Lack of propagating degrees of freedom in degenerate $f(R)$ models.

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- PRD 111 (2025) 4, 044030 (**arXiv:2412.09366**),
- PRD 108 (2023) 6, 064006 (**arXiv:2303.02103**).

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Introduction.

Metric $f(R)$ gravity: fundamentals.

Simplest possible generalisation of GR, with action

$$S[g_{\mu\nu},\Psi] = \frac{1}{16\pi G}\int_{\mathscr{M}} \mathrm{d}^4x\,\sqrt{-g}\,f(R) + S_{\mathrm{matter}}[g_{\mu\nu},\Psi],$$

leading to **fourth-order EOM**:

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f'(R) = 8\pi G T_{\mu\nu}.$$

Equivalence to **GR + an additional dynamical scalar** (scalaron) in the so-called 'Einstein frame:'

$$\phi(R) = \sqrt{\frac{3}{16\pi G}}\ln f'(R), \qquad V(\phi) = \frac{f'(R)R - f(R)}{16\pi G f'^2(R)}.$$

Metric $f(R)$ gravity: three GW polarisations.

The extra scalar mode was attested **early studies of GWs** in $f(R)$ gravity,[1] linearising around Minkowski space-time $\eta_{\mu\nu}$:

$$\begin{aligned} \Box\,\bar{h}_{\mu\nu} + \mathscr{O}(h^2) &= 0, \\ (\Box - m_{\text{eff}}^2)R^{(h)} + \mathscr{O}(h^2) &= 0, \end{aligned}$$

where $R = R^{(h)} + \mathscr{O}(h^2)$ and we have introduced:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \left[\frac{h}{2} + \frac{f''(0)}{f(0)}R^{(h)}\right]\eta_{\mu\nu}, \qquad m_{\text{eff}}^2 \equiv \frac{f'(0)}{3f''(0)}$$

Three polarisations: $+$, $\times$ (graviton) and scalar.

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[1] S. Capozziello, C. Corda, M. F. De Laurentis, PLB 669 (2008) 255-259.

Metric $f(R)$ gravity: three GW polarisations?

The previous picture has been put into question several times:

- Some authors claimed there was a second scalar mode (the **breathing mode**). Rigorously proven not to exist.
- Other authors showed that some particular $f(R)$ models (such as $f(R) \propto R^2$) **lack the graviton modes**.[2, 3]
- Later works confirmed that $\boldsymbol{f(R) \propto R^2}$ **does not propagate neither the graviton nor the scalaron** modes.[4, 5, 6]

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[2] L. Álvarez-Gaumé *et al.*, Fortsch. Phys. 64 (2016) 2-3, 176-189.

[3] ACT, Á. de la Cruz-Dombriz, A. Dobado, PRD 108 (2023) 6, 064006.

[4] A. Hell, D. Lüst, G. Zoupanos, JHEP 02 (2024) 039.

[5] A. Golovnev, IJTP 63 (2024) 8, 212.

[6] G. K. Karananas, PRD 111 (2025) 4, 044068; arXiv:2408.16818 [hep-th].

Our goals.[7]

- **Unify** and **extend** all previous results on the topic.
- Provide a complete picture of the number of degrees of freedom propagated by $f(R)$ gravities and their associated instabilities.
- Prove that **there are no propagating degrees of freedom** in **maximally-symmetric (MS) backgrounds** in the large class of so-called **degenerate $f(R)$ models** (including the aforementioned $f(R) \propto R^2$).
- Compare with the **non-degenerate models**.

[7] ACT, Á. de la Cruz-Dombriz, A. Dobado, PRD 111 (2025) 4, 044030.

A brief glossary: degenerate and non-degenerate $f(R)$ models.

Glossary: **Constant curvature; non-degenerate $f(R)$ models.**

- **Constant-curvature space-time**: one having constant Ricci scalar $R = R_0$ (e.g. Schwarzschild).

  For $R = R_0$, the $f(R)$ **EOM** become:

$$f'(R_0)R_{\mu\nu}^{(0)} = \frac{f(R_0)}{2} g_{\mu\nu}^{(0)} \quad \underset{\text{trace}}{\Longrightarrow} \quad f'(R_0)R_0 = 2f(R_0).$$

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- **R_0-non-degenerate $f(R)$ models**: $\boxed{f'(R_0) \neq 0}$.

  Constant-curvature solutions: only those of $\mathbf{GR} + \boldsymbol{\Lambda}$.

  $$R^{(0)}_{\mu\nu} = \Lambda g^{(0)}_{\mu\nu}, \qquad \Lambda = \frac{R_0}{4} = \frac{f(R_0)}{2f'(R_0)}.$$

  Example: $f(R) = R - 2\Lambda + \alpha R^2$ with $R_0 = 4\Lambda$.

Glossary: R_0-degenerate $f(R)$ models.

- **R_0-degenerate $f(R)$ models**: $\boxed{f'(R_0) = 0} \underset{\text{trace}}{\Longrightarrow} \boxed{f(R_0) = 0}$.

  Every metric with $R = R_0 = \text{const.}$ solves the EOM!

  $$f'(R_0) R^{(0)}_{\mu\nu} = \frac{f(R_0)}{2} g^{(0)}_{\mu\nu} \qquad \text{becomes} \qquad 0 = 0 \text{ identically!}$$

  Example: $f(R) \propto R^{1+\delta}$ for $\delta > 0$ and $R_0 = 0$.

[8] ACT, Á. de la Cruz-Dombriz, A. Dobado, PRD 108 (2023) 6, 064006.

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  Example: $f(R) \propto R^{1+\delta}$ for $\delta > 0$ and $R_0 = 0$.

  These models are **inherently pathological**:[8]

  – R_0-degenerate solutions are generically **unstable**.

  – Other unphysical features (**naked singularities**).

[8] ACT, Á. de la Cruz-Dombriz, A. Dobado, PRD 108 (2023) 6, 064006.

Main results.

Perturbations around MS backgrounds.

MS backgrounds are **constant-curvature space-times**:

$$g_{\mu\nu}^{(0)} = \begin{cases} \text{de Sitter (dS)} & \text{if } R_0 > 0, \\ \text{Minkowski} & \text{if } R_0 = 0, \\ \text{Anti-de Sitter (AdS)} & \text{if } R_0 < 0, \end{cases}$$

We consider small **perturbations around** $g_{\mu\nu}^{(0)}$:

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll |g_{\mu\nu}^{(0)}|;$$

$$R = R_0 + R^{(h)} + \mathscr{O}(h^2), \qquad R^{(h)} = -\Box h + \nabla_\mu \nabla_\nu h^{\mu\nu} - \frac{R_0}{4} h.$$

Note that $R^{(h)}$ is **gauge-invariant** ($x^\mu \to x'^\mu = x^\mu + \xi^\mu$).

MS backgrounds: **Linearised $f(R)$ gravity.**

At linear level around a MS $g^{(0)}_{\mu\nu}$, the $f(R)$ **EOM** become

$$
\begin{aligned}
0 &= f''(R_0)\left[\frac{R^{(0)}}{4}g^{(0)}_{\mu\nu} - \left(\nabla_\mu\nabla_\nu - g^{(0)}_{\mu\nu}\Box\right)\right]R^{(h)} \\
&+ f'(R_0)\left[R^{(h)}_{\mu\nu} - \frac{R^{(h)}}{2}g^{(0)}_{\mu\nu}\right] - \frac{f(R_0)}{2}h_{\mu\nu} + \mathcal{O}(h^2),
\end{aligned}
$$

while their **trace** reads:

$$
f''(R_0)\left(\Box + \frac{R_0}{3}\right)R^{(h)} - \frac{f'(R_0)}{3}R^{(h)} + \mathcal{O}(h^2) = 0.
$$

Note we have never divided by $f(R_0)$, $f'(R_0)$ or $f''(R_0)$!

MS backgrounds: **Scalaron propagation (deg. *and* non-deg.).**

$\boxed{f''(R_0) \neq 0}$ $R^{(h)}$ satisfies a **massive Klein-Gordon equation**:

$$(\Box - m_{\text{eff}}^2)R^{(h)} + \mathcal{O}(h^2) = 0, \qquad m_{\text{eff}}^2 = \frac{1}{3}\left[\frac{f'(R_0)}{f''(R_0)} - R_0\right].$$

Therefore, $R^{(h)}$ is a **propagating DOF** (the **scalaron**).

$\boxed{f''(R_0) = 0}$ $$\cancelto{0}{f''(R_0)}\left(\Box + \frac{R_0 - f'(R_0)}{3}\right)R^{(h)} + \mathcal{O}(h^2) = 0.$$

Result 1. MS backgrounds with $R = R_0$ are **strongly-coupled** in $f(R)$ models with $f''(R_0) = 0$.

MS backgrounds: R_0-degenerate case.

$\boxed{f'(R_0) = 0}$ **R_0-degenerate case.** (*Assuming* $f''(R_0) \neq 0$.)

The linearised EOM and their trace reduce to:

$$
\begin{aligned}
0 &= \left[\frac{R^{(0)}}{4} g^{(0)}_{\mu\nu} - \left(\nabla_\mu \nabla_\nu - g^{(0)}_{\mu\nu} \Box\right)\right] R^{(h)} + \mathcal{O}(h^2), \\
0 &= \left(\Box + \frac{R_0}{3}\right) R^{(h)} + \mathcal{O}(h^2).
\end{aligned}
$$

We notice that:

- The **graviton** has **disappeared**! (**Strong-coupling**.)
- The remaining **scalaron** $R^{(h)}$ is **constrained**.

MS backgrounds: R_0-degenerate case.

$\boxed{R_0 = 0}$ The general solution is:

$$R^{(h)} = C_\mu x^\mu + D.$$

$\boxed{R_0 > 0}$ The general solution is:

$$R^{(h)} = \mathscr{A}\, \mathrm{e}^{H_0 t}.$$

Neither represent a localised perturbation unless $C_\mu = 0$, $D = 0$ and $\mathscr{A} = 0$. In other words, the **only solution** is $\boxed{R^{(h)} = 0}$.

Result 2. R_0-degenerate $f(R)$ models with $f''(R_0) \neq 0$ **do not propagate any degrees of freedom** atop MS backgrounds. Only the graviton is truly strongly-coupled.

MS backgrounds: **Non-degenerate case, graviton.**

$\boxed{f'(R_0) \neq 0}$ **Non-degenerate case**. (*Assuming* $f''(R_0) \neq 0$.)

Introducing

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \left[\frac{h}{2} + \frac{f''(R_0)}{f'(R_0)} R^{(h)}\right] g^{(0)}_{\mu\nu},$$

the **linearised $f(R)$ EOM** become:

$$\left(\Box - \frac{R_0}{6}\right) \bar{h}_{\mu\nu} + \mathscr{O}(h^2) = 0.$$

Gauge symmetry: $\nabla^\mu \bar{h}_{\mu\nu} = 0, \quad \bar{h} = 0, \quad \bar{h}_{\mu 0} = 0.$

$\Longrightarrow$ There are **two polarisation modes** (+ the scalaron).

MS backgrounds: **Non-degenerate case, graviton.**

In **de Sitter** space-time ($R_0 \equiv 12H_0^2 > 0$),

$$\mathrm{d}s^2_{(0)} = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\vec{x}^2, \qquad a(t) = \mathrm{e}^{H_0 t},$$

simple solutions with wave vector $\vec{k}$ are:

$$\begin{aligned} \bar{h}^{\vec{k}}_{ij}(t,\vec{x}) &= A^{(1)}_{ij}(\vec{k})\,\mathrm{e}^{H_0 t/2}\,\psi^{(1)}_{\vec{k}}(t)\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} \\ &+ A^{(2)}_{ij}(\vec{k})\,\mathrm{e}^{H_0 t/2}\,\psi^{(2)}_{\vec{k}}(t)\,\mathrm{e}^{+\mathrm{i}\vec{k}\cdot\vec{x}}, \\ \psi^{(1,2)}_{\vec{k}=\vec{0}}(t) &= \mathrm{e}^{\mp 3H_0 t/2}, \\ \psi^{(1,2)}_{\vec{k}\neq\vec{0}}(t) &= H^{(1,2)}_{3/2}\left(\frac{|\vec{k}|}{H_0}\,\mathrm{e}^{-H_0 t}\right). \end{aligned}$$

MS backgrounds: **Non-degenerate case, graviton.**

All **graviton modes are tachyonic** regardless of $\vec{k}$:

$$\bar{h}_{ij}^{\vec{k}}(t,\vec{x}) \underset{t\to+\infty}{\sim} a^2(t) = \mathrm{e}^{2H_0 t} \to \infty.$$

However, **there is <u>no tachyonic instability</u>**, because the modes do *not* grow faster than the background:

$$g_{ij}^{(0)} = a^2(t)\,\delta_{ij} = \mathrm{e}^{2H_0 t}\,\delta_{ij} \quad \Longrightarrow \quad \frac{|h_{ij}^{\vec{k}}|}{|g_{ij}^{(0)}|} \ll 1 \text{ at all times.}$$

'Cosmic expansion dilutes the tachyonic (exponential) growth of perturbations.'

MS backgrounds: **Non-degenerate case, scalaron.**

In **de Sitter** space-time ($R_0 \equiv 12H_0^2 > 0$), the **scalaron EOM**,

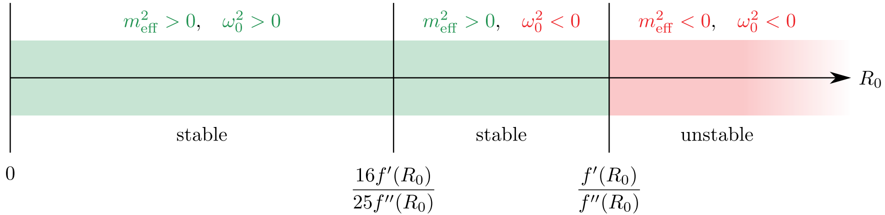
$$(\Box - m_{\rm eff}^2)R^{(h)} + \mathscr{O}(h^2) = 0, \qquad m_{\rm eff}^2 = \frac{1}{3}\left[\frac{f'(R_0)}{f''(R_0)} - R_0\right].$$

also admits **simple solutions** with wave vector $\vec{q}$:

$$\begin{aligned}
R_{\vec{q}}^{(h)}(t,\vec{x}) &= A^{(1)}(\vec{q})\,\mathrm{e}^{-3H_0t/2}\,\psi_{\vec{q}}^{(1)}(t)\,\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{x}} \\
&+ A^{(2)}(\vec{q})\,\mathrm{e}^{-3H_0t/2}\,\psi_{\vec{q}}^{(2)}(t)\,\mathrm{e}^{+\mathrm{i}\vec{q}\cdot\vec{x}},
\end{aligned}$$

$$\begin{aligned}
\psi_{\vec{q}=\vec{0}}^{(1,2)}(t) &= \mathrm{e}^{\pm\mathrm{i}\omega_0 t}, & \omega_0^2 &= m_{\rm eff}^2 - \frac{9H_0^2}{4}, \\
\psi_{\vec{q}\neq\vec{0}}^{(1,2)}(t) &= H_\nu^{(1,2)}\left(\frac{|\vec{q}|}{H_0}\,\mathrm{e}^{-H_0t}\right), & \nu^2 &= -\frac{\omega_0^2}{H_0^2}.
\end{aligned}$$

MS backgrounds: **Non-degenerate case, summary.**



Result 3. Non-degenerate models with $f''(R_0) \neq 0$ propagate **graviton** + **scalaron** atop MS backgrounds. The scalaron is **tachyonically unstable** if $m^2_{\text{eff}} < 0$.

Summary and conclusions.

Summary and conclusions.[9]

- Even in simple theories such as $f(R)$ gravity, determining the number of propagating DOF is convoluted, yet crucial.
- $\boldsymbol{R_0}$**-degenerate** $\boldsymbol{f(R)}$ **models** propagate:
  - $f''(R_0) \neq 0$: no degrees of freedom, strongly-coupled graviton.
  - $f''(R_0) = 0$: no degrees of freedom, strongly-coupled graviton and scalaron.
- **Non-degenerate** $\boldsymbol{f(R)}$**-models** propagate:
  - $f''(R_0) \neq 0$: graviton and scalaron (latter having a tachyonic instability if $m^2_{\mathrm{eff}} < 0$).
  - $f''(R_0) = 0$: graviton, strongly-coupled scalaron.

[9] ACT, Á. de la Cruz-Dombriz, A. Dobado, PRD 111 (2025) 4, 044030.

Thank you!

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