Revisiting the Gravitational Two-Body Problem in the Amplitudes Framework

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Based on:

Background material:

- 2306.16488: Report on the gravitational eikonal
  Paolo Di Vecchia, CH, Rodolfo Russo, Gabriele Veneziano
- 2312.07452, 2402.06361: Analysis of the NLO waveform
  In collaboration with Alessandro Georgoudis, CH, Rodolfo Russo
- 2406.03937: Angular momentum losses from the NLO waveform
  CH, Rodolfo Russo

New results:

- 2501.02904: Radiation-Reaction and Angular Momentum Loss at $\mathcal{O}(G^4)$
  CH

Outline

Introduction

Elastic Eikonal and Deflection Angle

Eikonal Operator and Gravitational Waveform

Energy and Angular Momentum Losses

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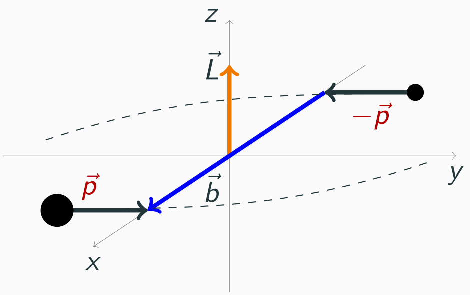
Two-Body Problem: Analytical Approximation Methods

- Post-Newtonian (PN): expansion
  "for small G and small v"
  $$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1\,.$$
- Post-Minkowskian (PM): expansion
  "for small G"
  $$\frac{Gm}{rc^2} \ll 1\,, \qquad \text{generic } \frac{v^2}{c^2}\,.$$



- Self-Force: expansion
  in the near-probe limit $m_2 \ll m_1$ or
  $$m = m_1 + m_2\,, \qquad \nu = \frac{m_1 m_2}{m^2} \ll 1\,.$$
- Soft limit: expansion
  in the limit of small frequencies
  $$\omega \ll \frac{v}{r}\,.$$

Key Idea: Extract the PM gravitational dynamics from scattering amplitudes.

- Weak-coupling expansion $\leftrightarrow$ PM expansion

Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\mathcal{A}_3 = \mathcal{O}(G^4)$
PM:	1PM	2PM	3PM	4PM State of the art: [[Driesse et al. '24; Bern et al. '24] 5PM, 1SF from WQFT]

- Lorentz invariance $\leftrightarrow$ generic velocities
- Study scattering events, then export to bound trajectories (V_{eff}, analytic continuation...) [Kälin, Porto '19; Saketh, Steinhoff, Vines, Buonanno '21; Cho, Kälin, Porto '21]

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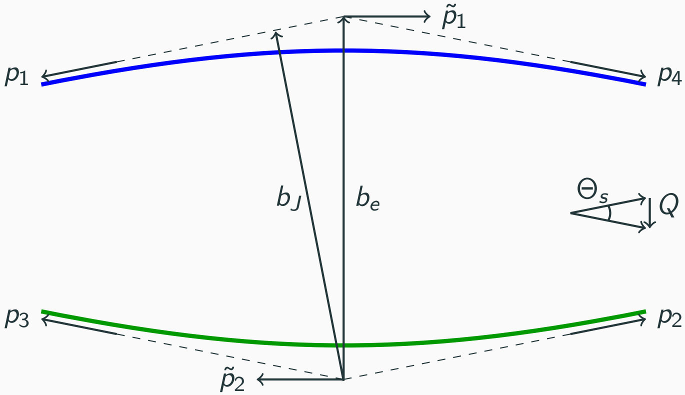
Kinematics of Classical Post-Minkowskian (PM) Scattering

$$\tilde{p}_1^\mu = m_1 \tilde{u}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$

$$\tilde{p}_2^\mu = m_2 \tilde{u}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$

$$Q^\mu = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$

$$b_e^\mu = b_J^\mu - \left(\frac{\check{v}_1^\mu}{2m_1} - \frac{\check{v}_2^\mu}{2m_2}\right) Qb$$



In this way, $v_1 \cdot b_J = v_2 \cdot b_J = 0$ and $\tilde{u}_1 \cdot b_e = \tilde{u}_2 \cdot b_e = 0$. Classical PM regime:

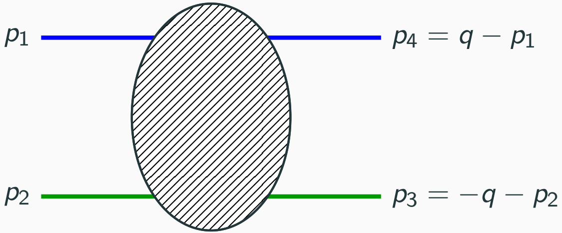
$$\frac{Gm^2}{\hbar} \underset{CL}{\gg} 1\,, \qquad \frac{Gm}{b} \underset{\mathrm{PM}}{\ll} 1\,, \qquad \boxed{\frac{\hbar}{m} \ll Gm \ll b} \qquad \sigma = \frac{1}{\sqrt{1-v^2}} \geq 1 \text{ (generic).}$$

Kinematics of the Elastic $2 \to 2$ Amplitude

$$\bar{p}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$

$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$

$$\boxed{q^\mu} = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$



Defining velocities by $p_1^\mu = -m_1 v_1^\mu$, $p_2^\mu = -m_2 v_2^\mu$

$$\boxed{\sigma} = -v_1 \cdot v_2 = \frac{1}{\sqrt{1 - v^2}}$$

with v the speed of either object as measured by the other one.

Dual velocities: $v_1^\mu = \sigma \check{v}_2^\mu + \check{v}_1^\mu$, $v_2^\mu = \sigma \check{v}_1^\mu + \check{v}_2^\mu$ obey $\check{v}_i \cdot v_j = -\delta_{ij}$.

The Elastic Eikonal

- From q to b: Fourier transform $[q \sim \mathcal{O}(\frac{\hbar}{b})]$

$$\tilde{\mathcal{A}}^{(4)}(b) = \frac{1}{4Ep} \int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q} \mathcal{A}^{(4)}(q)\,, \qquad \boxed{1 + i\tilde{\mathcal{A}}^{(4)}(b) = e^{2i\delta(b)}}$$

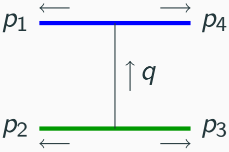
with $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots \sim \frac{Gm^2}{\hbar} \left(\log b + \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \cdots\right)$

- From b to Q: stationary-phase approximation $[Q \sim \mathcal{O}(p \cdot \frac{Gm}{b})]$

$$\int d^{D-2}b\, e^{-ib\cdot Q} e^{i2\delta(b)} \implies Q_\mu = \frac{\partial \operatorname{Re} 2\delta}{\partial b_e^\mu}$$

Tree-Level Amplitude and 1PM Impulse

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



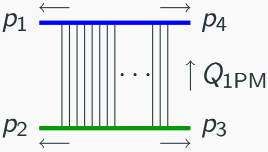
$$\mathcal{A}_0^{(4)}(q) = \frac{32\pi G m_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0^{(4)}(b) = \frac{4 G m_1 m_2 (\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2 - 1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} .$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al. '18]

$$e^{2i\delta_0} \underset{\text{“small } G\text{”}}{\longrightarrow} 1 + i\tilde{\mathcal{A}}_0^{(4)} \implies 2\delta_0 = \tilde{\mathcal{A}}_0^{(4)} .$$

- From $2\delta_0$, we obtain the leading-order deflection

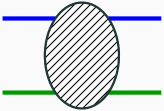


$$Q_{1\mathrm{PM}} = -\frac{\partial 2\delta_0}{\partial b} = \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2 - 1}}$$

$$\Theta_{1\mathrm{PM}} = \frac{4 G E \left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2 - 1)} .$$

Impulse from the Eikonal Phase up to One Loop

[Collado, Di Vecchia, Russo '19]

$1 + i\,\mathrm{FT}$  $\sim e^{2i\delta}\,, \qquad 2\delta = 2\delta_0 + 2\delta_1 + \cdots \qquad Q_\mu = \dfrac{\partial 2\delta}{\partial b_e^\mu}$

- Tree level: $i\tilde{\mathcal{A}}_0 = 2i\delta_0$, so

$$2\delta_0 = \tilde{\mathcal{A}}_0^{(4)} = \frac{2Gm^2\nu(\sigma^2 - \frac{1}{2-2\epsilon})}{\sqrt{\sigma^2-1}}\frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}}\,, \qquad Q^\mu_{\mathrm{1PM}} = -\frac{4Gm^2\nu(\sigma^2 - \frac{1}{2})}{b\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

- One loop: By the unitarity, $i\tilde{\mathcal{A}}_1 - \frac{1}{2!}(2i\delta_0)^2 = i\,\mathrm{Re}\,\tilde{\mathcal{A}}_1 = 2i\delta_1$, so

$$2\delta_1 = \mathrm{Re}\,\tilde{\mathcal{A}}_1^{(4)} = \frac{3\pi G^2 m^3\nu\left(5\sigma^2-1\right)}{4b\sqrt{\sigma^2-1}}\,, \qquad Q^\mu_{\mathrm{2PM}} = -\frac{3\pi G^2 m^3\nu\left(5\sigma^2-1\right)}{4b^2\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

The 3PM Eikonal in General Relativity [Di Vecchia, CH, Russo, Veneziano '20, '21]

[Related work at 3PM: Bern, Cheung, Roiban, Shen, Solon, Zeng '19; Damour '20; Herrmann, Parra-Martinez, Ruf, Zeng '21, Bjerrum-Bohr, Damgaard, Planté, Vanhove '21; Brandhuber, Chen, Travaglini, Wen '21]

- Eikonal phase:

$$\mathrm{Re}\, 2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left[\frac{s\left(12\sigma^4-10\sigma^2+1\right)}{2m_1 m_2\left(\sigma^2-1\right)^{\frac{3}{2}}}-\frac{\sigma\left(14\sigma^2+25\right)}{3\sqrt{\sigma^2-1}}-\frac{4\sigma^4-12\sigma^2-3}{\sigma^2-1}\,\mathrm{arccosh}\sigma\right]$$
$$+\,\mathrm{Re}\, 2\delta_2^{\mathrm{RR}}\,,$$

$$\mathrm{Re}\, 2\delta_2^{\mathrm{RR}} = \frac{G}{4} Q_{1\mathrm{PM}}^2 \mathcal{I}(\sigma)\,, \quad \mathcal{I}(\sigma) \equiv \frac{2(8-5\sigma^2)}{3(\sigma^2-1)} + \frac{2\sigma\left(2\sigma^2-3\right)}{(\sigma^2-1)^{3/2}}\,\mathrm{arccosh}\,\sigma\,.$$

- Infrared divergent exponential suppression:

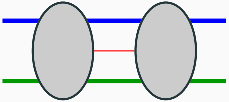
$$\mathrm{Im}\, 2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon}+\log(\sigma^2-1)\right]\mathrm{Re}\, 2\delta_2^{\mathrm{RR}} + \cdots$$

- $\mathrm{Re}\, 2\delta_2^{\mathrm{RR}}$ contributes half-odd-PN corrections (odd in velocity) to $\Theta_{3\mathrm{PM}}$

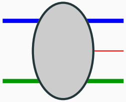
Unitarity and Analyticity Fix the Radiation-Reaction Contribution

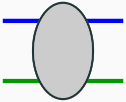
[Di Vecchia, CH, Russo, Veneziano '21]

- Unitarity determines the imaginary part of the two-loop eikonal,

$$2\,\mathrm{Im}\,2\delta_2 = \mathrm{FT}$$


- IR divergence comes from low frequencies, use the soft graviton theorem:



$$\sim \sqrt{8\pi G}\sum_a \frac{p_a^\mu p_a^\nu}{p_a\cdot k}$$


as $k^\alpha \to 0$

- Then, using the natural upper cutoff $\omega^* \simeq \frac{v}{b}$, we find

$$\mathrm{Im}\,2\delta_2 = \frac{G}{2\pi}\left[-\frac{1}{2\epsilon} + \log\sqrt{\sigma^2-1}\right] Q_{1\mathrm{PM}}^2\,\mathcal{I}(\sigma) + \cdots$$

- By analyticity, $i\log(1-\sigma^2-i0) = i\log(\sigma^2-1)+\pi$, hence

$$\mathrm{Re}\,2\delta_2^{\mathrm{RR}} = \lim_{\epsilon\to 0}\left[-\pi\epsilon\,\mathrm{Im}\,2\delta_2\right] = \frac{G}{4}\,Q_{1\mathrm{PM}}^2\,\mathcal{I}(\sigma)\,.$$

Smoothness and Universality of $\mathrm{Re}\, 2\delta_2$ at High Energy [Di Vecchia, CH, Russo, Veneziano '20, '21]

At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit:

- The *complete* eikonal phase is smooth, although the conservative and radiation-reaction parts separately diverge like $\log \sigma$
- Its expression is the same in $\mathcal{N} = 8$ supergravity and in GR,

$$\mathrm{Re}\, 2\delta_2 \sim Gs\, \frac{\Theta_s^2}{4}\,, \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

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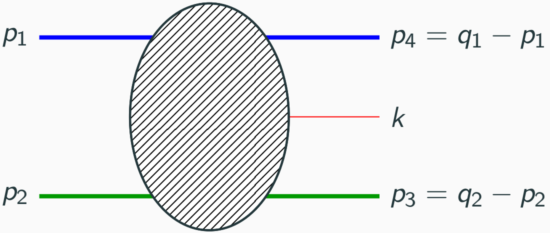
$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$
$$\boxed{q_1^\mu} = p_1^\mu + p_4^\mu$$
$$\boxed{q_2^\mu} = p_2^\mu + p_3^\mu$$
$$0 = q_1^\mu + q_2^\mu + k^\mu$$



More invariants, besides q_1^2, q_2^2, also

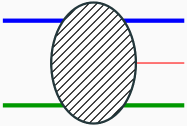
$$\boxed{\sigma} = -v_1 \cdot v_2\,, \qquad \boxed{\omega_1} = -v_1 \cdot k\,, \qquad \boxed{\omega_2} = -v_2 \cdot k\,.$$

We denote by E, ω the total energy and the graviton frequency in the CoM frame,

$$E = \sqrt{-(p_1 + p_2)^2}\,, \qquad \omega = \frac{1}{E}\,(p_1 + p_2) \cdot k = \frac{1}{E}(m_1\omega_1 + m_2\omega_2)\,, \qquad \alpha_{1,2} = \frac{\omega_{1,2}}{\omega}\,.$$

$2 \to 3$ Amplitude up to One Loop

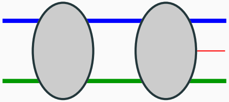
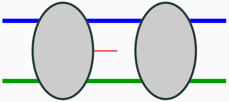
[Brandhuber et al. '23; Herderschee, Roiban, Teng 23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23] [Georgoudis, CH, Vazquez-Holm '23]

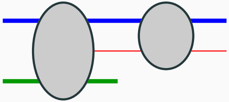
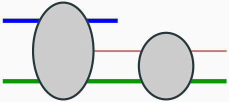
$\mathcal{A} =$  $= \mathcal{A}_0 + \mathcal{A}_1 + \cdots$

with $\mathcal{A}_0$ the tree-level amplitude, and

$$\mathcal{A}_1 = \mathcal{B}_1 + \frac{i}{2}(s + s') + \frac{i}{2}(c_1 + c_2)\,.$$

where $\mathcal{B}_1 = \mathrm{Re}\,\mathcal{A}_1$ and the unitarity cuts can be depicted as follows,

$s =$  $s' =$ 

$c_1 =$  $c_2 =$ 

Inelastic Final State [Di Vecchia, CH, Russo, Veneziano '22]

[cf. Kosower, Maybee, O'Connell '18; Damgaard, Planté, Vanhove '21; Cristofoli et al. '21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation:

$$e^{2i\hat{\delta}(b_1,b_2)} = e^{i\,\mathrm{Re}\,2\delta(b)}e^{i\int_k\left[\tilde{W}(k)a^\dagger(k)+\tilde{W}^*(k)a(k)\right]}\,.$$

- Final state, schematically:

$$|\mathrm{out}\rangle = e^{2i\hat{\delta}(b_1,b_2)}|\mathrm{in}\rangle$$

- Unitarity:

$$\langle\mathrm{out}|\mathrm{out}\rangle = \langle\mathrm{in}|\mathrm{in}\rangle = 1$$

- The asymptotic metric fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ sourced by the scattering (the waveform) is expressed formally as

$$h_{\mu\nu}(x) = \sqrt{32\pi G}\,\langle\mathrm{out}|\hat{H}_{\mu\nu}(x)|\mathrm{out}\rangle \sim \frac{4G}{\kappa r}\int_0^\infty e^{-i\omega U}\,\tilde{W}_{\mu\nu}(\omega n)\,\frac{d\omega}{2\pi} + (\mathrm{c.c.})$$

  where $\kappa = \sqrt{8\pi G}$, r is the distance from the observer and U the retarded time. Normalization $\tilde{W}^{\mu\nu} = \kappa\,\tilde{w}^{\mu\nu}$.

The Waveform Kernel

[See e.g. Herderschee, Roiban, Teng '23]

- Working with "eikonal" variables, we can use the following radiation kernel,

$$W = \mathcal{A}_0 + \left[\mathcal{B}_1 + \tfrac{i}{2}\left(c_1 + c_2\right)\right].$$

- Tree level: $\mathcal{A}_0$ is a relatively simple rational function
- One loop: We isolate the even and odd parts of $\mathcal{B}_1$ under $\omega_{1,2} \mapsto -\omega_{1,2}$,

$$\mathcal{B}_1 = \mathcal{B}_{1O} + \mathcal{B}_{1E}\,,$$

and $\mathcal{B}_{1O} = \mathcal{B}_{1O}^{(h)} + \mathcal{B}_{1O}^{(i)}$ is fixed by unitarity and analyticity in terms of the tree-level amplitude,

$$\mathcal{B}_{1O}^{(h)} = \pi G E \omega\, \mathcal{A}_0\,, \qquad \mathcal{B}_{1O}^{(i)} = -\frac{\sigma\left(\sigma^2 - \frac{3}{2}\right)}{(\sigma^2-1)^{3/2}}\,\pi G E \omega\, \mathcal{A}_0$$

while $\mathcal{B}_{1E}$ and c_1, c_2 represent new one-loop data.

Infrared Divergences Revisited

[Georgoudis, CH, Russo '24] [Bini et al. '24]

- IR divergences due to c_1, c_2,

$$\frac{i}{2}\,c_1 = 2iGm_1\omega_1\left(-\frac{1}{2\epsilon}+\log\frac{\omega_1}{\mu}\right)\mathcal{A}_0+\frac{i}{2}\,c_1^{(\text{reg})}$$

exponentiate in momentum space,

$$W = e^{-\frac{i}{\epsilon}GE\omega}\left[\mathcal{A}_0+\mathcal{B}_1+\tfrac{i}{2}\,\mathcal{C}\right] = e^{-\frac{i}{\epsilon}GE\omega}\,W^{\text{reg}}\,,$$

where $\frac{i}{2}\,\mathcal{C} = \sum_{a=1,2}\left(2iGm_a\omega_a\log\frac{\omega_a}{\mu}+\frac{i}{2}\,c_a^{(\text{reg})}\right)$

- The divergence can be canceled by redefining the origin of retarded time

$$h_{\mu\nu}(x)\sim\frac{4G}{\kappa r}\int_0^\infty e^{-i\omega U}\,\tilde{W}^{\text{reg}}_{\mu\nu}(\omega n)\,\frac{d\omega}{2\pi}+(\text{c.c.})$$

- It **resums velocity corrections** to the Einstein quadrupole formula up to $\mathcal{O}(G^3)$:
  - $\mathcal{A}_0$, $\mathcal{B}_{1O}^{(h)}$ and $\mathcal{B}_E$ give integer PN corrections (even powers of v)
  - $\mathcal{B}_{1O}^{(i)}$ and $c_1^{(\text{reg})}$, $c_2^{(\text{reg})}$ give half-odd PN corrections (odd powers of v)

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Emitted Energy-Momentum and Angular Momentum

[Herrmann, Parra-Martinez, Ruf, Zeng '21; Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo, Veneziano '22]

- We define for later convenience the notation

$$\boldsymbol{K}_\alpha[\tilde{X},\tilde{Y}] = D^{\mu\nu,\rho\sigma} k_\alpha \tilde{X}^*_{\mu\nu} \tilde{Y}_{\rho\sigma}\,, \qquad \boldsymbol{O}_{\alpha\beta}[\tilde{X},\tilde{Y}] = D^{\mu\nu,\rho\sigma} \tilde{X}^*_{\mu\nu} k_{[\alpha} \frac{\overleftrightarrow{\partial}}{\partial k^{\beta]}} \tilde{Y}_{\rho\sigma} + 2\tilde{X}^*_{\mu[\alpha} \tilde{Y}^{\mu}_{\beta]}$$

- The operator insertion for the energy-momentum $\langle \text{out}|\hat{P}^\alpha|\text{out}\rangle = P^\alpha$ leads to leads to

$$\boldsymbol{P}^\alpha = \int_k \boldsymbol{K}^\alpha[\tilde{W},\tilde{W}]\,, \qquad \int_k = \int 2\pi\theta(k^0)\,\delta(k^2)\,\frac{d^Dk}{(2\pi)^D}$$

- For the angular momentum, one has $\langle \text{out}|\hat{J}_{\alpha\beta}|\text{out}\rangle = \boldsymbol{J}_{\alpha\beta}$ with

$$\boldsymbol{J}^{\alpha\beta} = -i\int_k \boldsymbol{O}^{\alpha\beta}[\tilde{W},\tilde{W}]$$

Emitted Energy-Momentum at $\mathcal{O}(G^3)$ (Two Loops) [Heissenberg '25]

- To leading order

$$\boldsymbol{P}^{\alpha}_{\mathcal{O}(G^3)} = \int_k \boldsymbol{K}^{\alpha}_0\,, \qquad \boldsymbol{K}^{\alpha}_0 = \boldsymbol{K}^{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_0]$$

- Note that $\tilde{\mathcal{A}}_0^* = \tilde{\mathcal{A}}_0\big|_{b\mapsto -b}$ (the tree-level amplitude is real!). So,

$$\boldsymbol{K}^{\alpha}_0 = \boldsymbol{K}^{\alpha}_0\big|_{b\to -b}\,. \tag{1}$$

- Writing $\boldsymbol{K}^{\alpha}_0 = f_{u_1}\,\check{u}_1^{\alpha} + f_{u_2}\,\check{u}_2^{\alpha} + f_b\,b^{\alpha} + f_k\,k^{\alpha}$ (here $\check{u}_i \cdot u_j = -\delta_{ij}$) we deduce

$$f_{u_{1,2}}(-b\cdot k) = +f_{u_{1,2}}(b\cdot k)\,, \quad f_b(-b\cdot k) = -f_b(b\cdot k)\,, \quad f_k(-b\cdot k) = +f_k(b\cdot k)\,.$$

- Therefore, the integrand $b\cdot \boldsymbol{K}_0 = f_b\,b^2 + f_k\,b\cdot k$ is odd under $b\cdot k \mapsto -b\cdot k$,

$$b\cdot \boldsymbol{P}_{\mathcal{O}(G^3)} = 0$$

in agreement with the explicit result [Herrmann, Parra-Martinez, Ruf, Zeng '21]

Take-home message: Some components vanish by analyticity considerations.

Emitted Angular Momentum at $\mathcal{O}(G^3)$ (Two Loops) [Heissenberg '25]

- To leading order

$$\boldsymbol{J}^{\alpha\beta}_{\mathcal{O}(G^3)} = -i \int_k \boldsymbol{O}^{\alpha\beta}_0 , \qquad \boldsymbol{O}^{\alpha\beta}_0 = \boldsymbol{O}^{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_0]$$

- ... [intermediate steps left to the reader as an exercise!]
- We can show that the integrand $u_1 \cdot \boldsymbol{O}_0 \cdot u_2$ is odd under $b \cdot k \mapsto -b \cdot k$,

$$u_1 \cdot \boldsymbol{J}_{\mathcal{O}(G^3)} \cdot u_2 = 0$$

  in agreement with the explicit result [Manohar, Ridgway, Shen '22]

The nontrivial components: $u_{1,2} \cdot \boldsymbol{P}_{\mathcal{O}(G^3)}$ and $b \cdot \boldsymbol{J}_{\mathcal{O}(G^3)} \cdot u_{1,2}$ can be evaluated by reducing them to (cut) **two-loop** integrals

[Herrmann, Parra-Martinez, Ruf, Zeng '21; Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo, Veneziano '22].

Emitted Energy and Angular Momentum at $\mathcal{O}(G^4)$

[Heissenberg '25]

To next-to-leading order, we split $\boldsymbol{P}^{\alpha}_{\mathcal{O}(G^4)} = \boldsymbol{P}^{\alpha}_{1\text{rad}} + \boldsymbol{P}^{\alpha}_{2\text{rad}}$ and $\boldsymbol{J}^{\alpha\beta}_{\mathcal{O}(G^4)} = \boldsymbol{J}^{\alpha\beta}_{1\text{rad}} + \boldsymbol{J}^{\alpha\beta}_{2\text{rad}}$

- Integer-PN contributions (even in velocity):

$$\boldsymbol{P}^{\alpha}_{1\text{rad}} = 2\int_k \operatorname{Re}\boldsymbol{K}^{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(i)}_{1O} + \tilde{\mathcal{B}}_{1E}]\,, \qquad \boldsymbol{J}^{\alpha\beta}_{1\text{rad}} = 2\int_k \operatorname{Im}\boldsymbol{O}^{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(i)}_{1O} + \tilde{\mathcal{B}}_{1E}]$$

- Half-odd-PN contributions (odd in velocity):

$$\boldsymbol{P}^{\alpha}_{2\text{rad}} = \int_k \left(2\operatorname{Re}\boldsymbol{K}^{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(h)}_{1O}] - \operatorname{Im}\boldsymbol{K}^{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{C}}]\right),$$

$$\boldsymbol{J}^{\alpha\beta}_{2\text{rad}} = \int_k \left(2\operatorname{Im}\boldsymbol{O}^{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(h)}_{1O}] + \operatorname{Re}\boldsymbol{O}^{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{C}}]\right)$$

These (naively) involve **three-loop** integrals.

Can their analytic structure dictated by unitarity help us?

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- The integer-PN contributions (even in velocity) behave as the $\mathcal{O}(G^3)$ ones,

$$b \cdot \boldsymbol{P}_{1\text{rad}} = 0\,, \qquad u_1 \cdot \boldsymbol{J}_{1\text{rad}} \cdot u_2 = 0$$

  and $u_{1,2} \cdot \boldsymbol{P}_{1\text{rad}}$ and $b \cdot \boldsymbol{J}_{1\text{rad}} \cdot u_{1,2}$ indeed involve integrals at **tree loops**

- For the half-odd-PN contributions (odd in velocity) we find instead

$$u_{1,2} \cdot \boldsymbol{P}_{2\text{rad}} = 2u^{\alpha}_{1,2} \int_k \text{Re}\, \boldsymbol{K}_{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(h)}_{1O}] \qquad \textbf{two loops}$$

$$b \cdot \boldsymbol{P}_{2\text{rad}} = -b^{\alpha} \int_k \text{Im}\, \boldsymbol{K}_{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{C}}] \qquad \textbf{three loops}$$

$$u_{1,2} \cdot \boldsymbol{J}_{2\text{rad}} \cdot b = 2u^{\alpha}_{1,2} b^{\beta} \int_k \text{Im}\, \boldsymbol{O}_{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(h)}_{1O}] \qquad \textbf{two loops}$$

$$u_1 \cdot \boldsymbol{J}_{2\text{rad}} \cdot u_2 = u^{\alpha}_1 u^{\beta}_2 \int_k \text{Re}\, \boldsymbol{O}_{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{C}}] \qquad \textbf{three loops}$$

The 2rad energy and angular momentum (in the CM) only involve two-loop integrals!

Results: Half-Odd-PN Emitted Energy

[Heissenberg '25]

- **Warm-up**: 2rad emitted energy, $\boldsymbol{P}^\alpha_{\text{2rad}} = \boldsymbol{P}^\alpha_\parallel + (\cdots)b^\alpha$

$$\boldsymbol{P}^\alpha_\parallel = \frac{G^4 m_1^2 m_2^2}{b^4}\left[m_1(\mathcal{E}^{(1)}\check{u}_1^\alpha + \mathcal{E}^{(2)}\check{u}_2^\alpha) + (1 \leftrightarrow 2)\right]$$

  with

$$\mathcal{E}^{(i)} = \frac{f_1^{(i)}}{\sigma^2 - 1} + f_2^{(i)}\frac{\operatorname{arccosh}\sigma}{(\sigma^2-1)^{3/2}} + f_3^{(i)}\frac{(\operatorname{arccosh}\sigma)^2}{(\sigma^2-1)^2}$$

  for $i = 1, 2$ and polynomials in σ denoted by $f^{(i)}_{1,2,3}$ (here omitted for brevity)

- Perfectly matches [Dlapa, Kälin, Liu, Porto '22], where Q_1^α and Q_2^α where calculated up to $\mathcal{O}(G^4)$, using

$$\boldsymbol{P}^\alpha = -Q_1^\alpha - Q_2^\alpha\,.$$

Results: Half-Odd-PN Angular Momentum Loss

[Heissenberg '25]

- **New result**: 2rad emitted angular momentum, $\boldsymbol{J}_{\text{2rad}}^{\alpha\beta} = \boldsymbol{J}_{\perp}^{\alpha\beta} + (\cdots) u_1^{[\alpha} u_2^{\beta]}$

$$\boldsymbol{J}_{\perp}^{\alpha\beta} = \frac{G^4 m_1^2 m_2^2}{b^3} \left[m_1 (\mathcal{F}^{(1)} b^{[\alpha} u_1^{\beta]} + \mathcal{F}^{(2)} b^{[\alpha} u_2^{\beta]}) + (1 \leftrightarrow 2) \right]$$

with

$$\mathcal{F}^{(i)} = \frac{g_1^{(i)}}{(\sigma^2 - 1)^2} + g_2^{(i)} \frac{\operatorname{arccosh} \sigma}{(\sigma^2 - 1)^{5/2}} + g_3^{(i)} \frac{(\operatorname{arccosh} \sigma)^2}{(\sigma^2 - 1)^3}$$

for $i = 1, 2$ and polynomials in σ denoted by $g_{1,2,3}^{(i)}$ (here omitted for brevity)

- Adding the static contribution [CH, Russo '24]

$$J_{\text{2rad}} = \boldsymbol{J}_{\text{2rad}} + \mathcal{J}_{\text{2rad}}\,, \qquad \mathcal{J}_{\text{2rad}} = \frac{G^2 p}{2b} Q_{\text{1PM}}^2 \, \mathcal{I}(\sigma)^2$$

we obtain the 2rad angular momentum loss in the CM frame J_{2rad}

- The first few terms in its PN expansion agree with [Bini, Damour, Geralico '21, '22]

$$J_{\text{2rad}} = \frac{G^4 M^5}{b^3} \nu^2 p_\infty \left[\frac{448}{5} + \left(\frac{1184}{21} - \frac{220256\nu}{1575} \right) p_\infty^2 + \cdots \right]$$

Summary and Outlook

- The **eikonal approach** provides a framework to calculate scattering observables, including the **impulse**, the **waveform** and the emitted **energy and angular momentum**.
- The unitarity and analyticity properties of the waveform at next-to-leading order greatly simplify the calculation of half-odd-PN (2rad) contributions to the radiated energy and angular momentum at $\mathcal{O}(G^4)$ (**three loops** $\to$ **two loops**)

For the future:

- Calculate the integer-PN (1rad) contributions!
- Inclusion of tidal/spin effects
- NNLO waveform? Nonlinear memory effect