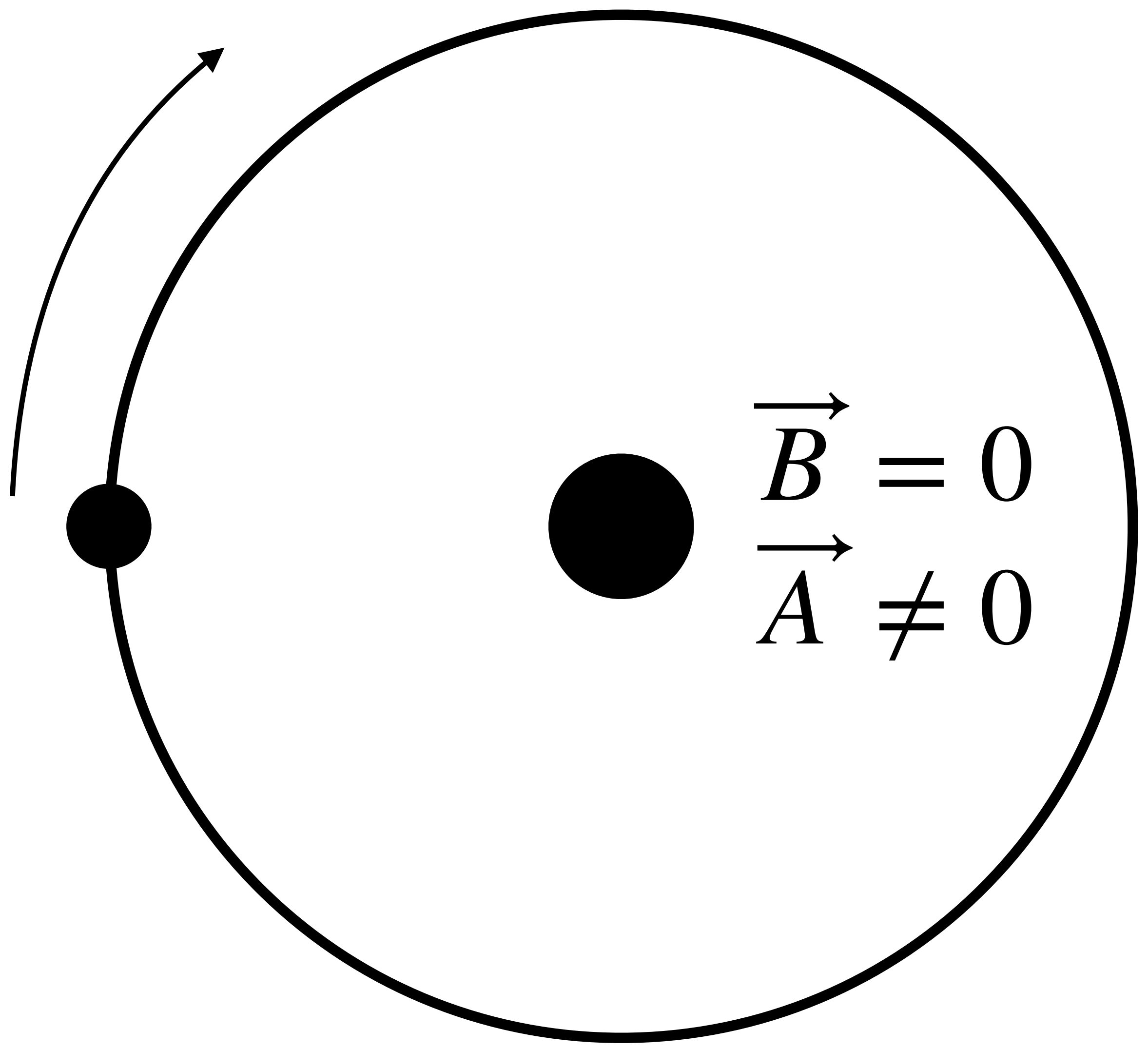
Topological actions

Topological terms

- *Class 1: Theta terms*

- Example: Aharonov-Bohm effect

Particle on a circle: $q(t)$



$$I = \underbrace{\int \mathcal{D}q(t)}_{\text{only one path}} \exp(iS) = \mathcal{N} \exp\left(ieA \int dq\right) = \mathcal{N} \exp\left(i\theta \int dq\right)$$

differential form*

theta-coefficient: $\mathbb{R}$ - valued.

Common source of topological actions:

$$S_{\text{top}} \sim \int_{\Sigma} (\text{diff . form})$$

Terms not involving the spacetime metric.

** Differential forms are mathematical objects - smooth antisymmetric tensors - ready to be integrated over curves, surfaces, volumes... E.g. a field strength tensor $F = F_{\mu\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}$ is a 2-form.*

Topological terms

- *Class 2: Wess-Zumino-Witten terms*

- Example: Low-energy QCD

$$\left.\begin{array}{c} G = SU(3)_L \times SU(3)_R \\ \downarrow \\ H = SU(3)_{L+R} \end{array}\right\} \quad U(x) = e^{\frac{2i}{f_\pi}\pi(x)^a T^a} \; : \; \Sigma_4 \to X = \frac{SU(3)_L \times SU(3)_R}{SU(3)_{L+R}} \simeq SU(3)$$

Chiral perturbation theory:

$$\mathscr{L} = \frac{f_\pi^2}{4} \mathrm{tr}\left(D_\mu U^\dagger D^\mu U\right) + \mathcal{O}(D_\mu^4)$$

Weinberg '68, CCWZ '69

Symmetries:

$P_0 : \vec{x} \to -\vec{x}$

$(-1)^{N_\pi} : U \to U^\dagger \implies \pi^a \to -\pi^a$

- CCWZ terms are invariant under both
- QCD preserves only $P = P_0(-1)^{N_\pi}$
- Are there terms missing?

Witten '83

Topological terms

- *Class 2: Wess-Zumino-Witten terms*

- Example: Low-energy QCD

Are there terms missing? Can one construct Lagrangian that is P_0 - odd?

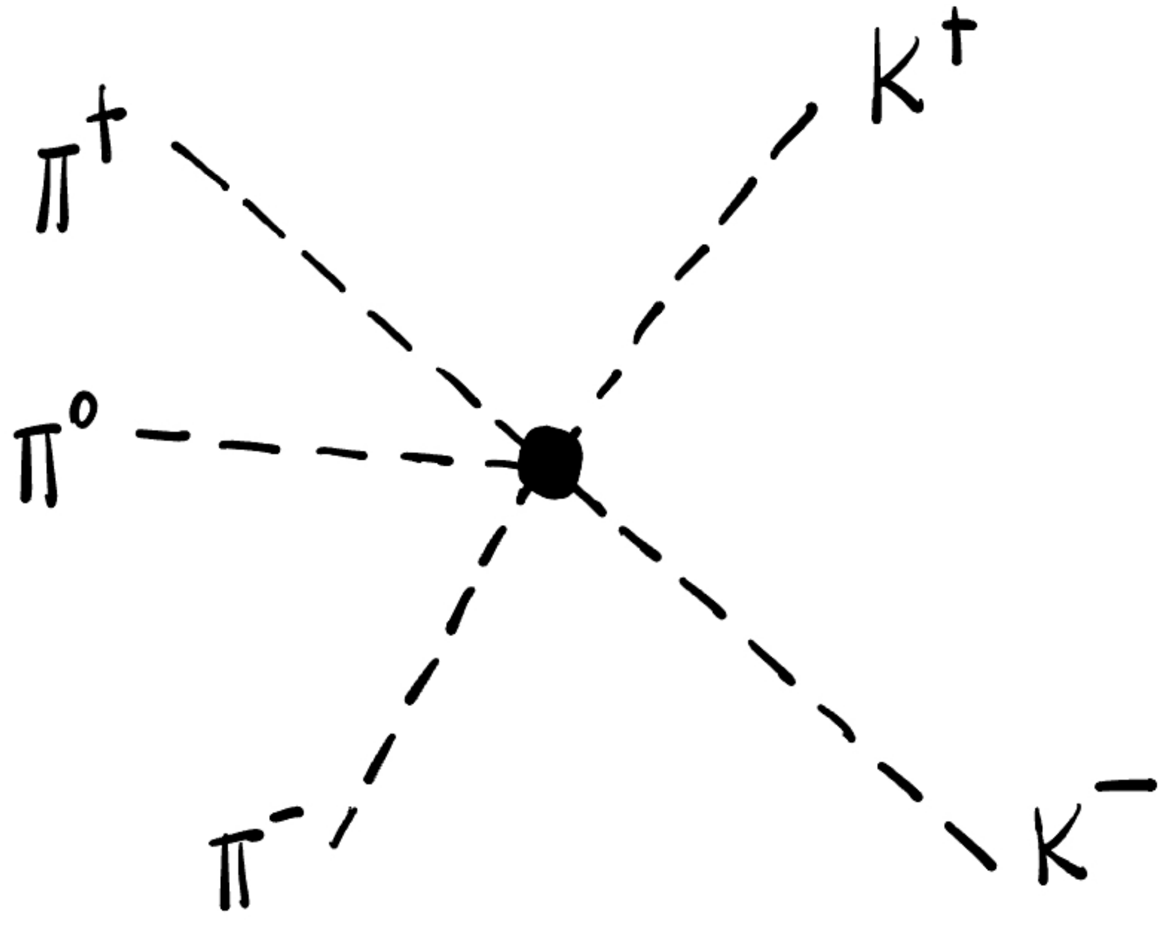
$$\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(U^\dagger\,(\partial_\mu U)U^\dagger\,(\partial_\nu U)U^\dagger\,(\partial_\rho U)U^\dagger\,(\partial_\sigma U)\right) = 0 \quad ?$$

However, equation of motion can be written: Witten '83

$$\frac{1}{2}f_\pi^2\,\partial_\mu(U^\dagger\partial^\mu U) = \frac{k}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\,U^\dagger\,(\partial_\mu U)U^\dagger\,(\partial_\nu U)U^\dagger\,(\partial_\rho U)U^\dagger\,(\partial_\sigma U)$$

Topological terms

- *Class 2: Wess-Zumino-Witten terms*



- Example: Low-energy QCD

The solution was to go to the higher dimension:

To write the action that gives the correct EOM, Witten noted the existence a 5-form:

$$\left.\begin{aligned}\omega_5 &= -\frac{1}{480\pi^3}\mathrm{Tr}\left[\left(U^{-1}\mathrm{d}U\right)^5\right] \\ &= -\frac{1}{480\pi^3}\,\mathrm{d}x^5\,\epsilon^{\mu\nu\rho\sigma\tau}\,\mathrm{Tr}\left[U^\dagger\left(\partial_\mu U\right)U^\dagger\left(\partial_\nu U\right)U^\dagger\left(\partial_\rho U\right)U^\dagger\left(\partial_\sigma U\right)U^\dagger\left(\partial_\tau U\right)\right]\end{aligned}\right\}\quad \boxed{S_{\mathrm{WZW}} = ik\int_{\Sigma_5}\omega_5}$$

Wess-Zumino '71
Witten '83

- Expanding locally:

$$U^\dagger\partial_\mu U = \frac{2i}{f_\pi}\partial_\mu\pi + \mathcal{O}(\pi^2)$$

WZW-coefficient: $\mathbb{Z}$ - valued

- Results in new interactions: $S_{\mathrm{WZW}} = \dfrac{2k}{45\pi^2 f_\pi^5}\displaystyle\int_{\Sigma_4=\partial\Sigma_5}\mathrm{d}^4x\,\epsilon^{\nu\rho\sigma\tau}\,\mathrm{Tr}\left[\pi\left(\partial_\nu\pi\right)\left(\partial_\rho\pi\right)\left(\partial_\sigma\pi\right)\left(\partial_\tau\pi\right)\right] + \mathcal{O}(\pi^6)$

QCD: $k = N_c = 3$

Topological terms

- *Summary:*

Common source of topological terms: $S_{\text{top}} \sim \int_\Sigma (\text{diff . form})$

Theta terms	*WZW terms*
• Obtained from closed d-forms, for d-dimensional theories. • $\mathbb{R}$-valued coefficients. • Do not affect classical EOMs. • No perturbative effects.	• Obtained from closed d+1-forms, for d-dimensional theories. • $\mathbb{Z}$-valued coefficients. • Affect classical EOMs. • Seen as perturbative effects in QFT.

Our topological portal/term is of the WZW type.

Construction of the portal

Invariant forms in QCD

$$\left.\begin{array}{c} G = SU(3)_L \times SU(3)_R \\ \downarrow \\ H = SU(3)_{L+R} \end{array}\right\} \quad U(x) = e^{\frac{2i}{f_\pi}\pi(x)^a T^a} \;:\; \Sigma_4 \to X = \frac{SU(3)_L \times SU(3)_R}{SU(3)_{L+R}} \simeq SU(3)$$

$\boxed{\text{There are } \textit{only} \text{ two } G\text{-invariant (closed) forms on } X.}$

$$\omega_5 \sim \mathrm{Tr}\,(U^{-1}dU)^5 \qquad \omega_3 \sim \mathrm{Tr}\,(U^{-1}dU)^3$$

WZW term

What is this?

Invariant forms in QCD

$$\left.\begin{array}{c} G = SU(3)_L \times SU(3)_R \\ \downarrow \\ H = SU(3)_{L+R} \end{array}\right\} \quad U(x) = e^{\frac{2i}{f_\pi}\pi(x)^a T^a} \;:\; \Sigma_4 \to X = \frac{SU(3)_L \times SU(3)_R}{SU(3)_{L+R}} \simeq SU(3)$$

There are *only* two G-invariant (closed) forms on X.

$$\omega_3 \sim \mathrm{Tr}\,(U^{-1}dU)^3$$

What is this?

Appears as a charge of the conserved current:

$$B^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,(U^\dagger(\partial_\nu U)\,U^\dagger(\partial_\rho U)\,U^\dagger\,\partial_\sigma U) \quad \Big| \quad \partial_\mu B^\mu = 0 \quad \Big| \quad B = \int_{\Sigma_3}\omega_3 = \int d^3x\,B^0$$

Invariant 3-form

$$\omega_3 \sim \mathrm{Tr}\,(U^{-1}dU)^3$$

- What else?

Can be used to construct:

$$\omega_5^{\mathrm{portal}} \sim \omega_3^{\mathrm{QCD}} \times \omega_2^{\mathrm{D}}$$

The main idea.

- ω_2^D : invariant 2-form on dark coset
- ω_3^{QCD}: already provided by QCD
- Which dark cosets provide ω_2^D? $\longrightarrow$ Unique* dark coset $\Longrightarrow$ unique portal!

QCD $\times$ Dark Topological Portal

- Collective non-linear sigma model on a product coset:

$$X = \frac{SU(3)_L \times SU(3)_R \times G_{\mathrm{D}}}{SU(3)_{L+R} \times H_{\mathrm{D}}} \simeq SU(3) \times \frac{G_{\mathrm{D}}}{H_{\mathrm{D}}}$$

$$\omega_5^{\mathrm{portal}} \sim \omega_3^{\mathrm{QCD}} \times \omega_2^{\mathrm{D}} \quad \text{on} \quad X = \frac{SU(3)_L \times SU(3)_R \times G_{\mathrm{D}}}{SU(3)_{L+R} \times H_{\mathrm{D}}} \simeq SU(3) \times \frac{G_{\mathrm{D}}}{H_{\mathrm{D}}}$$

- If dynamical assumption is the chiral symmetry breaking in the dark sector:

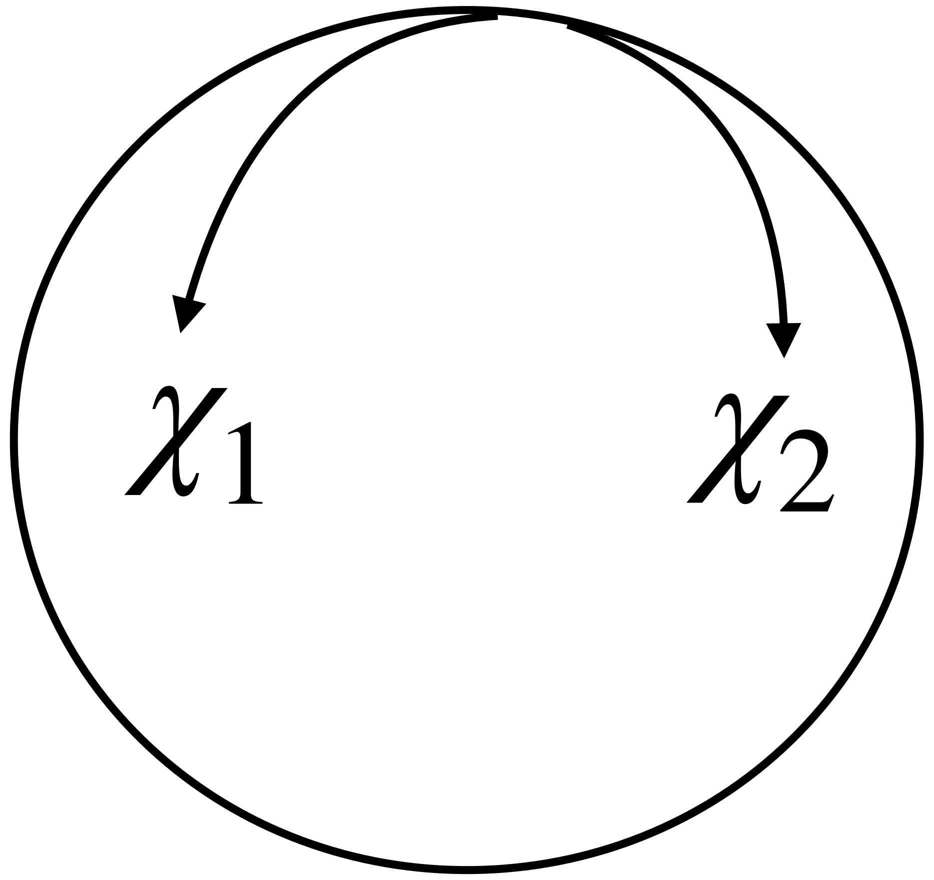
uniquely $\longrightarrow$ $G_{\mathrm{D}}/H_{\mathrm{D}} = SU(2)/SO(2) \simeq S^2$!

C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Am. Math. Soc. 63 (1948) 85–124

H. Cartan, D´emonstration homologique des th´eoremes de p´eriodicit´e de bott, ii. homologie et cohomologie des groupes classiques et de leurs espaces homogenes, S´eminaire Henri Cartan 12 (1959) 1–32.

J. Davighi and B. Gripaios, *Homological classification of topological terms in sigma models on homogeneous spaces*, JHEP 09 (2018) 155

QCD $\times$ Dark Topological Portal



- So $G_D/H_D = SU(2)/SO(2) \simeq S^2$ $\rightarrow$ a sphere.

$$\omega_2^{\mathrm{D}} = \frac{1}{4\pi f_D^2}\epsilon_{ij}\,\mathrm{d}\chi_i\,\mathrm{d}\chi_j$$ $\rightarrow$ a volume form, with χ_1/f_D and χ_2/f_D coordinates

$$\boxed{\omega_{\mathrm{portal}} = \frac{n}{96\pi^3 f_\pi^3 f_D^2} f_{abc}\epsilon_{ij} d\pi_a d\pi_b d\pi_c d\chi_i d\chi_j}$$

- Using the Stoke's theorem:

$$\boxed{\mathcal{L}_{\mathrm{portal}}^{e=0} = \frac{in\epsilon^{\mu\nu\rho\sigma}}{48\pi^2 f_\pi^3 f_D^2} f_{abc}\epsilon_{ij}\pi_a\partial_\mu\pi_b\partial_\nu\pi_c\partial_\rho\chi_i\partial_\sigma\chi_j}$$

Topological Portal

- Main phenomenology after gauging (same as for $\pi^0 \to \gamma\gamma$)

- Prescription (Yonekura: General anomaly matching by Goldstone bosons, JHEP 03 (2021) 057):

$$\frac{1}{f_\pi^2}\partial_\mu\pi^+\partial_\nu\pi^- \to eF_{\mu\nu}$$

$$\boxed{\tilde{\mathcal{L}}_{\mathrm{portal}} - \mathcal{L}_{\mathrm{portal}}^{e=0} = \frac{ne\,\epsilon^{\mu\nu\rho\sigma}}{16\pi^2 f_\pi f_D^2}\left(\pi^0 + \frac{\eta}{\sqrt{3}}\right)F_{\mu\nu}\partial_\rho\chi_1\partial_\sigma\chi_2}$$

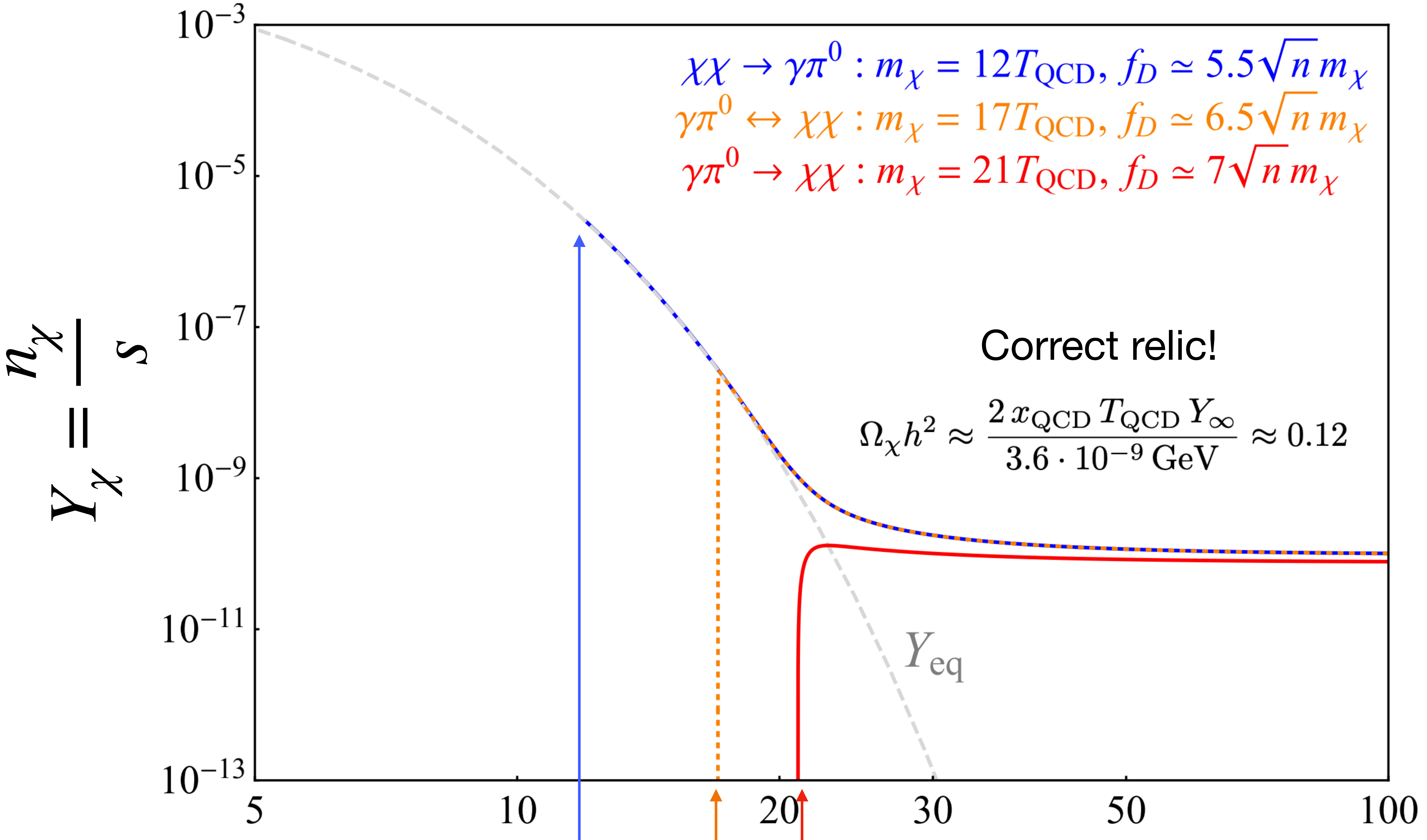
- The leading term in the EFT power counting, the next one being:

$$\frac{1}{f_\pi^2 f_D^2}(D_\mu\pi_a D^\mu\pi^a)(\partial_\nu\chi_i\partial^\nu\chi^i)$$

Phenomenology

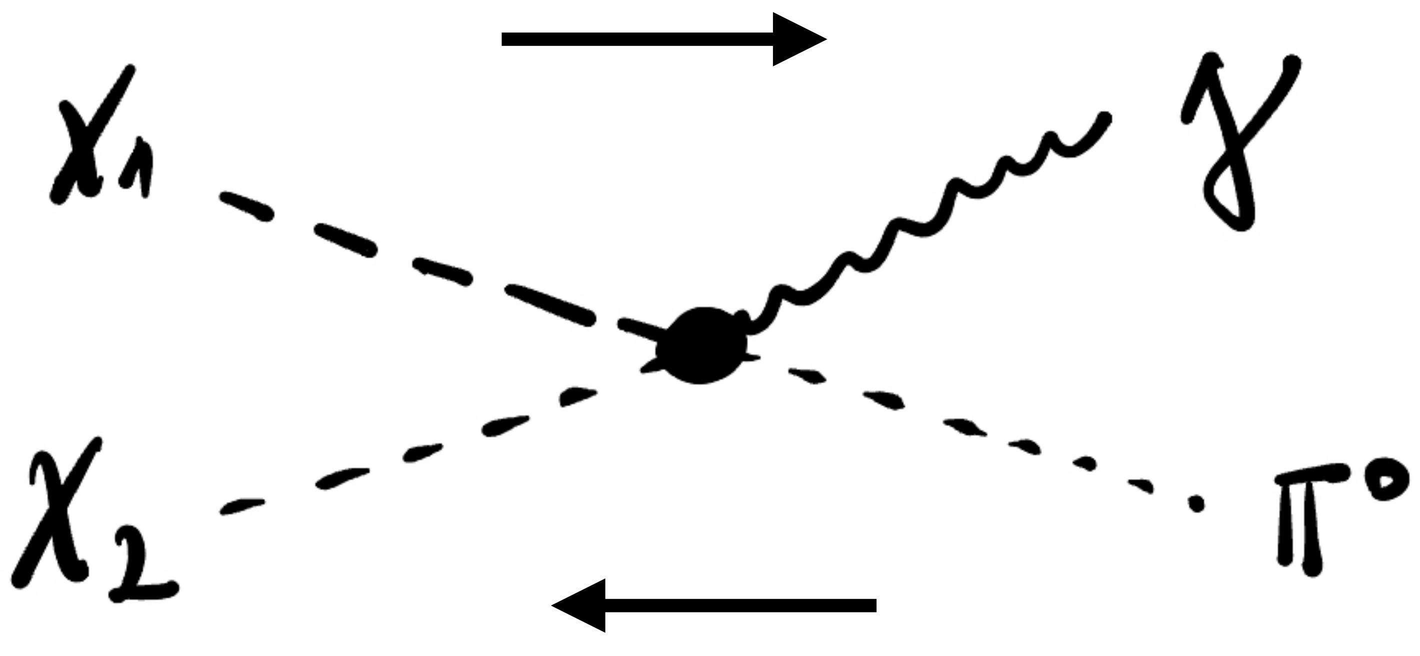
Relic Abundance

Can be robustly set.



QCD phase transition moment
(different m_χ values)

$x = \frac{m_\chi}{T}$
(time)



- *No dependence* on the cosmological history of the dark pions.
- Everything is fixed by m_χ and f_D .
- QCD phase transition should happen *no later* than $x = x_{\rm max} \approx 23$:

$$\boxed{m_\chi \lesssim 3.7\ {\rm GeV}}$$

$$f_D \sim \mathcal{O}(5-7)\sqrt{n}\, m_\chi$$

- Dark pions can be light thermal DM, and the mass range is such that pion-EFT description is reasonable!

Relic Abundance

Mass splitting effects.

- Define: $\Delta m_\chi = m_{\chi_2} - m_{\chi_1}$ and $\Delta := \Delta m_\chi / m_{\chi_1}$: χ_1 is the sole dark matter candidate
- Affects the thermally averaged cross section:

D'Agnolo, Mondino, Ruderman, Wang, Exponentially Light Dark Matter from Coannihilation, *JHEP* 08 (2018) 079

$$\langle \sigma v \rangle \to \langle \sigma v \rangle \, \exp(-x\Delta)$$

- Translates to f_D needed for the correct relic abundance:

$$f_D(\Delta) \to f_D(0) \, \exp\left(-\frac{x_{\max}\Delta}{4}\right)$$

- If $\Delta m_\chi \neq 0$, then $\Delta m_\chi > m_{\pi^0}$, otherwise $\chi_2 \to \chi_1 \gamma\gamma\gamma$ through π^0 with long lifetime $\tau > 1$ sec: problematic for BBN

Kawasaki et.al, Revisiting Big-Bang Nucleosynthesis Constraints on Long-Lived Decaying Particles *Phys.Rev.D* 97 (2018) 2, 023502

How to test this?

Indirect/Direct detection.

- Topological operator (a differential form) couples two *different* dark pions:

1. *Indirect detection*

Annihilation of $\chi_1\chi_1$ highly suppressed.
No late-time DM annihilation.

$\boxed{\text{Light thermal inelastic DM scenario}}$

David Tucker-Smith, Neal Weiner,
Inelastic dark matter, *Phys.Rev.D* 64 (2001) 043502

2. *Direct detection*

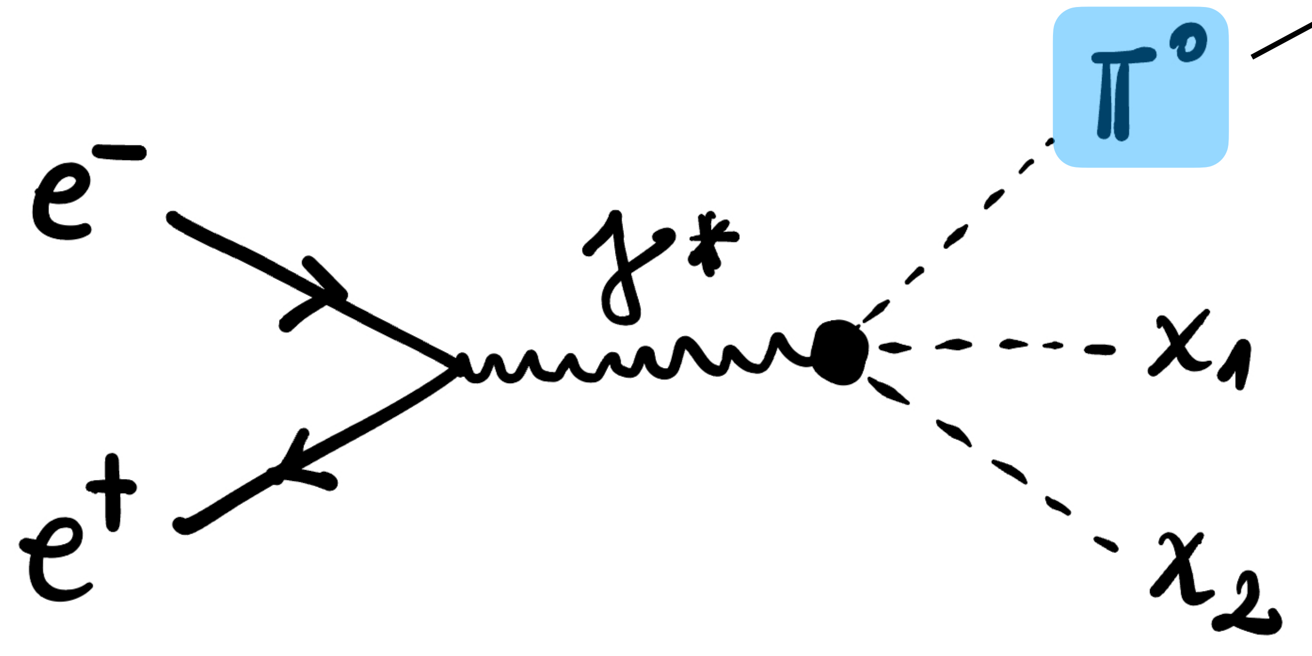
As we need $\Delta m_\chi > m_{\pi^0}$ for consistent BBN:

No sufficient energy for χ_1 to up-scatter: $\chi_1 \to \chi_2$, huge suppression $(\partial/f_D)^3$ for $v \ll 1$.
Direct detection cross-section effectively zero.

How to test this?

Novel collider signatures.

- *Belle II* could tell us a lot.



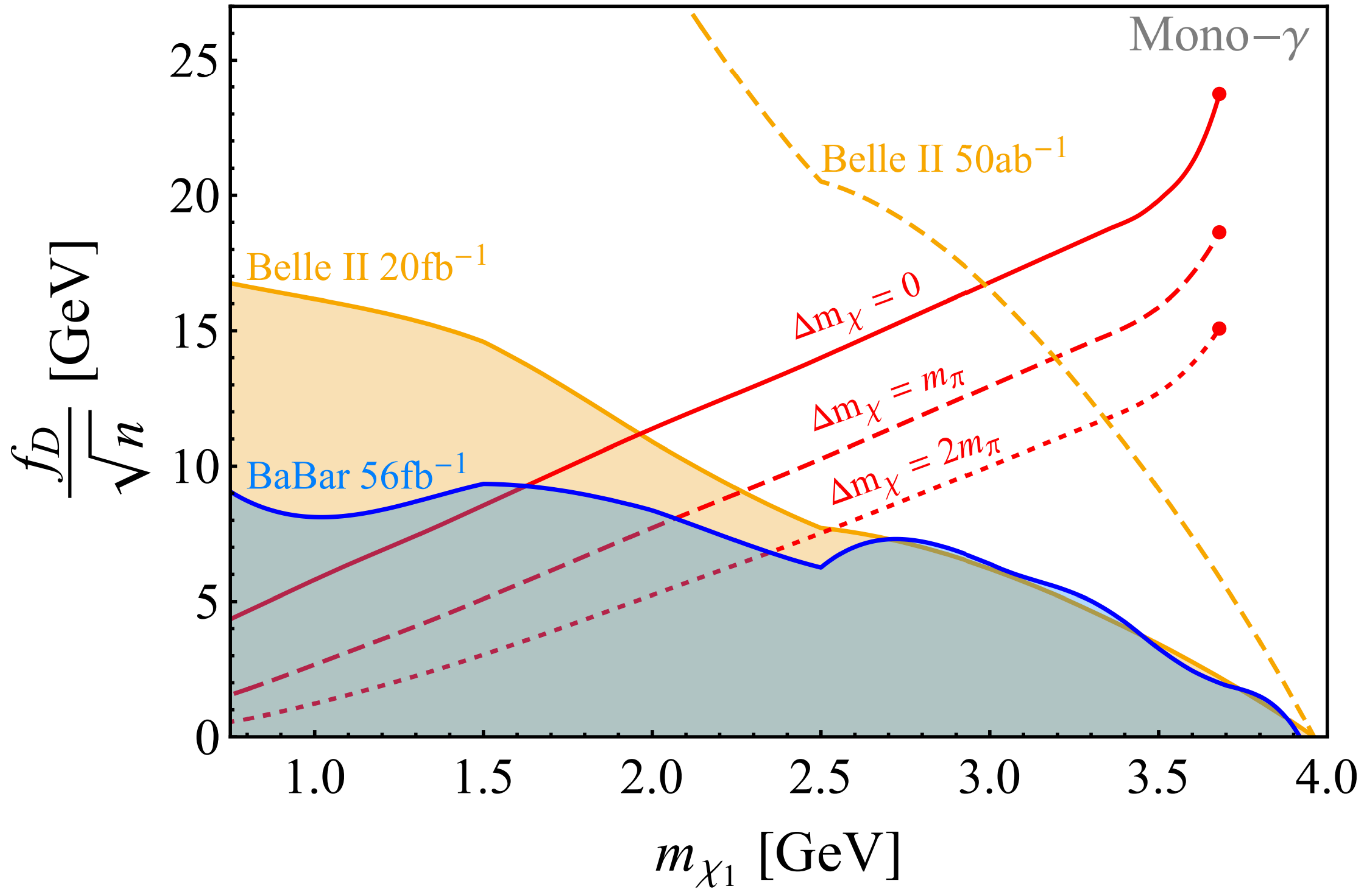
- Two regimes:
  1. χ_2 long lived (detector stable $c\tau > 10$ m)
  2. χ_2 "medium" lived (displaced vertex)

Δm_χ	$\lesssim 1.7 m_{\pi^0}$	$\gtrsim 1.7 m_{\pi^0}$
Signature	$\pi^0 + \not{E}_T$	$\pi^0 + \not{E}_T + \mathrm{DV}(\pi^0 \gamma \not{E}_T)$

Belle II is working on this now: work with Christopher Hearty and Guorui Lui

Belle II Physics Briefing Book 1808.10567

Boosted to a few GeV, reconstructed as a single photon at Belle II: Recast of the mono-γ proposed search.



(long-lived χ_2)

Conclusions

- A novel portal (with a unique dark coset) resulting in an elegant light thermal inelastic DM scenario.

- Topology explains why we haven't observed DM in direct/indirect detection experiments.

- Belle II could give us a definite answer!

  Relative rate for $e^+e^- \to \gamma* \to \pi^0\chi_1\chi_2$ *and* $e^+e^- \to \gamma* \to \eta\chi_1\chi_2$ *completely fixed.*

- Topological interactions offer new directions for DM model building.

  E.g. one can have a 3-form in the dark sector - coupled to the SM photon (2-form)
  WIP with Davighi, Murayama, Moldovsky, Scherb [2506.xxxxx]

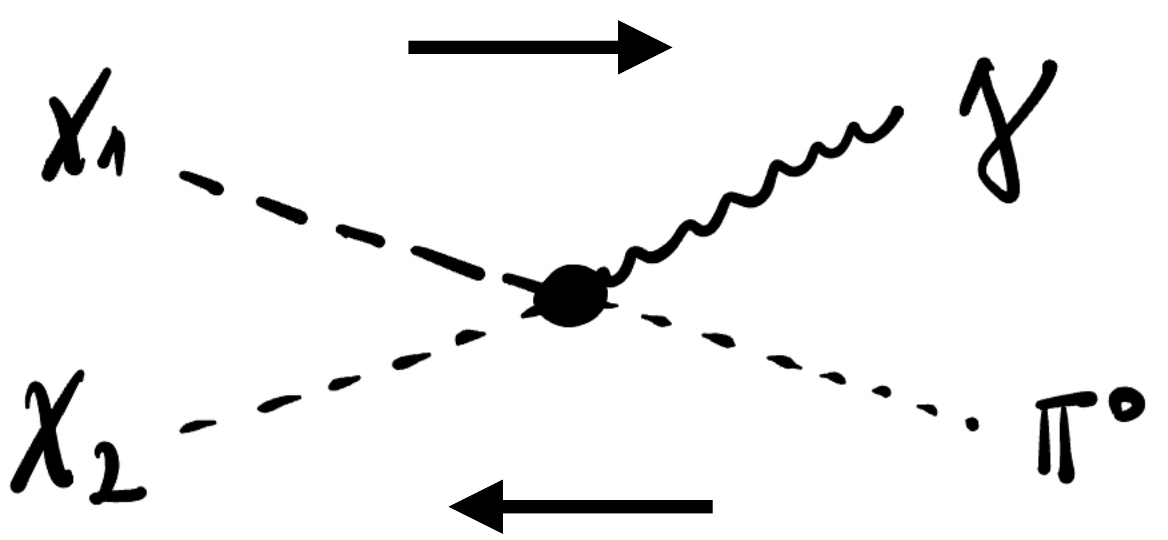
THANK YOU!

TH

$$\omega_5^{\mathrm{portal}} \sim \omega_3^{\mathrm{QCD}} \times \omega_2^{\mathrm{D}}$$

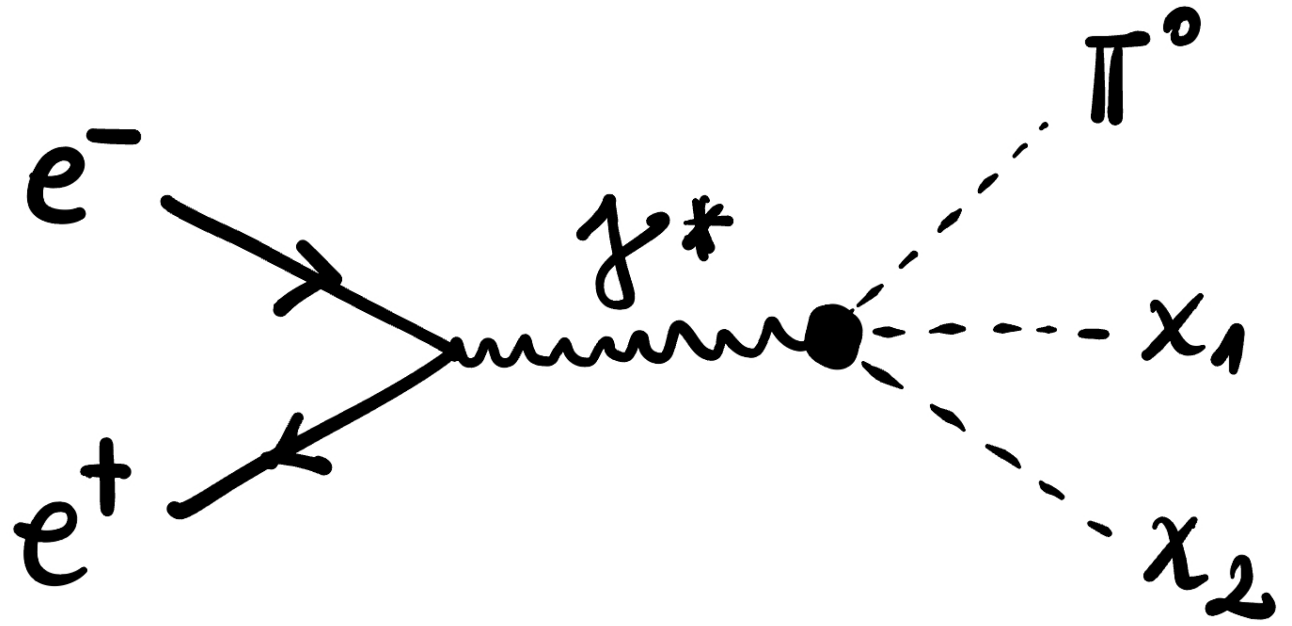
$$\tilde{\mathcal{L}}_{\mathrm{portal}} = \frac{n e\, \epsilon^{\mu\nu\rho\sigma}}{16\pi^2 f_\pi f_D^2} \left(\pi^0 + \frac{\eta}{\sqrt{3}} \right) F_{\mu\nu} \partial_\rho \chi_1 \partial_\sigma \chi_2$$

PH



Light thermal inelastic DM scenario

EXP



Belle II

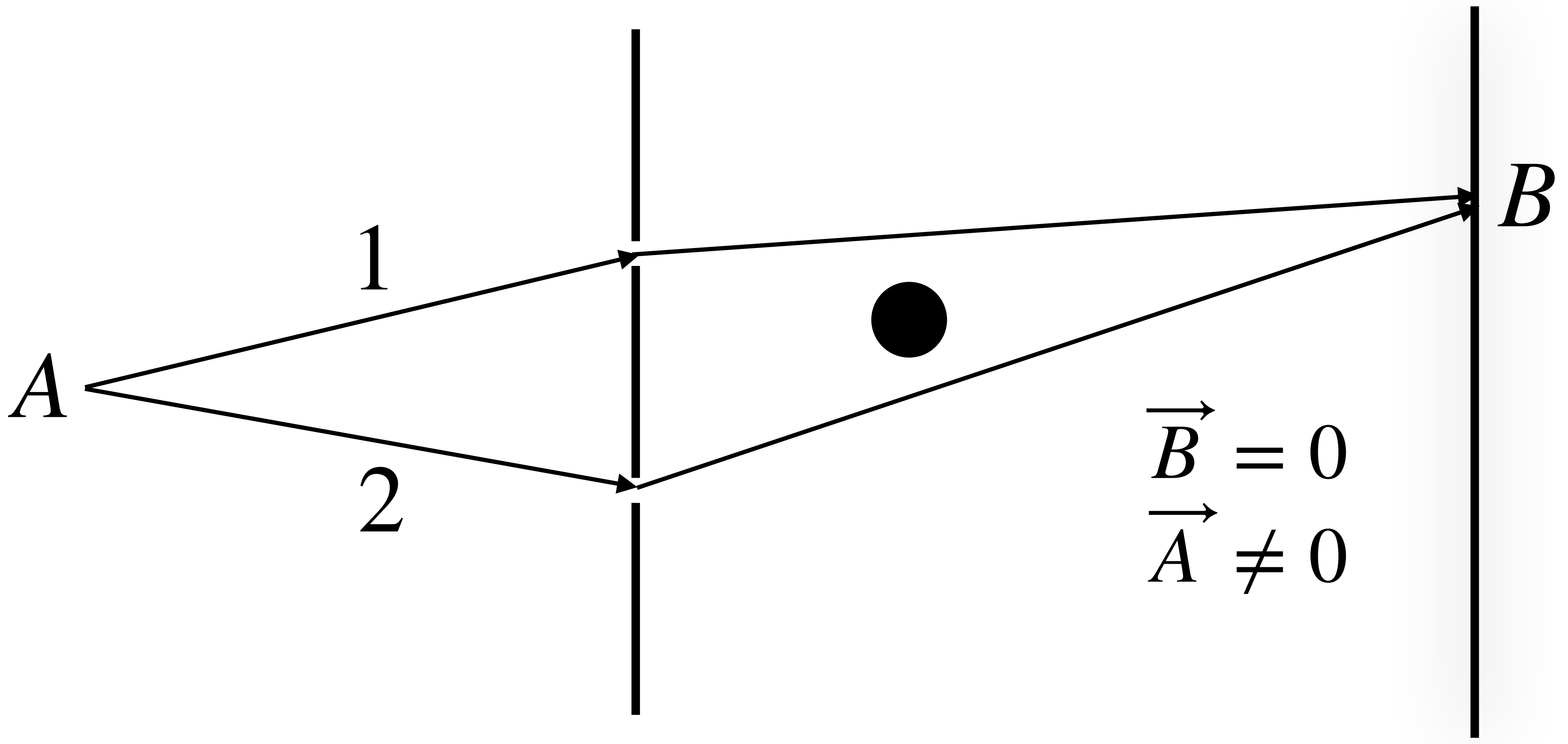
Backup

Topological terms

- *Class 1: Theta terms*

- Example 1: Aharonov-Bohm effect

Has physical consequences: double slit experiment with the solenoid



Akira Tonomura et.al '86

$$I^{(1)}_{A\to B} + I^{(2)}_{A\to B} = \mathcal{N} \, \exp\left(iq\oint A_i\, dx^i\right)$$

Stoke's theorem:

$$\oint A_i\, dx^i = \iint \vec{\nabla} \times \vec{A}\, ds = \iint \vec{B} \cdot \vec{n}\, ds = \Phi_B$$

Extra phase changes the interference pattern!

Topological terms

- *Class 1: Theta terms*

- Example 2: Instantons in 4d gauge theory

$$S_\theta = \theta \int d^4x \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = \theta N_{\text{inst}} \longrightarrow \theta \in \mathbb{R}$$

$$S_\theta = \theta \int d^4x \frac{g^2}{32\pi^2} \partial_\mu \mathrm{Tr} \left(A_\nu \partial_\alpha A_\beta + \frac{2}{3} A_\nu A_\alpha A_\beta \right) \longrightarrow \text{Total derivative}$$

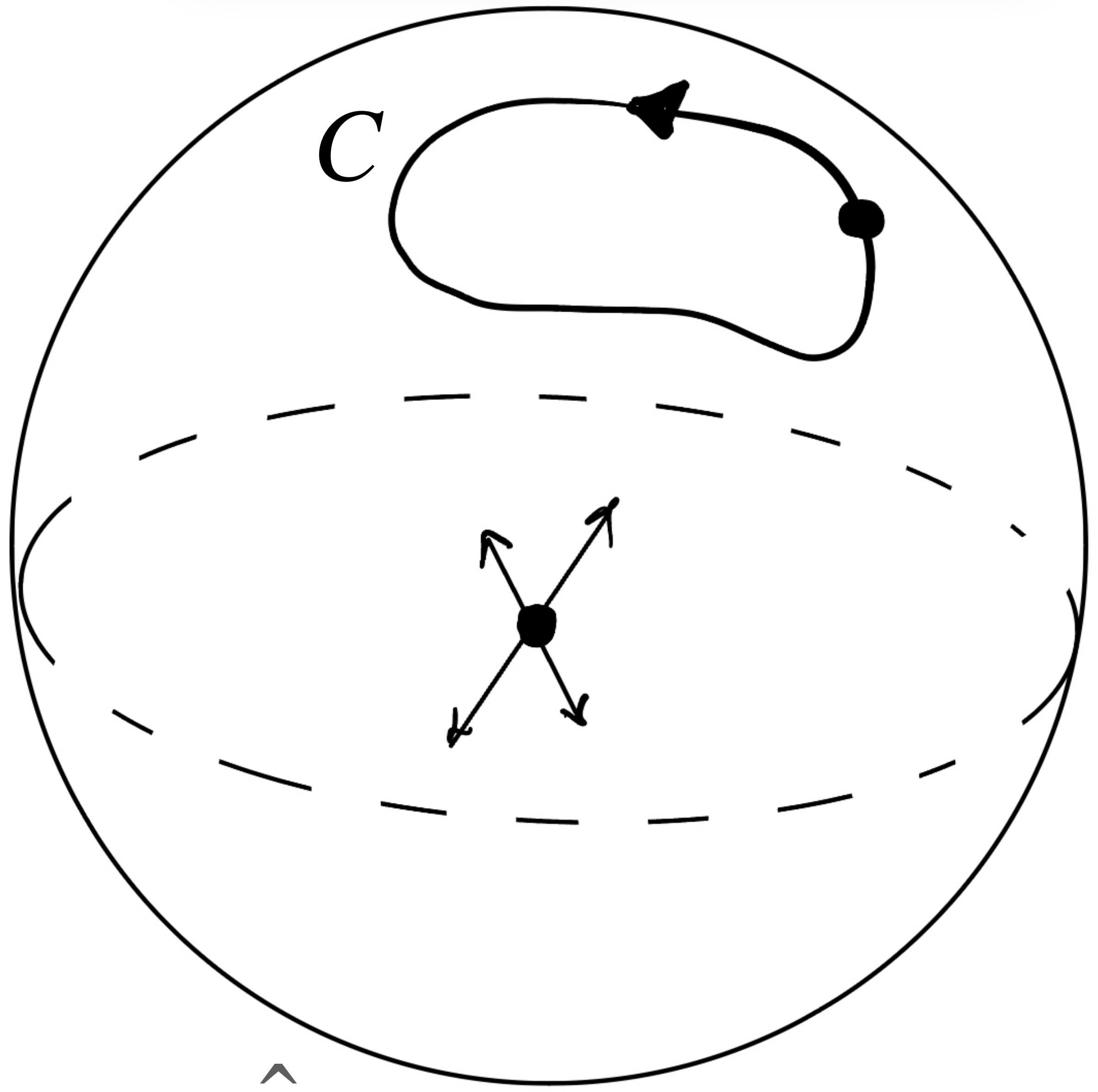
Effects not seen in perturbation theory: no Feynman diagrams.

Topological terms

- *Class 2: Wess-Zumino-Witten terms*

- Example 1: Particle on a sphere coupled to magnetic monopole.
  (More intuitive?)

Symmetries:
SO(3) rotations
LHS: ***P*** and ***T*** invariant
RHS: ***PT*** invariant



$$\vec{B} = \frac{g\hat{r}}{4\pi r^2}$$

$$m\vec{a} = q\vec{v} \times \vec{B} \longrightarrow m\ddot{x}_i = \lambda\epsilon_{ijk}x_k\dot{x}_k$$

Can we construct the action which would give this?

$$L \sim \epsilon_{ijk}x_i x_j \dot{x}_k = 0 \quad ?$$

First possibility: introduce the gauge potential $A_i(x)$

$$S = \int_C dt \left(\frac{1}{2}m\dot{x}_i^2 + \lambda A_i(x)\dot{x}^i\right)$$

$$A_\phi^N = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}$$

- No manifest SO(3) symmetry - violated by A_ϕ^N
- Dirac string along $\theta = \pi$

Witten '83
- Global aspects of current algebra

Topological terms

- *Class 2: Wess-Zumino-Witten terms*

Symmetries:
SO(3) rotations
LHS: ***P*** and ***T*** invariant
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- Example 1: Particle on a sphere coupled to magnetic monopole.
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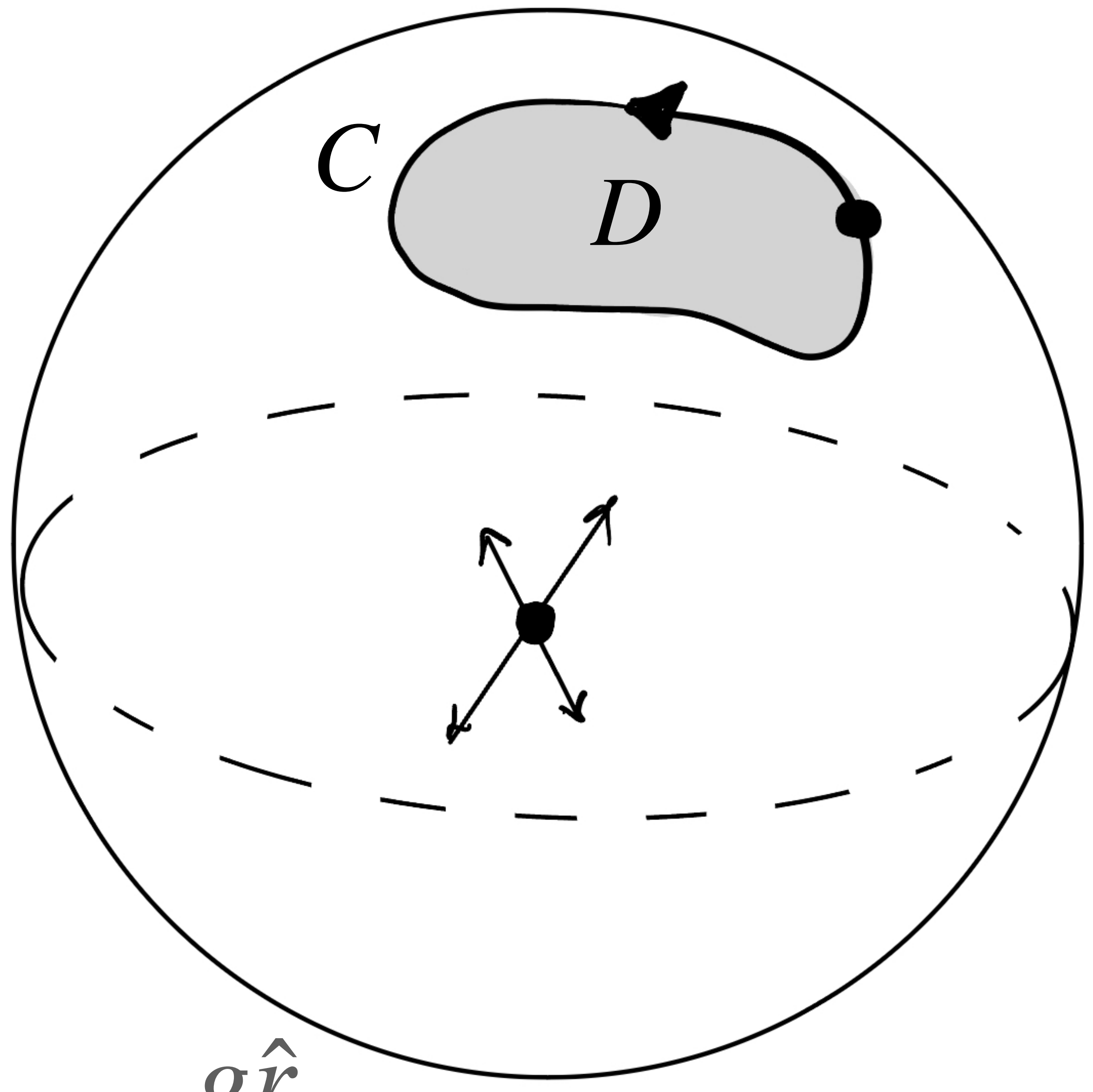


$$\overrightarrow{B} = \frac{g\hat{r}}{4\pi r^2}$$

$$m\vec{a} = q\vec{v} \times \overrightarrow{B} \longrightarrow m\ddot{x}_i = \lambda \epsilon_{ijk} x_k \dot{x}_k$$

Can we construct the action which would give this?

Second possibility: go one dimension higher

$$\int_C dt A_i(x)\dot{x}^i = \int_D dS^{ij} F_{ij}(x) \qquad C = \partial D$$

$$F_{ij} = \epsilon_{ijk} x^k / |x|^3$$

- Manifestly SO(3) symmetric
- Non-singular away from the origin

Witten '83
- Global aspects of current algebra

Topological terms

- *Class 2: Wess-Zumino-Witten terms*

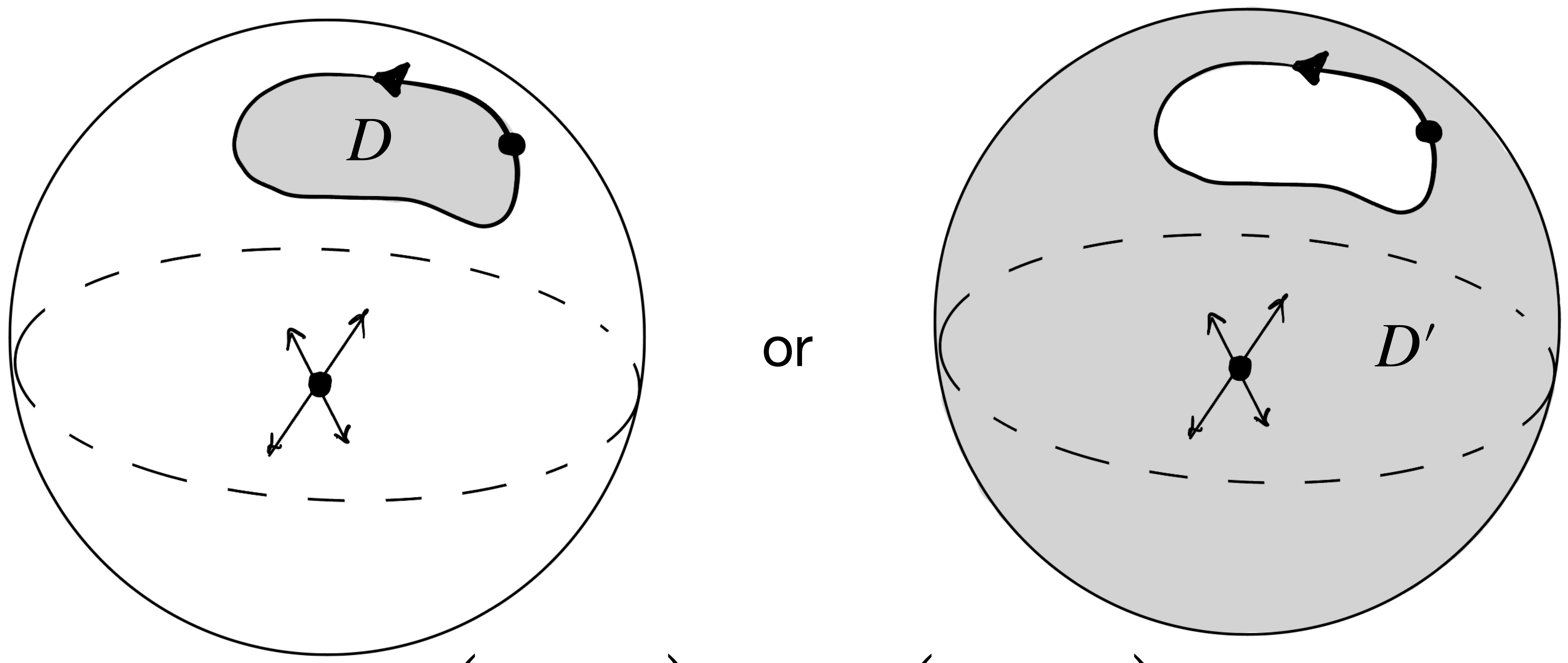
- Example 1: Particle on a sphere coupled to magnetic monopole.

  (More intuitive?)

  Again, a topological term is an integral of a differential form: 2-form ω_2

$$S = \exp\left(i\lambda \int_D dS^{ij} F_{ij} \right)$$

$$i\lambda \int_D \epsilon_{ijk} dx^i dx^j x^k / |x|^3 = i\lambda \int_D \omega_2$$



$$\exp\left(i\lambda \int_D \omega_2 \right) = \exp\left(i\lambda \int_{D'} \omega_2 \right)$$

$$\exp\left(i\lambda \int_{DUD'=S^2} \omega_2 \right) = 1 \quad \text{and} \quad \underbrace{\int_{S^2} \omega_2 = 2\pi n}_{\text{The magnetic flux through any closed surface is quantised}} \longrightarrow \lambda \in \mathbb{Z}$$

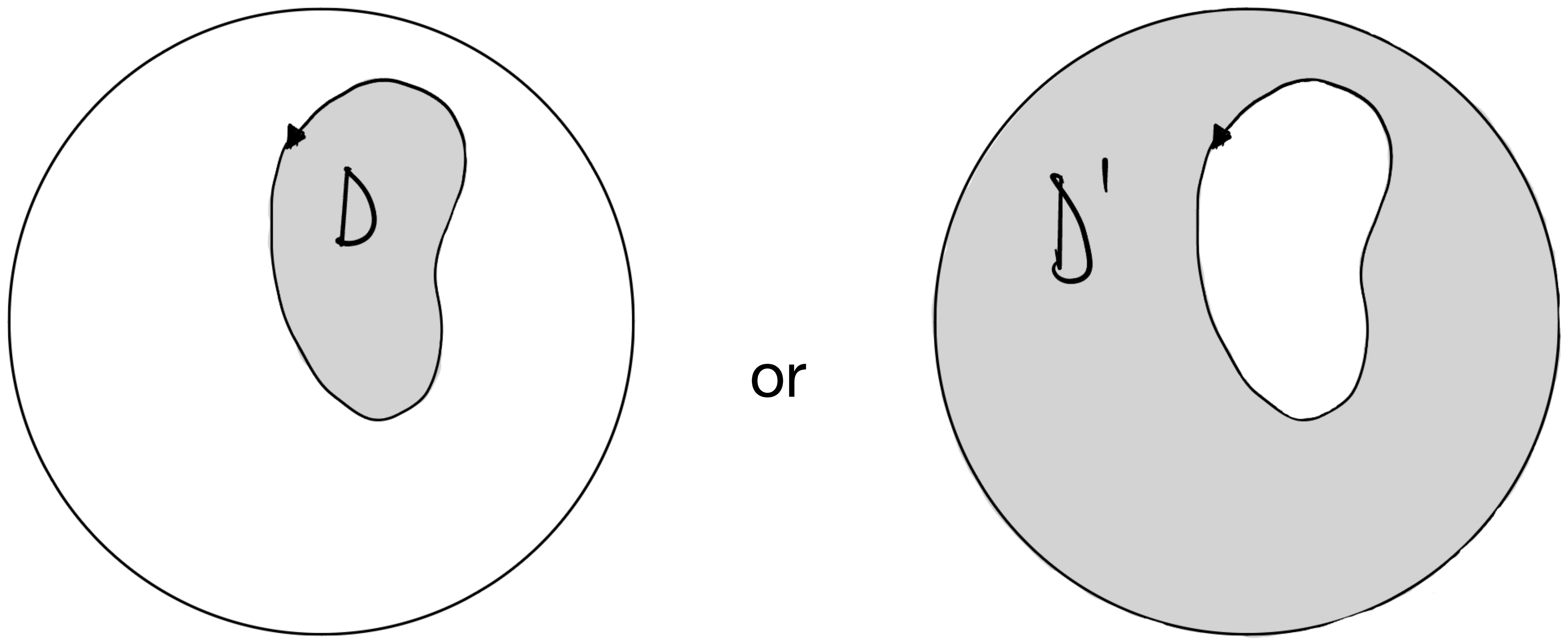
Topological terms

Witten '83
- Global aspects of current algebra

- *Class 2: Wess-Zumino-Witten terms*

- Example 2: Low-energy QCD

Integer coefficients:



$$\exp\left(ik\int_D \omega_5\right) = \exp\left(ik\int_{D'} \omega_5\right)$$

Mathematically, ω_5 is:

1. $G = SU(3)_L \times SU(3)_R$ - invariant;
2. *Closed*, $d\omega_5 = 0$;
3. *Integral*, integrates to $(\text{normalisation}) \times n$.

Complete analogy with Example 1.

$$\exp\left(ik\int_{DUD'=\Sigma_5} \omega_5\right) = 1 \quad \text{and} \quad \underbrace{\int_{\Sigma_5} \omega_5 = 2\pi n}_{} \longrightarrow k \in \mathbb{Z}$$

Integrality ($\Pi_5(SU(3)) = \mathbb{Z}$)

Topological terms

Witten '83
- Global aspects of current algebra

- *Class 2: Wess-Zumino-Witten terms*

- Example 2: Low-energy QCD

Chiral anomalies: gauging the WZW term $\quad U(1)_{\text{QED}} \supset SU(3)_{L+R}$

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

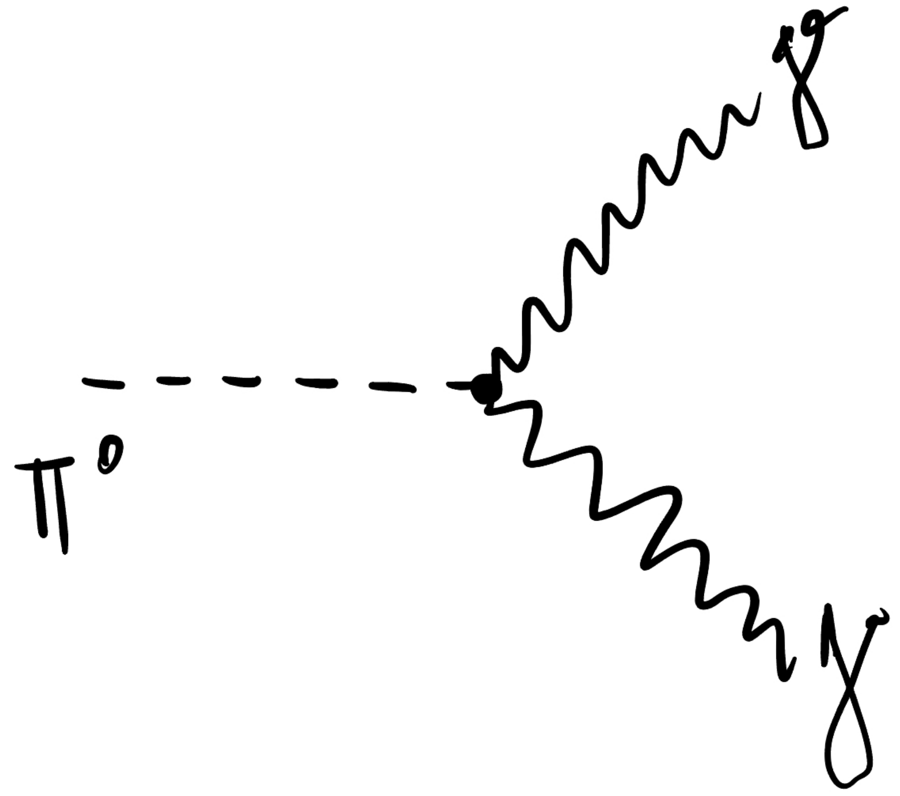
Non-trivial derivation using Noether method:

$$J^\mu = \frac{1}{48\pi^2}\,\epsilon^{\mu\rho\sigma\tau}\mathrm{tr}\left(\{Q, U^\dagger\}\,\partial_\rho U\, U^\dagger\partial_\sigma U\, U^\dagger\partial_\lambda U\right)$$

$$S_{WZW} = k\left[-e\int d^4x\; A_\mu(x)J^\mu + \frac{ie^2}{24\pi^2}\int d^4x\;\epsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)A_\rho \mathrm{tr}\left(\{Q^2, U^\dagger\}\,\partial_\sigma U + U^\dagger Q U Q U^\dagger \partial_\sigma U\right)\right]$$

$$L_{\text{WZW}}^{\text{gauged}} \supset \underbrace{\frac{ke^2}{96\pi^2 f_\pi}}_{\downarrow}\pi^0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$k = N_c$$



Invariant 3-form

$$\omega_3 \sim \mathrm{Tr}\,(U^{-1}dU)^3$$

David Tong
- Gauge theory

- Does not appear in the QCD action.
- However, it appears as the *topologically conserved* current:

Consider static field configurations in the chiral Lagrangian: $U(x) \to 1$ as $|x| \to \infty$

Fields are maps: $U(x) : S^3 \to SU(3)$

Characterised by the winding number: $\Pi_3(SU(3)) = \mathbb{Z}$, which can be computed as

$$B = \int_{\Sigma_3} \omega_3 = \frac{1}{24\pi^2} \int d^3x\, \epsilon_{ijk} \mathrm{Tr} \left(U^\dagger(\partial_i U) U^\dagger(\partial_j U) U^\dagger(\partial_k U) \right)$$

Also a charge of the conserved current:

$$B^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\left(U^\dagger(\partial_\nu U)\, U^\dagger(\partial_\rho U)\, U^\dagger\, \partial_\sigma U\right) \quad \Big| \quad \partial_\mu B^\mu = 0 \quad \Big| \quad B = \int d^3x\, B^0$$

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Same properties as ω_5:

1. $G = SU(3)_L \times SU(3)_R$ - invariant;
2. *Closed*, $d\omega_3 = 0$;
3. *Integral*, integrates to $(\text{normalisation}) \times n$.

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QCD × Dark Topological Portal

EFT description.

- Collective non-linear sigma model on a product coset:

$$X = \frac{SU(3)_L \times SU(3)_R \times G_\mathrm{D}}{SU(3)_{L+R} \times H_\mathrm{D}} \simeq SU(3) \times \frac{G_\mathrm{D}}{H_\mathrm{D}}$$

$$\omega_5^{\mathrm{portal}} \sim \omega_3^{\mathrm{QCD}} \times \omega_2^{\mathrm{D}} \quad \text{on} \quad X = \frac{SU(3)_L \times SU(3)_R \times G_\mathrm{D}}{SU(3)_{L+R} \times H_\mathrm{D}} \simeq SU(3) \times \frac{G_\mathrm{D}}{H_\mathrm{D}}$$

1. Closed: $\mathrm{d}\omega_5^{\mathrm{portal}} = 0$ implies $\mathrm{d}\omega_2^{\mathrm{D}} = 0$ since $\mathrm{d}\omega_3^{\mathrm{QCD}} = 0$.
2. $SU(3)_L \times SU(3)_R \times G_\mathrm{D}$ - invariance: Product structure implies ω_2^{D} is G_D - invariant.
3. Integrality: Cycles factorise; normalise ω_3^{QCD} and ω_2^{D} separately.

J. Davighi and B. Gripaios, *Homological classification of topological terms in sigma models on homogeneous spaces*, JHEP 09 (2018) 155

QCD $\times$ Dark Topological Portal

Which dark coset fits?

- Consider QCD-like theories*:

$$\frac{G_D}{H_D} = \left\{ \frac{SU(N)_L \times SU(N)_R}{SU(N)_{L+R}}, \frac{SU(N)}{SO(N)}, \frac{SU(2N)}{Sp(2N)} \right\}$$

- All are symmetric spaces:
  G_D-invariant forms on G_D/H_D are in 1-to-1 with cohomology classes (C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Am. Math. Soc. 63 (1948) 85–124.)

$$\boxed{\text{The portal exists iff } H^2(G_D/H_D) \neq 0.}$$

(de Rham cohomology $H^k(M)$ - the set of closed ***k***-forms on ***M.***)

**One could, in principle, consider other options, such as a complex projective space $\mathbb{C}P^n$, that go beyond the chiral symmetry-breaking dynamical assumption.*

QCD × Dark Topological Portal

Which dark coset fits?

H. Cartan, D´emonstration homologique des th´eoremes de p´eriodicit´e de bott, ii. homologie et cohomologie des groupes classiques et de leurs espaces homogenes, S´eminaire Henri Cartan 12 (1959) 1–32.

		Portal			SIMP
p	1	2	3	4	5
$H^p(SU(2))$	0	0	$\mathbb{R}$	–	–
$H^p(SU(n)),\ n \geq 3$	0	0	$\mathbb{R}$	0	$\mathbb{R}$
$H^p(SU(2)/SO(2))$	0	$\mathbb{R}$	–	–	–
$H^p(SU(3)/SO(3))$	0	0	0	0	$\mathbb{R}$
$H^p(SU(4)/SO(4))$	0	0	0	$\mathbb{R}$	$\mathbb{R}$
$H^p(SU(n)/SO(n)),\ n \geq 5$	0	0	0	0	$\mathbb{R}$
$H^p(SU(2n)/Sp(2n)),\ n \geq 2$	0	0	0	0	$\mathbb{R}$

QCD

Unique choice!

5 dark pions
- no portal

Hochberg et al,
1402.5143, 1411.3727

- UV completion?

First steps by Joe Davighi and Nakarin Lohitsiri in:
WZW terms without anomalies: generalised symmetries in chiral Lagrangians, 2407.20340

There is an immediate *puzzle*:

Topological term in QCD ($\omega_5 \sim \mathrm{Tr}\left(U^{-1}dU\right)^5$) matches all t'Hooft anomalies related to gauging any subgroup of $SU(3)_L$ and $SU(3)_R$: tells us about underlying quark content.

Here, however, there is no mixed anomaly between $SU(3)_{L/R}$ and $SU(2)_D$:

$$\mathrm{Tr}(F_{SU(3)}\, F_{SU(3)}\, f_{SU(2)}) = 0$$

regardless of the fermion content.

As realised by Joe and Nakarin in
WZW terms without anomalies: generalised symmetries in chiral Lagrangians, 2407.20340

There is an additional 1-form symmetry related to a conserved 2-form ω_2:

$$d\omega_2 = 0$$

Moreover, there is mixing between 0-form (global) and 1-form symmetry in a *2-group structure.* For example, a gauge transformation of the background gauge field related to the subgroup of $SU(3)_L$:
$A_L^{(1)} \to g_L A_L^{(1)} g_L^{-1} + g_L d g_L^{-1}$

$$\delta_L \int_{\Sigma_5} \omega_5 \sim n \int_{\Sigma_4} \mathrm{Tr}(g_L d g_L^{-1} F_L) \wedge j^{(2)}_{\mathrm{wind}} \quad \text{with} \quad j^{(2)}_{\mathrm{wind}} = \star\, \omega_2 \,.$$

As realised by Joe and Nakarin in
WZW terms without anomalies: generalised symmetries in chiral Lagrangians, 2407.20340

Crucially, 2-group structure is integer quantised (Cordova, Dumitrescu, Intriligator, 1802.04790):
preserved along RG flow.

The topological portal endows the IR with a non-vanishing 2-group algebra:
Same must be true for the UV!

No-Go theorem:
Dark QCD-like UV theory has no conserved 1-forms, and thus no 2-group symmetry. Therefore, it cannot give rise to an IR theory with the topological portal.

Lead to study UV completions that at least have 1-form symmetries, Abelian theories, even weakly coupled.

Future work with Joe and Admir.