Amplitudes and Perturbative Unitarity Bounds

Planck 2025

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May 29, 2025 — Padova, Italy

Based on `[2504.12855]` with GABRIELE LEVATI and PARIDE PARADISI

Why Unitarity Bounds Matter

Key question

What is the energy scale at which an effective field theory (EFT) breaks down?

- Unitarity violation $\rightsquigarrow$ New physics scale or strong dynamics
- Historical example: *no-lose Higgs theorem* $\quad m_H \lesssim 1\ \text{TeV}$

VOLUME 38, NUMBER 16 PHYSICAL REVIEW LETTERS 18 APRIL 1977

Strength of Weak Interactions at Very High Energies and the Higgs Boson Mass

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(Received 28 February 1977)

It is shown that if the Higgs boson mass exceeds $M_c = (8\pi\sqrt{2}/3G_F)^{1/2}$ partial-wave unitarity is not respected by the tree diagrams for two-body reactions of gauge bosons, and the weak interactions must become strong.

PHYSICAL REVIEW D VOLUME 16, NUMBER 5 1 SEPTEMBER 1977

Weak interactions at very high energies: The role of the Higgs-boson mass

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(Received 20 April 1977)

We give an S-matrix-theoretic demonstration that if the Higgs-boson mass exceeds $M_c = (8\pi\sqrt{2}/3G_F)^{1/2}$, parital-wave unitarity is not respected by the tree diagrams for two-body scattering of gauge bosons, and the weak interactions must become strong at high energies. We exhibit the relation of this bound to the structure of the Higgs-Goldstone Lagrangian, and speculate on the consequences of strongly coupled Higgs-Goldstone systems. Prospects for the observation of massive Higgs scalars are noted.

- Unitarity bounds are necessary for correct interpretation of experimental data (*e.g.*, tails of kinematical distributions, vector boson scattering, . . .)

Standard Approach to Unitarity Bounds

- **Standard approach**: $2 \to 2$ partial wave decomposition with Wigner d-matrix

$$\mathcal{A}_{h_1,h_2\to h_3,h_4}(s,\theta,\varphi) = 8\pi \sum_J (2J+1)\, a^J_{h_1,h_2\to h_3,h_4}(s)\, d^J_{h_1-h_2,h_3-h_4}(\theta)\, e^{i(h_1-h_2-h_3+h_4)\varphi}$$

h_i: helicities J: total angular momentum [Jacob, Wick '59]

- Two critical gaps:

  1. $N \to M$ amplitudes (*e.g.*, $2 \to 3$), particularly relevant for high-energy future colliders, do not admit such decomposition

  2. Spin-2 or higher-spin EFT of gravity: Feynman rules impractical

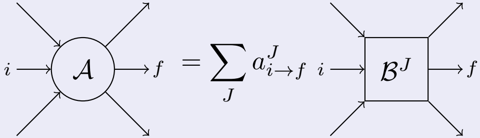
- **This work**: New *vectorial formalism* to generalize partial wave unitarity bounds

Amplitudes & Partial Wave Decomposition

Linear algebra problem

Project an amplitude $|\mathcal{A}_{i\to f}\rangle$ onto a *kinematic basis* $|\mathcal{B}^J_{i\to f}\rangle$ with definite angular momentum J

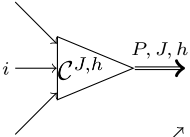
$$|\mathcal{A}_{i\to f}\rangle = \sum_J a^J_{i\to f}\,|\mathcal{B}^J_{i\to f}\rangle$$



- $|\mathcal{A}_{i\to f}\rangle\,,\ |\mathcal{B}^J_{i\to f}\rangle \in V_{i\to f}$ vector space
- $a^J_{i\to f}$: partial wave coefficients that encode the *dynamics*

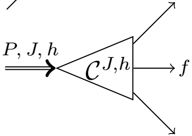
Poincaré Clebsch-Gordan Coefficients

$$\langle P, J, h | i \rangle = \mathcal{C}^{J,h}_{i \to *}\, \delta^{(4)}\left(P - \sum_{k \in i} p_k\right) =$$



$\in V_{i \to *}$

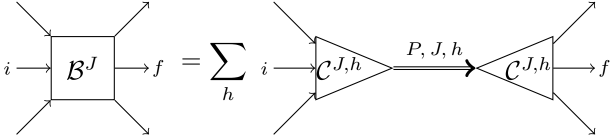
$$\langle f | P, J, h \rangle = \mathcal{C}^{J,h}_{* \to f}\, \delta^{(4)}\left(P - \sum_{k \in f} p_k\right) =$$



$\in V_{* \to f}$

$|P, J, h\rangle$: Poincaré irreducible multiparticle state [Jiang, Shu, Xiao, Zheng ’20]

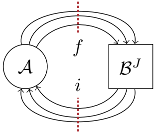
$$\left|\mathcal{B}^{J}_{i \to f}\right\rangle = \sum_h \left|\mathcal{C}^{J,h}_{i \to *}\right\rangle \otimes \left|\mathcal{C}^{J,h}_{* \to f}\right\rangle$$



Perturbative Unitarity Bounds

- *Inner product* via Lorentz-invariant phase-space integrals

$$a^J_{i\to f} = \frac{1}{2J+1}\langle \mathcal{B}^J_{i\to f}|\mathcal{A}_{i\to f}\rangle = \frac{1}{2J+1}\int \mathrm{d}\Phi_i\,\mathrm{d}\Phi_f\,\mathcal{A}_{i\to f}\,\left(\mathcal{B}^J_{i\to f}\right)^* = \frac{1}{2J+1}$$



$$\langle \mathcal{B}^J_{i\to f}|\mathcal{B}^{J'}_{i\to f}\rangle = (2J+1)\delta^{JJ'} \qquad \Longleftrightarrow \qquad \int \mathrm{d}\Phi_X\;|\mathcal{B}^J_{i\to X}\rangle \otimes |\mathcal{B}^{J'}_{X\to f}\rangle = |\mathcal{B}^J_{i\to f}\rangle\,\delta^{JJ'}$$

- *Generalized optical theorem*:

$$|\mathcal{A}_{i\to f}\rangle - |\mathcal{A}^*_{f\to i}\rangle = i\sum_X \int \mathrm{d}\Phi_X\;|\mathcal{A}_{i\to X}\rangle \otimes |\mathcal{A}^*_{f\to X}\rangle$$

$$a^J_{i\to f} - \left(a^J_{f\to i}\right)^* = i\sum_X a^J_{i\to X}\left(a^J_{f\to X}\right)^*$$

- *Partial wave unitarity bounds*:

$$\left|\mathrm{Re}\,a^J_{i\to i}\right| \leq 1 \qquad 0 \leq \mathrm{Im}\,a^J_{i\to i} \leq 2 \qquad \left|a^J_{i\to f}\right| \leq 1 \quad (f\neq i)$$

Determination of $|\mathcal{B}^J_{i\to f}\rangle$: a 3-Step Algorithm

1. Find a basis for $V_{i\to f}$ of kinematic monomials in spinor-helicity variables $\{\lambda_k, \widetilde{\lambda}_k\}_{k=1}^n$
   [Shadmi, Weiss '18] [De Angelis '22] [Li, Ren, Xiao, Yu, Zheng '22]
2. Find the eigenvectors with definite $J_{\mathcal{I}}$ of the *Pauli-Lubanski operator squared*

$$\mathbb{W}^2_{\mathcal{I}} = \frac{1}{8}\mathbb{P}^2_{\mathcal{I}}\left(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\mathbb{M}_{\mathcal{I},\alpha\beta}\mathbb{M}_{\mathcal{I},\gamma\delta} + \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\delta}\widetilde{\mathbb{M}}_{\mathcal{I},\dot\alpha\dot\beta}\widetilde{\mathbb{M}}_{\mathcal{I},\dot\gamma\dot\delta}\right) + \frac{1}{4}\mathbb{P}^{\alpha\dot\alpha}_{\mathcal{I}}\mathbb{P}^{\beta\dot\beta}_{\mathcal{I}}\mathbb{M}_{\mathcal{I},\alpha\beta}\widetilde{\mathbb{M}}_{\mathcal{I},\dot\alpha\dot\beta}$$

   where $\mathcal{I} = i$ or f and [Witten '03]

$$\mathbb{P}^{\alpha\dot\alpha}_{\mathcal{I}} = \sum_{i\in\mathcal{I}}\lambda^\alpha_i\widetilde{\lambda}^{\dot\alpha}_i$$

$$\mathbb{M}^{\alpha\beta}_{\mathcal{I}} = \sum_{i\in\mathcal{I}}\left(\lambda^\alpha_i\frac{\partial}{\partial\lambda_{i,\beta}} + \lambda^\beta_i\frac{\partial}{\partial\lambda_{i,\alpha}}\right) \qquad \widetilde{\mathbb{M}}^{\dot\alpha\dot\beta}_{\mathcal{I}} = \sum_{i\in\mathcal{I}}\left(\widetilde{\lambda}^{\dot\alpha}_i\frac{\partial}{\partial\widetilde{\lambda}_{i,\dot\beta}} + \widetilde{\lambda}^{\dot\beta}_i\frac{\partial}{\partial\widetilde{\lambda}_{i,\dot\alpha}}\right)$$

   The eigenvalues are $-P^2_{\mathcal{I}}J_{\mathcal{I}}(J_{\mathcal{I}}+1)$
3. Normalize them such that their norm is $\sqrt{2J_{\mathcal{I}}+1}$

Photon & Gravity EFT

Anomalous quartic couplings

For *generic* spin-S ($S \in \mathbb{N}$) massless particles

$$\frac{\mathcal{L}^{(S)}}{\sqrt{-g}} = c_1^{(S)} (\mathcal{Q}^{(S)})^2 + c_2^{(S)} (\widetilde{\mathcal{Q}}^{(S)})^2 + c_3^{(S)} \mathcal{Q}^{(S)} \widetilde{\mathcal{Q}}^{(S)}$$

$$\mathcal{Q}^{(1)} = F_{\mu\nu} F^{\mu\nu} \qquad \widetilde{\mathcal{Q}}^{(1)} = F_{\mu\nu} \widetilde{F}^{\mu\nu} \qquad \mathcal{Q}^{(2)} = M_{\mathrm{P}}^2 \, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \qquad \widetilde{\mathcal{Q}}^{(2)} = M_{\mathrm{P}}^2 \, R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma}$$

Photon & Gravity EFT

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Full $2 \to 2$ helicity amplitude:

$|\mathcal{A}_{i\to f}\rangle =$

	$(3^{+S}, 4^{+S})$	$(3^{+S}, 4^{-S})$	$(3^{-S}, 4^{-S})$
$(1^{+S}, 2^{+S})$	$8\, c_+^{(S)} \langle 12\rangle^{2S} [34]^{2S}$	0	$8\, c_-^{(S)} (\langle 12\rangle^{2S} \langle 34\rangle^{2S} + \langle 13\rangle^{2S} \langle 24\rangle^{2S} + \langle 14\rangle^{2S} \langle 23\rangle^{2S})$
$(1^{+S}, 2^{-S})$	0	$8\, c_+^{(S)} \langle 14\rangle^{2S} [23]^{2S}$	0
$(1^{-S}, 2^{-S})$	$8\, (c_-^{(S)})^* ([12]^{2S} [34]^{2S} + [13]^{2S} [24]^{2S} + [14]^{2S} [23]^{2S})$	0	$8\, c_+^{(S)} \langle 34\rangle^{2S} [12]^{2S}$

with $c_+^{(S)} = c_1^{(S)} + c_2^{(S)} \in \mathbb{R}$ and $c_-^{(S)} = c_1^{(S)} - c_2^{(S)} + i\, c_3^{(S)} \in \mathbb{C}$

Photon & Gravity EFT

1. Normalized eigenvectors of $\mathbb{W}_{12}^2$ corresponding to $J_{12} = 0$ (with $s = 2p_1 \cdot p_2$):

$$\left|\mathcal{B}^0_{i\to f}\right\rangle = \left(\begin{array}{c||ccc} & (3^{+S},4^{+S}) & (3^{+S},4^{-S}) & (3^{-S},4^{-S}) \\ \hline\hline (1^{+S},2^{+S}) & \frac{16\pi}{s^{2S}}\langle 12\rangle^{2S}[34]^{2S} & 0 & \frac{16\pi}{s^{2S}}\langle 12\rangle^{2S}\langle 34\rangle^{2S} \\ (1^{+S},2^{-S}) & 0 & 0 & 0 \\ (1^{-S},2^{-S}) & \frac{16\pi}{s^{2S}}[12]^{2S}[34]^{2S} & 0 & \frac{16\pi}{s^{2S}}\langle 34\rangle^{2S}[12]^{2S} \end{array}\right)$$

2. Partial wave coefficients:

$$a^0_{i\to f} = \langle \mathcal{B}^0_{i\to f}|\mathcal{A}_{i\to f}\rangle = \left(\begin{array}{c||ccc} & (3^{+S},4^{+S}) & (3^{+S},4^{-S}) & (3^{-S},4^{-S}) \\ \hline\hline (1^{+S},2^{+S}) & \frac{s^{2S}}{2\pi}c_+^{(S)} & 0 & \frac{s^{2S}}{2\pi}\frac{2S+3}{2S+1}c_-^{(S)} \\ (1^{+S},2^{-S}) & 0 & 0 & 0 \\ (1^{-S},2^{-S}) & \frac{s^{2S}}{2\pi}\frac{2S+3}{2S+1}(c_-^{(S)})^* & 0 & \frac{s^{2S}}{2\pi}c_+^{(S)} \end{array}\right)$$

3. Non-zero eigenvalues:

$$\frac{s^{2S}}{2\pi}\left(c_+^{(S)} \pm \frac{2S+3}{2S+1}|c_-^{(S)}|\right) \implies \frac{s^{2S}}{2\pi}\left|c_+^{(S)} \pm \frac{2S+3}{2S+1}|c_-^{(S)}|\right| \leq 1$$

Positivity Bounds

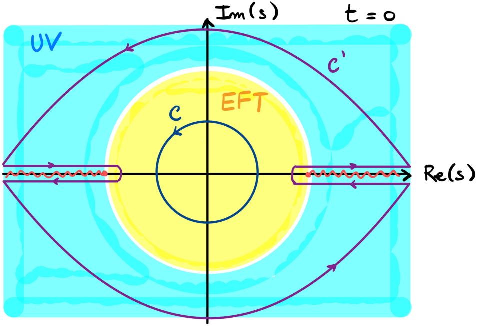
[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '06]

- Certain signs of Wilson coefficients violate infrared principles:
  - Unitarity, Causality & Analyticity

- Forward and crossing symmetric $2 \to 2$ amplitude

  $\mathcal{A}(s) = \lim_{t\to 0} \mathcal{A}(s,t) = \sum_{n\geq 0} c_n s^n$

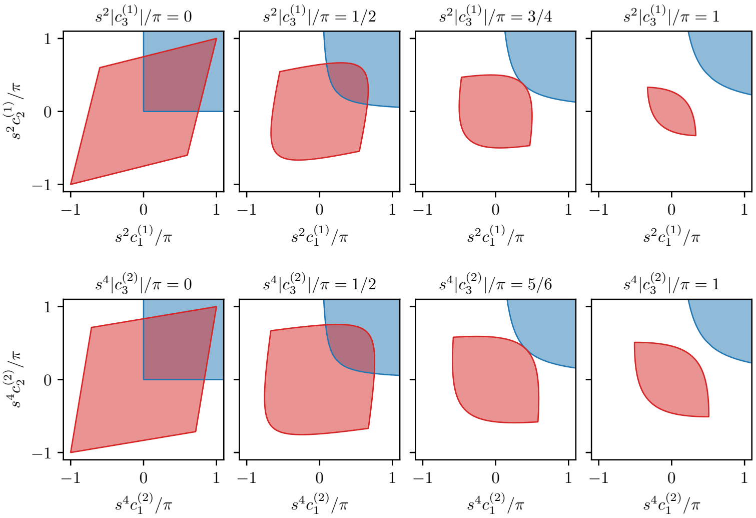
  - Froissart bound:
    $|\mathcal{A}(s)| = o(|s|^n)$ as $|s| \to \infty$
  - Schwartz reflection principle:
    $\mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon) = 2i\,\mathrm{Im}\,\mathcal{A}(s)$
  - Optical theorem: $\mathrm{Im}\,\mathcal{A}(s) = s\,\sigma(s)$



imply $$c_n = \frac{1+(-1)^n}{\pi}\int_{\Lambda^2}^{\infty} \mathrm{d}s\, s^{-n}\sigma(s) \geq 0$$

- In our example: $|c_-^{(S)}| \leq c_+^{(S)}$

Synergy of Perturbative Unitarity & Positivity Bounds



Partial wave unitarity ($\mathscr{U}$)

$$|c_+^{(S)}| + \frac{2S+3}{2S+1}|c_-^{(S)}| \leq \frac{2\pi}{s^{2S}}$$

Positivity ($\mathscr{P}$)

$$c_1^{(S)} \geq 0 \qquad c_2^{(S)} \geq 0$$

$$(c_3^{(S)})^2 \leq 4\, c_1^{(S)} c_2^{(S)}$$

$$\frac{\mathrm{Vol}(\mathscr{U} \cap \mathscr{P})}{\mathrm{Vol}(\mathscr{U})} = \frac{1}{32}\left(\frac{2S+3}{S+1}\right)^2$$

$2 \to 2$ *vs.* $2 \to 3$: SMEFT Examples

Dimension-6 example

$$\mathcal{L}^{(6)} \supset C_{eH}^{pr} \left(H^\dagger H\right) \bar{\ell}_p e_r H + C_{dH}^{pr} \left(H^\dagger H\right) \bar{q}_p d_r H + C_{uH}^{pr} \left(H^\dagger H\right) \bar{q}_p u_r \widetilde{H} + \text{h.c.}$$

$$\sqrt{\text{Tr}\left[3C_{uH}C_{uH}^\dagger + 3C_{dH}C_{dH}^\dagger + C_{eH}C_{eH}^\dagger\right]} \leq \frac{32\pi^2}{\sqrt{3s}}$$

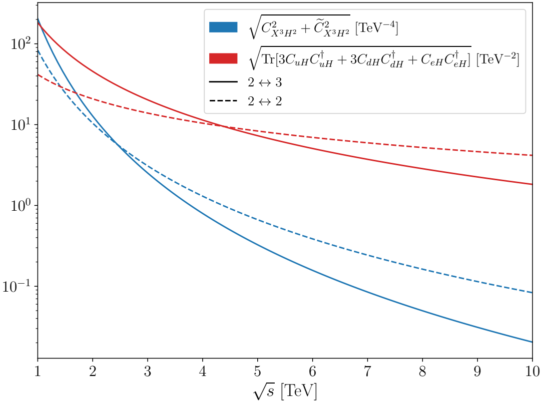
$$\sqrt{\text{Tr}\left[3C_{uH}C_{uH}^\dagger + 3C_{dH}C_{dH}^\dagger + C_{eH}C_{eH}^\dagger\right]} \leq \frac{4\sqrt{2}\pi}{\sqrt{3sv^2}}$$

Dimension-8 example

$$\mathcal{L}^{(8)} \supset C_{X^3H^2} \left(H^\dagger H\right) f^{ABC} X_\mu^{A\nu} X_\nu^{B\rho} X_\rho^{C\mu} + \widetilde{C}_{X^3H^2} \left(H^\dagger H\right) f^{ABC} X_\mu^{A\nu} X_\nu^{B\rho} \widetilde{X}_\rho^{C\mu}$$

$$\sqrt{C_{X^3H^2}^2 + \widetilde{C}_{X^3H^2}^2} \leq \frac{32\sqrt{10}\pi^2}{s^2} \frac{1}{\sqrt{C_2(G)d(G)}}$$

$$\sqrt{C_{X^3H^2}^2 + \widetilde{C}_{X^3H^2}^2} \leq \frac{8\sqrt{2}\pi}{vs^{3/2}} \frac{1}{\sqrt{C_2(G)}}$$



With gauge group $G = SU(3)$, $d(G) = 8$ and $C_2(G) = 3$

Conclusions

- New *vectorial formalism* establishes unitarity bounds for:
  - Arbitrary $N \to M$ processes $(N, M \geq 2)$
  - Higher-spin EFTs
- Applications:
  - EFT of gravity and light-by-light scattering
  - SMEFT: $\psi^2 H^3$ dim-6 operators and $X^3 H^2$ dim-8 operators
- Provided an angular momentum basis for $2 \to 3$ amplitudes
- *Synergy* of positivity & perturbative unitarity bounds

Thank you for your attention!

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