Modular invariant Inflation, reheating and leptogenesis

Wenbin Zhao

Bethe Center for Theoretical Physics and Physikalisches Institut,
Universität Bonn

27/05/2025, Padua, PLANCK2025

[2405.06497] + [2411.18603] with G. Ding, S. Jiang, Y. Xu.

Outline

Motivation

- Modular symmetry has been successfully used as a guiding principle to explain several puzzles in the SM:
  - Fermion mass hierarchy,
  - Flavor mixing,
  - CP violation,

  where a scalar (modulus) field, determines the Yukawa coupling.

Motivation

- Modular symmetry has been successfully used as a guiding principle to explain several puzzles in the SM:
  - Fermion mass hierarchy,
  - Flavor mixing,
  - CP violation,

  where a scalar (modulus) field, determines the Yukawa coupling.
- Vev of modulus is determined by parameter fitting.
- Dynamically fix the vev by its potential, can also realize inflation.
- Combination of Modular inflation and flavor model can be used to discuss reheating and leptogenesis.

Modular Symmetry I

Modular Group SL(2,$\mathbb{Z}$)

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}.$$

Modular Transformation

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \mathrm{Im}\tau > 0\,.$$

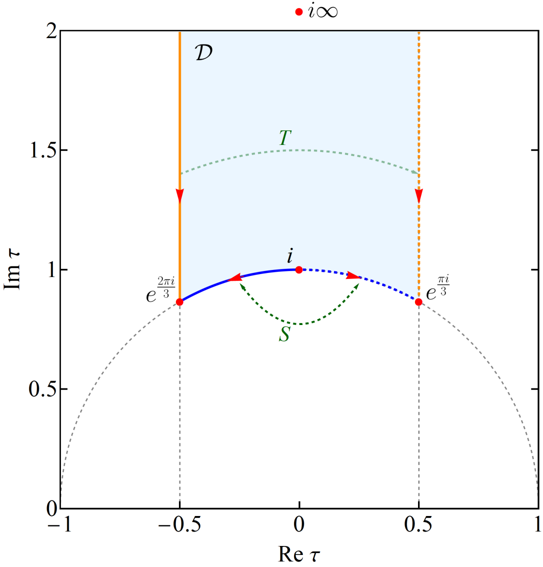
$$\mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \tau \to -\frac{1}{\tau}\,, \quad \mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \tau \to \tau + 1\,,$$

Modular Forms

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma\,,$$

where the weight k is a generic non-negative integer.

Modular Symmetry II:Fundamental domain



Modular Symmetry III

The derivative of a weight k modular form f satisfies:

$$f'(\gamma\tau) = (c\tau + d)^{k+2} f'(\tau) + \frac{k}{2\pi i} c (c\tau + d)^{k+1} f(\tau), \quad \gamma \in \Gamma\,,$$

For a weight 0 modular form, its derivative is a weight 2 modular form. There are 3 fixed points (under $\mathcal{S}$ or $\mathcal{T}$ or their combinations) in the fundamental domain:

$$i, \omega = e^{\frac{2\pi i}{3}}, i\infty$$

Derivatives of wight 0 modular form have to vanish there.

i and ω are natural candidates for vacuum.

ω and $i\infty$ have been used for inflation.

Modular Symmetry from String Theory

$D = 10$ Superstring $\longrightarrow$ Dialton S and other fields

Compactifycation $\longrightarrow$ modulus fields: massless scalar fields with no potential

$\mathcal{M}_{10} = \mathcal{M}_6 \times \mathcal{M}_4$

$\downarrow$

moduli stabilization: generate potential, positive mass and vev

$$\mathcal{M}_6 = T^2 \times T^2 \times T^2$$

Kähler modulus τ and complex structure modulus

Modular group is the mapping class group of torus

$\downarrow$

$D = 4, N = 1$ SUGRA $\longrightarrow$ Modular invariant scalar potential for S and τ

SuperGravity framework

In SUGRA, scalar potential is determined by Kähler potential $\mathcal{K}$ and superpotential $\mathcal{W}$ in a combined way:

$$\mathcal{G}(\tau,\bar{\tau},S,\bar{S}) = \mathcal{K}(\tau,\bar{\tau},S,\bar{S}) + \ln|\mathcal{W}(\tau,S)|^2\,,$$

And the scalar potential reads:

$$\begin{aligned} V(\tau,S) &= e^{\mathcal{K}}(\mathcal{K}^{\alpha\overline{\beta}}D_\alpha\mathcal{W}\overline{D_\beta\mathcal{W}} - 3|\mathcal{W}|^2) \\ &= e^{\mathcal{G}}(\mathcal{G}_\alpha\mathcal{G}^{\alpha\bar{\beta}}\mathcal{G}_{\bar{\beta}} - 3) \end{aligned}$$

where the covariant derivative is defined by $D_\alpha\mathcal{W} \equiv \partial_\alpha\mathcal{W} + \mathcal{W}(\partial_\alpha\mathcal{K})$ and $\mathcal{K}^{\alpha\overline{\beta}}$ is the inverse of the Kähler metric $\mathcal{K}_{\alpha\overline{\beta}} = \partial_\alpha\partial_{\overline{\beta}}\mathcal{K}$. The total bosonic action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}\mathcal{R} - g^{\mu\nu}\mathcal{K}_{\alpha\overline{\beta}}\partial_\mu\phi^\alpha\partial_\nu\overline{\phi^\beta} - V(\phi)\right]\,,$$

Potential setup I

$$\mathcal{K}(\tau,\bar{\tau},S,\bar{S}) = K(S,\bar{S}) - 3\ln(-i(\tau-\bar{\tau}))\,,$$
$$\mathcal{W}(S,\tau) = \Lambda_W^3 \frac{\Omega(S)H(\tau)}{\eta^6(\tau)}\,,$$

- We assume dialton S is stabilized.
- η is the Dedekind eta function with a modular *weight* $1/2$:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q \equiv e^{2\pi i\tau}\,,$$

- Under Modular transformation, they reads:

$$-3\ln\left[-i(\tau-\bar{\tau})\right] \to -3\ln\left[-i(\tau-\bar{\tau})\right] + 3\ln(c\tau+d) + 3\ln(c\bar{\tau}+d)\,.$$
$$\mathcal{W} \to e^{i\delta(\gamma)}(c\tau+d)^{-3}\mathcal{W}\,,$$

- $\mathcal{G}(\tau,\bar{\tau},S,\bar{S})$ and potential are modular invariant.

Potential setup II

The most general form without singularity inside the fundamental domain:

$$H(\tau) = (j(\tau) - 1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau)) \,, \quad m, n \in \mathbb{N} \,,$$

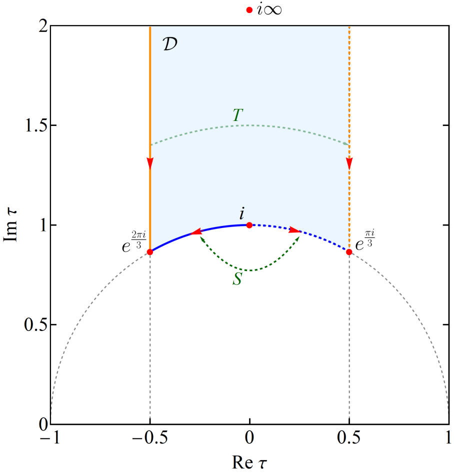
where j is called Klein j invariant.

$$j(i\infty) = +\infty \,, \qquad j(\omega) = 0 \,, \qquad j(i) = 1728 = 12^3 \,.$$

m, n determine vacua of the potential and we choose:

- $m = 0, n \geq 2$, slow roll from i (saddle point) to ω (Minkowski minimum) along the arc.
- $m \geq 2, n \geq 2$, we consider slow roll from $i\infty$ to the fixed point ω (Minkowski minimum) along the left boundary.
- $m = n = 0$, slow roll from i (saddle point) to ω (dS minimum) along the arc (King, Wang, 2405.08924).

Inflation in the Fundamental domain



Modular symmetry + Reality of potential
stabilize the orthogonal direction of inflation!

Full potential

We choose the following polynomial:

$$\mathcal{P}(j(\tau)) = 1 + \beta\left(1 - \frac{j(\tau)}{1728}\right) + \gamma\left(1 - \frac{j(\tau)}{1728}\right)^2 ,$$

and the full potential reads:

$$V(\tau) = \frac{\Lambda_S^4}{i(\tau-\bar{\tau})^3|\eta(\tau)|^{12}}\left[(A(S,\bar{S})-3)|H(\tau)|^2 + \hat{V}(\tau,\bar{\tau})\right] ,$$

$$A(S,\bar{S}) = \frac{K^{S\bar{S}}D_S W D_{\bar{S}}\bar{W}}{|W|^2} = \frac{K^{S\bar{S}}|\Omega_S + K_S\Omega|^2}{|\Omega|^2} ,$$

$$\hat{V}(\tau,\bar{\tau}) = \frac{-(\tau-\bar{\tau})^2}{3}\left|H_\tau(\tau) - \frac{3i}{2\pi}H(\tau)\widehat{G}_2(\tau,\bar{\tau})\right|^2 ,$$

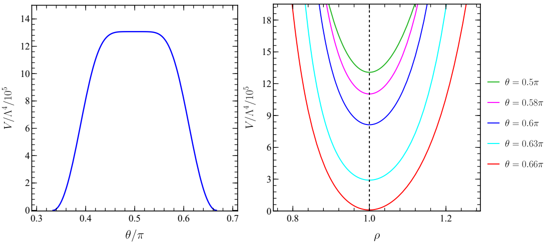
$$Z(\tau,\bar{\tau}) = \frac{1}{i(\tau-\bar{\tau})^3|\eta(\tau)|^{12}} ,$$

In short, 3 parameter sets: (m,n), (β,γ), $A(S,\bar{S})$

Slow roll along the unit arc

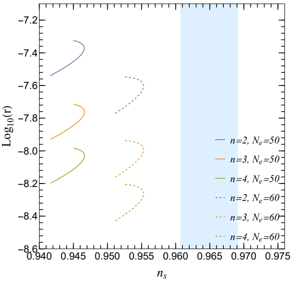
$m = 0, n \geq 2$: $\tau = \rho e^{i\theta}$ and $\tau = i$ is the start point of inflation:

$$V > 0 \quad \Rightarrow \quad A(S, \bar{S}) > 3\,,$$
$$\varepsilon_V = \frac{1}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \quad \Rightarrow \quad \text{modular symmetry}\,,$$
$$\eta_V = \frac{V''}{V} \ll 1 \quad \Rightarrow \quad (\beta, \gamma)\,.$$

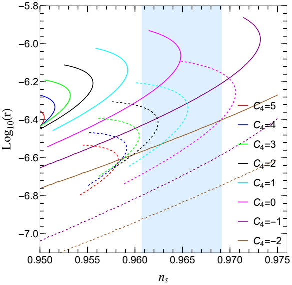


Example: $m = 0$, $n = 2$, $A = 24.3091$ and $\beta = 0.126425, \gamma = 0$.

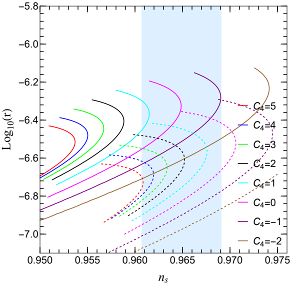
Slow roll along the unit arc



(a) $\mathcal{P}(j) = 1$.



(b) with β.



(c) with γ.

- Taylor expansion: $V(\phi) = V_0(1 - \sum_{k=1}^{\infty} C_{2k}\phi^{2k})$,
- The simplest case, $P(j) = 1$ gives too small spectral index.
- The rest: $r < 10^{-6}$, $\alpha \approx -10^{-4}$.

Slow roll at Infinity

A series of work in realizing α attractors with $SL(2,Z)$ symmetry.

2408.05203, 2411.07552, 2503.13682, 2503.14904

At large $\mathrm{Im}(\tau)$, the squared norm of modular j-invariant can be approximated as

$$|j(\tau)|^2 \approx e^{4\pi\,\mathrm{Im}(\tau)}\,.$$

One can define

$$I(\tau,\bar\tau) = \frac{\ln\left(|j(\tau)|^2+\beta^2\right)}{\ln\beta^2}\,,\quad I(\tau,\bar\tau)\Big|_{\mathrm{Im}(\tau)\to\infty} \approx \frac{4\pi}{\ln(\beta^2)}\,\mathrm{Im}(\tau) \equiv c\,\mathrm{Im}(\tau)\,,$$

and construct an inflation potential:

$$V(\tau,\bar\tau) = V_0\left(\frac{I(\tau,\bar\tau)-1}{I(\tau,\bar\tau)+1}\right)^2\,,\quad V(\tau,\bar\tau)\Big|_{\mathrm{Im}(\tau)\to\infty} \approx V_0\left(1-\frac{8}{c\,\mathrm{Im}(\tau)}+\ldots\right)$$

Other possibilities:

Starobinsky inflation: 2407.12081

Higgs-Modular Inflation: 2504.01622

Modular approach to lepton flavor problem

Assign different weight and representations under finite modular group Γ_N to supermultiplets, we choose A_4 as an example :

$$L \sim \mathbf{3}\,,\quad e^c \sim \mathbf{1}'\,,\quad \mu^c \sim \mathbf{1}'\,,\quad \tau^c \sim \mathbf{1}''\,,\quad N^c \equiv \{N_1^c, N_2^c, N_3^c\} \sim \mathbf{3}$$
$$k_L = 1\,,\quad k_{e^c} = 1\,,\quad k_{\mu^c} = 5\,,\quad k_{\tau^c} = 5\,,\quad k_N = 1\,.$$

We use a Type-I seesaw to generate neutrino masses:

$$\begin{aligned}\mathcal{W}_\nu \;&=\; y_1^D\left((LN^c)_{\mathbf{3}_S}Y_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}}H_u + y_2^D\left((LN^c)_{\mathbf{3}_A}Y_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}}H_u \\ &\quad + y_1^N\Lambda\left((N^cN^c)_{\mathbf{3}_S}Y_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}}\,.\end{aligned}$$

Expanding the modulus field around its minimum gives the mass matrix and coupling among the fields. Has a best fit value(on the arc):

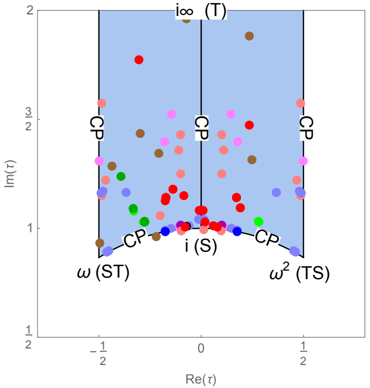
$$\tau_0 = 0.48 + 0.87i$$

Modular approach to lepton flavor problem

vev of τ in various models

.

Feruglio, Ferruccio, 2211.00659



A small deviation of fixed point can be used as the Froggatt-Nielsen charge.

P. P. Novichkov, J. T. Penedo, S. T. Petcov, 2102.07488

Modular approach to lepton flavor problem

Vacuum of the potential:

$$H(\tau) = (j(\tau) - 1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau)) \,, \quad m, n \in \mathbb{N} \,,$$

Make an analogy:

$$H(\tau) = (j(\tau) - j(\tau_0))^2 \left(1 + \beta \left(1 - \frac{j(\tau)}{1728}\right) + \gamma \left(1 - \frac{j(\tau)}{1728}\right)^2\right) \,,$$

For large field: $I(\tau, \bar{\tau}) = \frac{\ln\left(|j(\tau)|^2 + \beta^2\right)}{\ln\left(|j(\tau_0)|^2 + \beta^2\right)}$

Minkowski minimum at $\tau = \tau_0$, Taylor expansion of Canonical field ϕ:
Zero Order: Right handed neutrino (RHN) mass, Yukawa coupling.
First Order: Interactions between Inflaton and other fields.

$$\Gamma(\phi \to N_i^c N_j^c) = \left(\frac{m_N \lambda_1^{ij}}{M_{pl}}\right)^2 \frac{(m_\phi^2 - 4m_{N_i}^2)^{3/2}}{8\pi m_\phi^2}$$

$$\Gamma(\phi \to \tilde{N}_i^c \tilde{N}_j^c) = \left(\frac{(m_N)^2 \lambda_2^{ij}}{M_{pl}}\right)^2 \frac{1}{8\pi m_\phi} \left(1 - \frac{4m_{N_i}^2}{m_\phi^2}\right)$$

Reheating and leptogenesis

Consider all possible channels:

$$\Gamma_{\rm tot} = \Gamma(\phi \to N_i^c N_j^c) + \Gamma(\phi \to \tilde{N}_i^c \tilde{N}_j^c) + \Gamma(\phi \to L_i H N_j^c) + \Gamma(\phi \to \text{others})$$

The reheating temperature:

$$T_{\rm reh} \sim \left(\frac{90}{g_* \pi^2}\right)^{\frac{1}{4}} \sqrt{M_{\rm pl}\Gamma_{\rm tot}}$$

Non-thermal production of RHN can generate lepton asymmetry:

$$\frac{n_L}{s} \simeq \frac{3}{4}\frac{T_{\rm rh}}{m_\phi}\sum_i \epsilon_i \times (2\text{BR}_2 + \text{BR}_1)\ , \quad \frac{n_B}{s} \simeq -\frac{8}{23}\frac{n_L}{s} \simeq 8.6 \times 10^{-11}$$

where the ϵ_i measures the asymmetry in the right handed neutrino decays:

$$\epsilon_i = \frac{\Gamma(N_i \to H_u + L) - \Gamma(N_i \to \overline{H}_u + \overline{L})}{\Gamma(N_i \to H_u + L) + \Gamma(N_i \to \overline{H}_u + \overline{L})}$$

Reheating and Leptogenesis

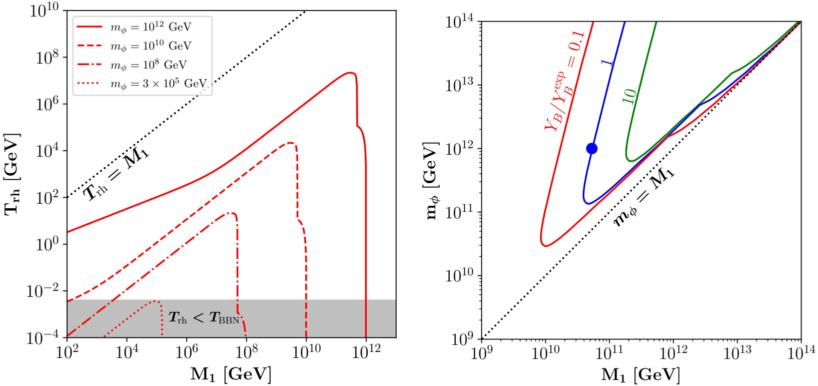
Required lepton asymmetry:

$$\frac{n_L}{s} \simeq \frac{3}{4}\frac{T_{\rm rh}}{m_\phi}\sum_i \epsilon_i \times (2{\rm BR}_2 + {\rm BR}_1) \simeq 2.5\times 10^{-10}\,,$$

Several points to get the target asymmetry:

- We need sphaleron process to be active: $T_{\rm reh} > 100$ GeV
- Non-thermal RHNs: $T_{\rm reh} < m_N$
  otherwise we have to take account the wash-out effects.
- For this model, Degenerate RHNs $m_{N_2} = 1.03\, m_{N_1}$,
  enhancement in ϵ_i, resonant leptogenesis [hep-ph/0309342]
- Need $m_\phi > 10^{11}$GeV, without introducing additional coupling.

Reheating and Leptogenesis



Caveat:

We have fixed the expansion direction along the arc; the results do not directly apply to other cases.

The inflation mass is not high enough for leptogenesis for our specific inflation model.

Summary

- It is interesting to combine modular symmetry with inflation.
- Modular symmetry is a strong constraint as well as a useful handle.
- Three parameter sets: $A(S, \bar{S})$, (m, n), (β, γ).
- Two inflationary trajectories: Along the arc or left boundary.
- One Baryon asymmetry: Non-thermal RHNs decay.
- More to explore:
  - Maybe fine-tuned. A more natural way (multi-field inflation)?
  - Inside the fundamental domain?
  - Other inflationary models?
    1604.02995, 2303.02947, 2403.02125, 2407.12081, 2408.05203
  - Dynamics of dilaton field?
    Stabilization at a higher energy scale.
  - DM in the framework?
    1904.03937, 2108.09984, 2409.02178.
  - CP-problem?
    2305.08908, 2404.08032, 2406.01689, 2504.03506.

Summary

- It is interesting to combine modular symmetry with inflation.
- Modular symmetry is a strong constraint as well as a useful handle.
- Three parameter sets: $A(S,\bar{S})$, (m,n), (β,γ).
- Two inflationary trajectories: Along the arc or left boundary.
- One Baryon asymmetry: Non-thermal RHNs decay.
- More to explore:
  - Maybe fine-tuned. A more natural way (multi-field inflation)?
  - Inside the fundamental domain?
  - Other inflationary models?
    1604.02995, 2303.02947, 2403.02125, 2407.12081, 2408.05203
  - Dynamics of dilaton field?
    Stabilization at a higher energy scale.
  - DM in the framework?
    1904.03937, 2108.09984, 2409.02178.
  - CP-problem?
    2305.08908, 2404.08032, 2406.01689, 2504.03506.

Thanks for your attention!

Eisenstein series

The Eisenstein series $G_{2k}(\tau)$ of weight $2k$ for integer $k > 1$ is defined as:

$$G_{2k}(\tau) = \sum_{\substack{n_1,n_2\in\mathbb{Z}\\ n_1,n_2\neq(0,0)}} (n_1 + n_2\,\tau)^{-2k}\,,$$

and the Fourier series of Eisenstein series read:

$$G_{2k}(q) = 2\zeta(2k)\left(1 + c_{2k}\sum_{i=1}^{\infty}\sigma_{2k-1}(i)q^i\right)\,,$$

where the coefficients c_{2k} are given by

$$c_{2k} = \frac{(2\pi i)^{2k}}{(2k-1)!\zeta(2k)} = -\frac{-4k}{B_{2k}} = \frac{2}{\zeta(1-2k)}\,. \tag{1}$$

Here B_n are the Bernoulli numbers, $\zeta(z)$ is the Riemann's zeta function and $\sigma_p(n)$ is the divisor sum function,

$$\sigma_p(n) = \sum_{d|n} d^p\,. \tag{2}$$

j invariant

The Klein j-invariant function is a modular form of weight zero, defined in terms of Dedekind eta function and Eisenstein series as follows:

$$j(\tau) \equiv \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\eta^{24}(\tau)} = \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\Delta(\tau)}\,, \quad \Delta(\tau) \equiv \eta^{24}(\tau)\,,$$

For convenience, the q-expansion of j-function is given by

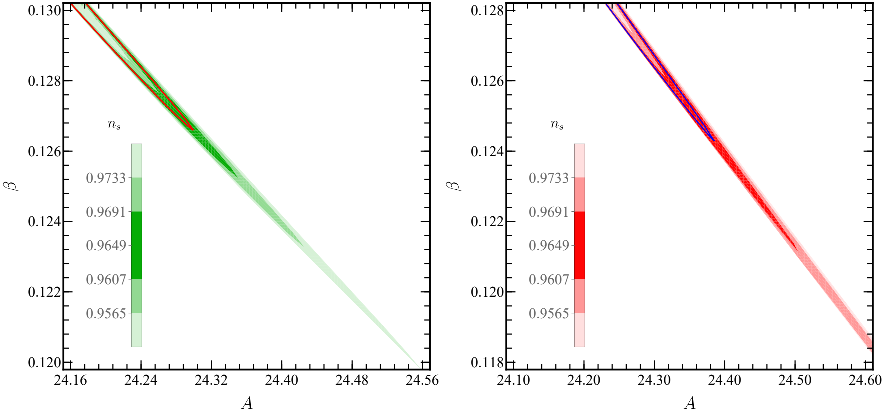
$$\begin{aligned} j(\tau) \;\; = \;\; & 744 + \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 \\ & + 20245856256q^4 + 333202640600q^5 + 4252023300096q^6 \\ & + 44656994071935q^7 + \mathcal{O}(q^8)\,. \end{aligned}$$

Vacuum structure of the potential

The vacuum structure of this potential at $\tau = i$ and at $\tau = \omega = e^{i2\pi/3}$ has been extensively studied in 2212.03876, where they find the following results based on the choice of (m, n):

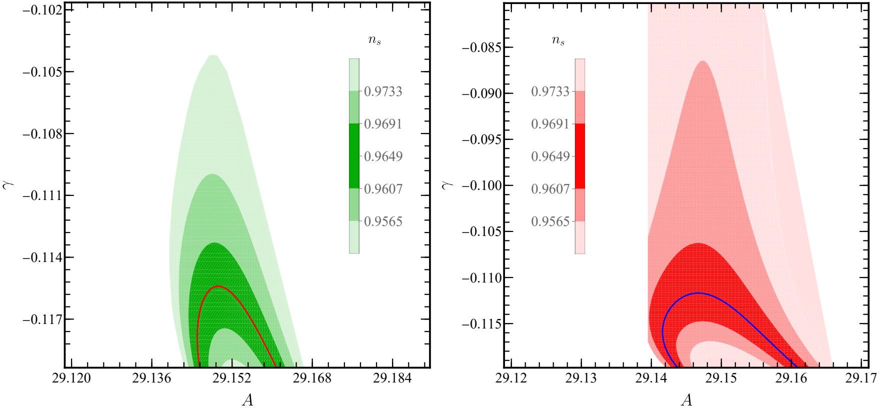
- If $m = n = 0$, then both fixed points can have a de Sitter (dS) vacuum.
- If $m > 1, n = 0$, then $\tau = \omega$ is a dS minimum, while $\tau = i$ is Minkowski minimum.
- If $m = 0, n > 1$, then $\tau = i$ is a conditional dS minimum, which depends on the value of $A(S, \bar{S})$. $\tau = \omega$ is always a Minkowski minimum.
- If $m = 1, n > 0$ or $n = 1, m > 0$, the vacuum is unstable.
- If $m > 1, n > 1$, then we always have Minkowski extrema in these two fixed points.

Slow roll along the unit arc



$$\mathcal{P}(j) = 1 + \beta(1 - j/1728)\,.$$

Slow roll along the unit arc



$$\mathcal{P}(j) = 1 + \gamma(1 - j/1728)\,.$$