Bubble-Wall Velocity from Hydrodynamics

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Outline

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Cosmological first order phase transitions

Let us consider theory of scalar field given by Lagrangian density:

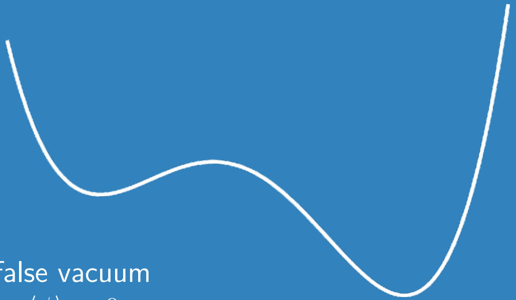
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi, T),$$

leading to the equation of motion in the form:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} = \frac{\partial V(\phi, T)}{\partial \phi},$$

where T is temperature.

Scalar effective potential $V(\phi, T)$



false vacuum
$\langle \phi \rangle = 0$

true vacuum
$\langle \phi \rangle \neq 0$

Cosmological first order phase transitions

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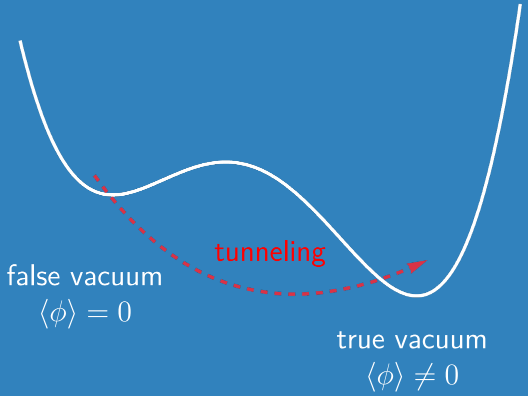


Tunneling bubbles

Nucleation rate:

$$\Gamma(T) = A(T) \cdot \exp\left(-S\right)$$

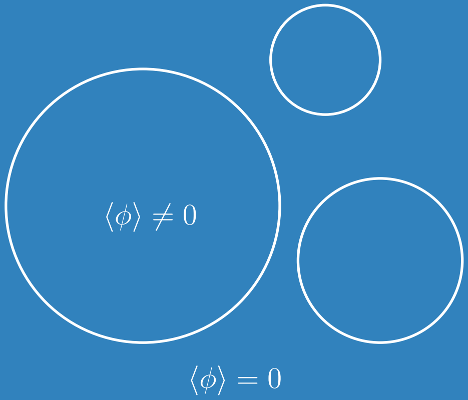
For tunneling in finite temperatures:

$$S = \frac{S_3}{T} \qquad A(T) = T^4 \left(\frac{S_3}{2\pi T}\right)^{\frac{3}{2}}$$

where S_3 is an action of $O(3)$-symmetric solution of the eom.

Nucleation condition:

$$\frac{\Gamma(T_n)}{H^4} \approx 1$$

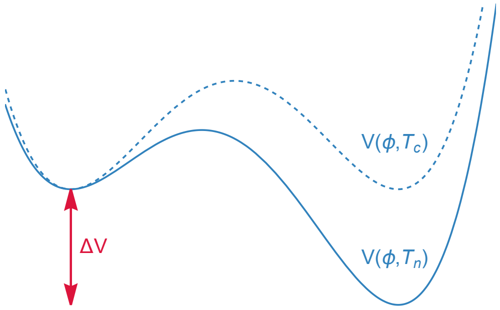


Phase transition parameters

- Critical and nucleation temperatures: T_c, T_n
- Level of supercooling: T_n/T_c
- Transition strength: $\alpha \sim \Delta V/\rho_r$
  In this work:

$$\alpha_{\bar{\theta}} = \frac{\Delta\bar{\theta}}{3w_s}, \qquad \text{with} \qquad \bar{\theta} = e - \frac{p}{c_b^2}$$

  with the speed of sound in the broken phase c_b and model-dependent energy e, pressure p and enthalpy w.
- Bubble-wall velocity: v_w



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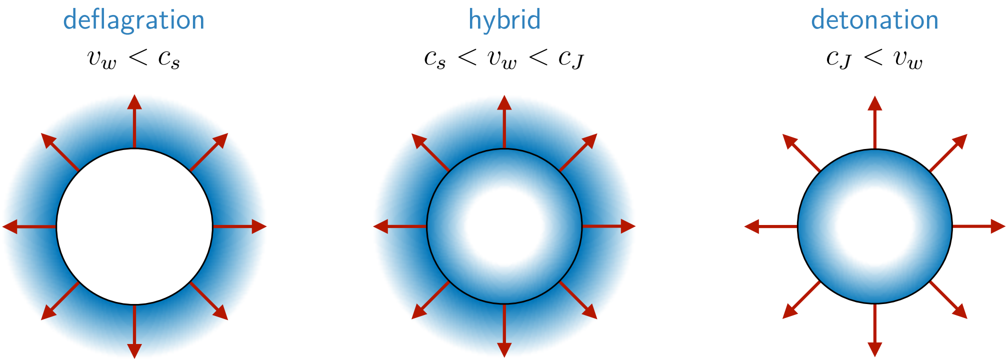
Expansion of bubbles

Different modes depending on bubble-wall velocity v_w and transition strength α_θ:



Where Jouget velocity is $c_J = \frac{1}{\sqrt{3}} \frac{1+\sqrt{3\alpha^2+2\alpha}}{1+\alpha}$.

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Dynamics of the steady state expansion

Integrated EoM of the growing bubble:

$$\int \mathrm{d}z \frac{\mathrm{d}\phi}{\mathrm{d}z}\left(\Box\phi + \frac{\partial V_{\mathrm{eff}}}{\partial\phi} + \sum_i \frac{\mathrm{d}m_i^2(\phi)}{\mathrm{d}\phi}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_i}\delta f_i(p,x)\right) = 0$$

$$\Big\downarrow \quad \frac{\mathrm{d}\phi}{\mathrm{d}z}\frac{\partial V_{\mathrm{eff}}}{\partial\phi} = \frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}z} - \frac{\partial V_{\mathrm{eff}}}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}z}$$

$$\Delta V_{\mathrm{eff}} = \boxed{\int \mathrm{d}z \frac{\partial V_{\mathrm{eff}}}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}z}} - \boxed{\sum_i \int \mathrm{d}\phi \frac{\mathrm{d}m_i^2(\phi)}{\mathrm{d}\phi}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_i}\delta f_i(p,x)}$$

driving force $=$ hydrodynamic backreaction $+$ non-equilibrium friction

- Boltzmann eq. $+$ EoM (different approaches: e.g fluid ansatz)
- LTE approximation (only backreaction) and beyond (including effective friction)
- Numerical simulations with effective friction η parametrizing δf

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Stationary profiles

Energy-momentum tensor for the plasma is given by

$$T^{\mu\nu} = \omega u^{\mu} u^{\nu} - g^{\mu\nu} p$$

Conservation of $T^{\mu\nu}$ along the flow and its projection orthogonal to the flow leads to

$$\partial_\mu (u^\mu \omega) - u_\mu \partial^\mu p = 0 \qquad \bar{u}^\nu u^\mu \omega \partial_\mu u_\nu - \bar{u}^\nu \partial_\mu p = 0.$$

Spherical symmetry + scale invariance:

$$u_\mu \partial^\mu = -\frac{\gamma}{t}(\xi - v)\partial_\xi \qquad \bar{u}_\mu \partial^\mu = \frac{\gamma}{t}(1 - \xi v)\partial_\xi$$

Hydrodynamic equation

$$2\frac{v}{\xi} = \gamma^2 (1 - v\xi) \left[\frac{\mu^2}{c_s^2} - 1 \right] \partial_\xi v,$$

with $\mu(\xi, v) = \frac{\xi - v}{1 - \xi v}$ and $\xi = r/t$.

Stationary profiles

Hydrodynamic equation

$$2\frac{v}{\xi} = \gamma^2(1 - v\xi)\left[\frac{\mu^2}{c_s^2} - 1\right]\partial_\xi v,$$

Matching equations

1 $\omega_-\gamma_-^2 v_- = \omega_+\gamma_+^2 v_+$

2 $\omega_-\gamma_-^2 v_-^2 + p_- = \omega_+\gamma_+^2 v_+^2 + p_+$

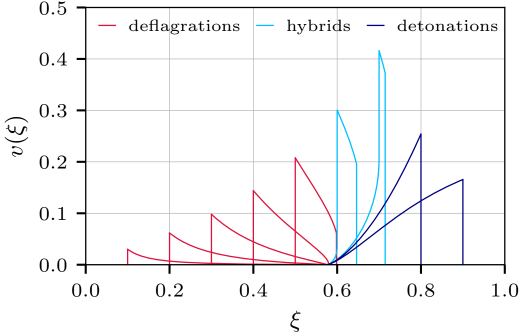
3 $s_-\gamma_- v_- = s_+\gamma_+ v_+$ (if $\delta f = 0$)

Bag equation of state

$\epsilon_s = 3a_s T_s^4 + \theta_s$ $\qquad$ $\epsilon_b = 3a_b T_b^4 + \theta_b$

$p_s = a_s T_s^4 - \theta_s$ $\qquad$ $p_b = a_b T_b^4 - \theta_b$

Solving hydrodynamic equation assuming bag model and proper boundary conditions (1 and 2), we get analytical profiles $v(\xi)$ depending on ξ_w, α_θ.



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LTE approximation $(\delta f = 0)$[1]

Matching method: conservation of entropy accross the bubble-wall (matching eq. 3)

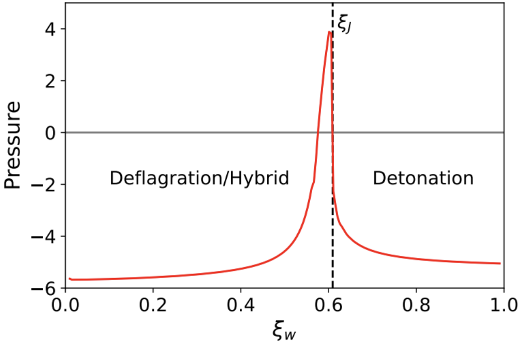


Figure: Total pressure acting on the wall

Evaluation of bubble-wall velocity based on

- transition strength α_θ
- enthalpy ratio $\Psi_N = \frac{\omega_b}{\omega_s}$
- speed of sound in the plasma c_s, c_b

Stationary state (deflagration or hybrid) can be typically found for not too large α_θ
There are no stable detonations in LTE.

[1]Wen-Yuan Ai, Benoit Laurent, Jorinde van de Vis, *Model-independent bubble wall velocities in local thermal equilibrium*, JCAP 07 (2023) 002

Beyond LTE approximation $(\delta f \neq 0)^2$

Entropy production:

$$\partial_\mu (u^\mu s) = \frac{\eta}{T} (u^\mu \partial_\mu \phi)^2$$

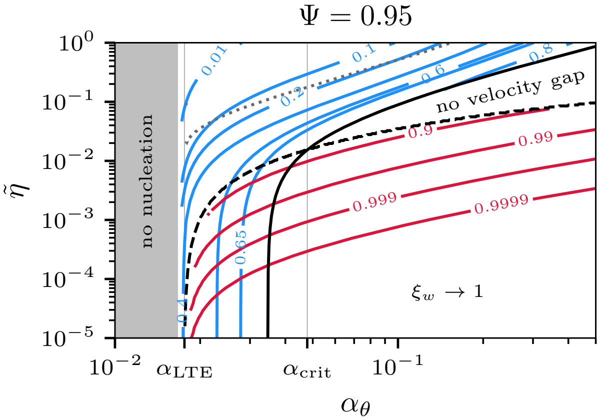
Integrating over the field profile

$$\phi(z,t) = \frac{v_0}{2} \left[1 - \tanh\left(\frac{z}{L_w} \right) \right]$$

we get new matching equation:

$$\frac{T_+}{T_-} = \frac{\gamma_-}{\gamma_+} \left(1 + \tilde{\eta}\, \gamma_+ v_+\right)$$

with $\tilde{\eta} \equiv \frac{v_0^2 \eta}{3 w_+ L_w}$



[2] T. Krajewski, M. Lewicki, I. Nałęcz, M. Zych
Steady-state bubbles beyond local thermal equilibrium, arXiv:2411.16580

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Outline

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Scalar fields and perfect fluid

The system consists of

- relativistic perfect fluid
- real scalar fields h, s.

The fields acquires a temperature-dependent effective potential V_{eff}.

Equation of state

$$p(h,s,T) = -V_{\text{eff}}(h,s,T),$$

$$e(h,s,T) = V_{\text{eff}}(h,s,T) - T\frac{\mathrm{d}V_{\text{eff}}(h,s,T)}{\mathrm{d}T},$$

$$w(h,s,T) = -T\frac{\mathrm{d}V_{\text{eff}}(h,s,T)}{\mathrm{d}T}.$$

Energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\text{field}} + T^{\mu\nu}_{\text{fluid}}$$

$$T^{\mu\nu}_{\text{field}} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi\right)$$

$$T^{\mu\nu}_{\text{fluid}} = wu^\mu u^\nu + g^{\mu\nu}p$$

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Equations of motion

Total energy-momentum tensor in conserved, but both contributions are not, due to the extra coupling term parametrized by effective non-equilibrium friction η

$$\nabla_\mu T^{\mu\nu}_{\text{field}} = \frac{\partial V_{\text{eff}}}{\partial \phi}\partial^\nu \phi + \eta u^\mu \partial_\mu \phi \partial^\nu \phi, \quad \nabla_\mu T^{\mu\nu}_{\text{fluid}} = -\frac{\partial V_{\text{eff}}}{\partial \phi}\partial^\nu \phi - \eta u^\mu \partial_\mu \phi \partial^\nu \phi$$

Local thermal equilibrium: $\eta = 0$

EoM - scalar fields

$$-\partial_t^2 h + \frac{1}{r^2}\partial_r (r^2 \partial_r h) - \frac{\partial V_{\text{eff}}}{\partial h} = \eta\gamma(\partial_t h + v\partial_r h)$$

$$-\partial_t^2 s + \frac{1}{r^2}\partial_r (r^2 \partial_r s) - \frac{\partial V_{\text{eff}}}{\partial s} = 0$$

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Equations of motion

Total energy-momentum tensor in conserved, but both contributions are not, due to the extra coupling term parametrized by effective non-equilibrium friction η

$$\nabla_\mu T^{\mu\nu}_{\text{field}} = \frac{\partial V_{\text{eff}}}{\partial \phi}\partial^\nu \phi + \eta u^\mu \partial_\mu \phi \partial^\nu \phi, \quad \nabla_\mu T^{\mu\nu}_{\text{fluid}} = -\frac{\partial V_{\text{eff}}}{\partial \phi}\partial^\nu \phi - \eta u^\mu \partial_\mu \phi \partial^\nu \phi$$

Local thermal equilibrium: $\eta = 0$

EoM - plasma

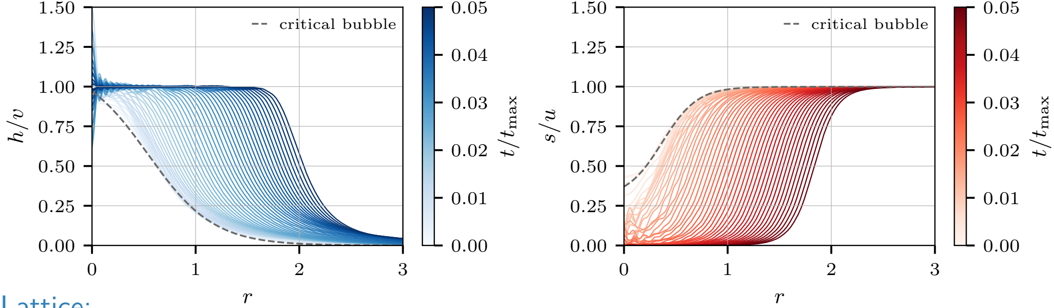
$$\partial_t \tau + \frac{1}{r^2}\partial_r (r^2(\tau + p)v) = \frac{\partial V_{\text{eff}}}{\partial h}\partial_t h + \frac{\partial V_{\text{eff}}}{\partial s}\partial_t s + \eta\gamma(\partial_t h + v\partial_r h)\partial_t h$$

$$\partial_t Z + \frac{1}{r^2}\partial_r \left(r^2 Z v\right) + \partial_r p = -\frac{\partial V_{\text{eff}}}{\partial h}\partial_r h - \frac{\partial V_{\text{eff}}}{\partial s}\partial_r s - \eta\gamma(\partial_t h + v\partial_r h)\partial_r h$$

where $Z := w\gamma^2 v$ and $\tau := w\gamma^2 - p$

Evolution: initial conditions

- Evolution is initialized with $h(r)$ and $s(r)$ profiles correspond to the critical bubble at nucleation temperature T_n.
- Plasma initially remains at rest $v(r) = 0$ with $T(r) = T_n$
- Bubble-wall quickly achieves constant velocity v_w

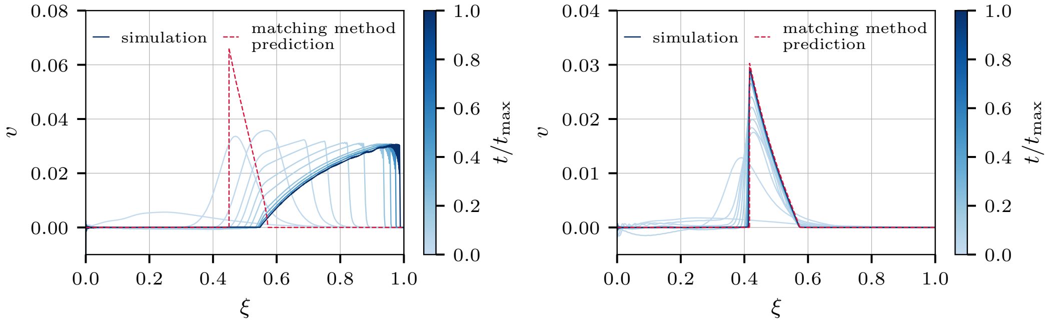


Lattice:

$t_{\max} = 100\ \mathrm{GeV}^{-1} \quad r_{\max} = ct_{\max} \quad \delta r = 10 \quad \delta t = 10^{-2}\ \mathrm{GeV}^{-1}$

Evolution: late stages $(\eta = 0)$

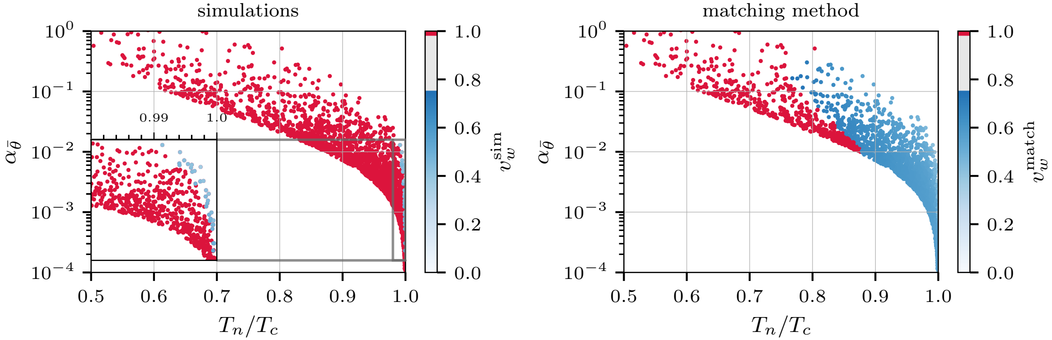
Self-similar profiles: $\xi = r/t$



Two possible scenarios for the growing bubble in LTE:

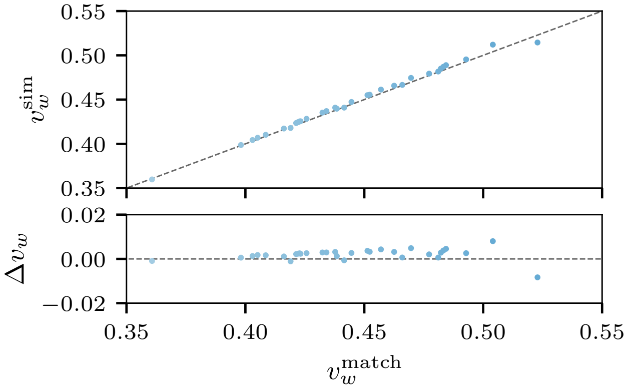
- rapid expansion beyond Chapman-Jouguet velocity leading to a runaway scenario
- evolution toward a stationary state predicted by matching conditions

Analytical treatment vs real-time simulations $(\eta = 0)$



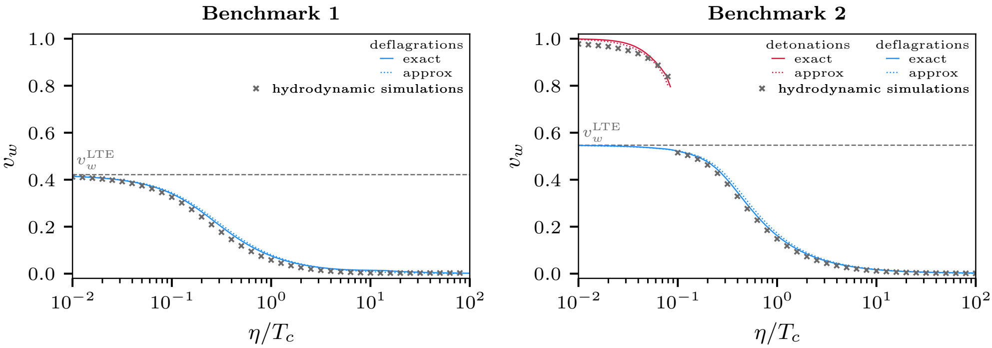
While matching equations predict significant number of stationary deflagrations and hybrids, in real-time simulations only few indeed evolve towards stationary state.

Analytical treatment vs real-time simulations $(\eta = 0)$



If the stationary state is achieved for a given model, bubble-wall velocity is very accurately predicted by the matching equations.

Analytical treatment vs real-time simulations $(\eta \neq 0)$[3]

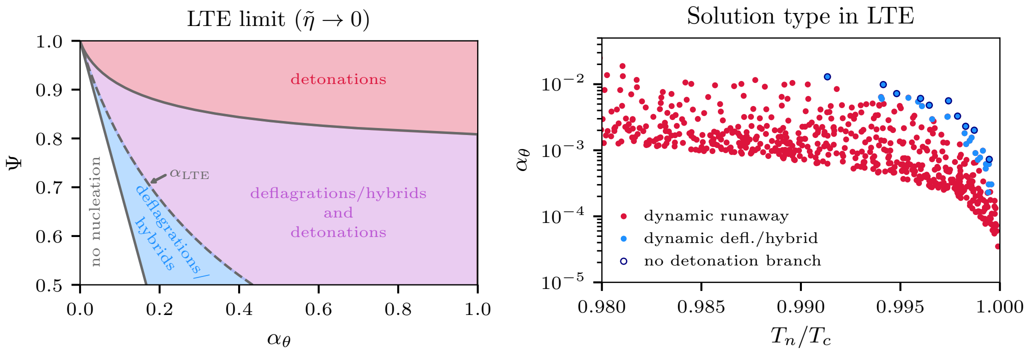


New matching equation allows to determine bubble-wall velocity as a function of η.

[3]T. Krajewski, M. Lewicki, I. Nałęcz, M. Zych
Steady-state bubbles beyond local thermal equilibrium, arXiv:2411.16580

LTE selection rule

$$\alpha_\theta > \alpha_{\rm LTE} \implies \text{detonation (runaway)}$$
$$\alpha_\theta < \alpha_{\rm LTE} \implies \text{deflagration or hybrid}$$



Presence of the detonation branch explains the runaway behaviour in the LTE limit.

Outline

Conclusions

We investigated fluid solutions in the presence of growing bubbles of the scalar field in cosmological FOPT using numerical lattice simulations:

- Without non-equilibrium friction bubbles generically expand as runaways.
- Stationary profiles are dynamically achieved only for tiny supercooling ($T_n/T_c \lesssim 1$).
- If steady state is achieved, it matches to equilibrium prediction with high precision.
- Beyond LTE matching condition correctly predicts bubble-wall velocity as a function of entropy production.
- Presence of the detonation branch may serve as a selection rule for the LTE results.

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- Presence of the detonation branch may serve as a selection rule for the LTE results.

Thank you for your attention!

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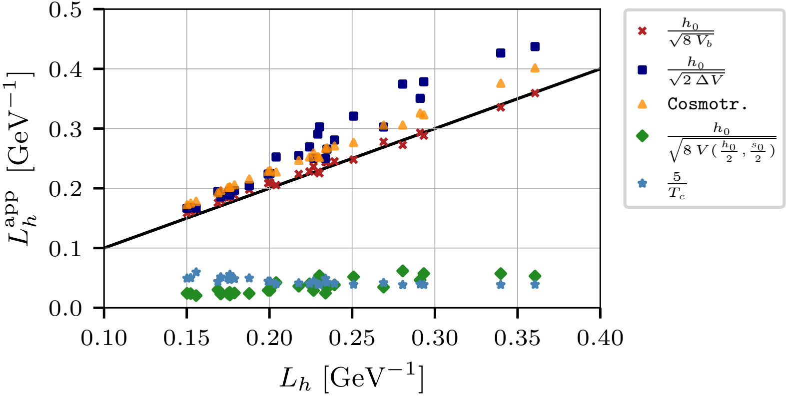
Bubble wall width

We suggest to use the following approximation:

$$L_w = \frac{v_0}{\sqrt{8 V_b}},$$

with the barrier high V_b evaluated at critical temperature T_c, on the tunneling path.



Spatial discretization

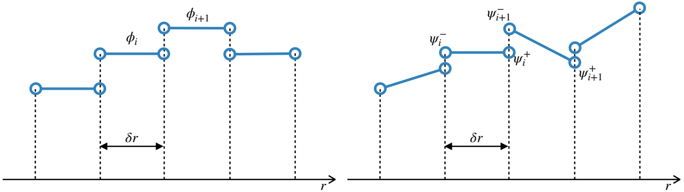
Spatial discretization of the field: discontinous Galerkin method

$$0 = \forall_q \int_0^\infty dr r^2 f(r,\phi(r),\psi(r))q \quad \rightarrow \quad \forall_q \sum_{i=0}^{N-1} \int_{r_i}^{r_{i+1}} dr r^2 f(r,\phi(r),\psi(r))q$$

with an auxiliary variable $\psi := \partial_r \phi$. We introduce following interpolations:

$$\phi(t,r)\big|_{I_i} = \phi_i(t)$$

$$\psi(t,r)\big|_{I_i} = \psi_i^+(t)\frac{r_{i+1}-r}{r_{i+1}-r_i} + \psi_{i+1}^-(t)\frac{r-r_i}{r_{i+1}-r_i}.$$



Boundary conditions:

$$\psi|_{r=0} = 0$$

$$\phi|_{r=\infty} = 0$$

Similar approach for the thermodynamic variables: Z, τ

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Temporal discretization for the fields

Temporal discretization of the fields: Strömer-Verlet method

$$\phi_{i,j+1/2} = \phi_{i,j} + \frac{1}{2}\delta t \dot{\phi}_{i,j}$$

$$\dot{\phi}_{i,j+1} = \dot{\phi}_{i,j} - \delta t \left(\frac{\partial V}{\partial \phi}(\phi_{i,j+1/2}) - \Delta_d \phi_{i,j+1/2} \right)$$

$$\phi_{i,j+1} = \phi_{i,j+1/2} + \frac{1}{2}\delta t \dot{\phi}_{i,j+1}$$

Can be interpreted as discontinuous Galerkin method in time.

Temporal discretization for plasma

For high order spacial discretization we use explicit midpoint method:

$$U_{i,j+1/2} = U_{i,j} + \frac{t_{j+1} - t_j}{2} \left[\mathcal{F}_{i+1/2}(U_{\cdot,j}) - \mathcal{F}_{i-1/2}(U_{\cdot,j}) + \mathcal{G}(U_{i,j}, \phi_{i,j}, \nabla_d \phi_{i,j}, \dot{\phi}_{i,j}, r_i)\right],$$

$$U_{i,j+1} = U_{i,j} + (t_{j+1} - t_j) \Big[\mathcal{F}_{i+1/2}(U_{\cdot,j+1/2}) - \mathcal{F}_{i-1/2}(U_{\cdot,j+1/2}) + \mathcal{G}\left(U_{i,j+1/2}, \phi_{i,j+1/2}, \nabla_d \phi_{i,j+1/2}, \dot{\phi}_{i,j+1/2}, r_i\right)\Big].$$

Temporal discretization of low order scheme: forward/backward Euler method:

$$U_{i,j+1} = U_{i,j} + \delta t \Big[\theta \left(\mathcal{F}_{i+1/2}(U_{\cdot,j+1}) - \mathcal{F}_{i-1/2}(U_{\cdot,j+1}) + \mathcal{G}(U_{i,j+1})\right) + (1-\theta) \left(\mathcal{F}_{i+1/2}(U_{\cdot,j}) - \mathcal{F}_{i-1/2}(U_{\cdot,j}) + \mathcal{G}(U_{i,j})\right)\Big]$$

We use implicit ($\theta = 1$) for low order spacial discretization since explicit one ($\theta = 0$) turned out to be unstable.

Flux corrected transport

1. Compute F^L using low order method guaranteed not to generate unphysical values.
2. Compute F^H using high order method accurate in smooth regions of the solution.
3. Compute the "antidiffusive fluxes":

$$A = F^H - F^L$$

4. Compute numerical solution U^L with low order method.
5. Limit the "antidiffusive fluxes":

$$A_\alpha = \alpha A, \qquad 0 \leq \alpha \leq 1$$

   such that $\alpha \sim 1$ in the smooth regions of the solution and $\alpha \sim 0$ around shocks.
6. Apply the limited "antidiffusive fluxes" to U^L in order to obtain final solution reproducing high order scheme in the smooth regions of the solution.