Extended Local Ergotropy

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A fundamental problem in quantum thermodynamics is to properly quantify the work extractable from out-of-equilibrium systems. While for closed systems, maximum quantum work extraction is defined in terms of the ergotropy functional, this question is unclear in open systems interacting with an environment. The concept of local ergotropy has been proposed, but it presents several problems, such as it is not guaranteed to be nonincreasing in time. Here, we introduce the concept of *extended local ergotropy* by exploiting the free evolution of the system-environment compound. At variance with the local ergotropy, the extended local ergotropy is greater, is nonincreasing in time, and activates the potential of work extraction in many cases. We then concentrate on specific schemes in which we alternate repeated local unitaries and free system-environment evolution. We provide examples based on the Jaynes-Cummings model, presenting practical protocols and analytic results that serve as proof of principle for the aforementioned advantages.

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How much work can be extracted from an out-of-equilibrium quantum system? For an isolated quantum system S, described by Hamiltonian H_S and state ρ_S, the ergotropy functional [1] is commonly accepted as the figure of merit for the maximum work extractable under cyclic protocols [2–5]. It is defined as

$$\mathcal{E}(\rho_S, H_S) := \max_{U_S \in \mathcal{U}_S} \mathrm{tr}\left[H_S(\rho_S - U_S \rho_S U_S^\dagger)\right], \qquad (1)$$

where $\mathcal{U}_S$ is the set of unitaries on S. Remarkably, the optimal unitary has a closed expression in terms of the eigenvectors of H_S and ρ_S [6].

However, in practical experimental settings, the system of interest S interacts (weakly or not) with external degrees of freedom, the environment E, on which we typically have very limited control. Hence, it becomes crucial to properly include the environment E and its interaction with the system S in a work extraction task. Regarding this, a series of approaches have been proposed so far. Ergotropy extraction (EE) via thermal operations has been proposed in Ref. [7]. Thermal operations mimic classical interaction between S and E, i.e., they keep unchanged the sum of the local energies of S and E [8]. This construction, however, requires in general a detailed engineering of the $S-E$ interaction. One can instead consider all possible quantum channels, defined by completely positive trace preserving (CPTP) maps, acting locally on S as allowed operations [9]. Consequently, states whose energy cannot be decreased by any local CPTP map are named local *CP* passive, and semidefinite programming techniques can be used to characterize them and compute upper bounds on the extractable energy [10].

However, in this case, the energy extracted is not unequivocally accepted as work as it typically implies an entropy change. More recently, in Ref. [11], and in analogy with ergotropy, the set of allowed operations is restricted to all unitaries acting locally on the S subsystem. Such notion of local ergotropy (LE) is defined as

$$\mathcal{E}_S(\rho_{SE}, H_{SE}) := \max_{U_S \in \mathcal{U}_S} \mathrm{tr}\left[H_{SE}(\rho_{SE} - U_S \rho_{SE} U_S^\dagger)\right], \quad (2)$$

where ρ_{SE} is the joint state of SE, H_{SE} is the full interacting Hamiltonian

$$H_{SE} = H_S + H_E + V_{SE}, \qquad (3)$$

with local terms H_S, H_E, and interaction V_{SE}. Eq. (2) expresses the fact that it is not possible to control the

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environmental degrees of freedom and the interaction between the systems. Nonetheless, the energy functional is evaluated with respect to the full interacting Hamiltonian H_{SE} making SE correlations a resource in several cases. Both the definitions introduced in [9,11], namely CP passivity and LE, share a notable operational problem: they are not guaranteed to be nonincreasing in time. As the time evolution of the joint system is not local, both quantities are time-dependent when the state is evolving under the compound Hamiltonian H_{SE}. This leads to exotic situations, such as having passive states [1] according to these quantities (that is states with zero extractable work) evolving into nonpassive ones by the natural SE dynamics with no active intervention on S (see, e.g., Fig. 1 of Ref. [11]). A second important element to take into account is the adiabatic approximation underlying the definitions of CP- passivity or LE. In order to eventually generate a local map on S, it is necessary to assume that the local manipulation is carried out on a much shorter timescale than the one governing the dynamics of the joint compound SE.

The objective of this work is to introduce the concept of extended local ergotropy (ELE), a figure of merit for work extraction in open systems that solves all the previous issues: (i) it is nonincreasing under the natural SE dynamics, (ii) it does not require any adiabatic approximation, and (iii) it does not involve any control on the environmental degrees of freedom. The main point behind our construction is to take into account the free evolution of the SE compound and its entangling potential to enlarge the set of implementable unitary operations.

Extended local ergotropy—To start, we define the set $\bar{\mathcal{U}}_{\mathrm{ex}}(H_{\mathrm{SE}})$ (or $\bar{\mathcal{U}}_{\mathrm{ex}}$ for compactness) of extended local unitaries as the closure [12] of the set

$$\mathcal{U}_{\mathrm{ex}}(H_{\mathrm{SE}}) \coloneqq \left\{ U s.t. U = \mathcal{T} \exp\left[-i \int_0^{t_f} H_{\mathrm{SE}} + H_{\mathrm{S}}(t) dt \right] \right\}. \tag{4}$$

Here, we do not allow for Hamiltonian control of E, but only on S. These unitaries have a clear operational character: they are not strictly local on S, but the nonlocality is provided only by the natural SE dynamics, in particular, by its interacting term V_{SE}. ELE then reads [13]

$$\mathcal{E}_{\mathrm{ex}}(\rho_{\mathrm{SE}}, H_{\mathrm{SE}}) \coloneqq \sup_{U \in \bar{\mathcal{U}}_{\mathrm{ex}}} \mathrm{tr}\left[H_{\mathrm{SE}} (\rho_{\mathrm{SE}} - U \rho_{\mathrm{SE}} U^{\dagger}) \right]. \tag{5}$$

When the SE system is finite dimensional, the sup can be replaced by a max. We notice that, by construction, ELE is always greater or equal to LE and smaller or equal to the global ergotropy (GE) $\mathcal{E}(\rho_{\mathrm{SE}}, H_{\mathrm{SE}})$ of the SE compound which measures the work extractable when global operations are granted in the model, i.e.,

$$\mathcal{E}_{\mathrm{S}}(\rho_{\mathrm{SE}}, H_{\mathrm{SE}}) \leq \mathcal{E}_{\mathrm{ex}}(\rho_{\mathrm{SE}}, H_{\mathrm{SE}}) \leq \mathcal{E}(\rho_{\mathrm{SE}}, H_{\mathrm{SE}}). \tag{6}$$

Also, the functional (5) is convex in the state ρ_{SE} and, in contrast with local ergotropy, nonincreasing under time evolution induced by the free Hamiltonian H_{SE},

$$\mathcal{E}_{\mathrm{ex}}(\rho_{\mathrm{SE}}(t), H_{\mathrm{SE}}) \leq \mathcal{E}_{\mathrm{ex}}(\rho_{\mathrm{SE}}(0), H_{\mathrm{SE}}), \quad \forall t \geq 0. \tag{7}$$

Indeed, since time evolution is part of the $\bar{\mathcal{U}}_{\mathrm{ex}}$ set, ELE can only decrease in time, proving (7). This is a property generically demanded to a quantum resource. For instance, entanglement is nonincreasing under LOCC operations. Analogously, ELE is nonincreasing under free SE evolution. Notably, the equality

$$\mathcal{E}_{\mathrm{ex}}(\rho_{\mathrm{SE}}(t), H_{\mathrm{SE}}) = \mathcal{E}_{\mathrm{ex}}(\rho_{\mathrm{SE}}(0), H_{\mathrm{SE}}), \quad \forall t \geq 0, \tag{8}$$

holds if the H_{SE} is bounded and has discrete spectrum. The proof [14] relies on the quasirecurrence of unitary evolutions [15,16]. In [17] estimates and upper bounds for such recurrence time are reported. Nevertheless, these values are excessively long to be relevant in any realistic scenario. Consequently, in any practical situation, one should consider ELE as a quantity that strictly decreases over time. Interestingly, it can also be shown [14] that a continuous Hamiltonian allows for irreversible flows of energy from S to E, making ELE strictly decreasing in time.

Bang-bang representation—Bang-bang control procedures are obtained by abruptly alternating between two different types of Hamiltonian drivings. Such evolutions proved to be very effective in generating optimal control pulses in different contexts of quantum information [18,19]. In our case we can show that, as long as the SE compound is finite dimensional, all possible elements of $\bar{\mathcal{U}}_{\mathrm{ex}}$ can be generated via bang-bang sequences formed by free evolutions of the system and by strong driving pulses on S [14]. Indeed, if the dimension of the Hilbert of SE is finite, quantum control theory assesses that $\bar{\mathcal{U}}_{\mathrm{ex}}$ is a compact and connected Lie group given by the exponential of the dynamical Lie algebra [20,21]. Accordingly, it follows that given $U \in \bar{\mathcal{U}}_{\mathrm{ex}}$ it can be expressed as

$$U = U_{\mathrm{S}}^{(\mathcal{N}-1)} U_0(\delta t_{\mathcal{N}-1}) \ldots U_{\mathrm{S}}^{(0)} U_0(\delta t_0), \tag{9}$$

obtained by alternating unitary operations on the system, $U_{\mathrm{S}}^{(0)}, \ldots, U_{\mathrm{S}}^{(\mathcal{N}-1)} \in \mathcal{U}_{\mathrm{S}}$ with time intervals δt of free-time evolution, defined by the operator $U_0(\delta t) \coloneqq \exp(-i \delta t H_{\mathrm{SE}})$. Further, the number of necessary unitaries $\mathcal{N}$ is uniformly bounded. This result does not always apply to infinite-dimensional models. For completeness, however, as an example of the treatment in the infinite-dimensional discrete case, we will analyze the Jaynes-Cummings model. Systems with continuous spectra, instead, warrant a separate analysis that goes beyond the scope of this work.

Saturation of the global entropy—While under general conditions the gap between ELE and GE is expected to be a strict one, for some models the two values may coincide. An example is provided by a 1D chain of M 1/2-spins with Heisenberg-like nearest neighbor interaction,

$$H_{\mathrm{SE}} = \sum_{n=1}^{M-1} \gamma \left(\sigma_x^{(n)} \sigma_x^{(n+1)} + \sigma_y^{(n)} \sigma_y^{(n+1)} + \Delta \sigma_z^{(n)} \sigma_z^{(n+1)} \right),$$

with $\sigma_{x,y,z}^{(i)}$ describing the Pauli operators of the ith chain element. In this case, identifying the first spin element with S and the remaining ones with the environment E, using the results of quantum control [22], one can show that $\mathcal{E}_{\mathrm{ex}}(\rho_{\mathrm{SE}}, H_{\mathrm{SE}}) = \mathcal{E}(\rho_{\mathrm{SE}}, H_{\mathrm{SE}})$ via appropriate manipulations of the local magnetic field acting on S [14]. Interestingly, for $\Delta = 0$, controlling instead the first two spins even allows efficient control (i.e., quadratically longer than for direct control) [23].

A further example of a system for which ELE and GE can coincide is represented by the case where S is a two-level (qubit) system interacting with a single electromagnetic cavity mode via a Jaynes-Cummings (JC) interaction [24]. In this case the Hamiltonian of the SE compound is given by (setting $\hbar = 1$)

$$H_{\mathrm{JC}} \coloneqq \omega_{\mathrm{S}} \frac{\sigma_z + \mathbb{1}}{2} + \omega_{\mathrm{E}} a^\dagger a + \frac{\Omega}{2} (\sigma^+ \otimes a + \sigma^- \otimes a^\dagger), \tag{10}$$

where $a^\dagger$ (a) is the bosonic creation (annihilation) operator of the cavity mode, ω_{S} and ω_{E} are, respectively, the atom's gap energy and the cavity's frequency, and $\sigma^\pm \coloneqq (\sigma_x \pm i\sigma_y)/2$. Let $\Delta\omega \coloneqq \omega_{\mathrm{S}} - \omega_{\mathrm{E}}$ be the detuning and $\phi_n \coloneqq \frac{1}{2}\arctan(\sqrt{n+1}\Omega/\Delta\omega)$. For $\sqrt{\Delta\omega^2 + \Omega^2} < (\omega_{\mathrm{E}} + \omega_{\mathrm{S}})$, the Hamiltonian's ground state is $|00\rangle_{\mathrm{SE}}$ with eigenvalue $E_0 = 0$, where $|0\rangle_{\mathrm{S}}$ and $|1\rangle_{\mathrm{S}}$ are the eigenstates of σ_z corresponding to eigenvalues -1 and 1, and $\{|n\rangle_{\mathrm{E}}\}_{n\in\mathbb{N}}$ are bosonic Fock states. The excited states are

$$\begin{aligned} |n+\rangle &= \cos\phi_n |1\rangle_{\mathrm{S}} \otimes |n\rangle_{\mathrm{E}} + \sin\phi_n |0\rangle_{\mathrm{S}} \otimes |n+1\rangle_{\mathrm{E}}, \\ |n-\rangle &= \sin\phi_n |1\rangle_{\mathrm{S}} \otimes |n\rangle_{\mathrm{E}} - \cos\phi_n |0\rangle_{\mathrm{S}} \otimes |n+1\rangle_{\mathrm{E}}, \end{aligned} \tag{11}$$

with corresponding eigenvalues

$$E_{n,\pm} = \omega_{\mathrm{S}}/2 + \omega_{\mathrm{E}}(n + 1/2) \pm \Delta\omega_n, \tag{12}$$

and $\Delta\omega_n \coloneqq \frac{1}{2}\sqrt{\Delta\omega^2 + (n+1)\Omega^2}$. For this model, we have shown [14] that local controls on the spin degrees of freedom are enough to have approximate density matrix controllability on the joint system. More precisely, for each pair of unitarily equivalent states $\rho_{\mathrm{SE}}, \sigma_{\mathrm{SE}}$, $\forall\ \epsilon > 0$ and for almost all values of the coupling constant Ω [25–29]:

$$\exists\, U \in \mathcal{U}_{\mathrm{ex}}(H_{\mathrm{JC}}) \text{ s.t. } \mathrm{tr}[(U\rho_{\mathrm{SE}}U^\dagger - \sigma_{\mathrm{SE}})^2] \le \epsilon. \tag{13}$$

This means we can always bring the initial state arbitrarily close to its passive state $\rho_{\mathrm{SE}}^{\mathrm{P}}$. However, since H_{JC} is unbounded this does not guarantee that the energy of the final state can be brought arbitrarily close to $\mathrm{tr}[H_{\mathrm{JC}}\rho_{\mathrm{SE}}^{\mathrm{P}}]$. Despite this, for any finite-dimensional approximation of the compound, we have $\mathcal{E}(\rho_{\mathrm{SE}}, H_{\mathrm{SE}}^{(N)}) = \mathcal{E}_{\mathrm{ex}}(\rho_{\mathrm{SE}}, H_{\mathrm{SE}}^{(N)})$, where $H_{\mathrm{SE}}^{(N)}$ is a truncation of H_{JC} obtained by only allowing up to N photons in the cavity. However, it is not clear how to construct such a unitary, if there exist upper bounds on the number of operations or on the time needed to implement it. For this reason, in the next section we show for different classes of initial states the amount of work that can be extracted with minimal protocols, involving a small number of operations.

Practical protocols for the JC model—As a first example, we consider the case where the two-level system of the JC model is in the ground state $|0\rangle$ and the cavity mode is in a Fock state $|n+1\rangle$. Here, for zero detuning $\Delta\omega \coloneqq 0$, we can alternate work extraction unitaries on the system (bit flips) with free time evolutions to reexcite the atom and extract one by one all the photons stored in the cavity. This makes the bang-bang protocol saturating the ELE and the GE of SE. Let $\delta t_n \coloneqq (\pi/2\Delta\omega_n)$ and $U_{\mathrm{S}}^{(\mathrm{bf})}$ be the bit-flip on qubit S, then [14]

$$U_{\mathrm{S}}^{(\mathrm{bf})} U_0(\delta t_0) U_{\mathrm{S}}^{(\mathrm{bf})} \ldots U_{\mathrm{S}}^{(\mathrm{bf})} U_0(\delta t_n) |\psi_{\mathrm{in}}\rangle_{\mathrm{SE}} = |00\rangle_{\mathrm{SE}}. \tag{14}$$

This implies

$$\mathcal{E}_{\mathrm{ex}}(|0\rangle_{\mathrm{S}} \otimes |n+1\rangle_{\mathrm{E}}) = \mathcal{E}(|0\rangle_{\mathrm{S}} \otimes |n+1\rangle_{\mathrm{E}}) = (n+1)\omega_{\mathrm{S}}, \tag{15}$$

proving that ELE can saturate the ergotropy of SE. We remark that, instead, for any n and value of the detuning, the LE is zero, $\mathcal{E}_{\mathrm{S}}(|0\rangle_{\mathrm{S}}|n\rangle_{\mathrm{E}}) = 0$. All this is depicted in Fig. 1 where we plot the work extracted in terms of the number of steps $\mathcal{N}$ in the bang-bang protocol for input state $|0, n\rangle$. ELE reaches the GE value for $\mathcal{N} = n$. For completeness, we report work extracted with a generalization of the protocol described in Eq. (14) [14] for different input states. We consider qubit in the ground state and cavity in a coherent state $|0, \alpha\rangle$ and the eigenstate of the Hamiltonian of SE, $|m+\rangle$, with initial parameters (n, α, and m) such that initial states have approximately the same GE [30]. Also for these two examples, the final value of the ELE is considerably high, despite not reaching GE performances.

Thermal states—We now move on to the case in which the qubit and the cavity of the JC model are in a tensor product of two thermal states. We then set

$$\rho_{\mathrm{in}} = \gamma_{\beta_{\mathrm{S}}} \otimes \gamma_{\beta_{\mathrm{E}}}, \tag{16}$$

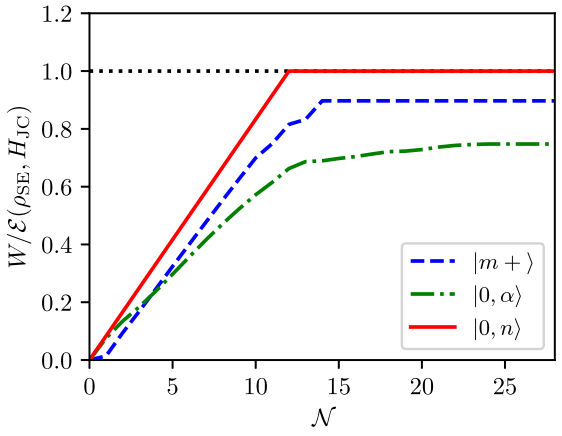

FIG. 1. Lower bound W for extended local ergotropy in the case of the Jaynes-Cummings model, obtained using a slight generalization of the bang-bang unitary of Eq. (14) (see [14] for details). We plot the ratio between W and the GE, as a function of the number of steps $\mathcal{N}$. We consider different input states: $|0,n\rangle$, qubit in the ground state and cavity in a Fock state (red); $|0,\alpha\rangle$, qubit in the ground state and cavity in a coherent state (green dash-dotted); eigenstate $|m+\rangle$ of the Hamiltonian of SE (blue dashed). Initial resources are such that initial states have approximately the same GE [30]. With input $|0,n\rangle$ the bang-bang protocol is optimal even in saturating the GE bound. We have set $n = |\alpha|^2 = 12$, $m = 11$, $\omega_S = \omega_E = 1$, $\Omega = 0.1$.

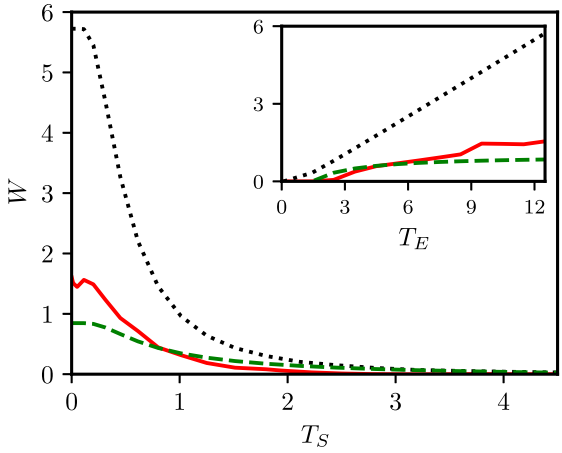

FIG. 2. Work W extracted via $U_{bb}(T_S, T_E)$ (red) and maximum extractable ergotropy, as defined in Eq. (52) of Ref. [7] (green dashed) for a qubit at temperature T_S and bath at temperature T_E. Both curves are computed for varying T, while we fixed T_E to the value corresponding to 12 average photons. Inset: work extracted via $U_{bb}(0, T_E)$ (red) and maximum extractable ergotropy (green dashed) for a qubit in the ground state as a function of T_E. The work extracted via the considered bang-bang protocols (red curves) represents a lower bound for ELE. We also report GE values (black dotted). We have set $k_B = 1$ and $\omega_S = \omega_E = 1$, $\Omega = 0.1$. Each $U_{bb}(T_S, T_E)$ requires less than 100 local operations.

where the two thermal states refer to the respective local Hamiltonians and $\beta_X := 1/(k_B T_X)$ are the inverse temperatures. First of all, as for the Fock initial state, we have zero LE, $\mathcal{E}_S(\gamma_{\beta_S} \otimes \gamma_{\beta_E}, H_{SE}) = 0$ [31]. This means that any observed nonzero ELE implies an advantage with respect to LE. In this context it is also natural to compare ELE performances with respect to the ergotropic extraction framework defined in Ref. [7] which is based on thermal operations.

In order to fairly make comparisons with the latter, we will suppose that we can neglect the interaction when evaluating the energy functionals, this makes the energy functional depending only on the local states. However, this does not mean that the existence of the interaction is not important. Indeed, the set of allowed unitary operations $\mathcal{U}_{ex}(H_{SE})$ is independent of the strength of the interaction [32]. Moreover, it is important to note that there is, in general, no trivial ordering between the optimal protocols of the ELE and EE settings, because the set of unitary energy-preserving maps on SE is neither bigger nor smaller than $\bar{\mathcal{U}}_{ex}(H_{SE})$.

For the considered JC model, we show in Fig. 2 that ELE exceeds for large intervals of temperatures the extracted ergotropy even for simple suboptimal protocols. We plot as a function of the qubit temperature T_S the work extracted with an appropriate unitary operation $U_{bb}(T_S, T_E) \in \bar{\mathcal{U}}_{ex}$ (with fixed cavity temperature T_E). In the inset we plot the work extracted by $U_{bb}(0, T_E) \in \bar{\mathcal{U}}_{ex}$ as a function of the cavity temperature T_E, while $T_S = 0$. The unitaries $U_{bb}(T_S, T_E)$ of Fig. 2 where obtained via a bang-bang protocol of the form (9). Here, time steps δt_k are aimed to maximize the ergotropy $\mathcal{E}(\rho_S, H_S)$ of the subsystem S via free SE evolution given by $U_0(\delta t_k)$. Local terms $U_S^{(k)}$ implement either the corresponding local unitary that maximizes (local) work extraction or a random energy-preserving local unitary. The latter is implemented only in case free evolution did not act (or equivalently acted with a $\delta t_k = 0$) because it could not enhance the ergotropy of the subsystem S on the given input state [14]. We emphasize that unitaries $U_{bb}(T_S, T_E)$ are not optimal, meaning that the red lines in Fig. 2 represent a lower bound for $\mathcal{E}_{ex}(\gamma_{\beta_S} \otimes \gamma_{\beta_E})$. In this setting, GE (and hence ELE) becomes zero when the system and the bath reach the same temperature. This comes as a consequence of the weak-coupling assumption and from the passivity of the compound thermal state with respect to the noninteracting Hamiltonian $H_S + H_E$. At lower temperatures T_S, the main plot in Fig. 2 shows how ELE is greater than EE for a considerable temperature interval. Furthermore, we notice that in the assumed weak-coupling regime, we can define work and heat as follows. Given a certain protocol for energy extraction, i.e., given a $U \in \bar{\mathcal{U}}_{ex}$, the work extracted is $W = \mathrm{tr}[H_{SE}(\rho_{in} - U\rho_{in}U^\dagger)]$ and the heat is the energy difference of the thermal bath $Q = \mathrm{tr}[H_E(\rho_{in} - U\rho_{in}U^\dagger)]$. Protocols implemented in Fig. 2 are also characterized by considerably high W/Q ratio (in our case W and Q are both positive). Indeed, the final local state $\rho_{out,S} := \mathrm{tr}_E(U\rho_{in}U^\dagger)$

is a (completely) passive state, which implies $\mathrm{tr}(\rho_{\mathrm{out,S}} H_{\mathrm{S}}) \leq \omega_{\mathrm{S}}/2$. This means that the heat exchanged is always smaller than $W + \omega_{\mathrm{S}}/2$. For example, we get $W/Q \geq 0.763$ for a cavity temperature corresponding to 12 photons on average and the qubit initially in the ground state (last point in the inset of Fig. 2).

Discussion—We have investigated maximal work extraction from a system interacting with an environment via manipulation of the local Hamiltonian on S. We introduced an extended version of local ergotropy that exploits the free evolution of the SE compound. We showed that extended local ergotropy has conceptual and practical advantages with respect to local ergotropy [11] and ergotropy extraction via thermal operations [7]. Most notably, it is not increasing under free time evolution and rules out arbitrary control of the environment or its interaction with the system. We then quantified the performances and efficiency of extended local ergotropy in the Jaynes-Cummings model, using both analytical tools from quantum control theory and numerical calculations on practical suboptimal protocols.

Finally, we notice that our work leaves open questions addressable via quantum control methods, such as identifying the shortest amount of time or number of local unitaries needed to perform the optimal protocol or those Hamiltonians for which ELE saturates the global ergotropy.

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[12] The topological closure of a set X, here denoted as $\bar{X}$, is defined as the smallest closed set that contains X. In our case it means that the unitaries that can be approximated arbitrarily well by unitaries of $\mathcal{U}_{\mathrm{ex}}$ belong to $\bar{\mathcal{U}}_{\mathrm{ex}}$. For instance, local unitaries are part of the $\bar{\mathcal{U}}_{\mathrm{ex}}$. Indeed, if $U_{\mathrm{S}} = e^{-i\tau H'_{\mathrm{S}}} = \lim_{n\to\infty} \mathcal{T} \exp[-i \int_0^{\tau/n} H_{\mathrm{SE}} + n H'_{\mathrm{S}} dt]$.
[13] It is worth mentioning that in a real experiment, Hamiltonian driving should also respect the cyclic condition $H_{\mathrm{S}}(0) = H_{\mathrm{S}}(t_f) = 0$. However, this constraint is irrelevant to the mathematical definition of $\mathcal{U}_{\mathrm{ex}}$.
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[30] For the coupling strength used to generate Fig. 1, $\Omega = 0.1$ we have $\mathcal{E}(|0\rangle_{\mathrm{S}} \otimes |12\rangle_{\mathrm{E}}) = \mathcal{E}(|0, \alpha = \sqrt{12}\rangle_{\mathrm{SE}}) = 12$ and $\mathcal{E}(|11+\rangle_{\mathrm{SE}}) = 12 + \sqrt{3}/10$.
[31] For factorized states one can prove [11] that $\mathcal{E}_{\mathrm{S}}(\rho_{\mathrm{S}} \otimes \rho_{\mathrm{E}}, H_{\mathrm{S}} + H_{\mathrm{E}} + V_{\mathrm{SE}}) = \mathcal{E}(\rho_{\mathrm{S}}, H'_{\mathrm{S}})$, where $H'_{\mathrm{S}} = H_{\mathrm{S}} + \mathrm{tr}_{\mathrm{E}}[V_{\mathrm{SE}} \rho_{\mathrm{E}}]$. In our case the interaction is linear in creation and destruction operators, therefore $\mathrm{tr}_{\mathrm{E}}[V_{\mathrm{SE}} \gamma_{\beta_{\mathrm{E}}}] = 0$, leading to $\mathcal{E}_{\mathrm{S}}(\gamma_{\beta_{\mathrm{S}}} \otimes \gamma_{\beta_{\mathrm{E}}}, H_{\mathrm{S}} + H_{\mathrm{E}} + V_{\mathrm{SE}}) = \mathcal{E}(\gamma_{\beta_{\mathrm{S}}}, H_{\mathrm{S}}) = 0$.
[32] The dynamical Lie algebra $\mathcal{L}$ generated by λV_{AB} and local Hamiltonians on S is independent of λ as long as $\lambda \neq 0$, see [14].