CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

Small-Instanton induced Flavor Invariants and the Axion Potential

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PLANCK Conference 2025
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Preliminary & Outline of this talk

- Axion potential:

  How to make small(UV)-instanton becoming relevant?

  QCD contribution & UV contributions via Instanton effects

- Small-instantons & Axion potential:

  Topological susceptibilities & Flavour invariants

  UV completions of small-instanton

  Bounds from neutron EDM

Preliminary & Outline of this talk

- Instanton #101:

QCD θ-vacuum = Superposition of n-vacua (energy degenerate but topologically distinct)

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta}|n\rangle = \cdots |0\rangle + e^{-i\theta}|1\rangle + \cdots$$

Instanton describes the tunnelling effect between degenerate n-vacua

Localised objects in Euclidean spacetime, satisfying the Euclidean equation of motion with non-trivial topologies and therefore minimise the Euclidean action.

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Explicit SU(2) BPST instanton solution with Q = 1:

(Background field configuration)

$$\frac{g^2}{32\pi^2}\int d^4x\, G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}(x)\Big|_{\text{inst.}} = Q\,, \text{ where } Q \in \mathbb{Z}\,.$$

$$G^a_\mu(x)\Big|_{1-\text{inst.}} = 2\,\eta_{a\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}$$

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Characterised by a set of collective coordinates => zero-modes (family of equivalent solutions)

x_0 — Location of instanton

ρ — Size of instanton

Preliminary & Outline of this talk

- Instanton #101: Path Integral with Instanton configurations

$$S_{\rm YM}^{\rm inst.} = \int d^4x \left(\frac{1}{4} G_{\mu\nu}^A G^{A,\,\mu\nu} + i\theta_{QCD} \frac{g^2}{32\pi^2} G_{\mu\nu}^A \tilde{G}^{A,\,\mu\nu} \right) \Big|_{\rm (a.-)inst.} = \frac{8\pi^2}{g^2} |Q| + i\theta_{QCD}\, Q$$

Estimating instanton effects => vacuum-to-vacuum transition amplitude:

$$\langle 0|0\rangle \,\big|_{\rm 1\text{-}inst.} \sim \int \mathcal{D}\varphi_I^{(0)} \int \mathcal{D}\varphi_I^{(\neq 0)}\, e^{-\left[S_{\rm YM}^{\rm 1\text{-}inst.} + \int d^4x\, \varphi_I^\dagger \left(\frac{\delta^2 \mathcal{L}}{\delta \varphi_I^2}\right) \varphi_I\right]}$$

Zero modes measure   Non-zero modes measure

't Hooft 76
Shifman, Vainshtein, Zakharov 79

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Integrating out non-zero modes

$$\langle 0|0\rangle\, \Big|_{\rm 1\text{-}inst.} = e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5}\, d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho\, d\xi_f^{(0)} d\bar{\xi}_f^{(0)} \right) e^{\int d^4x\, (-\bar{\psi} J \psi + \text{h.c.})}$$

Instanton density

't Hooft 76
Shifman, Vainshtein, Zakharov 79

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$$d_N(\rho) \sim e^{-8\pi^2/g^2(\Lambda = 1/\rho)}$$

Strongly suppressed at hight energy scale
(for asymptotic free theory)

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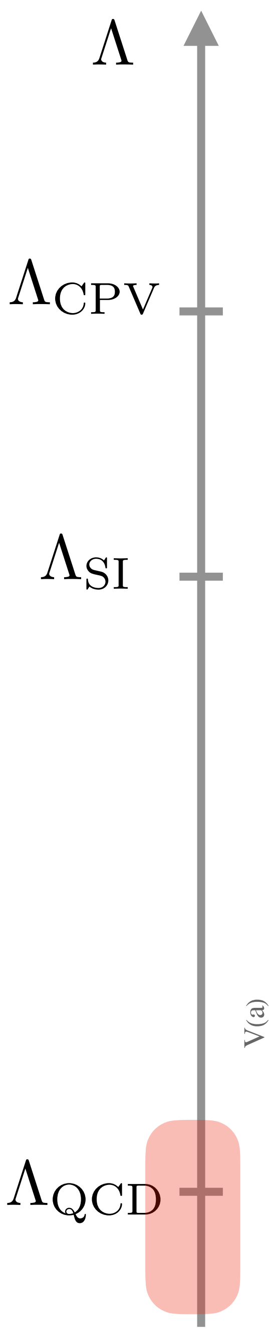
When small-instanton effects become relevant?
=> Boost the QCD coupling at high energy scale

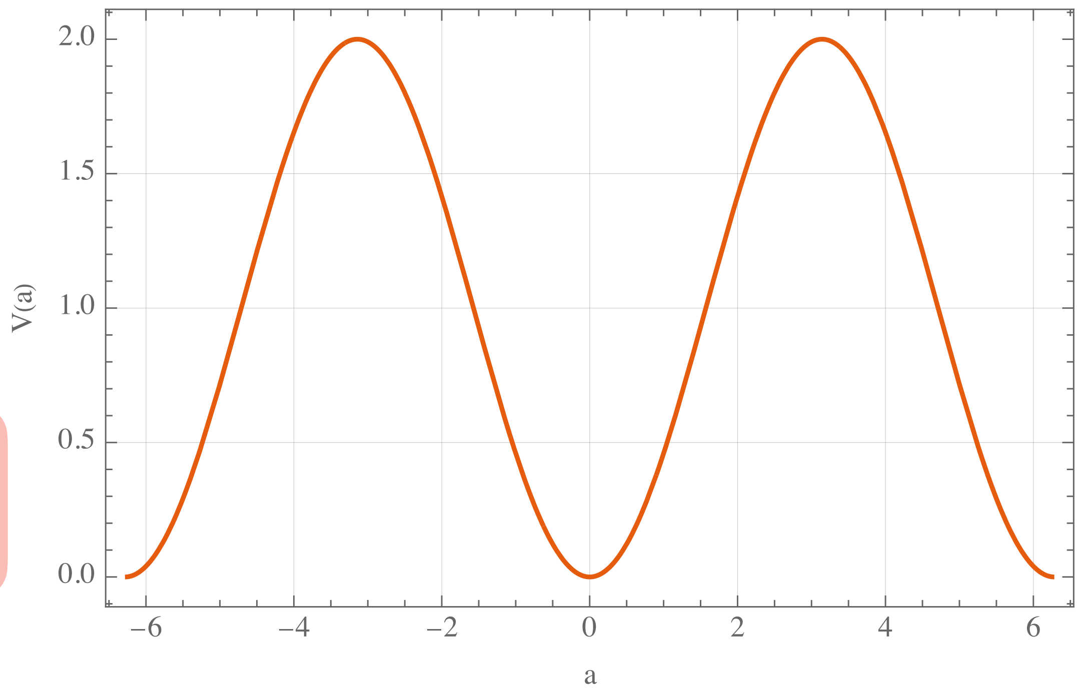
- Non-trivial embedding of QCD in UV theories: $SU(3)_1 \times SU(3)_2 \times \cdots \times SU(3)_k \to SU(3)_{\rm QCD}$
- Extra-dimensions (5D instantons)

Agrawal and Howe (1710.04213)
C. Csáki, M. Ruhdorfer, Y. Shirman (1912.02197)
T. Gherghetta, V. V. Khoze, A. Pomarol, Y. Shirman (2001.05610)

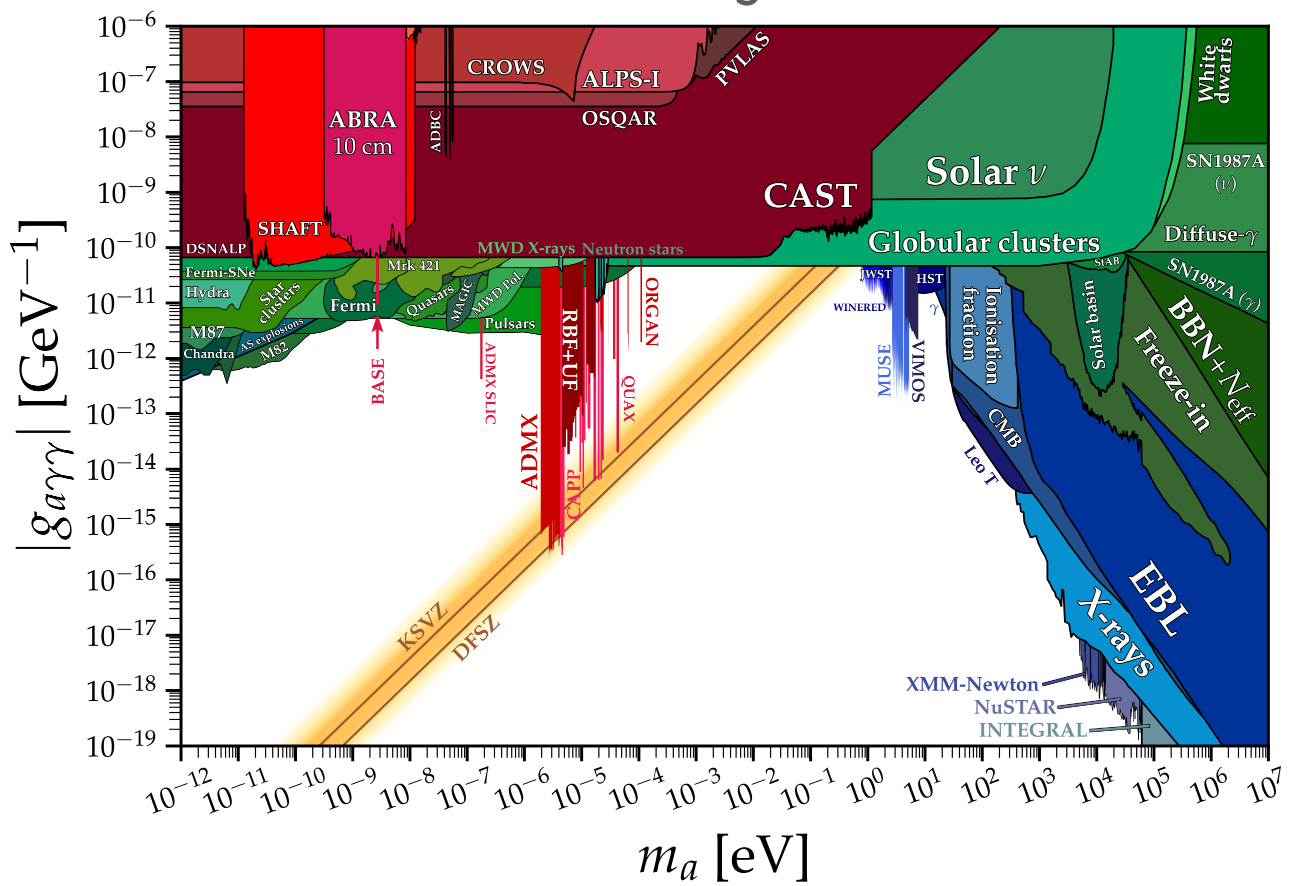
Preliminary & Outline of this talk

- Axion potential: QCD contribution





cajohare.github.io/AxionLimits/

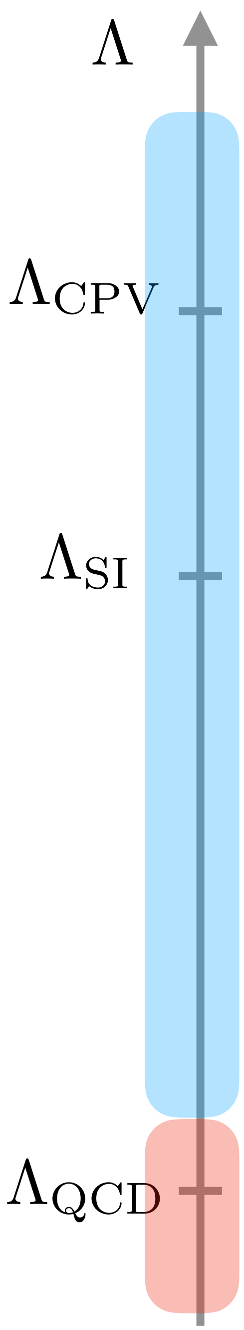


$$V(a) \sim m_\pi^2 f_\pi^2 \left[1 - \cos\frac{a}{f_a}\right] \rightarrow \langle \frac{a}{f_a} \rangle = 0$$

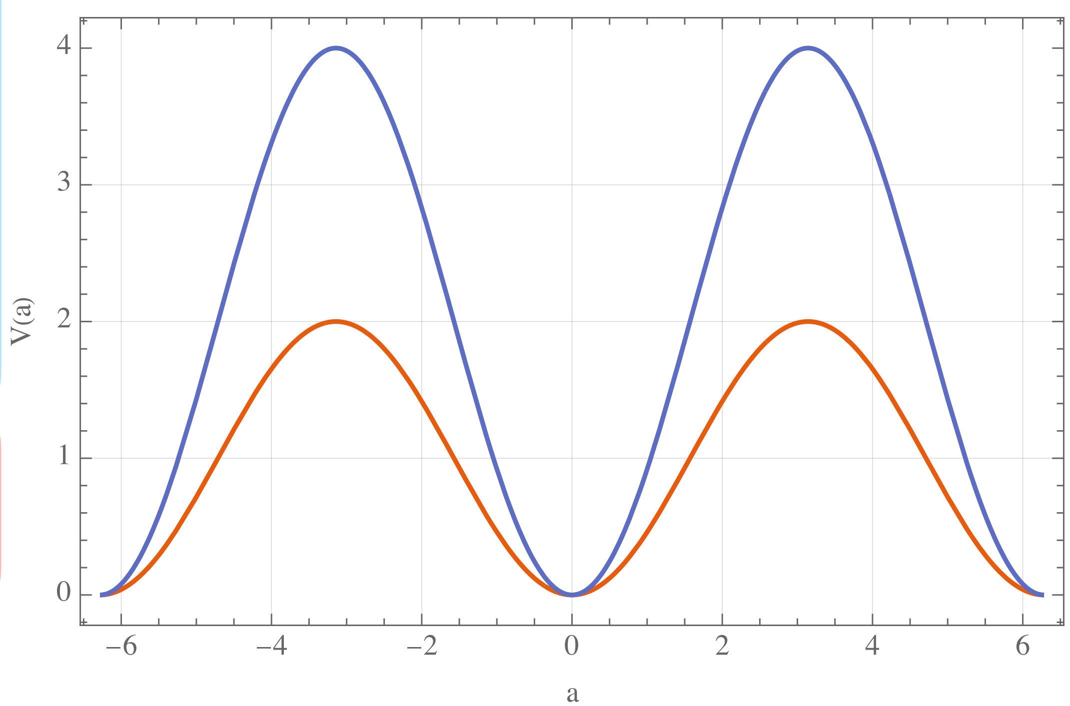
$$m_a^2 f_a^2 = \frac{m_u m_d}{(m_u + m_d)^2} m_\pi^2 f_\pi^2$$

Preliminary & Outline of this talk

- Axion potential: UV aligned contribution



Small instantons $\rho = 1/\Lambda$



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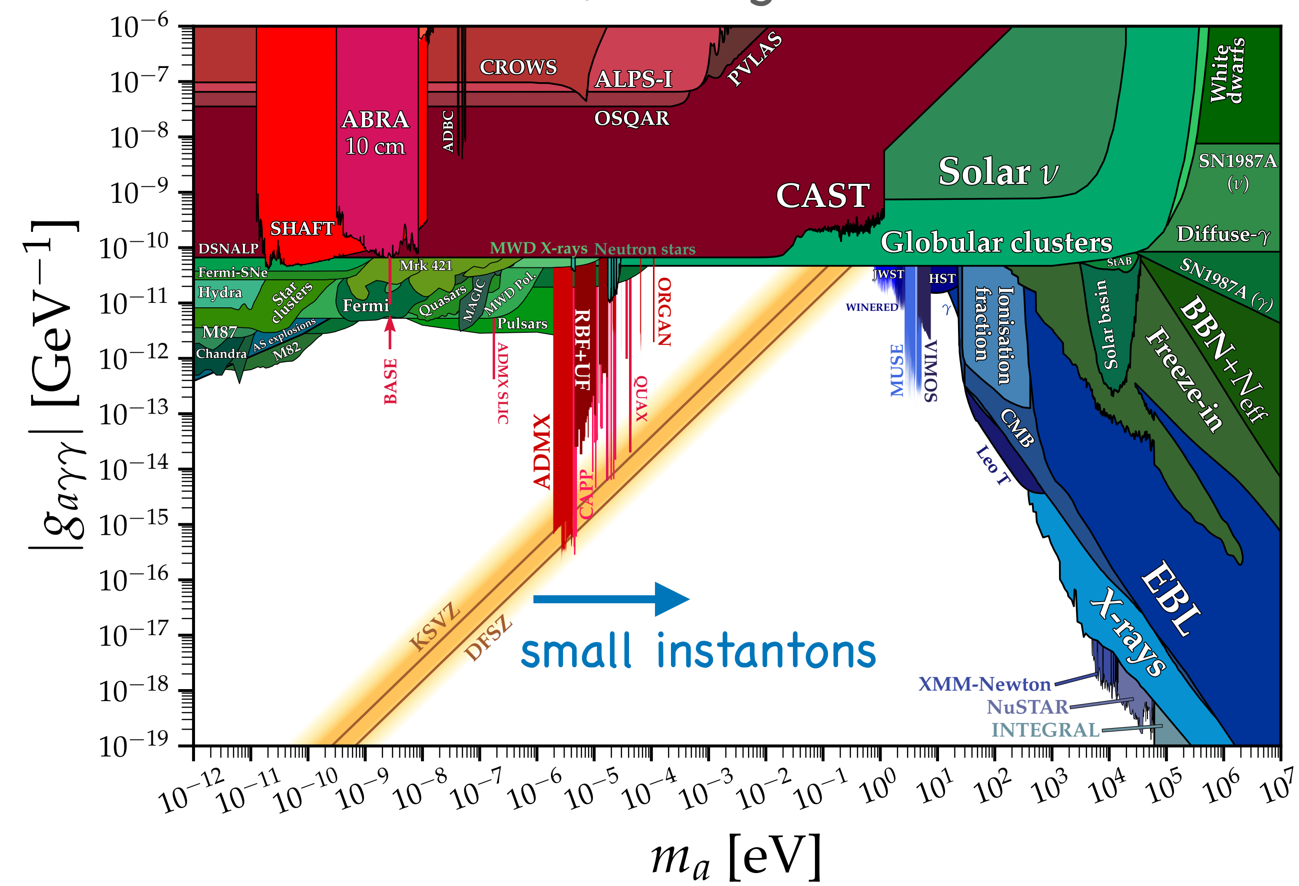
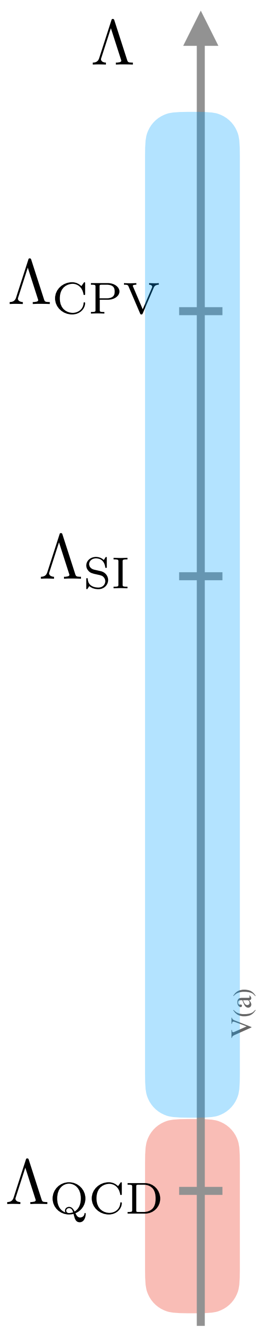


$$V(a) \sim (m_\pi^2 f_\pi^2 + \Lambda_{UV}^4)\left[1 - \cos\frac{a}{f_a}\right]$$

Enhance axion mass &
solve strong CP problem

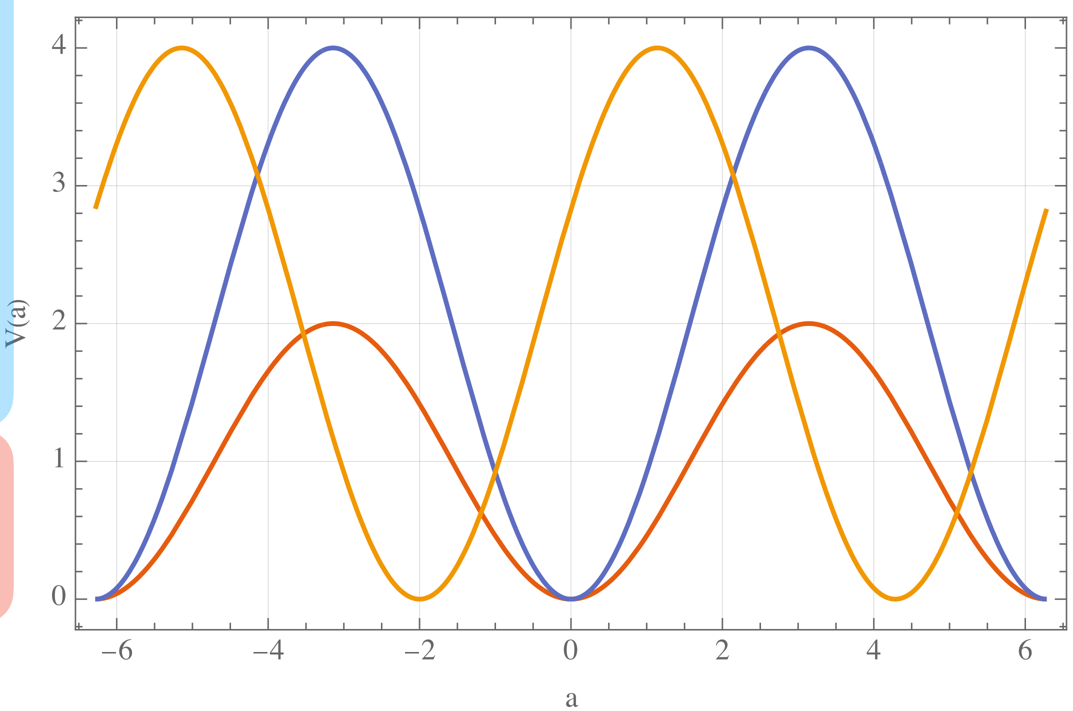
Preliminary & Outline of this talk

- Axion potential: UV misaligned contribution



New source of CPV, e.g. SMEFT CPV effective operators

Small instantons $\rho = 1/\Lambda$



$$V(a) \sim (m_\pi^2 f_\pi^2 + \Lambda_{UV}^4)\left[1 - \cos\left(\frac{a}{f_a} + \delta_{CPV}\right)\right]$$

Can potentially destroy the Axion solution

Preliminary & Outline of this talk

- CP-violation: The case of Standard Model (SM)

Jarlskog '85

Bernabeu, Branco, Gronau '86

CPV is parametrised by Jarlskog invariant:

$$J_4 = \mathrm{Im}\mathrm{Tr}\left([Y_u Y_u^\dagger, Y_d Y_d^\dagger]^3\right)$$

=> CP is conserved iff $J_4 = 0$ (neglecting $\bar{\theta}$)

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- CPV in the SM will not misalign the axion potential: Appear at 4-loop (from threshold corrections) and 7-loop (from radiative corrections) level

Ellis, Gaillard '79

Khriplovich '86

Georgi, Randall '86

$$\bar{\theta}_{\mathrm{ind}}^{(\mathrm{SM})} \sim 10^{-19}$$

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- BSM CP-violation: The case of SMEFT

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{C_i \mathcal{O}_i^{(6)}}{\Lambda^2}$$

Contain 1149 CP-odd couplings !!!

=> Generalise Jarlskog invariant to study CPV in the SMEFT systematically

Bonnefoy, Gendy, Grojean, Ruderman

2112.03889, 2302.07288

Preliminary & Outline of this talk

- CP-violation: The case of SMEFT

Considering non-perturbative effects => Use θ_{QCD} as a spurion:

	$U(3)_{\mathrm{Q}}$	$U(3)_{\mathrm{u}}$	$U(3)_{\mathrm{d}}$	$U(3)_{\mathrm{L}}$	$U(3)_{\mathrm{e}}$
$e^{i\theta_{\mathrm{QCD}}}$	$\mathbf{1}_{+6}$	$\mathbf{1}_{-3}$	$\mathbf{1}_{-3}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
Y_{u}	$\mathbf{3}_{+1}$	$\bar{\mathbf{3}}_{-1}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
Y_{d}	$\mathbf{3}_{+1}$	$\mathbf{1}_{0}$	$\bar{\mathbf{3}}_{-1}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
Y_{e}	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{3}_{+1}$	$\bar{\mathbf{3}}_{-1}$

SM has one more CP-odd flavour invariant:

$$J_\theta = \mathrm{Im}[e^{-i\theta_{\mathrm{QCD}}} \det(Y_{\mathrm{u}} Y_{\mathrm{d}})]$$

Built flavour invariants featuring θ_{QCD} for CP-violating SMEFT operators:

$$\mathcal{O}^{(1)}_{quqd} = \bar{Q}u\bar{Q}d$$

$$\mathcal{I}(C^{(1,8)}_{\mathrm{quqd}}) = \mathrm{Im}\left[e^{-i\theta_{\mathrm{QCD}}}\epsilon^{ABC}\epsilon^{abc}\epsilon^{DEF}\epsilon^{def}Y_{\mathrm{u},Aa}Y_{\mathrm{u},Bb}C^{(1,8)}_{\mathrm{quqd},CcDd}Y_{\mathrm{d},Ee}Y_{\mathrm{d},Ff}\right]$$

Note: $\bar{Q}u\bar{Q}d$ has 81 CP-odd phases

Bonnefoy, Gendy, Grojean, Ruderman
2112.03889, 2302.07288

Preliminary & Outline of this talk

- Small instanton & Axion potential: UV misaligned contributions

Small instantons generate axion potential of the form:

$$V(a) = \chi_{\mathcal{O}}(0)\frac{a}{f_a} + \frac{1}{2}\chi(0)\left(\frac{a}{f_a}\right)^2 \longrightarrow \langle\frac{a}{f_a}\rangle \equiv \theta_{\rm ind} = -\frac{\chi_{\mathcal{O}}(0)}{\chi(0)}$$

Induced by CP-violating operator

This talk

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Coefficients in the potential can be computed from following correlators: Witten '79

$$\chi(0) = -i\lim_{k\to 0}\int d^4x\, e^{ikx}\left\langle 0\left|T\left\{\frac{g^2}{32\pi^2}G\tilde{G}(x)\,,\,\frac{g^2}{32\pi^2}G\tilde{G}(0)\right\}\right|0\right\rangle\Bigg|_{\text{1-(a.-)inst.}}$$

$$\chi_{\mathcal{O}}(0) = -i\lim_{k\to 0}\int d^4x\, e^{ikx}\left\langle 0\left|T\left\{\frac{g^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a(x)\,,\,\frac{C^{ij\cdots}_{\mathcal{O}}}{\Lambda^{D-4}_{\not{CP}}}\mathcal{O}^{D,ij\cdots}(0)\right\}\right|0\right\rangle\Bigg|_{\text{1-(a.-)inst.}}$$

Evaluating these correlation functions within perturbative regime and one-(anti)instanton approximation. Making connection with SMEFT flavour invariants => Simplify the calculations

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

- **Key points:** Fermion zero mode & Grassmann integral

Fermion eigenmode expansion & fermion zero-mode solutions:

$$\psi_f(x) = \sum_k \xi_f^{(k)} \psi^{(k)}(x)\,; \quad \bar{\psi}_f(x) = \sum_k \bar{\xi}_f^{(k)} \bar{\psi}^{(k)}(x)$$

$$-i\not{D}\big|_{\text{1-inst.}} \psi^{(0)}(x) = 0. \longrightarrow \psi^{(0)}(x)\Big|_{\text{1-inst.}} = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \frac{1}{\pi} \frac{\rho}{\left[(x-x_0)^2 + \rho^2\right]^{3/2}} \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \quad \varphi_{\alpha m} = \epsilon^{\alpha m}$$

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Fermion zero modes & Grassmann integral give rise to determinant-like structures:

- The well-know Grassmann integration identity:

$$\int d^3\xi_1 d^3\xi_2 \; e^{\xi_1 A \xi_2} = \det A$$

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- Generalise the Grassmann integration identity for operator insertion:

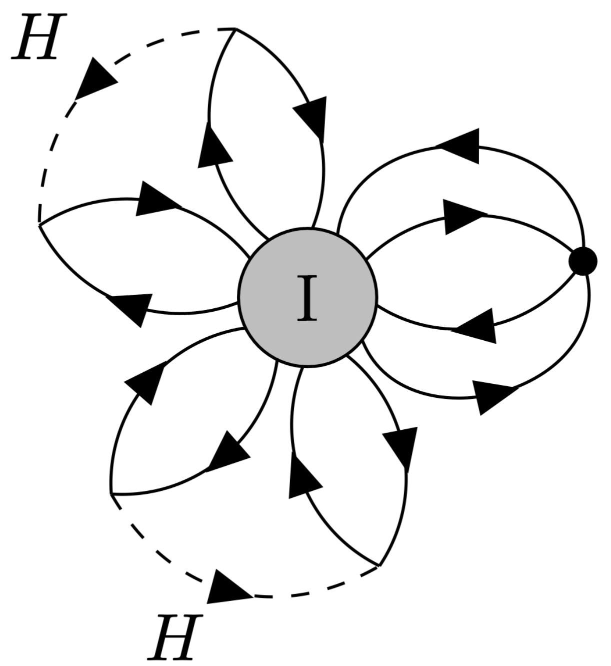
Example:

$$\int d^3\xi_1 d^3\xi_2 \; e^{\xi_1 A \xi_2} \xi_1 B \xi_2 = \frac{1}{2} \epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} A_{i_1 j_1} A_{i_2 j_2} B_{i_3 j_3}$$

$$\int d^3\xi_1 d^3\xi_2 d^3\xi_3 d^3\xi_4 \; e^{\xi_1 A \xi_2 + \xi_3 B \xi_4} \xi_1 C \xi_2 = \frac{1}{2} \epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} A_{i_1 j_1} A_{i_2 j_2} C_{i_3 j_3} \det B$$

Determinant-like contraction

Topological Susceptibilities & Flavor invariants: Four-quark operator



$$\chi^{(1)}_{\rm ququd}(0)^{1-\rm inst.} = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \left| T\left\{ \frac{1}{32\pi^2} G\widetilde{G}(x), \frac{C^{(1)}_{\rm ququd}}{\Lambda^2_{\not{\rm CP}}} \mathcal{O}^{(1)}_{\rm ququd}(0) \right\} \right| 0 \right\rangle$$

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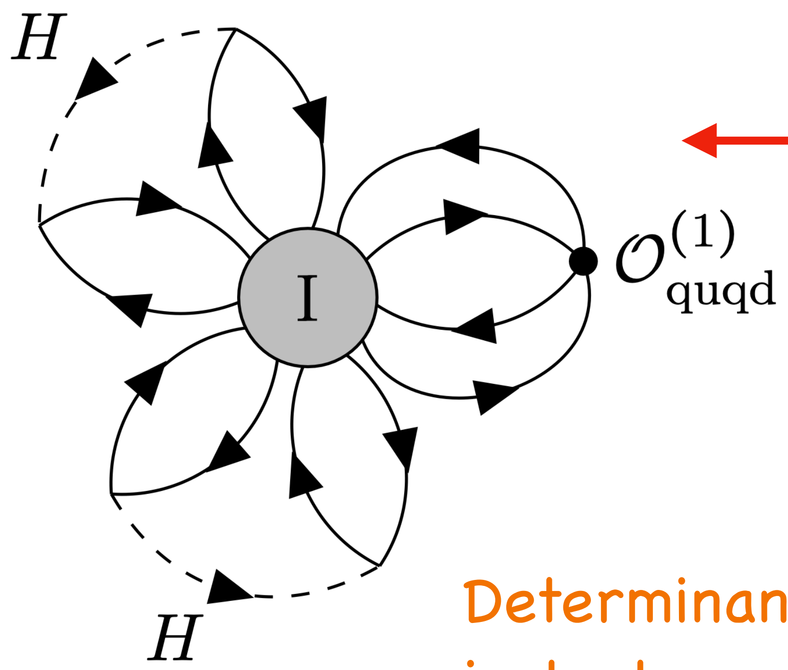

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Determinant-like flavour invariants naturally arise in the instanton calculations

$$\chi^{(1)}_{\rm ququd}(0)^{1-\rm inst.} = \frac{1}{4\Lambda^2_{\not{CP}}}\Big[e^{-i\theta_{\rm QCD}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n} Y_{{\rm u},i_1 j_1} Y_{{\rm u},i_2 j_2} C^{(1)}_{{\rm ququd},mnop}\epsilon^{k_1 k_2 o}\epsilon^{l_1 l_2 p} Y_{{\rm d},k_1 l_1} Y_{{\rm d},k_2 l_2}$$

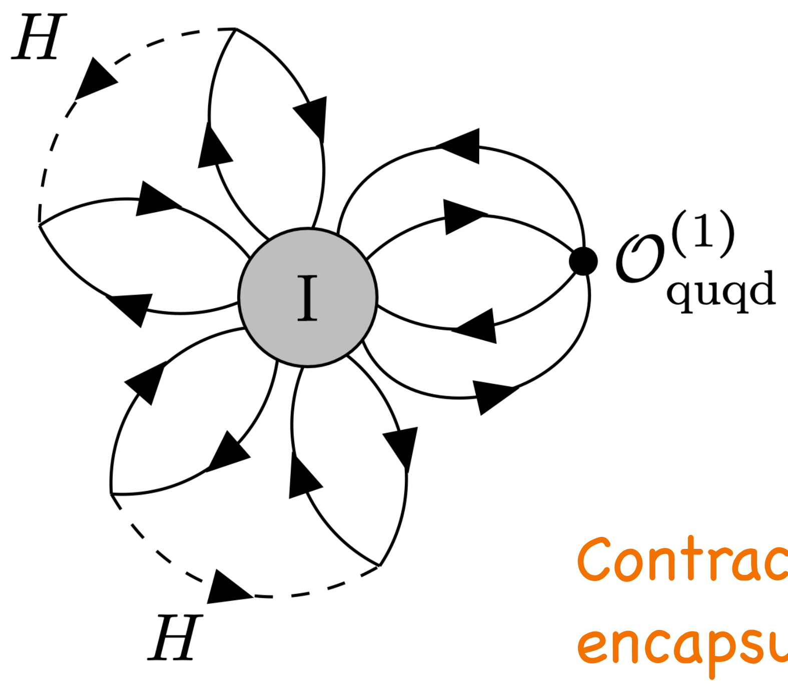
$$+e^{-i\theta_{\rm QCD}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n} Y_{{\rm u},i_1 j_1} Y_{{\rm u},i_2 j_2} C^{(1)}_{{\rm ququd},onmp}\epsilon^{k_1 k_2 o}\epsilon^{l_1 l_2 p} Y_{{\rm d},k_1 l_1} Y_{{\rm d},k_2 l_2}\Big] \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho)\rho^6$$

$$\times \underbrace{\int \mathcal{D}H\mathcal{D}H^\dagger\, e^{-S_0[H,H^\dagger]}\left[\int d^4x_1 d^4x_2 (\bar\psi^{(0)} H^\dagger_I \epsilon^{IJ} P_R \psi^{(0)})(x_1)\,(\bar\psi^{(0)}\epsilon_{JK} H^K P_R\psi^{(0)})(x_2)\right]^2}_{=2!\left[\int d^4x_1 d^4x_2 (\bar\psi^{(0)} P_R\psi^{(0)})(x_1)\Delta_H(x_1-x_2)\epsilon_{IJ}\epsilon^{JI}(\bar\psi^{(0)}P_R\psi^{(0)})(x_2)\right]^2 \equiv 2!\,\mathcal{I}^2}$$

$$\times\left(\epsilon_{MN}\epsilon^{MN}\bar\psi^{(0)}P_R\psi^{(0)}\,\bar\psi^{(0)}P_R\psi^{(0)}\right)(0)\int d^4x \frac{G\widetilde{G}(x)}{32\pi^2}\,.$$

Fermion zero modes

Topological Susceptibilities & Flavor invariants: Four-quark operator



Plugging explicit form of fermion zero modes
Integrate over loop momenta,
collective coordinates

Contraction of Yukawa matrices
encapsulated in the Flavour invariants

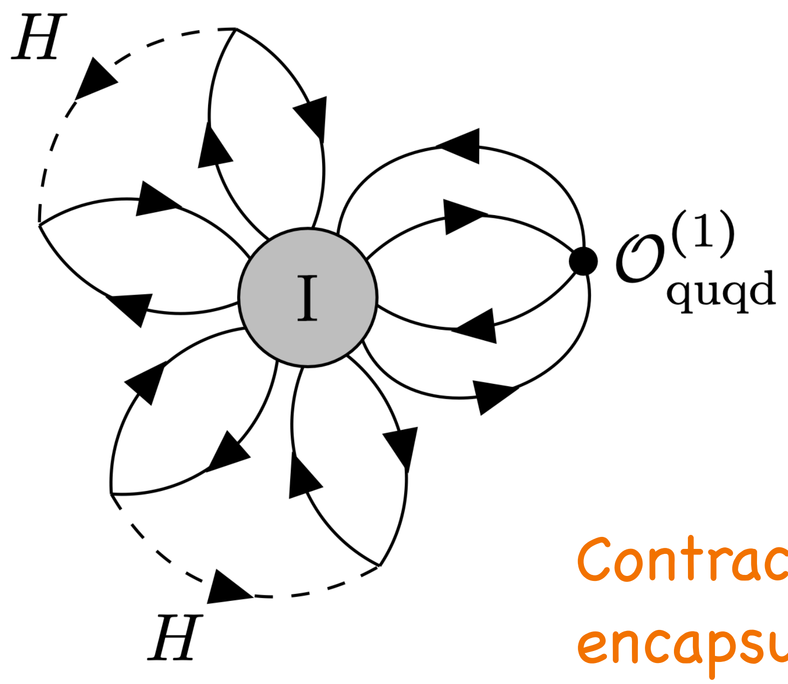
$$\chi^{(1)\,({\rm UV})}_{\rm ququd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

$$\mathrm{Im}(A^{(1)}_{\rm ququd}) = \mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)$$

$$\mathrm{Im}(B^{(1)}_{\rm ququd}) = \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)$$

$$A^{(1)}_{\rm ququd} = e^{-i\theta_{\rm QCD}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n} Y_{{\rm u},i_1 j_1} Y_{{\rm u},i_2 j_2} C^{(1)}_{{\rm ququd},mnop}\epsilon^{k_1 k_2 o}\epsilon^{l_1 l_2 p} Y_{{\rm d},k_1 l_1} Y_{{\rm d},k_2 l_2}$$
$$B^{(1)}_{\rm ququd} = e^{-i\theta_{\rm QCD}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n} Y_{{\rm u},i_1 j_1} Y_{{\rm u},i_2 j_2} C^{(1)}_{{\rm ququd},onmp}\epsilon^{k_1 k_2 o}\epsilon^{l_1 l_2 p} Y_{{\rm d},k_1 l_1} Y_{{\rm d},k_2 l_2}$$

Topological Susceptibilities & Flavor invariants: Four-quark operator



Can also use Instanton Naive Dimensional Analysis (NDA), result up to $\mathcal{O}(1)$

Csáki, D'Agnolo, Kuflik, Ruhdorfer (2311.09285)

Contraction of Yukawa matrices encapsulated in the Flavour invariants

$$\chi^{(1)\,(\rm UV)}_{\rm ququd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\int \frac{d\rho}{\rho^5} d_N(\rho) \frac{1}{(256\pi^6)\rho^2}$$

$$A^{(1)}_{\rm ququd} = e^{-i\theta_{\rm QCD}} \epsilon^{i_1 i_2 m} \epsilon^{j_1 j_2 n} Y_{{\rm u},i_1 j_1} Y_{{\rm u},i_2 j_2} C^{(1)}_{{\rm ququd},mnop} \epsilon^{k_1 k_2 o} \epsilon^{l_1 l_2 p} Y_{{\rm d},k_1 l_1} Y_{{\rm d},k_2 l_2}$$
$$B^{(1)}_{\rm ququd} = e^{-i\theta_{\rm QCD}} \epsilon^{i_1 i_2 m} \epsilon^{j_1 j_2 n} Y_{{\rm u},i_1 j_1} Y_{{\rm u},i_2 j_2} C^{(1)}_{{\rm ququd},onmp} \epsilon^{k_1 k_2 o} \epsilon^{l_1 l_2 p} Y_{{\rm d},k_1 l_1} Y_{{\rm d},k_2 l_2}$$

Combining Flavour invariants & Instanton NDA =>
quickly estimate 't Hooft flower diagrams

Topological Susceptibilities & Flavor invariants: Semi-leptonic operator

$\mathcal{O}^{(1)}_{\rm lequ} = \bar{L} e \bar{Q} u + \text{h.c.}$

- Now start from the Flavour invariants:

$$\text{Im}\left(I^{(1)}_{\rm lequ}\right) = \text{Im}\left[e^{-i\theta_{\rm QCD}} \epsilon^{i_1 i_2 m} \epsilon^{j_1 j_2 n} Y_{{\rm u},i_1 j_1} Y_{{\rm u},i_2 j_2} C^{(1)}_{{\rm lequ},opmn} Y^{\dagger}_{{\rm e},po} \det Y_{\rm d}\right] = \mathcal{I}^{0}_{0000}\left(C^{(1)}_{\rm lequ}\right)$$

Trace-like contraction

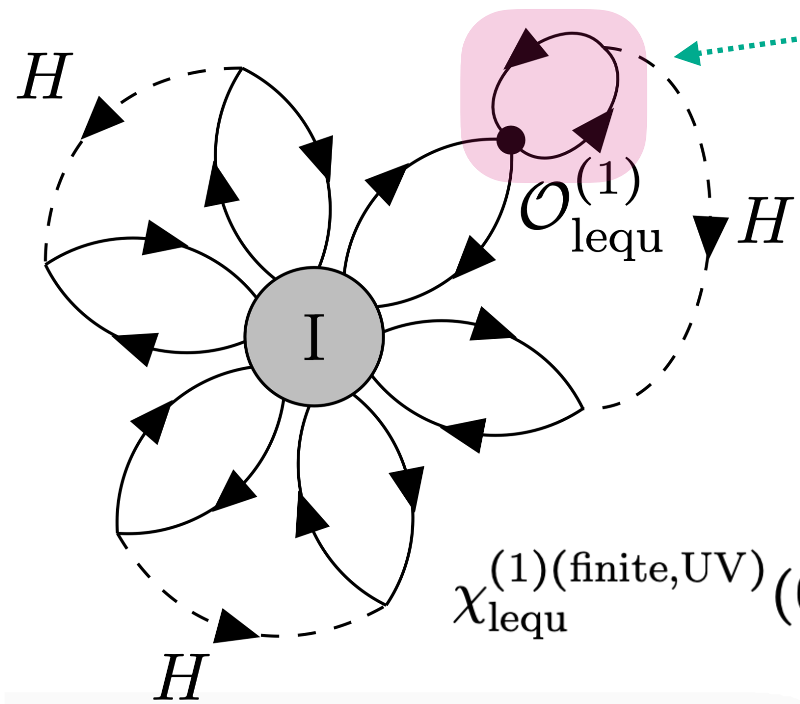
Topological Susceptibilities & Flavor invariants: Semi-leptonic operator

$\mathcal{O}^{(1)}_{\text{lequ}} = \bar{L}e\bar{Q}u + \text{h.c.}$

- Now start from the Flavour invariants:

$$\text{Im}\left(I^{(1)}_{\text{lequ}}\right) = \text{Im}\left[e^{-i\theta_{\text{QCD}}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n} Y_{\text{u},i_1 j_1} Y_{\text{u},i_2 j_2} C^{(1)}_{\text{lequ},opmn} Y^{\dagger}_{\text{e},po} \det Y_{\text{d}}\right] = \mathcal{I}^{0}_{0000}\left(C^{(1)}_{\text{lequ}}\right)$$

Trace-like contraction

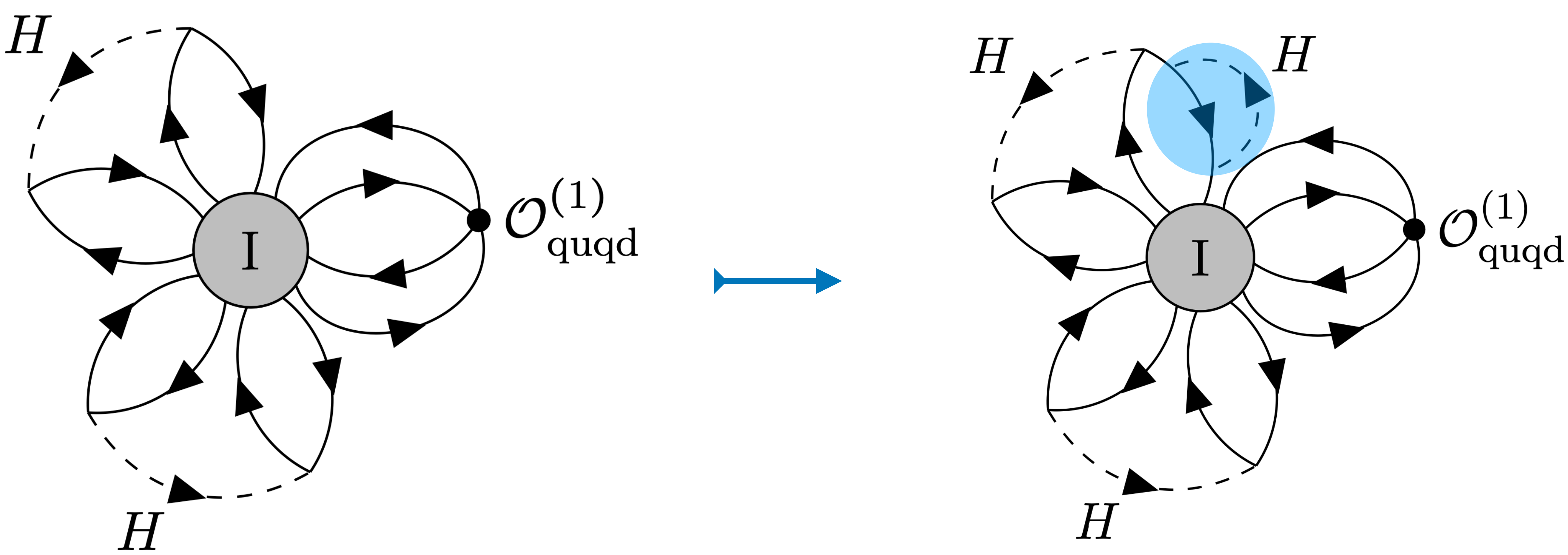


1-loop suppression induced by leptonic fields

$$\chi^{(1)(\text{finite,UV})}_{\text{lequ}}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\mathcal{I}^{0}_{0000}\left(C^{(1)}_{\text{lequ}}\right)\int \frac{d\rho}{\rho^5} d_N(\rho)\frac{3!}{(6\pi^2)^2}\frac{11 + 30\left(\log\left(\rho\Lambda_{\not{CP}}\right) + \gamma_{\text{E}} - \log 2\right)}{600\pi^4\rho^2}$$

- Anticipating how CPV SMEFT operators participate in the instanton computations
- Classifying the leading effects from the Wilson coefficients

Topological Susceptibilities & Flavor invariants: Higher-order Invariants



$$\mathcal{A}^{a_1,b_1,c_1,d_1}_{a_2,b_2,c_2,d_2}(C^{(1,8)}_{\rm quqd}) = \mathrm{Im}\left[e^{-i\theta_{\rm QCD}}\epsilon^{ABC}\epsilon^{abc}\epsilon^{DEF}\epsilon^{def}Y_{{\rm u},Aa}Y_{{\rm u},Bb}\left(X^{a_1}_{\rm u}X^{b_1}_{\rm d}X^{c_1}_{\rm u}X^{d_1}_{\rm d}\right)_C^{\;\;C'} \times C^{(1,8)}_{{\rm quqd},C'cD'd}\left(X^{a_2}_{\rm u}X^{b_2}_{\rm d}X^{c_2}_{\rm u}X^{d_2}_{\rm d}\right)_D^{\;\;D'}Y_{{\rm d},Ee}Y_{{\rm d},Ff}\right],$$

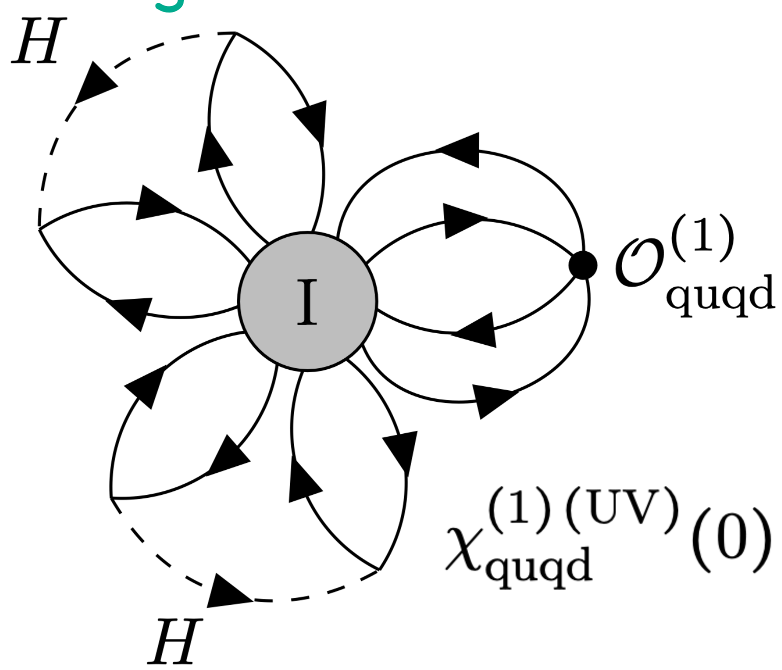
$$X_{\rm u,d} = Y_{\rm u,d}Y^\dagger_{\rm u,d}$$

=> Set X=1 for the lowest order flavour invariants

Topological Susceptibilities & Flavor invariants: Four-quark operator

- Integration over the size of instanton

$$\mathcal{O}^{(1)}_{\rm quqd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi^{(1)\,({\rm UV})}_{\rm quqd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right)\right) \int \frac{d\rho}{\rho^5}\, d_N(\rho)\, \frac{2!}{(6\pi^2)^2}\, \frac{2}{5\pi^2\rho^2}$$

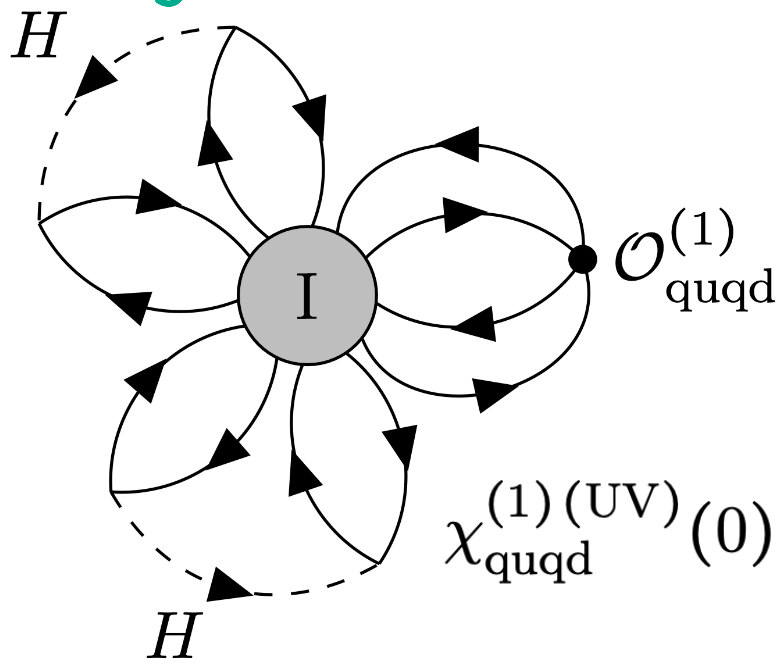
ρ-integral is IR divergent

=> Need a physical IR cut-off

Topological Susceptibilities & Flavor invariants: Four-quark operator

- Integration over the size of instanton

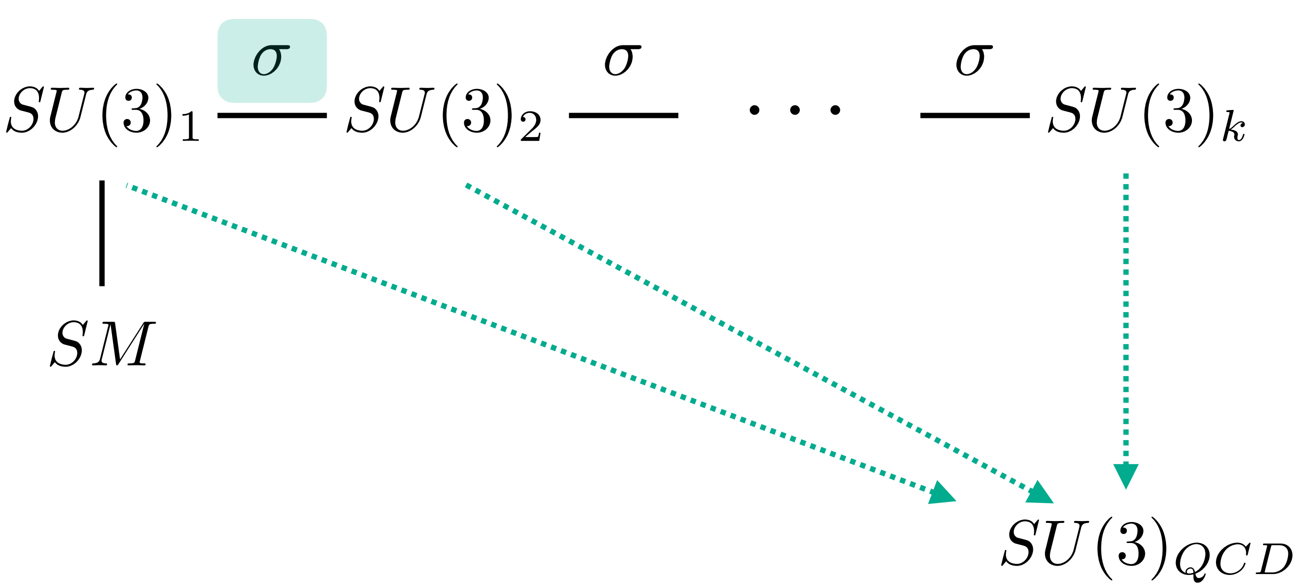
$$\mathcal{O}^{(1)}_{\rm quqd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi^{(1)\,(\rm UV)}_{\rm quqd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\, \frac{2!}{(6\pi^2)^2}\, \frac{2}{5\pi^2\rho^2}$$

- Possible UV completion of small-instantons:

  Product of Gauge groups

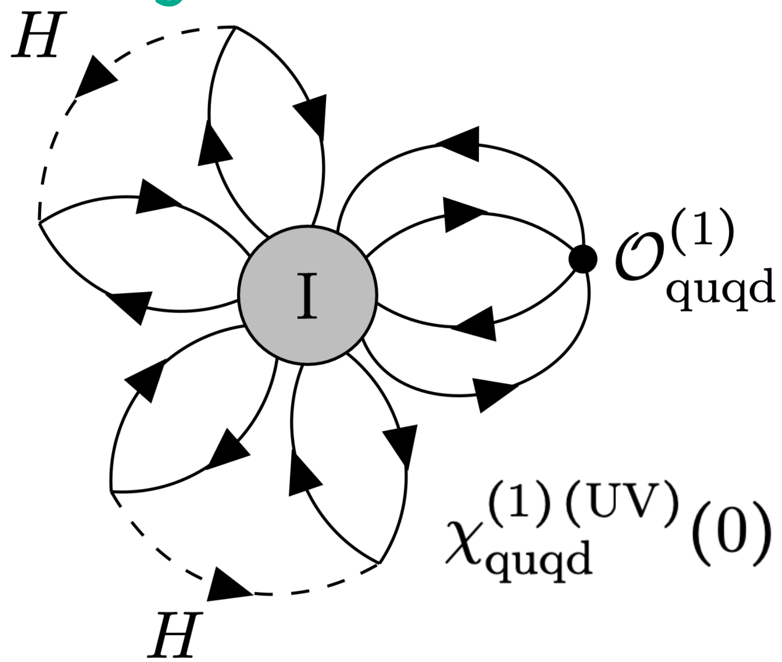


Agrawal and Howe (1710.04213)
C. Csáki, M. Ruhdorfer, Y. Shirman (1912.02197)

Topological Susceptibilities & Flavor invariants: Four-quark operator

- Integration over the size of instanton

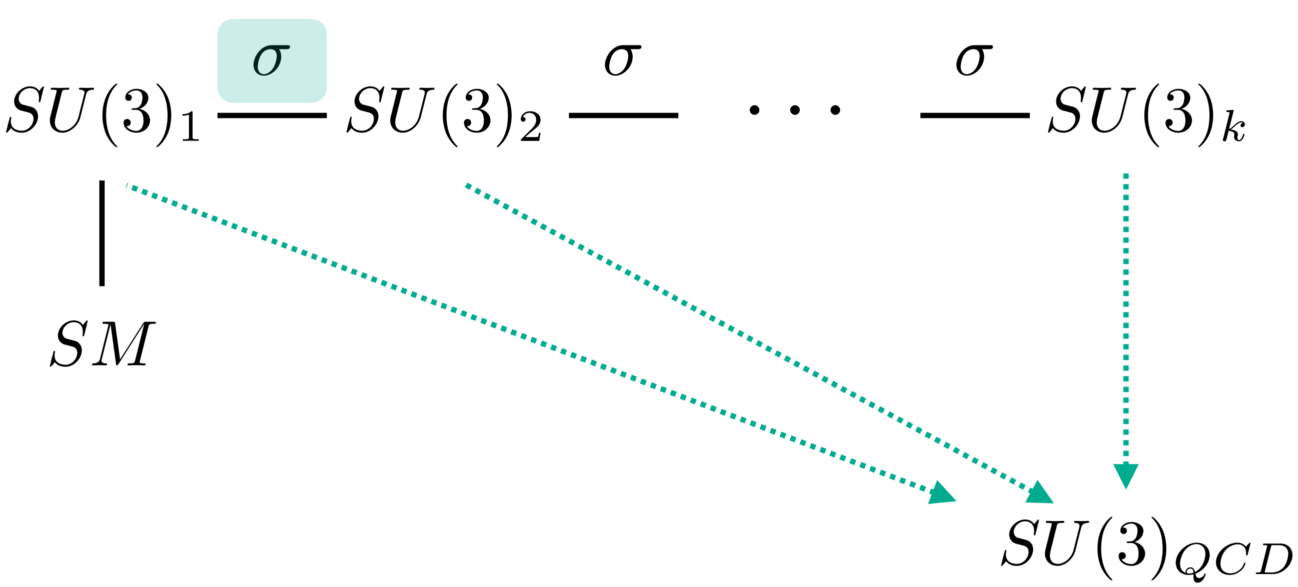
$$\mathcal{O}^{(1)}_{\rm ququd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi^{(1)\,(\rm UV)}_{\rm ququd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

- Possible UV completion of small-instantons:

Product of Gauge groups



Boost the coupling of each QCD subgroup:

$$\frac{1}{g^2_{\rm QCD}(\mu)} = \frac{1}{g_1^2(\mu)} + \frac{1}{g_2^2(\mu)} + \cdots + \frac{1}{g_k^2(\mu)}$$

Provide a physical cut-off:

$$d_N(\rho) \to d_N(\rho)\, e^{-2\pi^2\rho^2\sum|\langle\sigma\rangle|^2}$$

Agrawal and Howe (1710.04213)
C. Csáki, M. Ruhdorfer, Y. Shirman (1912.02197)

Topological Susceptibilities & Flavor invariants: Four-quark operator

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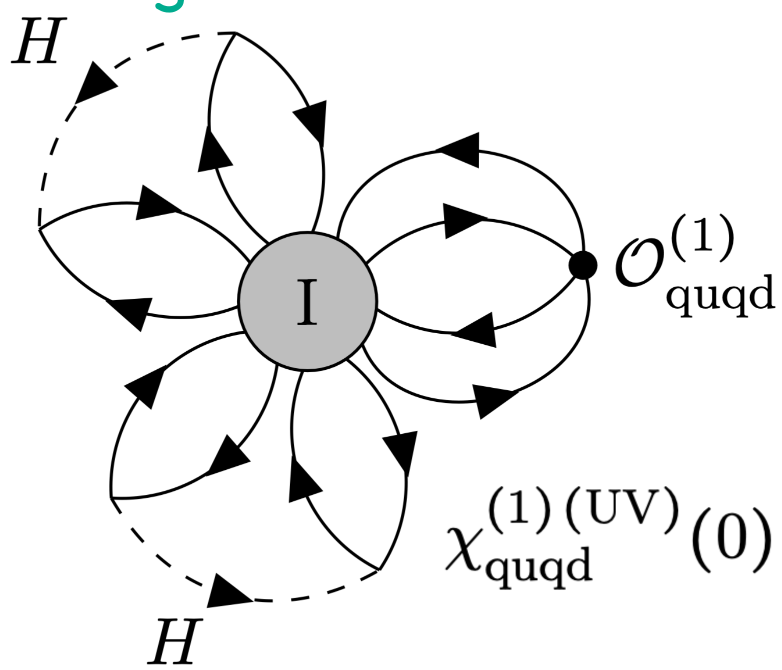
$$\mathcal{O}^{(1)}_{\rm ququd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi^{(1)\,(\rm UV)}_{\rm ququd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

- Possible UV completion of small-instantons:

  5D instantons

Uplift BPST instanton to a compact extra dimension of size R

$$\int d_N(\rho) \to \int_{1/\Lambda_5}^{R} d_N(\rho) e^{R/\rho}$$

T. Gherghetta, V. V. Khoze, A. Pomarol, Y. Shirman (2001.05610)

Bounds from non-measurement of theta-induced: Four-quark operator

$$\mathcal{O}^{(1)}_{\rm ququd} = \bar{Q}u\bar{Q}d + \text{h.c.}$$

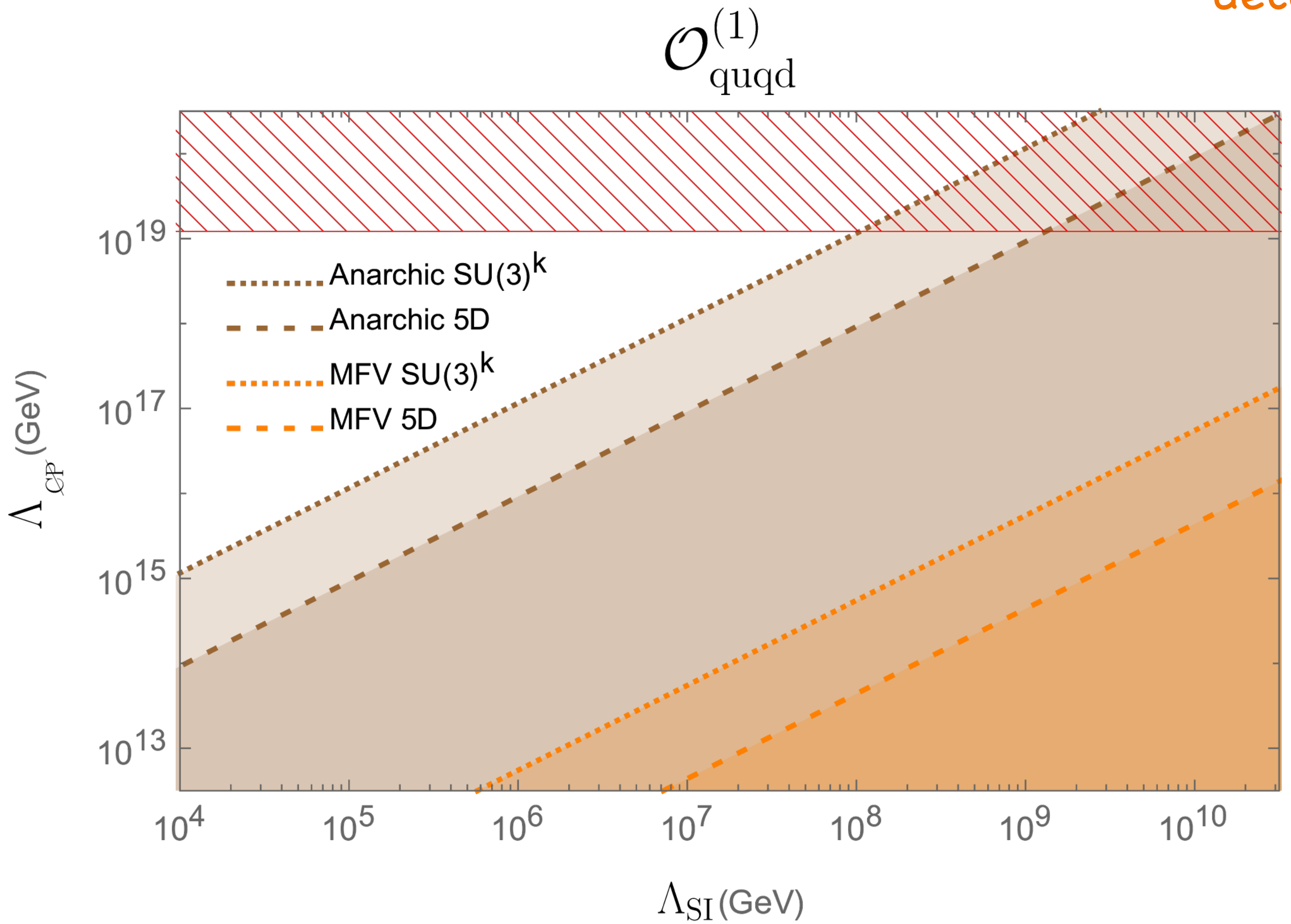
- Example: Product of gauge groups

$$\theta_{\rm ind} = \frac{16\pi^2}{5(b_0-6)K_\theta}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\frac{\Lambda^2_{\rm SI}}{\Lambda^2_{\not{CP}}}$$

$$K_\theta = \text{Re}\left[e^{-i\theta_{\rm QCD}}\det\left(Y_{\rm u}Y_{\rm d}\right)\right]$$

Test different flavour scenarios

Finite ratio in the decoupling limit



Summary & Outlook

- Enhancing the axion mass via small-instanton also (accidentally) enhances CPV effects that misalign potential
  => Dangerous effects which spoil Axion solution
  => The quality of Axion solution depends on UV scenarios

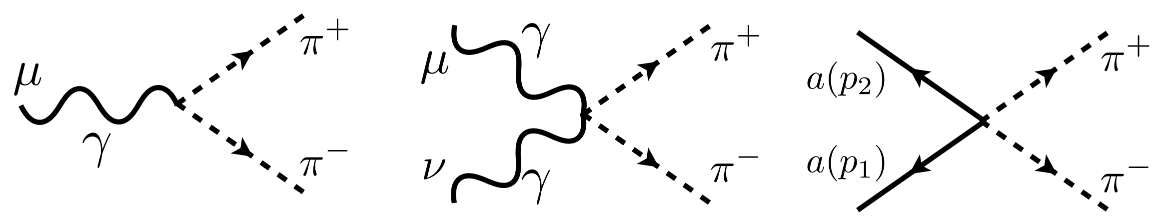
- The estimation of these effects can be made easier with the help of Determinant-like Flavour Invariants and Instanton Naive Dimensional Analysis
  => Allow to study the contributions of any SMEFT operators to the axion potential

Summary & Outlook

- It is natural to expect that the dynamics that endow an axion with a mass at the UV scale also lead to non-shift symmetric couplings of axions

Example: 1-loop contribution from QCD Chiral Lagrangian

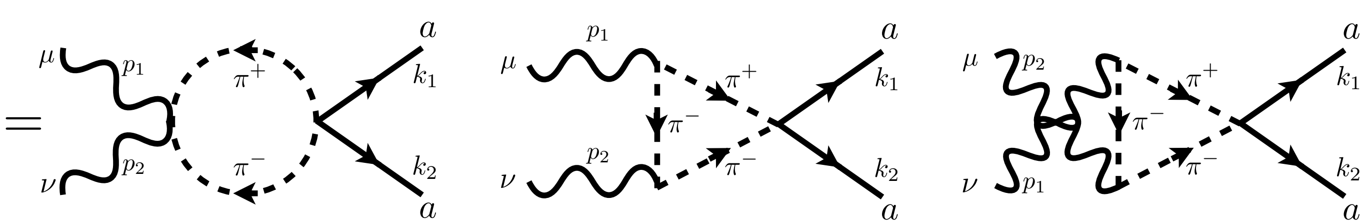
$\mathcal{L}_{\chi PT}^{(p^2)} \supset$



$-i\,e\,(p_+ - p_-)_\mu$ $\quad 2\,i\,e^2\,g_{\mu\nu}$ $\quad \frac{i}{f_a^2}\left[\frac{m_u\,m_d\,m_\pi^2}{(m_u+m_d)^2} - \frac{2(m_d-m_u)^2 p_1\cdot p_2}{(m_u+m_d)^2}\right]$

1-loop matching

$\mathcal{A}(\gamma\gamma \to aa) =$



$$\mathcal{L}_{a^2F^2} \supset c_{F^2}\,\frac{\alpha}{16\pi^2}\left(\frac{a^2}{f_a^2}\right) F_{\mu\nu}F^{\mu\nu} \quad \longrightarrow \quad \alpha \simeq \alpha_0\left(1 + \frac{\alpha_0\, m_u\, m_d\, a^2}{12\,\pi\, f_a^2\,(m_u+m_d)^2}\right)$$

$$c_{F^2} = \pi\, m_u\, m_d / 3\,(m_u+m_d)^2 \sim 0.2$$

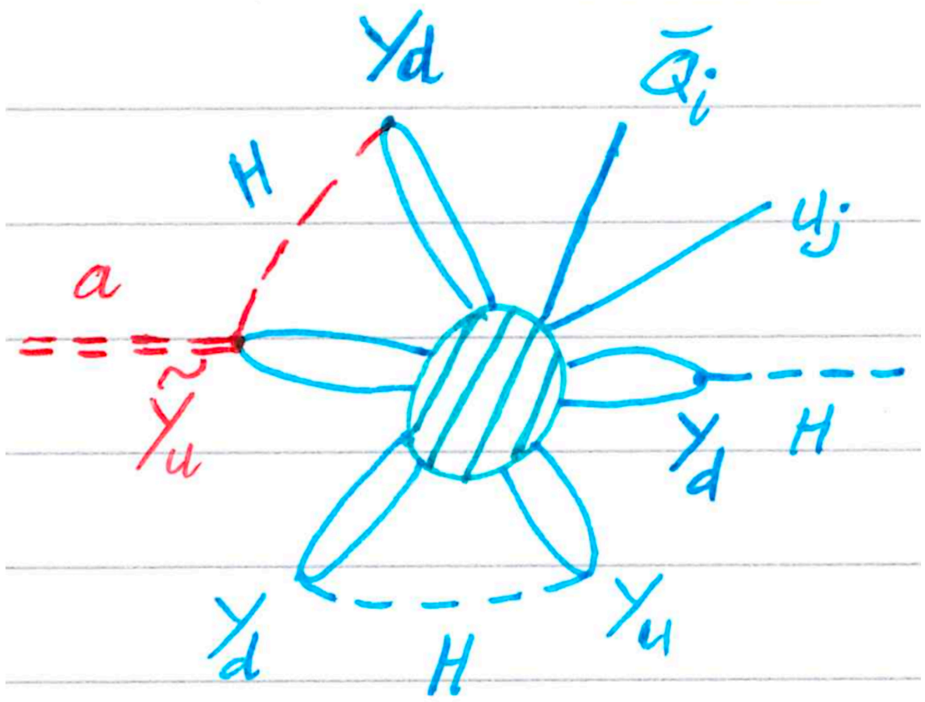
C. Beadle, S. A. R. Ellis, J. Quevillon, P. N. H. Vuong (2307.10362)

Summary & Outlook

- It is natural to expect that the dynamics that endow an axion with a mass at the UV scale also lead to non-shift symmetric couplings of axions

When small-instantons enhance axion mass, will they generate/enhance non-shift symmetric couplings of axions?

Example:
(work in progress)



Bonus slides

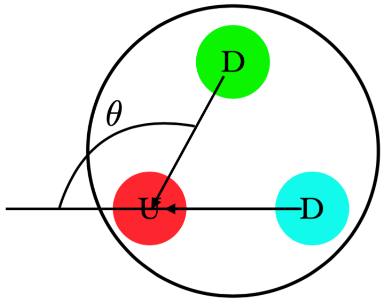
Preliminary

- Strong CP problem & Axion solution

1.) QCD vacuum allows an effective(CP violating) term in the Lagrangian:

$$\mathcal{L} \supset \bar{\theta}\frac{g_s^2}{32\pi^2}G_{\mu\nu}\tilde{G}^{\mu\nu}$$

#Key feature: $\bar{\theta} = \theta_{\rm QCD} - \arg(\det M_q)$ received contributions from both Strong & Electroweak sectors => theta-bar expected to be O(1)



2.) Bound from Neutron EDM: $\bar{\theta} < 10^{-10}$

Strong CP problem: Why is theta-bar so small?

Alternative questions: why no CP-violation in QCD? What make theta-bar so small? (any mechanism behind?)

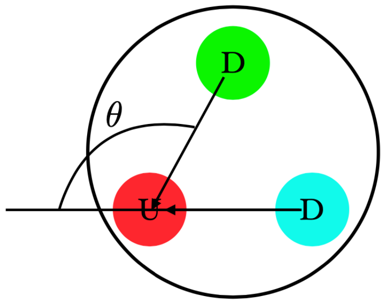
Preliminary

Strong CP problem & Axion solution

1.) QCD vacuum allows an effective(CP violating) term in the Lagrangian:

$$\mathcal{L} \supset \bar{\theta}\frac{g_s^2}{32\pi^2}G_{\mu\nu}\tilde{G}^{\mu\nu}$$

#Key feature: $\bar{\theta} = \theta_{\rm QCD} - \arg(\det M_q)$ received contributions from both Strong & Electroweak sectors => theta-bar expected to be O(1)



2.) Bound from Neutron EDM: $\bar{\theta} < 10^{-10}$

Strong CP problem: Why is theta-bar so small?

Alternative questions: why no CP-violation in QCD? What make theta-bar so small? (any mechanism behind?)

3.) Axion solution: dynamically relaxes theta-bar to zero

QCD confinement gives axion potential:

$V_{\chi PT}(a) \sim 1 - cos(a/f_a)$

minimised at <a>=0

$$\mathcal{L} \supset \left(\bar{\theta} + \frac{a}{f_a}\right)\frac{g_s^2}{32\pi^2}G_{\mu\nu}\tilde{G}^{\mu\nu}$$

- Axion as Goldstone Boson of $U(1)_{PQ}$ anomalous symmetry
- Shift symmetry $\frac{a}{f_a} \rightarrow \frac{a}{f_a} + \epsilon$

=> absorb $\bar{\theta}$ effects

Instanton density in SU(N) theory

$$d_N(\rho) = C[N] \left(\frac{8\pi^2}{g^2}\right)^{2N} e^{-8\pi^2/g^2(1/\rho)}$$

$$C[N] = \frac{C_1\, e^{-C_2 N}}{(N-1)!(N-2)!}\, e^{0.292 N_f}$$

$$\frac{8\pi^2}{g^2(1/\rho)} = \frac{8\pi^2}{g_0^2(\Lambda_{\rm UV})} - b_0 \log \rho\Lambda_{\rm UV}\,, \quad b_0 = \frac{11}{3}N - \frac{2}{3}N_f$$

't Hooft 76
Shifman, Vainshtein, Zakharov 79

Preliminary

- Small instanton & Axion potential: UV misaligned contributions

Small instantons generate axion potential of the form:

$$V(a) = \chi_{\mathcal{O}}(0)\frac{a}{f_a} + \frac{1}{2}\chi(0)\left(\frac{a}{f_a}\right)^2 \longrightarrow \langle\frac{a}{f_a}\rangle \equiv \theta_{\rm ind} = -\frac{\chi_{\mathcal{O}}(0)}{\chi(0)}$$

Induced by CP-violating operator — This talk

Coefficients in the potential can be computed from following correlators: Witten "79

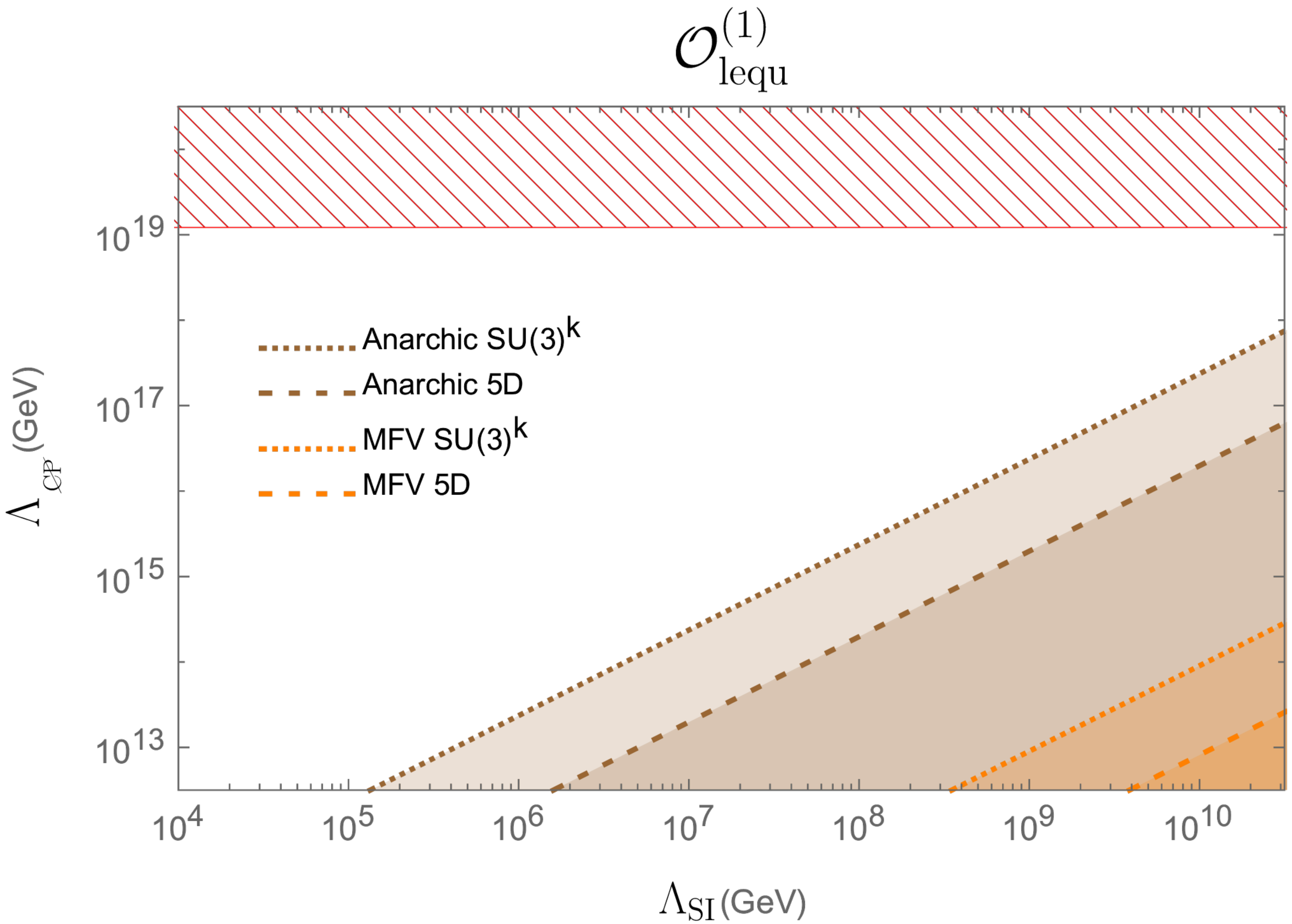
$$\chi(0) = -i\lim_{k\to 0}\int d^4x\, e^{ikx}\left\langle 0\left|T\left\{\frac{g^2}{32\pi^2}G\tilde{G}(x)\,,\,\frac{g^2}{32\pi^2}G\tilde{G}(0)\right\}\right|0\right\rangle$$

$$\chi_{\mathcal{O}}(0) = -i\lim_{k\to 0}\int d^4x\, e^{ikx}\left\langle 0\left|T\left\{\frac{g^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a(x)\,,\,\frac{C^{ij\cdots}_{\mathcal{O}}}{\Lambda^{D-4}_{\not{CP}}}\mathcal{O}^{D,ij\cdots}(0)\right\}\right|0\right\rangle$$

Note: In strongly coupled regime, one should use non-perturbative methods => Current algebra, QCD light cone sum rules

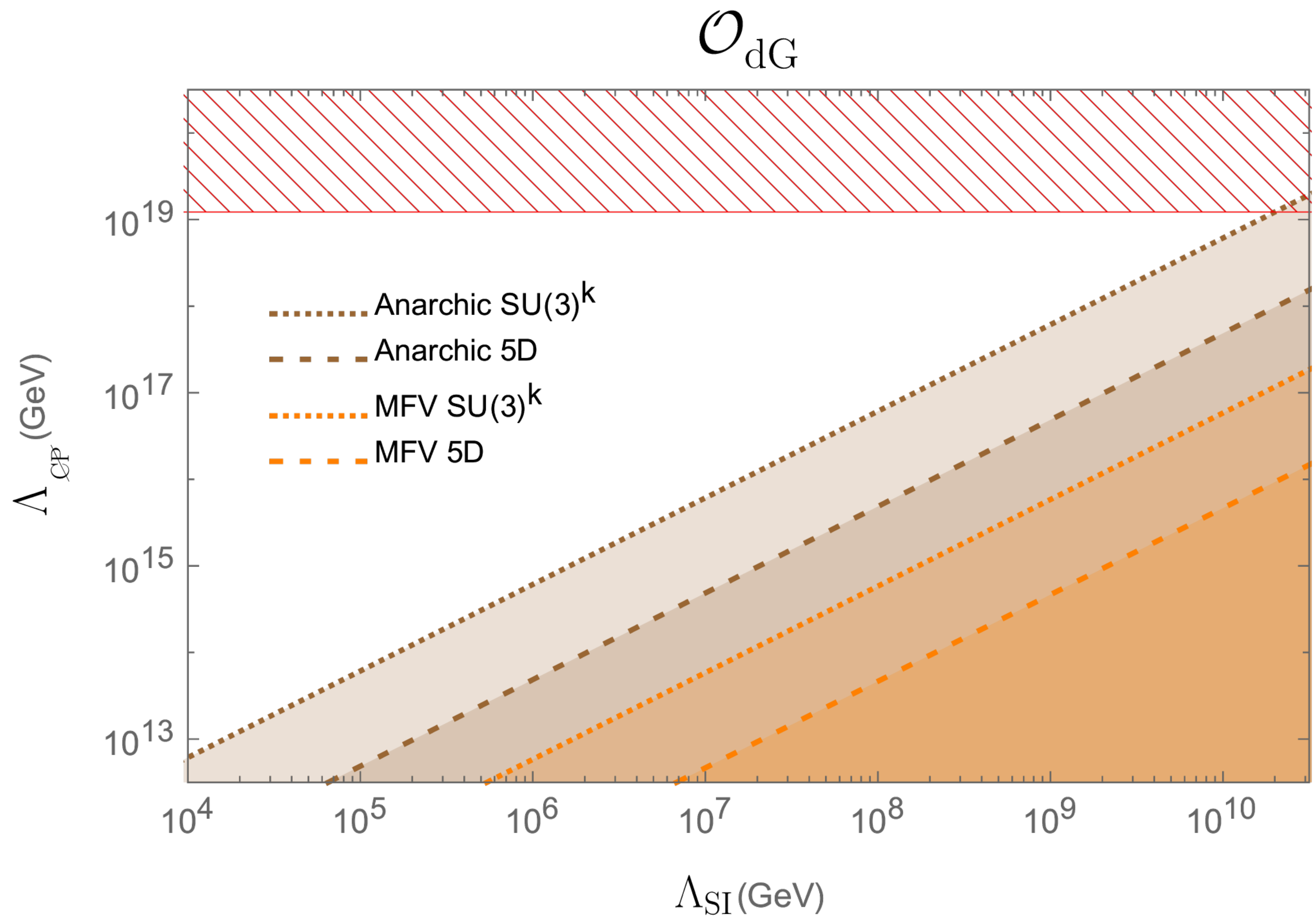
Bounds from non-measurement of theta-induced: Semi-leptonic operator

$$\mathcal{O}^{(1)}_{\rm lequ} = \bar{L}e\bar{Q}u + \text{h.c.}$$

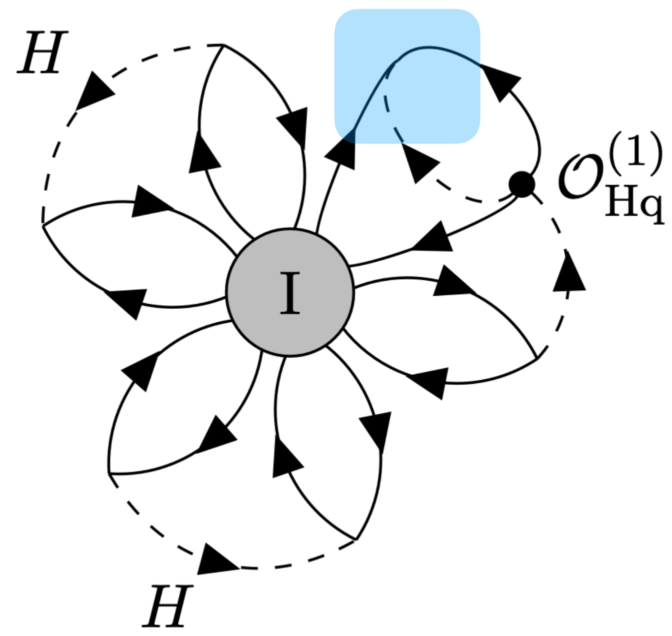


Bounds from non-measurement of theta-induced: Gluon dipole operator

$$\mathcal{O}_{\rm uG} = (\bar{Q}\sigma^{\mu\nu}T^A u)\tilde{H}G^A_{\mu\nu} + \text{h.c.}$$



Topological Susceptibilities & Flavor invariants: Higher-order Invariants



(c) Instanton diagram with an insertion of a non-chirality-flipping effective operator $\mathcal{O}^{(1)}_{\rm Hq}$.

$$\mathcal{I}_{abcd}(C^{(1,3)}_{\rm Hq}) \equiv {\rm Im}\left[e^{-i\theta_{\rm QCD}}\epsilon^{IJK}\epsilon^{ijk}Y_{{\rm u},Ii}Y_{{\rm u},Jj}\left(X^a_{\rm u}X^b_{\rm d}X^c_{\rm u}X^d_{\rm d}\,C^{(1,3)}_{\rm Hq}\,Y_{\rm u}\right)_{Kk}\det Y_{\rm d}\right]$$

$$X_{\rm u,d} = Y_{\rm u,d}Y^\dagger_{\rm u,d}$$

=> Set X=1 for the lowest order flavour invariants

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

- Key points: Path Integral & Instanton background

$$\chi_{\mathcal{O}}(0) = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \middle| T\left\{ \frac{g^2}{32\pi^2} G\tilde{G}(x)\,,\ \frac{C_{\mathcal{O}}}{\Lambda^2_{\not{CP}}} \mathcal{O}[\varphi_{\rm I}, \varphi](0) \right\} \middle| 0 \right\rangle ,$$

$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho\, d\xi_f^{(0)}\, d\bar{\xi}_f^{(0)}\right)$$

$$\times \int \mathcal{D}\varphi\, e^{-S_0[\varphi]-S_{\rm int}[\varphi_{\rm I},\varphi]} \int d^4x \frac{g^2}{32\pi^2} G\tilde{G}(x) \times \frac{C_{\mathcal{O}}}{\Lambda^2_{\not{CP}}} \mathcal{O}[\varphi_{\rm I}, \varphi](0) \Big|_{1\text{-(a.-)inst.}}$$

Fields with instanton solutions (e.g. gluon, quark): φ_I

=> Expand the fields in their eigenmodes, replace zero mode wave function by instanton solutions, and integrate out non-zero modes:

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

- Key points: Path Integral & Instanton background

$$\chi_{\mathcal{O}}(0) = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \left| T\left\{ \frac{g^2}{32\pi^2} G\tilde{G}(x)\,,\, \frac{C_{\mathcal{O}}}{\Lambda^2_{\not{CP}}} \mathcal{O}[\varphi_{\rm I},\varphi](0) \right\} \right| 0 \right\rangle ,$$

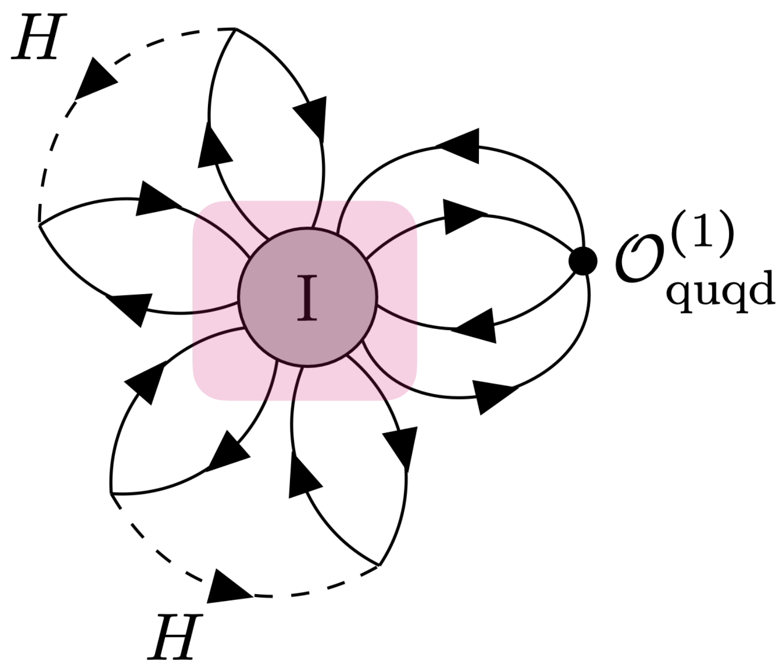
$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho\, d\xi_f^{(0)} d\bar{\xi}_f^{(0)}\right)$$

$$\times \int \mathcal{D}\varphi\, e^{-S_0[\varphi]-S_{\rm int}[\varphi_{\rm I},\varphi]} \int d^4x \frac{g^2}{32\pi^2} G\tilde{G}(x) \times \frac{C_{\mathcal{O}}}{\Lambda^2_{\not{CP}}} \mathcal{O}[\varphi_{\rm I},\varphi](0) \Big|_{\text{1-(a.-)inst.}}$$

Fields without instanton solutions: φ

=> Integrate over without performing the eigenmode expansion

Topological Susceptibilities & Flavor invariants: Four-quark operator



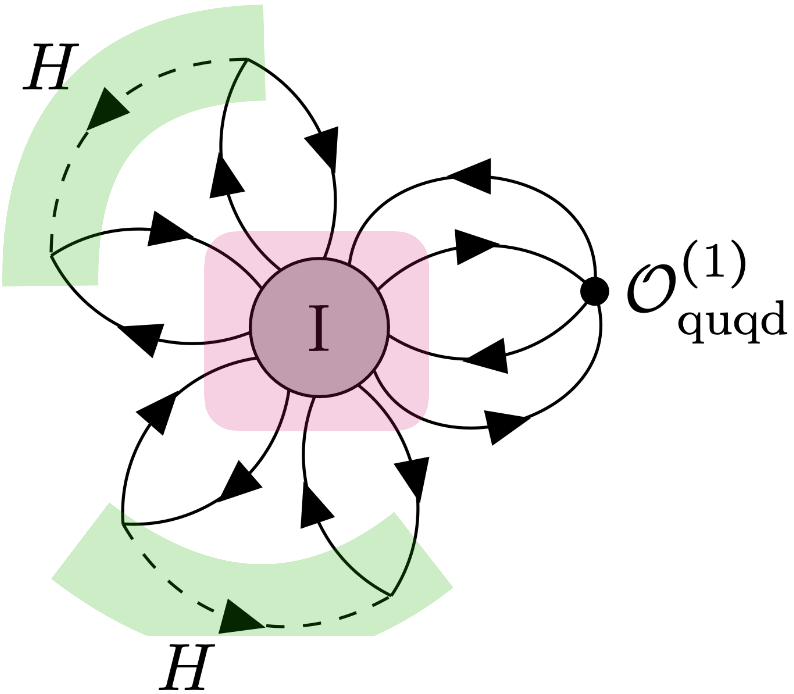
$$\chi^{(1)}_{\text{quqd}}(0)^{1-\text{inst.}} = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \middle| T\left\{ \frac{1}{32\pi^2} G\widetilde{G}(x),\, \frac{C^{(1)}_{\text{quqd}}}{\Lambda^2_{\not{CP}}} \mathcal{O}^{(1)}_{\text{quqd}}(0) \right\} \middle| 0 \right\rangle ,$$

$$= e^{-i\theta_{\text{QCD}}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H\mathcal{D}H^\dagger\, e^{-S_0[H,H^\dagger]} \int \prod_{f=1}^{3} \left(\rho^2\, d\xi^{(0)}_{u_f} d\xi^{(0)}_{d_f} d^2\bar{\xi}^{(0)}_{Q_f} \right)$$

$$\times\, e^{\int d^4x (\bar{Q}Y_{\text{u}}\widetilde{H}u + \bar{Q}Y_{\text{d}}Hd + \text{h.c.})(x)} \frac{1}{32\pi^2} \int d^4x\, G\widetilde{G}(x) \left(\frac{C^{(1)}_{\text{quqd}}}{\Lambda^2_{\not{CP}}} \bar{Q}u\bar{Q}d(0) + \text{h.c.} \right) ,$$

Q = 1

Topological Susceptibilities & Flavor invariants: Four-quark operator



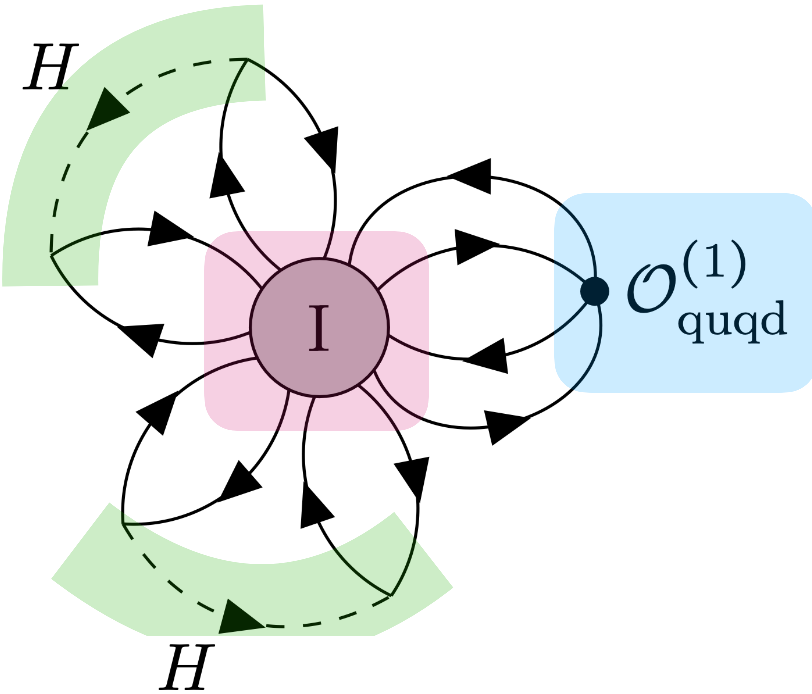
$$\chi^{(1)}_{\text{quqd}}(0)^{1-\text{inst.}} = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \left| T\left\{ \frac{1}{32\pi^2} G\widetilde{G}(x), \frac{C^{(1)}_{\text{quqd}}}{\Lambda^2_{\not{CP}}} \mathcal{O}^{(1)}_{\text{quqd}}(0) \right\} \right| 0 \right\rangle ,$$

$$= e^{-i\theta_{\text{QCD}}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H\mathcal{D}H^\dagger\, e^{-S_0[H,H^\dagger]} \int \prod_{f=1}^{3} \left(\rho^2\, d\xi^{(0)}_{u_f} d\xi^{(0)}_{d_f} d^2\bar{\xi}^{(0)}_{Q_f}\right)$$

$$\times\, e^{\int d^4x (\bar{Q} Y_u \widetilde{H} u + \bar{Q} Y_d H d + \text{h.c.})(x)} \frac{1}{32\pi^2} \int d^4x\, G\widetilde{G}(x) \left(\frac{C^{(1)}_{\text{quqd}}}{\Lambda^2_{\not{CP}}} \bar{Q} u \bar{Q} d(0) + \text{h.c.} \right) ,$$

Q = 1

Topological Susceptibilities & Flavor invariants: Four-quark operator



$$\chi^{(1)}_{\rm quqd}(0)^{1-\rm inst.} = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \left| T\left\{ \frac{1}{32\pi^2} G\widetilde{G}(x), \frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \mathcal{O}^{(1)}_{\rm quqd}(0) \right\} \right| 0 \right\rangle ,$$

$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H\mathcal{D}H^\dagger\, e^{-S_0[H,H^\dagger]} \int \prod_{f=1}^{3} \left(\rho^2\, d\xi^{(0)}_{u_f} d\xi^{(0)}_{d_f} d^2\bar{\xi}^{(0)}_{Q_f} \right)$$

$$\times\, e^{\int d^4x (\bar{Q} Y_{\rm u} \widetilde{H} u + \bar{Q} Y_{\rm d} H d + {\rm h.c.})(x)} \frac{1}{32\pi^2} \int d^4x\, G\widetilde{G}(x) \left(\frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \bar{Q} u \bar{Q} d(0) + {\rm h.c.} \right) ,$$

$Q = 1$