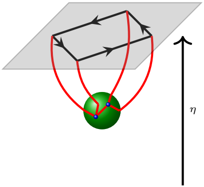
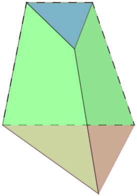
The Timeless Primordial Universe & Its Combinatorial Origin



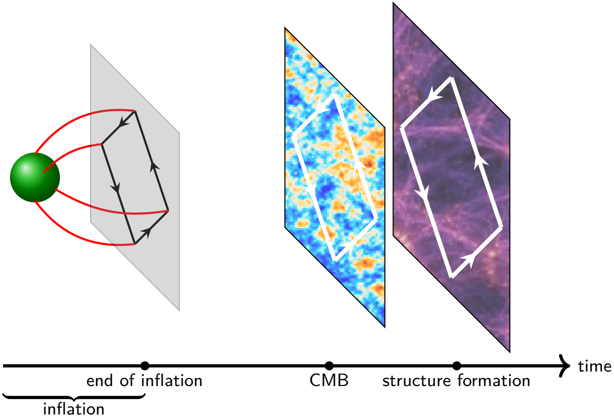


Paolo Benincasa

Instituto Galego de Física de Altas Enerxías

27 May 2025 – Planck 2025

The Timeless Priomordial Universe



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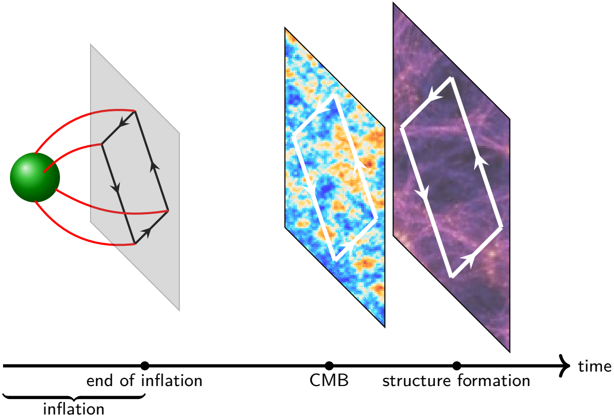

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1. Can we put constraints on which state can propagate during inflation in a completely model independent way?

2. What is the imprint of the inflationary physics in the analytic structure of the relevant observables?

3. What are the rules governing physical processes at energies as large as $H|_{\text{infl}} \sim 10^{14}\ GeV$?

Model independent constraints?

What does it even mean?

1. A theory is defined by..

Model independent constraints?

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   Computables Enjoy *sufficiently* physical features

   (e.g.: physical dofs, gauge inv.,...)

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2. Basic guiding principles that *any reasonable theory* ought to satisfy

Model independent constraints?

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3. Constraints $\equiv$ Consequences of the simultaneous validity of these principles

Model independent constraints? The particle physics lesson

Guiding Principles

Poincaré invariance

Cluster decomposition

Unitarity

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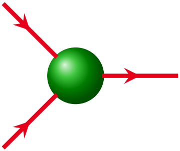
Particles with spin:
$0, \frac{1}{2}, \ldots, \frac{9}{2}, 5, \ldots$



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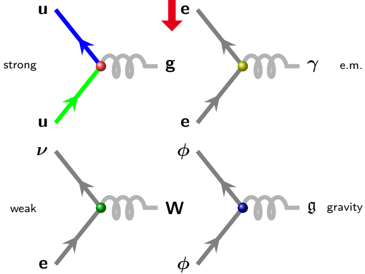

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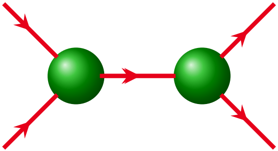

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Theorems

Spin 1
Self-interaction just for different species

Spin 2
Graviton uniqueness
$\mathcal{N} = 1$ Sugra

Spin >2
No self-interactions
No interactions with $s \leq 2$
No elementary massive particles

Model independent constraints? The particle physics lesson

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Theorems

Spin 1:
Charge conservation

Spin 2:
Equivalence principle

The particle physics lesson

1. Guiding principles + computables → Simplest processes + constraints

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Poincaré invariant operator: $\hat{\mathcal{O}}$

$\langle 3,4|\hat{\mathcal{O}}|1,2\rangle := \delta\left(p_1+p_2-p_3-p_4\right)$

Unitary S-matrix: $\hat{S} \,|\, \hat{S}\hat{S}^\dagger = \hat{\mathbb{I}} = \hat{S}^\dagger\hat{S}$

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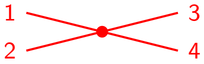

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Anything wrong?

?

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Anything wrong? → Wrong analytic structure
Non causal!

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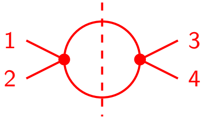

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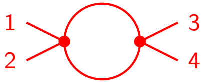

The particle physics lesson

1. Guiding principles + computables → Simplest processes + constraints

2. Language:

Fields:
redundant organisation of redundant degrees of freedom

Riemannian geometry:
mathematical tool for the explotation of the equivalence principle
[S. Weinberg, Gravitation and Cosmology]

The particle physics lesson

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[S. Weinberg, Gravitation and Cosmology]

3. Problems as our inability to extrapolate high energy knowledge from our low energy one:

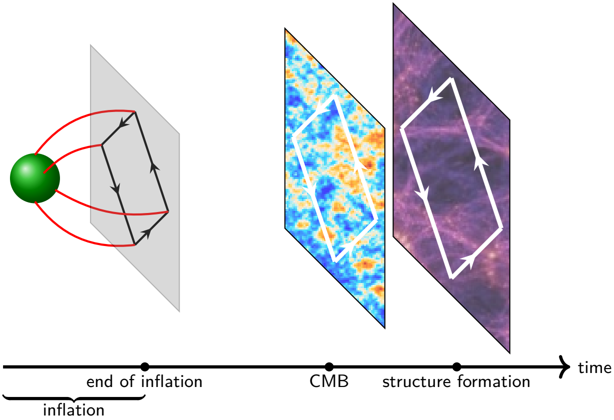
   (Quantum) gravity;
   Higgs: it is necessary to extrapolate high energy physics from the low energy one

Why Combinatorics?

Deeper understanding of the physics encoded
into cosmological observables

Novel rules which can allow to go beyond the regime
in which the combinatorial description has been formulated

Observables & Their Analytic Structure



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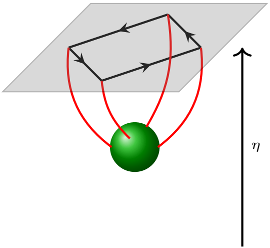

$$\langle \Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\rangle \;=\; \int \mathcal{D}\Phi\,\mathfrak{P}[\Phi]\Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)$$

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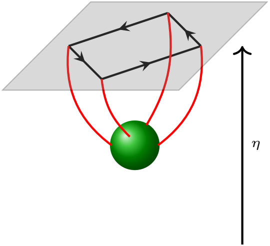

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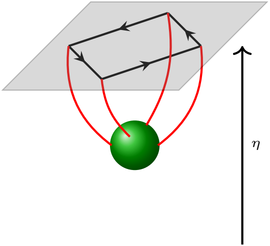
Probability distribution

Observables & Their Analytic Structure

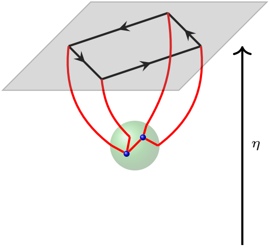


$$\langle \Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\rangle = \frac{\int \mathcal{D}\Phi\, \Psi^\dagger[\Phi]\Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\Psi[\Phi]}{\int \mathcal{D}\Phi\, |\Psi[\Phi]|^2}$$

$$\Psi[\Phi] := \langle \Phi|\hat{\mathcal{T}}\mathrm{exp}\left\{-i\int_{-\infty}^{0} d\eta\, H(\eta)\right\}|0\rangle$$

Wavefunction of the universe
(transition amplitude from $|0\rangle$ to $\langle\Phi|$)

Observables & Their Analytic Structure



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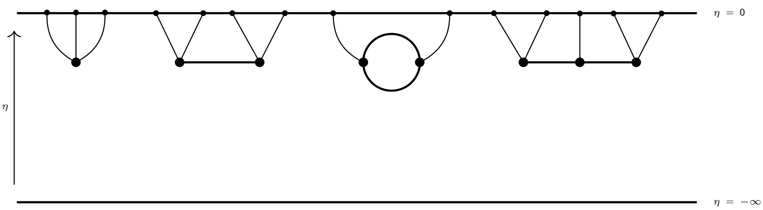
Perturbation theory



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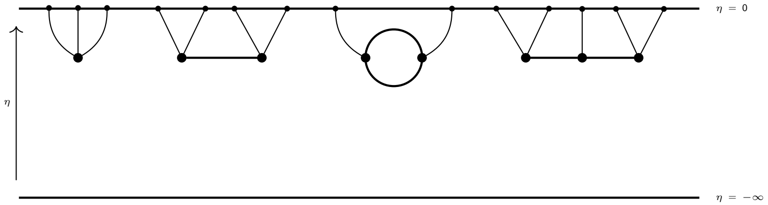

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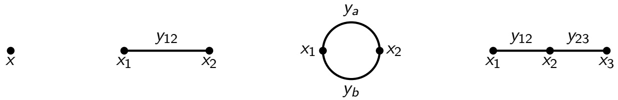

[N. Arkani-Hamed, P.B., A.Postnikov; '17]
[P.B.; '19]

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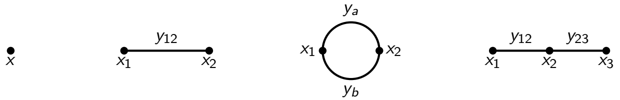
Cosmology

Loop integration

Universal integrand

Observables & Their Analytic Structure



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Power-law FRW

$\tilde{\lambda}(x_s - X_s) \sim (x_s - X_s)^{\alpha-1}$

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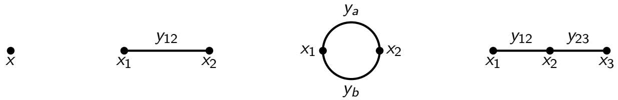

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Cosmology → Power-law FRW

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External kinematics: $X_s := \sum_{j\in s} |\vec{p}^{(j)}|,\; y_e := \left|\sum_{j\in s_e} \vec{p}^{(j)}\right| \; (e \in \mathcal{E} \setminus \{\mathcal{E}^{(L)}\})$

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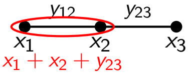


In this talk

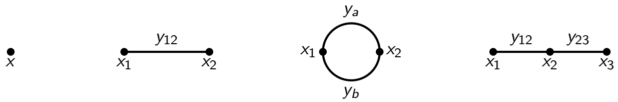
1. Cosmological integrands: Singularities, combinatorics & computation

2. The IR/UV structure of cosmological integrals

3. One loop corrections without integration

Cosmological integrands:
Singularities, combinatorics & computation

Cosmological Integrals



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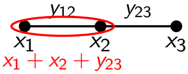
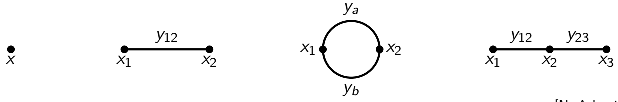


Cosmological Integrals



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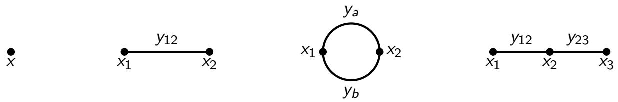
(weighted) cosmological polytope

[N. Arkani-Hamed, P.B., A. Postnikov; '17]
[P.B.; '19]
[P.B., G. Dian; '24]

Cosmological Integrals



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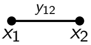
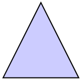
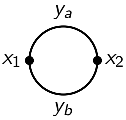


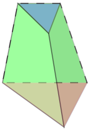
$$\omega\left(\mathcal{Y},\mathcal{P}_{\mathcal{G}}\right) = \frac{\mathfrak{n}_\delta(x,y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} q_{\mathfrak{g}}(x,y)}\,\frac{\prod_{s\in\mathcal{V}} dx_s \prod_{e\in\mathcal{E}} dy_e}{\text{Vol}\{GL(1)\}}$$



(weighted) cosmological polytope



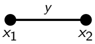
(Weighted) cosmological polytopes capture the singularity structure of $\mathcal{I}_{\mathcal{G}}$



[N. Arkani-Hamed, P.B., A. Postnikov; '17]
[P.B.; '19]
[P.B., G. Dian; '24]

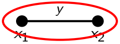
From Graphs to Polytopes

A flavour of cosmological polytopes



From Graphs to Polytopes

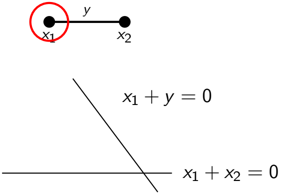
A flavour of cosmological polytopes



$x_1 + x_2 = 0$

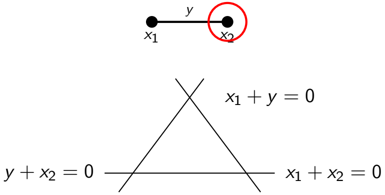
From Graphs to Polytopes

A flavour of cosmological polytopes



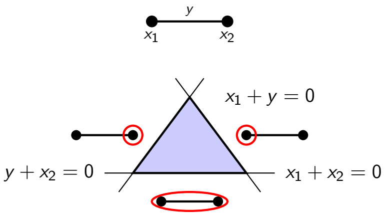
From Graphs to Polytopes

A flavour of cosmological polytopes



From Graphs to Polytopes

A flavour of cosmological polytopes

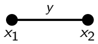


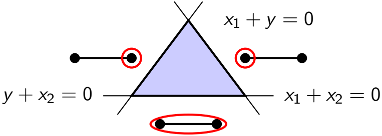
The singularities form a bounded region to which a function Ω is naturally associated

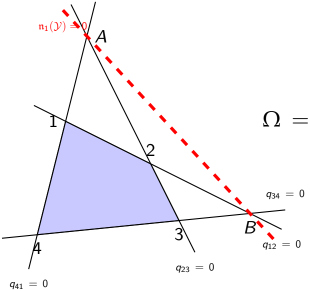
$$\Omega = \frac{1}{(x_1 + x_2)(x_1 + y)(y + x_2)} \equiv \frac{\mathfrak{n}_\delta}{q_{\mathcal{G}}\, q_{\mathfrak{g}_1}\, q_{\mathfrak{g}_2}}$$

From Graphs to Polytopes

A flavour of cosmological polytopes







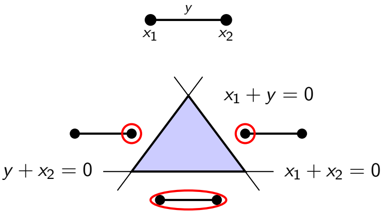
$$\Omega = \frac{\mathfrak{n}_1}{q_{12}q_{23}q_{34}q_{41}} = \frac{1}{q_{12}q_{34}}\left[\frac{1}{q_{23}} + \frac{1}{q_{41}}\right]$$

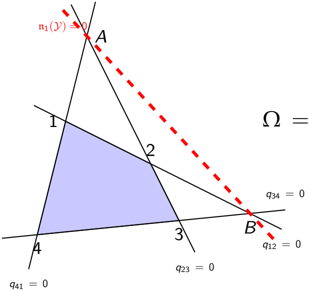
Linear relation
$q_{12} + q_{34} = q_{23} + q_{41}$

Triangulation of the polytope
≡
Representation for the integrand

From Graphs to Polytopes

A flavour of cosmological polytopes





Point B: $\begin{cases} q_{12} = 0 \\ q_{34} = 0 \end{cases}$

$$\Omega = \frac{\mathfrak{n}_1}{q_{12}q_{23}q_{34}q_{41}} = \frac{1}{q_{12}q_{34}}\left[\frac{1}{q_{23}} + \frac{1}{q_{41}}\right]$$

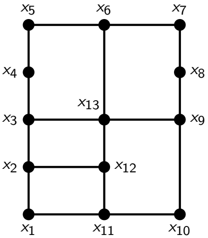
Linear relation
$q_{12} + q_{34} = q_{23} + q_{41}$

Compatibility conditions:

$$\text{Res}_{q_{12}=0}\text{Res}_{q_{34}=0}\boldsymbol{\Omega} = \mathbf{0} = \text{Res}_{q_{23}=0}\text{Res}_{q_{41}=0}\boldsymbol{\Omega}$$

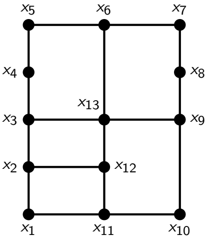
From Graphs to Polytopes

[P.B., W. Torres Bobadilla; '21]



From Graphs to Polytopes

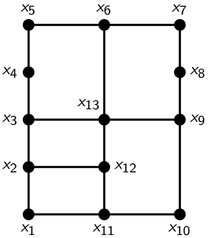
[P.B., W. Torres Bobadilla; '21]



$$\Omega = \prod_{\mathfrak{g}\in\mathfrak{G}_{\circ}} \frac{1}{q_{\mathfrak{g}}(x,y)} \sum_{\{\mathfrak{G}_c\}} \prod_{\mathfrak{g}'\in\mathfrak{G}_c} \frac{1}{q_{\mathfrak{g}'}(x,y)}$$

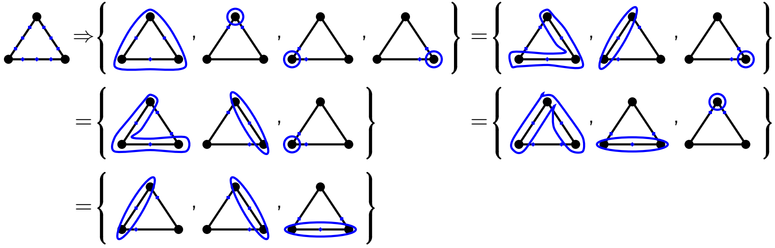
From Graphs to Polytopes

[P.B., W. Torres Bobadilla; '21]



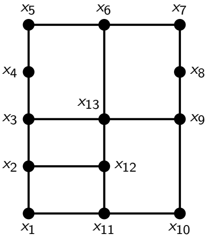
$$\Omega = \prod_{\mathfrak{g}\in\mathfrak{G}_\circ} \frac{1}{q_{\mathfrak{g}}(x,y)} \sum_{\{\mathfrak{G}_c\}} \prod_{\mathfrak{g}'\in\mathfrak{G}_c} \frac{1}{q_{\mathfrak{g}'}(x,y)}$$

Example: $\{\mathfrak{G}_\circ\}$ for a triangle graph



From Graphs to Polytopes

[P.B., W. Torres Bobadilla; '21]



$$\Omega = \prod_{\mathfrak{g}\in\mathfrak{G}_\circ} \frac{1}{q_{\mathfrak{g}}(x,y)} \sum_{\{\mathfrak{G}_c\}} \prod_{\mathfrak{g}'\in\mathfrak{G}_c} \frac{1}{q_{\mathfrak{g}'}(x,y)}$$

Compatibility conditions allow to:

1. write all the possible representations without spurious singularities
2. make manifest the symmetries that maps a simplex into another one
3. improve analytical/numerical efficiency of the integration.

(Weighted) Cosmological Polytopes & $\mathcal{I}_{\mathcal{G}}$: A dictionary

Cosmological Polytope $\mathcal{P}_{\mathcal{G}}$	Cosmological Integral $\mathcal{I}_{\mathcal{G}}$
Canonical form ω	Integrand of $\mathcal{I}_{\mathcal{G}}$
Triangulations	Representations for the integrand
Boundaries (Faces)	Residues of the integrands
Canonical form preserving transformations	Symmetries of the integrand
Paths along contiguous vertices	Symbols for $\mathcal{I}_{\mathcal{G}}$

(Weighted) Cosmological Polytopes & $\mathcal{I}_\mathcal{G}$: A dictionary

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Paths along contiguous vertices	Symbols for $\mathcal{I}_\mathcal{G}$

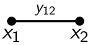
Trascendental function $\quad f_k = \int_a^b d\log R_1 \circ \ldots \circ d\log R_k \quad$ Iterated integral

Symbols $\quad \mathcal{S}(f_k) := R_1 \otimes \ldots \otimes R_k$

Towards a combinatorial RG:
The IR/UV structure of cosmological integrals

Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

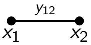
[P.B., F. Vazão; 24]



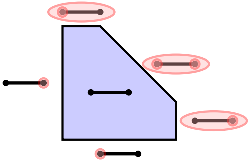
$$= \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j} \, x_j^{\alpha} \right] \frac{1}{(x_1 + x_2 + \mathcal{X}_{\mathcal{G}})(x_1 + \mathcal{X}_{\mathfrak{g}_1})(x_2 + \mathcal{X}_{2_2})}$$

Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]



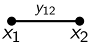
$$= \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j}\, x_j^{\alpha} \right] \frac{1}{(x_1^1 x_2^0 + x_1^0 x_2^1 + \mathcal{X}_{\mathcal{G}} x_1^0 x_2^0)(x_1^1 x_2^0 + \mathcal{X}_{\mathfrak{g}_1} x_1^0 x_2^0)(x_1^0 x_2^1 + \mathcal{X}_{\mathfrak{g}_2} x_1^0 x_2^0)}$$



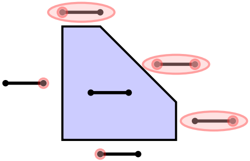
The integral converges for values of α that identifies points inside the Newton polytope

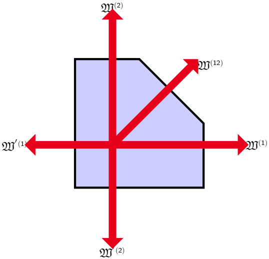
Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]



$$= \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j} x_j^{\alpha} \right] \frac{1}{(x_1^{1}x_2^{0} + x_1^{0}x_2^{1} + \mathcal{X}_{\mathcal{G}} x_1^{0}x_2^{0})(x_1^{1}x_2^{0} + \mathcal{X}_{\mathfrak{g}_1} x_1^{0}x_2^{0})(x_1^{0}x_2^{1} + \mathcal{X}_{\mathfrak{g}_2} x_1^{0}x_2^{0})}$$





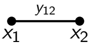
The integral converges for values of α that identifies points inside the Newton polytope

$$\mathfrak{W}^{(12)} = \begin{pmatrix} 2\alpha - 3 \\ 1 \\ 1 \end{pmatrix}, \quad \mathfrak{W}^{(1)} = \begin{pmatrix} \alpha - 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathfrak{W}^{(2)} = \begin{pmatrix} \alpha - 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathfrak{W}'^{(1)} = \begin{pmatrix} -\alpha \\ -1 \\ 0 \end{pmatrix}, \quad \mathfrak{W}'^{(2)} = \begin{pmatrix} -\alpha \\ 0 \\ -1 \end{pmatrix},$$

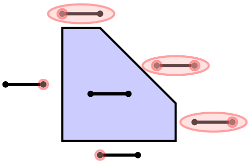
The integral diverges in the direction $\mathfrak{e}$ if the related λ is ≥ 0

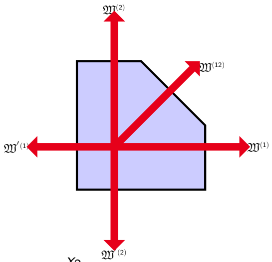
Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]



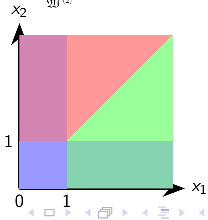
$$= \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j} \, x_j^{\alpha} \right] \frac{1}{(x_1^1 x_2^0 + x_1^0 x_2^1 + \mathcal{X}_{\mathcal{G}} x_1^0 x_2^0)(x_1^1 x_2^0 + \mathcal{X}_{\mathfrak{g}_1} x_1^0 x_2^0)(x_1^0 x_2^1 + \mathcal{X}_{\mathfrak{g}_2} x_1^0 x_2^0)}$$





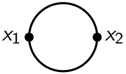
E.g.: if $\lambda^{(12)} \longrightarrow 0$: sector decomposition

$$\mathcal{I}^{\text{div}}_{\Delta_{j,12}} = \int_0^1 \frac{d\zeta_j}{\zeta_j} \frac{(\zeta_j)^{-\lambda^{(j)}}}{1+\zeta_j} \times \int_0^1 \frac{d\zeta_{12}}{\zeta_{12}} \, (\zeta_{12})^{-\lambda^{(12)}}$$



Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

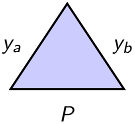
[P.B., F. Vazão; 24]



$$= \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j} x_j^{\alpha} \right] \int_{\Gamma} \prod_{e \in \mathcal{E}} \left[\frac{dy_e}{y_e} y_e^{\beta} \right] \mu(y) \times$$

$$\times \frac{2(x_1 + x_2 + y_a + y_b + \mathcal{X}_{\mathcal{G}})}{(x_1 + x_2 + \mathcal{X}_{\mathcal{G}})(x_1 + x_2 + y_a + \mathcal{X}_{\mathcal{G}})(x_1 + x_2 + y_b + \mathcal{X}_{\mathcal{G}})(x_1 + y_a + y_b + \mathcal{X}_1)(x_2 + y_a + y_b + \mathcal{X}_2)}$$

1. $\mu(y) \sim \left[\frac{\mathrm{Vol}^2 \Sigma_2(y_e^2, P^2)}{\mathrm{Vol}^2 \Sigma_1(P^2)} \right]^{\frac{d-3}{2}}$



2. $\Gamma \implies$ Volume of the triangle, and all its side, are positive

$$(y_a + y_b + P)(y_a + y_b - P)(y_a - y_b + P)(-y_a + y_b + P) \geq 0,$$

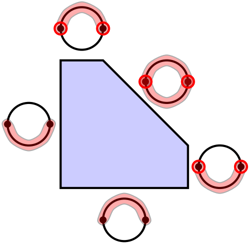
$$y_a \geq 0, \quad y_b \geq 0, \quad P \geq 0$$

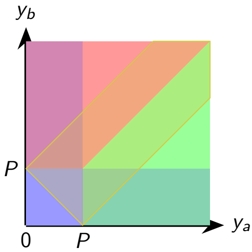
Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]

$$x_1 \bigcirc x_2 = \int_{\mathbb{R}_+} \prod_{j=1}^{2} \left[\frac{dx_j}{x_j} x_j^{\alpha} \right] \int_{\Gamma} \prod_{e \in \mathcal{E}} \left[\frac{dy_e}{y_e} y_e^{\beta} \right] \mu(y) \times$$

$$\times \frac{2(x_1 + x_2 + y_a + y_b + \mathcal{X}_{\mathcal{G}})}{(x_1 + x_2 + \mathcal{X}_{\mathcal{G}})(x_1 + x_2 + y_a + \mathcal{X}_{\mathcal{G}})(x_1 + x_2 + y_b + \mathcal{X}_{\mathcal{G}})(x_1 + y_a + y_b + \mathcal{X}_1)(x_2 + y_a + y_b + \mathcal{X}_2)}$$





Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]

$$\mathcal{I}_{\mathcal{G}} = \int_0^{+\infty} \prod_{s \in \mathcal{V}} \left[\frac{dx_s}{x_s} \, x_s^{\alpha_s} \right] \int_{\Gamma} \prod_{e \in \mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e} \, y_e^{\beta_e} \right] \mu_d(y) \, \frac{\mathfrak{n}_{\delta}(x + X, y)}{\prod_{\mathfrak{g} \subseteq \mathcal{G}} \left[q_{\mathfrak{g}}(x + X, y) \right]^{\tau_{\mathfrak{g}}}}$$

Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]

$$\mathcal{I}_{\mathcal{G}} = \int_0^{+\infty} \prod_{s\in\mathcal{V}} \left[\frac{dx_s}{x_s}\, x_s^{\alpha_s}\right] \int_{\Gamma} \prod_{e\in\mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e}\, y_e^{\beta_e}\right] \mu_d(y)\, \frac{\mathfrak{n}_\delta(x+X, y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} \left[q_{\mathfrak{g}}(x+X, y)\right]^{\tau_{\mathfrak{g}}}}$$

The asymptotic structure of $\mathcal{I}_{\mathcal{G}}$ is captured by:

1. a nestohedron, which is determined by the underlying cosmological polytope $\mathcal{P}_{\mathcal{G}}$, and whose facets are fixed via subgraphs

$$\mathfrak{W}^{\left(j_1\cdots j_{n_s^{(\mathfrak{g})}+n_e^{(\mathfrak{g})}}\right)} = \begin{pmatrix} \lambda^{\left(j_1\cdots j_{n_s^{(\mathfrak{g})}+n_e^{(\mathfrak{g})}}\right)} \\ \mathfrak{e}_{\left(j_1\cdots j_{n_s^{(\mathfrak{g})}+n_e^{(\mathfrak{g})}}\right)} \end{pmatrix}, \qquad \lambda^{\left(j_1\cdots j_{n_s^{(\mathfrak{g})}+n_e^{(\mathfrak{g})}}\right)} = \sum_{s\in\mathcal{V}_{\mathfrak{g}}} \alpha_s + \sum_{e\in\mathcal{E}^{(L)}} \beta_e - \sum_{\mathfrak{g}'\in(\text{tubings})} \tau_{\mathfrak{g}'}$$

The integral diverges in the direction $\mathfrak{e}$ if
the related λ is ≥ 0

2. the contour of the loop integration Γ, which selects the divergent directions among the $\mathfrak{W}$'s of the nestohedron

Towards a combinatorial RG: The IR/UV structure of $\mathcal{I}_{\mathcal{G}}$

[P.B., F. Vazão; 24]

$$\mathcal{I}_{\mathcal{G}} = \int_0^{+\infty} \prod_{s\in\mathcal{V}} \left[\frac{dx_s}{x_s} x_s^{\alpha_s}\right] \int_{\Gamma} \prod_{e\in\mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e} y_e^{\beta_e}\right] \mu_d(y) \frac{\mathfrak{n}_{\delta}(x+X, y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} \left[q_{\mathfrak{g}}(x+X, y)\right]^{\tau_{\mathfrak{g}}}}$$

This combinatorial picture allows to:

1. straightforwardly determine both the directions along which $\mathcal{I}_{\mathcal{G}}$ can diverge and their degree of divergence;
2. straightforwardly compute leading and subleading divergences (both in the IR and in the UV) via sector decomposition;
3. the leading divergence in the IR are associated to the restriction of the underlying cosmological polytope onto special hyperplanes;
4. write a systematic substraction that produces IR-finte quantities.

Understanding the space of functions

[**P.B.**, G. Brunello, M. K. Mandal, P. Mastrolia, F. Vazão; 24]

$$\mathcal{I}_{\mathcal{G}} = \int_0^{+\infty} \prod_{s \in \mathcal{V}} \left[\frac{dx_s}{x_s} \, x_s^{\alpha_s} \right] \int_\Gamma \prod_{e \in \mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e} \, y_e^{\beta_e} \right] \mu_d(y) \, \frac{\mathfrak{n}_\delta(x + X, y)}{\prod_{\mathfrak{g} \subseteq \mathcal{G}} \left[q_{\mathfrak{g}}(x + X, y) \right]^{\tau_{\mathfrak{g}}}}$$

Understanding the space of functions

[P.B., G. Brunello, M. K. Mandal, P. Mastrolia, F. Vazão; 24]

$$\mathcal{I}_{\mathcal{G}} = \int_0^{+\infty} \prod_{s\in\mathcal{V}} \left[\frac{dx_s}{x_s}\, x_s^{\alpha_s}\right] \int_\Gamma \prod_{e\in\mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e}\, y_e^{\beta_e}\right] \mu_d(y)\, \frac{\mathfrak{n}_\delta(x+X,y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} [q_{\mathfrak{g}}(x+X,y)]^{\tau_{\mathfrak{g}}}}$$

can be expressed in terms of

(twisted period integrals)

$$\mathcal{I}^{(j)}_{\{\tau_{\mathfrak{g}}\}} := \int_\Gamma \mu_d\, \varphi \quad , \qquad \varphi := \frac{\prod_{e\in\mathcal{E}^{(L)}} dy_e}{\prod_{\mathfrak{g}\in\mathfrak{G}^{(j)}\cup\{e\}} [q_{\mathfrak{g}}(y)]^{\tau_{\mathfrak{g}}}},$$

Each of these integrals can be expressed as a *finite* linear combination of master integrals

$$\mathcal{I}^{(j)}_{\{\tau_{\mathfrak{g}}\}} := \sum_{j=1}^{\nu} c_j \mathcal{J}_j, \qquad d\mathcal{J} = d\mathbb{A}\,\mathcal{J}$$

Canonical form: $d\mathcal{J} = \varepsilon d\mathbb{A}\,\mathcal{J} \Rightarrow \mathcal{J} = \mathbb{P}\mathrm{exp}\left\{\varepsilon \int_\Gamma d\mathbb{A}\right\} \mathcal{J}_\circ$

Understanding the space of functions

[P.B., G. Brunello, M. K. Mandal, P. Mastrolia, F. Vazão; 24]

1. The system of differential equations has a block-triangular form, and for each block can be rewritten in terms of a higher order differential equation for a single $\mathcal{J}_j$;

Understanding the space of functions

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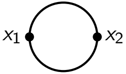

Loop integration $\log,\ \mathrm{Li}_2$ $\Longrightarrow$ Site-weight integration ${}_2F_1,\ {}_3F_2$ General power-law FRW cosmologies

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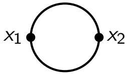

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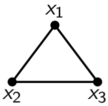

Loop integration   Polylogs, Elliptics

Conclusion

First clues on constraints on cosmological processes:
perturbative unitarity, flat-space limit,
factorisations, higher-codimensions singularities

General framework to have a direct formulation
with IR safe observables
& a novel formulation of the RG

We scratched the surface of the one-loop structure:
first glimpses of its analytic structure and its space of functions.

Summer School on:

The Disordered Universe

21 July - 01 August 2025, As Barreiras (Castro Caldelas, Spain)

Lecturers


Dionysios Anninos


Tarek Anous


Thomas Colas


Andrew McLeod

Topics

- Features of a $\Lambda > 0$ Universe
- Disorder & Expanding Universes
- Cosmology & Open Systems
- Landau analysis & Observables

Outreach & Education


Martí Berenguer Mimó


Rafael Carrasco Carmona


Lucas Vicente García-Consuegra


Silvia Plá García

New Venue

- Optimization of economical resources
- Lowering economical barriers to participation
- Informal & stimulating environment
- Backreaction in the area