The Two-Body Problem in Classical General Relativity: PM, PN, MPM, MAE, ADM, GSF, EOB, EFT, NRGR, FWF, TF, ACV, BCRSSZ and All That!

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EFT Methods from Bound States to Binary Systems

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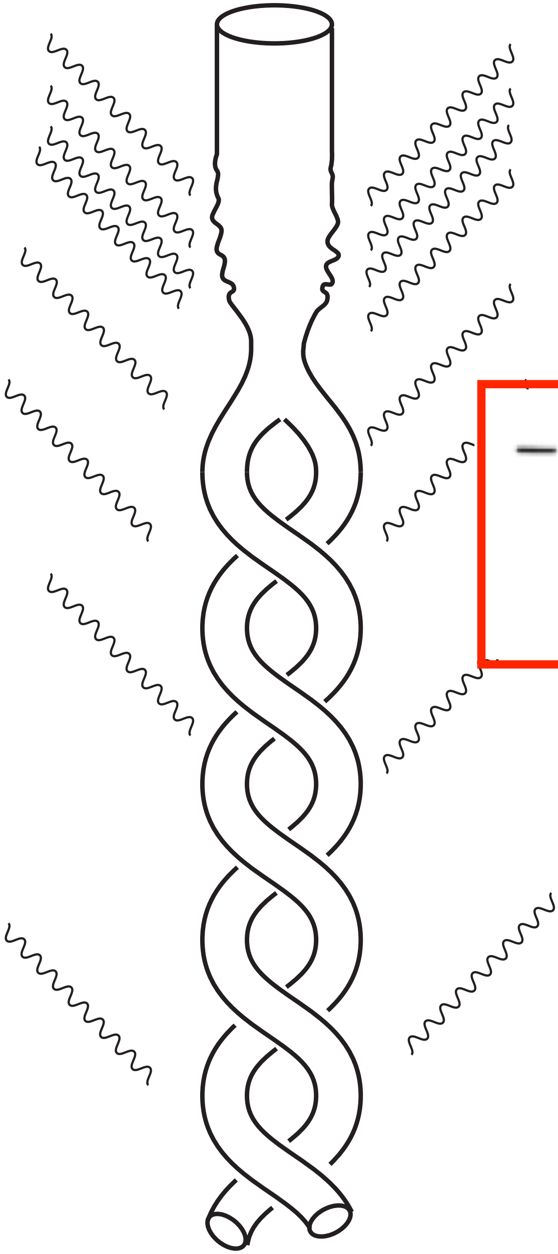
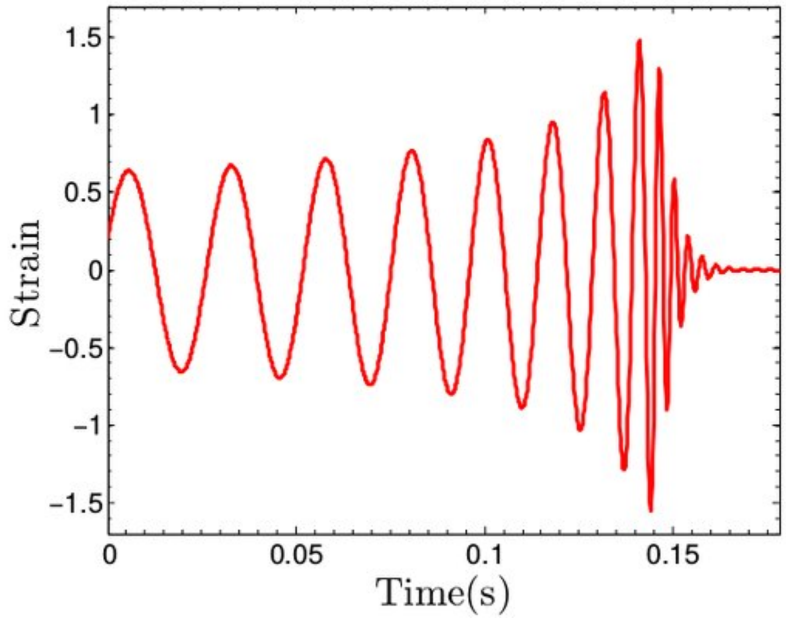
$$ds^2 = g_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$



$$\boxed{R_{\mu\nu} = 0}$$

$$\boxed{\begin{aligned} -g^{\mu\nu} g_{\alpha\beta,\,\mu\nu} + g^{\mu\nu} g^{\rho\sigma} \big(& g_{\alpha\mu,\,\rho}\, g_{\beta\nu,\,\sigma} - g_{\alpha\mu,\,\rho}\, g_{\beta\sigma,\,\nu} \\ & + g_{\alpha\mu,\,\rho}\, g_{\nu\sigma,\,\beta} + g_{\beta\mu,\,\rho}\, g_{\nu\sigma,\,\alpha} - \frac{1}{2} g_{\mu\rho,\,\alpha}\, g_{\nu\sigma,\,\beta} \big) = 0 \end{aligned}}$$



Analytical Tools used for the 2-body pb in classical GR

PM=Post-Minkowskian expansion in G/c^2

PN=Post-Newtonian expansion in 1/c^2= G/c^2, v^2/c^2 or c^(-2)d^2/dt^2

MPM= Multipolar post-Minkowskian expansion+ PN-matching

MAE= Matched Asymptotic Expansion

ADM= Arnowitt-Deser-Misner Hamiltonian approach to GR

GSF= Gravitational Self Force: expansion in m1/m2

EOB= Effective One-Body approach

EFT= Effective Field Theory approach

NRGR= Non-Relativistic General Relativity

TF= Tutti Frutti approach

FWF= Fokker-Wheeler-Feynman approach
ACV= Amati-Ciafaloni-Veneziano
BCRSSZ= Bern, Cheung, Roiban, Shen, Solon, Zeng

The GR two-body problem (1)

1912-1916: Einstein introduced both the **PM, nonlinearity expansion**:

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + G h_{1\mu\nu} + G^2 h_{2\mu\nu} + \cdots}$$

and the **PN expansion**: v/c <<1, T^ij << T^0i << T^00; hence h_0i << h_00,…

Droste 1912-1916 develops the PN expansion, using

$$\frac{1}{c}\frac{\partial}{\partial t} \ll \frac{\partial}{\partial x}$$

$$\Box = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Delta + O\left(\frac{1}{c^2}\right)$$

Droste-Lorentz 1917, Einstein-Infeld-Hoffmann 1938:
1PN-accurate dynamics (and Lagrangian) of 2-body systems

$$\mathrm{L_{PN}}(\boldsymbol{r}(t), \boldsymbol{r}'(t), \boldsymbol{v}(t), \boldsymbol{v}'(t)) = \mathrm{L_N} + \frac{1}{c^2}\mathrm{L_2} \qquad (2.1\,a)$$

with

$$\mathrm{L_N} = \frac{1}{2}mv^2 + \frac{1}{2}m'v'^2 + \frac{Gmm'}{\mathrm{R}} \qquad (2.1\,b)$$

$$\mathrm{L_2} = \frac{1}{8}mv^4 + \frac{1}{8}m'v'^4 + \frac{Gmm'}{2\mathrm{R}}\left[3v^2 + 3v'^2 - 7(vv') - (\mathrm{N}v)(\mathrm{N}v') - \mathrm{G}\frac{m+m'}{\mathrm{R}}\right] \qquad (2.1\,c)$$

The GR two-body problem (2)

Higher PN approximations ca.1970

(Chandrasekhar-Nutku'69, Chandraskhar-Esposito'70, Burke'69-70,Thorne'69, Ohta-Okamura-Kimura-Hiida'73)

IR difficulties at 2PN (v^4/c^4) and 2.5PN (v^5/c^5): incomplete and inconclusive results at 2PN and 2.5PN

Root of IR difficulties: general retarded wave

$$\boxed{\begin{aligned} &\Box\phi = 0 \\ &\phi(t,\mathbf{x}) = \sum_\ell \partial_x^\ell \left(\frac{S(t-\frac{r}{c})}{r} \right) \\ &= \sum_\ell \partial_x^\ell \left(\frac{S(t))}{r} - \frac{1}{c}\dot{S}(t) + \frac{1}{2c^2}\ddot{S}(t)r - \frac{1}{6c^3}\dddot{S}(t)r^2 \cdots \right) \end{aligned}}$$

Burke (69-70) suggested to use **Matched Asymptotic Expansions** to have a well-defined matching between nearzone and wavezone gravitationa fields, and to derive the Radiation-Reaction force acting on the system. However, his implementation was flawed (see Blanchet-TD'84)

The GR two-body problem: PM comes back

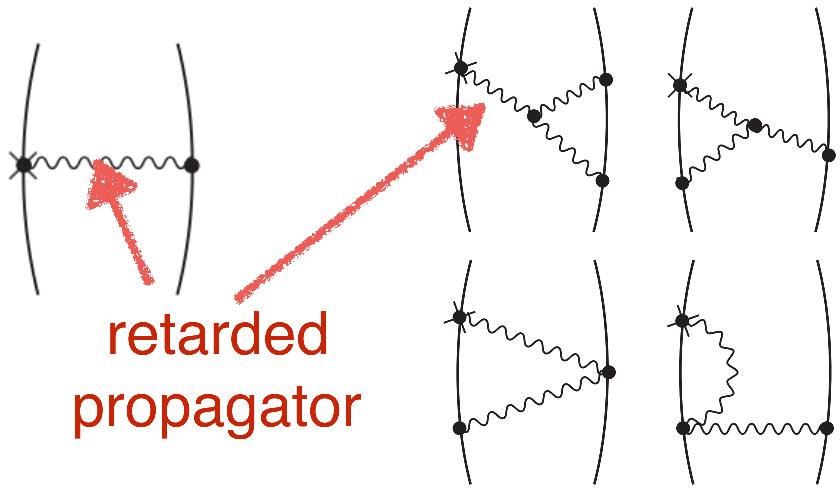
September 1974: Discovery binary pulsar PSR1913+16 (Hulse-Taylor'75)
An observational handle on gravitational radiation-reaction (Wagoner'75)

December 1978: 9th Texas Symposium (Munich):
J. H. Taylor announces that the orbital-period of PSR1913+16 decreases as:
dP_b/dt = (1.33 +/- 0.25) [dP_b/dt]_Quadrupole Formula

Unsatisfactory aspects of the then-existing « derivations » of the dynamics of binary systems in GR (emphasized by J. Ehlers and others):

Divergences appear in the 2.5PN expansion (Chandrasekhar-Esposito'70)
Incomplete treatment of nonlinear effects in the NZ-WZ matching (Burke-Thorne'69)
Inapplicability of weak-field PN to compact objects
No explicit derivation of the (conservative) 2PN eqs dynamics
Lack of clear proof of a balance between system's mechanical energy loss and GW flux

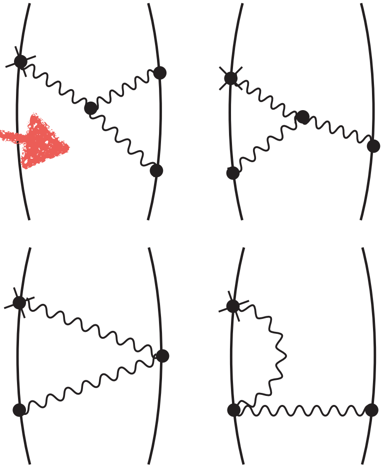
Motivates a PM-based approach to 2-body dynamics including radiation-reaction
(Rosenblum'78, Westpfahl'79, Bel-TD-Deruelle-Ibanez-Martin'81)



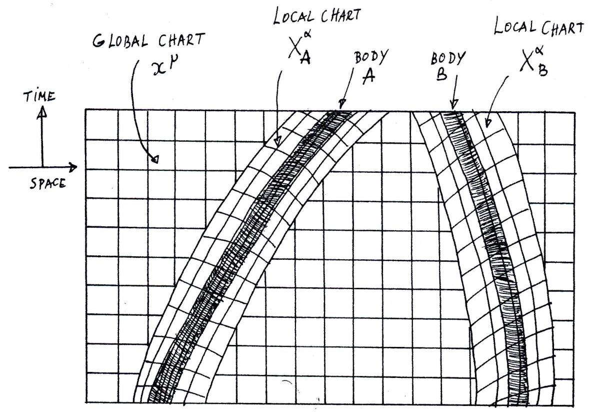
Eclectic approach to the 2-body pb at G^3 and 1/c^5

(TD-Deruelle'81, TD'82, using Bel et al.'81)

$$G_{\rm ret} = \Box^{-1}_{\rm ret}$$



Use of PM approximation: G^2 + part of G^3
Eqs of motion (because non conservative)
Followed by PN expansion of PM
Proof that the O(G^2/c^5) mechanical angular momentum loss agrees with the radiated ang. mom. (TD-Deruelle'81)



Matched Asymptotic Expansion for compact bodies
Skeletonization + Use of analytic regularization (proven to be equivalent to dim.reg)
Introduction of Love number k of compact bodies
Proof that k_BH=0 —> Effacing Property
Presence of a pole in the harmonic-coord metric at 4PM-3PN: G^3 du/dt/epsilon

$$L_4(z,v,a) = M_4 + N_4 ,$$

$$\begin{aligned} M_4(z,v,a) = {} & \Sigma \left(\tfrac{1}{16} m v^6\right) + \\ & + \Sigma\, G m m' R^{-1} \left(\tfrac{7}{8} v^4 + \tfrac{15}{16} v^2 v'^2 - 2 v^2 (vv') + \tfrac{1}{8} (vv')^2 - \tfrac{7}{8} (Nv)^2 v'^2 + \right. \\ & \qquad \left. + \tfrac{3}{4} (Nv)(Nv')(vv') + \tfrac{3}{16} (Nv)^2 (Nv')^2 \right) + \\ & + \Sigma\, G^2 m^2 m' R^{-2} \left(\tfrac{1}{4} v^2 + \tfrac{7}{4} v'^2 - \tfrac{7}{4} (vv') + \tfrac{7}{2} (Nv)^2 + \tfrac{1}{2} (Nv')^2 - \tfrac{7}{2} (Nv)(Nv') \right) + \\ & + \Sigma\, G m m' \left((Na) \left(\tfrac{7}{8} v'^2 - \tfrac{1}{8} (Nv')^2 \right) - \tfrac{7}{4} (v'a)(Nv') \right) , \end{aligned}$$

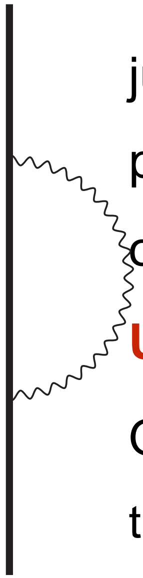
$$N_4(z) = \frac{G^3 m m'}{R^3} \left(\tfrac{1}{2} m^2 + \tfrac{1}{2} m'^2 + \frac{19}{4} m m' \right).$$

First obtention of the 2PN+2.5PN eom
2PN acceleration-dept Lagrangian
Derivation of the observable dP_b/dt from the dynamics without assuming energy balance

$$\dot{P}_0 \equiv k_1 \frac{\partial P(c_1{}^0, c_2{}^0)}{\partial c_1{}^0} + k_2 \frac{\partial P(c_1{}^0, c_2{}^0)}{\partial c_2{}^0}$$

Practical Techniques for Computing the Motion of Compact Bodies (NS or BH)

Skeletonization : $T_{\mu\nu} \longrightarrow$ **point-masses** (Mathisson '31) (**1st level of EFT**)



delta-functions in GR : Infeld '54, Infeld-Plebanski '60

justified by Matched Asymptotic Expansions (« Effacing Principle » Damour '83

possible internal-structure dependence in strong self-gravity objects (NSs, BHs)

only arise at 5PN= 5-loop level)

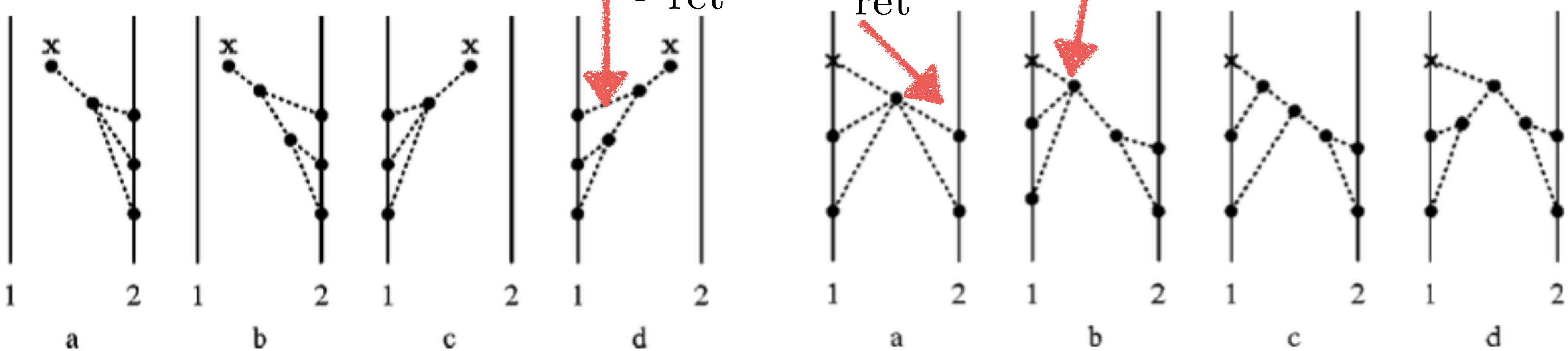
UV divergences linked to self-field effects (loops on external lines) [Dirac, 1938]

QFT's **analytic** (Riesz '49) **or dimensional regularization** (Bollini-Giambiagi '72, t'Hooft-Veltman '72) imported in GR (Damour '80, Damour-Jaranowski-Schäfer '01, …)

Feynman-like diagrams for iteratively solving Einstein's eqs; for iteratively computing the **retarded** eqs of motion; and for iteratively computing the FWF-type action

$$G_{\rm ret} = \Box^{-1}_{\rm ret}$$

Vertices defined by Einstein's action



Reduced Worldline Action in Electrodynamics

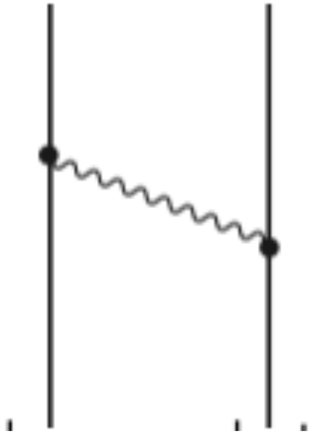
(Fokker 1929; Wheeler-Feynman 1949)

$$S_{\rm tot}[x_a^\mu, A_\mu] = -\sum_a \int m_a ds_a + \sum_a \int e_a dx_a^\mu A_\mu(x_a) - \int d^D x \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + S_{\rm gf}$$

« Integrate out » the field A_mu in the total (particle+field) action

$$\boxed{\begin{aligned} S_{\rm eff}^{\rm class}[x_a(s_a)] = &-\sum_a m_a \int {\rm d}s_a \\ &+ \frac{1}{2} \sum_{a,b} e_a e_b \iint {\rm d}x_a^\mu \, {\rm d}x_{b\mu} \, \delta\left((x_a - x_b)^2\right). \end{aligned}}$$

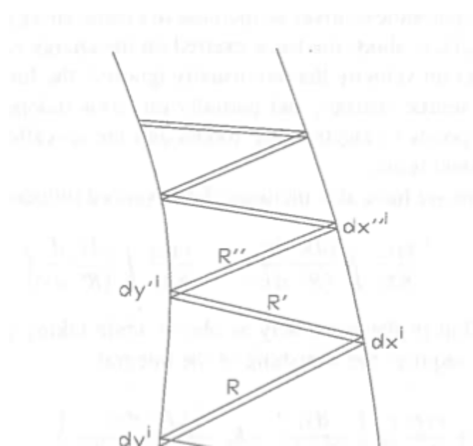
One-photon-exchange diagram

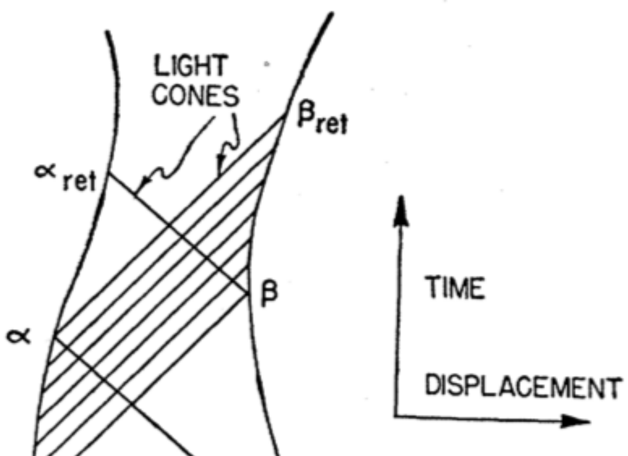


time-symmetric Green's function G.

$$G(x) = \delta(-\eta_{\mu\nu} x^\mu x^\nu) = \frac{1}{2r}\left(\delta(t-r) + \delta(t+r)\right) \; ; \; \Box G(x) = -4\pi\delta^4(x)$$

The effective action S_eff(x_a) was heavily used in the (second) **Wheeler-Feynman** paper (1949) together with similar diagrams to those used by Fokker





Reduced Action in Gravity and its PM Diagrammatic Expansion

PN: Infeld-Plebanski '60
PM:TD-Esposito-Farese '96

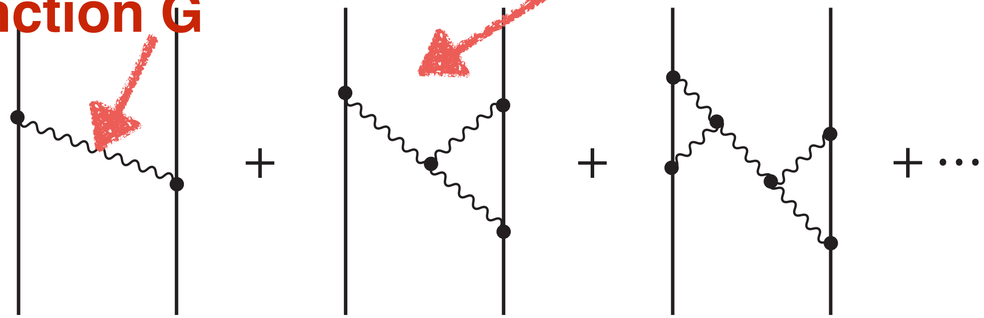
$$S_{\rm eff}^{\rm class}[x_a(s_a)] = [S_{\rm pm} + S_{\rm EH} + S_{\rm gf}]_{g_{\mu\nu}(x) \to g^{\rm gf}_{\mu\nu}[x_a(s_a)]}$$

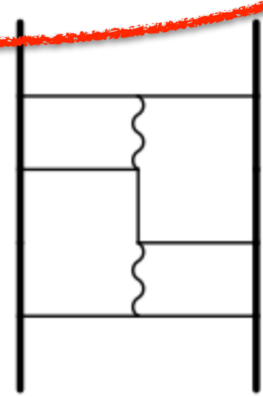
Needs gauge-fixed* action and time-symmetric Green function G.
*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.
Perturbatively solving (in dimension D=4 + eps) Einstein's equations
to get the equations of motion and the action for the conservative dynamics

$$g = \eta + h$$
$$S(h,T) = \int \left(\frac{1}{2} h \Box h + \partial\partial hhh + \ldots + (h + hh + \ldots) T \right)$$
$$\Box h = -T + \ldots \to h = GT + \ldots$$
$$S_{\rm red}(T) = \frac{1}{2} TGT + V_3(GT, GT, GT) + \ldots$$

$$\int ds_1 \int ds_2 \int ds'_2 (m_1 u_1 u_1)(m_2 u_2 u_2)(m_2 u'_2 u'_2)$$
$$\times \int d^4x \, \partial_x \partial_x \, G(x_1 - x) G(x - x_2) G(x - x'_2)$$

time-symmetric Green's function G





O(G)= 1PM= Newtonian + (v/c)^n corrections

O(G^2)=2PM =1 loop -> 1PN

O(G^3)=3PM =2 loop-> 2PN

O(G^5)=5PM =4 loop-> 4PN

Separating Conservative and Radiation-Reaction Effects

Within the PM approach: one used a PN-expansion of the PM dynamics to separate conservative and radiation-reaction effects; e.g. at the 2.5PN (v^5/c^5) accuracy it was explicitly shown that the 2PN dynamics derives (in harmonic coordinates) from a higher-derivative Lagrangian, while the G^2/c^5 + G^3/c^5 EOM terms where causing losses of mechanical E and L that agreed with the radiative losses (TD-Deruelle'81,TD'82).

$$a^i = A_0^i(z-z') + c^{-2}A_2^i(z-z',v,v') + c^{-4}A_4^i(z-z',v,v',S,S') + \\ + c^{-5}A_5^i(z-z',v-v') + O(c^{-6}),$$

dissipative

conservative 2PN

$$L(z,v,a) = L_0(z,v) + c^{-2}L_2(z,v) + c^{-4}L_4(z,v,a).$$

acceleration-dependent Lagrangian

dissipative
G^2/c^5 + G^3/c^5

$$A_o^i = -Gm'R^{-2}N^i, \tag{7}$$

$$A_2^i = Gm'R^{-2}(N^i(-v^2-2v'^2+4(vv')+\tfrac{3}{2}(Nv')^2+5(Gm/R)+4(Gm'/R))+(v^i-v'^i)(4(Nv)-3(Nv'))), \tag{8}$$

$$A_4^i = B_4^i + C_4^i, \tag{9}$$

with :

$$\begin{aligned} B_4^i = Gm'R^{-2}(N^i(&-2v'^4+4v'^2(vv')-2(vv')^2+\tfrac{3}{2}v^2(Nv')^2+\tfrac{9}{2}v'^2(Nv')^2-6(vv')(Nv')^2-\tfrac{15}{8}(Nv')^4 + \\ &+(Gm/R)(-\tfrac{15}{4}v^2+\tfrac{5}{4}v'^2-\tfrac{5}{2}(vv')+\tfrac{39}{2}(Nv)^2-39(Nv)(Nv')+\tfrac{17}{2}(Nv')^2) + \\ &+(Gm'/R)(4v'^2-8(vv')+2(Nv)^2-4(Nv)(Nv')-6(Nv')^2)) + \\ &+(v^i-v'^i)(v^2(Nv')+4v'^2(Nv)-5v'^2(Nv')-4(vv')(Nv) + \\ &+4(vv')(Nv')-6(Nv)(Nv')^2+\tfrac{9}{2}(Nv')^3 + \\ &+(Gm/R)(-\tfrac{63}{4}(Nv)+\tfrac{55}{4}(Nv'))+(Gm'/R)(-2(Nv)-2(Nv')))), \end{aligned} \tag{10}$$

$$C_4^i = G^3m'R^{-4}N^i(-\tfrac{57}{4}m^2-9m'^2-\tfrac{69}{2}mm'), \tag{11}$$

$$\begin{aligned} A_5^i = \tfrac{4}{5}G^2mm'R^{-3}(&V^i(-V^2+2(Gm/R)-8(Gm'/R)) + \\ &+N^i(NV)(3V^2-6(Gm/R)+\tfrac{52}{3}(Gm'/R))). \end{aligned} \tag{12}$$

Separating Conservative and Radiation-Reaction Effects

Within the ADM approach (Schaefer'85, Jaranowski-Schaefer'97)

Hamiltonian for matter + radiative dof obtained by integrating out the potential-mode-interactions by solving the constraints in a Coulomb-like gauge

$$\boxed{g^{-1/2}\left[gR+\frac{1}{2}(g_{ij}\pi^{ij})^2-\pi_{ij}\pi^{ij}\right]=\sum_a (g^{ij}p_{ai}p_{aj}+m_a^2)^{1/2}\delta_a \qquad -2\pi^{ij}{}_{|j}=\sum_a g^{ij}p_{aj}\delta_a}$$

$$g_{ij}=\left(1+\frac{1}{8}\phi\right)^4\delta_{ij}+h^{\mathrm{TT}}_{ij}$$

gravity analog of Coulomb gauge $\nabla\cdot\mathbf{A}=0 \leftrightarrow \partial_j h^{\mathrm{TT}}_{ij}=0$

$$\pi^{ii}=0,$$

Poincare invariance constraints

$$\boxed{H[\mathbf{x}_a,\mathbf{p}_a,h^{\mathrm{TT}}_{ij},\pi^{ij\mathrm{TT}}]=-\int d^3x\,\Delta\phi[\mathbf{x}_a,\mathbf{p}_a,h^{\mathrm{TT}}_{ij},\pi^{ij\mathrm{TT}}]}$$

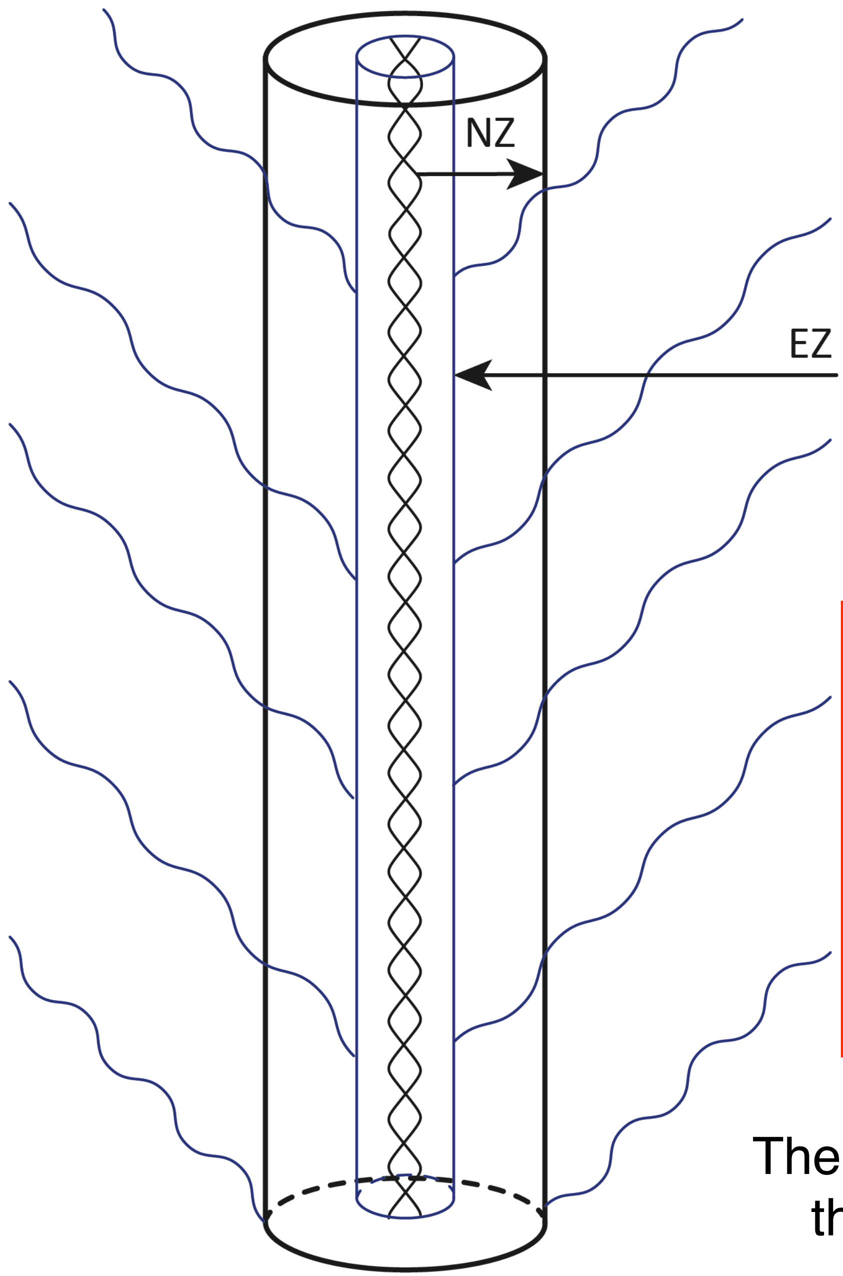
matter dof radiative dof

The 2 radiative dof h_ij^TT pi^{ijTT} satisfy wavelike equations:

$$\boxed{\Box_{d+1}h^{\mathrm{TT}}_{ij}=S^{\mathrm{TT}}_{ij}}$$

$$\boxed{\begin{aligned} S^{\mathrm{TT}}_{ij} = \delta^{\mathrm{TT}kl}_{ij}\Big\{ & S_{(4)kl}+2B_{(6)kl} \\ & +\frac{1}{d-1}\left(\phi_{(2)}\Delta_d h^{\mathrm{TT}}_{kl}+\Delta_d(\phi_{(2)}h^{\mathrm{TT}}_{kl})\right) \\ & +\frac{2(d-2)}{d-1}\partial_t(\phi_{(2)}\tilde{\pi}^{kl}_{(3)})\Big\}+\mathcal{O}(\epsilon^8). \end{aligned}}$$

MULTIPOLAR POST-MINKOWSKIAN FORMALISM (BLANCHET-DAMOUR-IYER)



Decomposition of space-time in various overlapping regions:

1. **near-zone: r << lambda : PN**
2. **exterior zone: r >> r_source: MPM**
3. **far wave-zone: Bondi-type expansion**

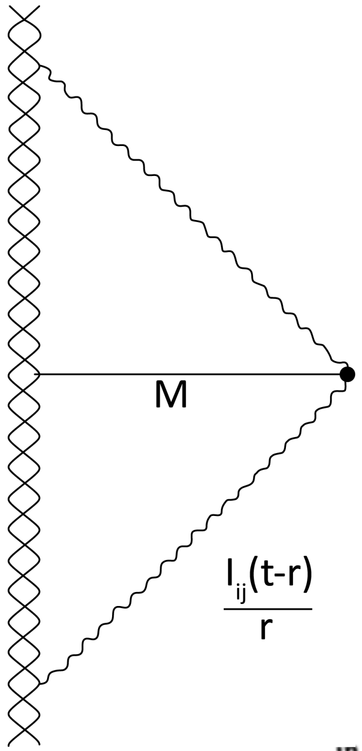
then matching between the zones

in exterior zone, iterative solution of Einstein's vacuum field equations by means of a double expansion in non-linearity and in multipoles, with crucial use of analytic continuation (complex B) for dealing with formal UV divergences at r=0

$$
\begin{aligned}
g &= \eta + G h_1 + G^2 h_2 + G^3 h_3 + \ldots, \\
\Box h_1 &= 0, \\
\Box h_2 &= \partial\partial h_1 h_1, \\
\Box h_3 &= \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2, \\
h_1 &= \sum_\ell \partial_{i_1 i_2 \ldots i_\ell} \left(\frac{M_{i_1 i_2 \ldots i_\ell}(t - r/c)}{r} \right) + \partial\partial \ldots \partial \left(\frac{\epsilon_{j_1 j_2 k} S_{k j_3 \ldots j_\ell}(t - r/c)}{r} \right), \\
h_2 &= FP_B \Box^{-1}_{\mathrm{ret}} \left(\left(\frac{r}{r_0} \right)^B \partial\partial h_1 h_1 \right) + \ldots, \\
h_3 &= FP_B \Box^{-1}_{\mathrm{ret}} \ldots
\end{aligned}
$$

The PN-matched MPM formalism has allowed to compute the GW emission to very high accuracy (Blanchet et al)

Nonlocality in time: Tail-transported hereditary effects (Blanchet-Damour ’88)



Hereditary (time-dissymetric) modification of the quadrupolar radiation-damping force, signalling a **breakdown of a basic tenet of PN expansion at the 4PN level**: (v/c)^8 fractional

$$g^{\rm in}_{00}(\mathbf{x},t) = -1+\frac{1}{c^2}\left[2\int \frac{d^3\mathbf{y}\,\rho(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}\right]+\frac{1}{c^4}\left[\partial_t^2 X - 2U^2 + 4\int \frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\rho\left[\mathbf{v}^2+U+\frac{\Pi}{2}+\frac{3p}{2\rho}\right]\right]$$

$$+\frac{1}{c^6}\,{}_6\hat{\Phi}_{00}+\frac{1}{c^7}\left[-\tfrac{2}{5}x_{ab}\,{}^{(5)}I_{ab}(t)\right]+\frac{1}{c^8}\,{}_8\hat{\Phi}_{00}+\frac{1}{c^9}\,{}_9\hat{\Phi}_{00}$$

$$+\frac{1}{c^{10}}\left[-\tfrac{8}{5}x_{ab}I(t)\int_0^{+\infty}dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}I_{ab}(t-v)+{}_{10}\hat{\Phi}_{00}\right]+\cdots .$$

generates a time-symmetric nonlocal-in-time 4PN-level action (Damour-Jaranowski-Schaefer’14) which was uniquely matched to the local-zone metric via the Regge-Wheeler-Zerilli-Mano-Suzuki-Takasugi- based work of Bini-Damour’13

$$H^{\rm nonloc}_{\rm 4PN}(t) = -\frac{1}{5}\frac{G^2M}{c^8}I^{(3)}_{ij}(t)\times \mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|}I^{(3)}_{ij}(t+v),$$

NRGR EFT approach (Goldberger-Rothstein'06…Gilmore-Ross, Foffa-Sturani, Porto, Levi,…)

The EFT approach has brought a new perspective on some important aspects of the 2-body problem (e.g. Goldberger-Ross'10, Foffa-Sturani'13, Cheung-Rothstein-Solon'18,…). It would be interesting to better delineate the advantages (and disadvantages) wrt the other approaches.

Use **v<<1** $h_{\mu\nu}(x) = \bar{h}_{\mu\nu}(x) + H_{\mu\nu}(x),$ $H_{\mu\nu}(x) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} H_{\mathbf{k}\mu\nu}(x^0),$

long-wavelength radiation field: omega \sim k \sim **v**/r12

potential gravitons

k \sim 1/r12

$$\boxed{\exp[iS_{\rm NRGR}[x_a, \bar{h}]] = \int \mathcal{D}H_{\mu\nu} \exp[iS[\bar{h} + H, x_a] + iS_{\rm GF}],}$$

matter dof

radiation field

potential gravitons with 2-point function

$$\mathcal{P}(\mathbf{k}^2, t_a, t_b) = \frac{i}{\mathbf{k}^2 - \partial_{t_a}\partial_{t_b}} \simeq \frac{i}{\mathbf{k}^2}\left(1 + \frac{\partial_{t_a}\partial_{t_b}}{\mathbf{k}^2} + \frac{\partial^2_{t_a}\partial^2_{t_b}}{\mathbf{k}^4} + \dots\right)$$

both nearzone expanded and time-symmetric ??

Then ?

$$\boxed{\exp(iS_{\rm eff}[x_a]) = \int \mathcal{D}\bar{h} \exp[iS_{\rm NRGR}(x_a, \bar{h})]}$$

Complex eff. action

with Feynman propagator ? (Foffa-Sturani'13)

A tale of many Green's functions

$$G_{\rm ret}(x) = \frac{\delta(t - r/c)}{r} \qquad\qquad G_{\rm ret} = \mathrm{P}\frac{1}{k^2} + i\pi \mathrm{sign}(k^0)\delta(k^2)$$

$$G_{\rm sym}(x) = \frac{\delta(t - r/c) + \delta(t + r/c)}{2r} \qquad\qquad G_{\rm sym} = \mathrm{P}\frac{1}{k^2}$$

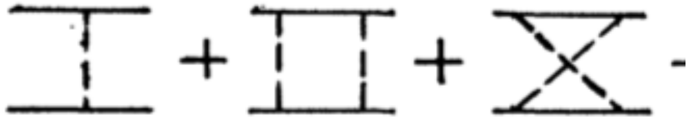
$$G^{\rm PN}_{\rm sym}(x) = \frac{\delta(t)}{r} + \frac{r}{2c^2}\ddot{\delta}(t) + \cdots \qquad\qquad G^{\rm PN}_{\rm sym} = \frac{1}{\mathbf{k^2}} + \frac{\omega^2}{c^2\mathbf{k^4}} + \cdots$$

$$G_{\rm F}(x) = \frac{i}{\pi(t^2 - r^2 + i0)} \qquad\qquad G_{\rm F} = \mathrm{P}\frac{1}{k^2} + i\pi\delta(k^2)$$

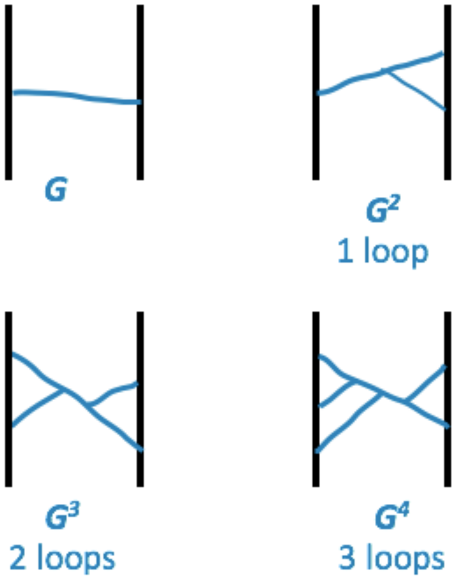
+ issues of: <in,out>; <in,in>, FWF, Schwinger-Keldysh,…

Effective One-Body (EOB) approach: H + Rad-Reac Force

Historically rooted in QM: Brezin-Itzykson-ZinnJustin'70
eikonal scattering amplitude+ Wheeler's:`Think quantum mechanically'

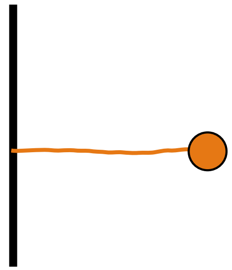


Real 2-body system
(in the c.o.m. frame)



An effective particle of mass mu in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$



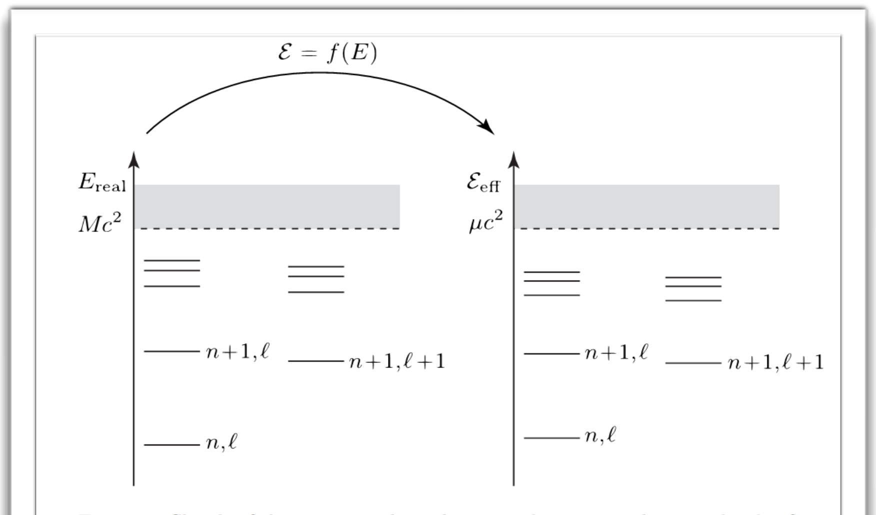
$$\boxed{g_{\rm eff}^{\mu\nu}(X)}$$

mass-shell constraint

$$\boxed{0 = g_{\rm eff}^{\mu\nu}(X) P_\mu P_\nu + \mu^2 + Q(X,P)}$$

Level correspondence
in the semi-classical limit:
Bohr-Sommerfeld ->
identification of
quantized action variables

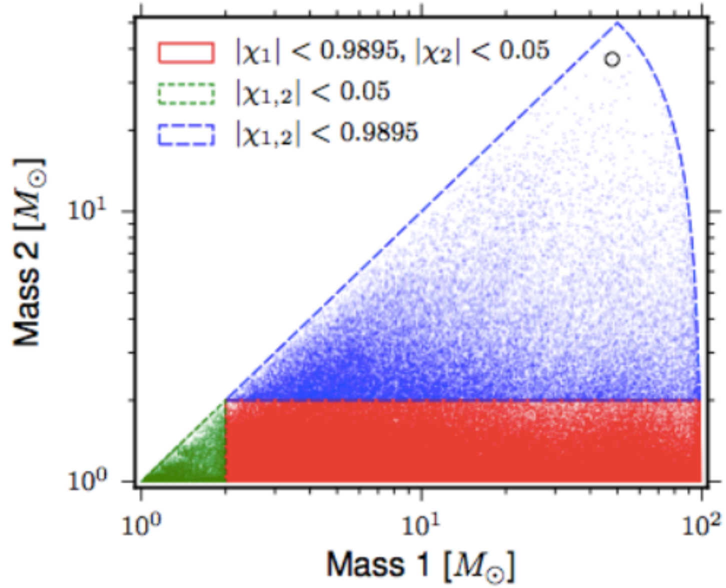
$$J = \ell\hbar = \frac{1}{2\pi}\oint p_\varphi d\varphi$$
$$N = n\hbar = I_r + J$$
$$I_r = \frac{1}{2\pi}\oint p_r dr$$



Crucial energy map

$$\boxed{\mathcal{E} = f(E)}$$

LIGO's bank of EOB search templates
(Taracchini et al.'14, Bohé et al.'17, Ossokine et al.'20, Nagar et al.'20)



SF

$m_1 \ll m_2$

NR

MPM

PN

$v \ll c$

$R \gg GM/c^2$

PM

$R \gg GM/c^2$

EOB

LISA's templates via EOB[SF] ?

Classical Scattering

Experimental Mathematics, Harmonic Polylogarithms
Bini-TD-Geralico-Laporta-Mastrolia'20

EFT

$v \sim c$

$R \sim GM/c^2$

QFT
perturbation theory

STRING
perturbation theory

Quantum Scattering Amplitudes

EOB= Effective-One-Body
Buonanno-Damour 1999, 2000;
Damour-Jaranowski-Schaefer 2000;
Damour 2001

Recent Advances

5PN

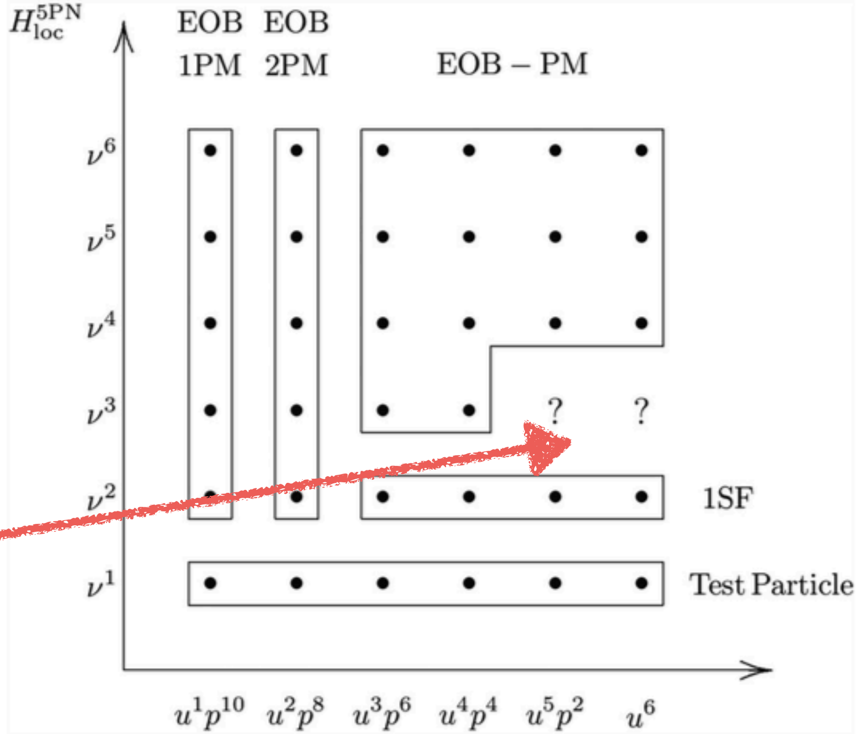
Tutti Frutti

(Bini-TD-Geralico):

combines many tools:
PN,PM,MPM,GSF,EFT,
EOB,Delaunay

determines 5PN and 6PN dynamics modulo a few undetermined parameters

$$\boxed{a_6^{\nu^2}, \bar{d}_5^{\nu^2}}$$



Bluemlein-Maier-Marquard-Schaefer 2010.13672
5PN potential contributions from EFT approach

$$\boxed{\begin{aligned} \bar{d}_5 &= r_{\bar{d}_5} + \frac{306545}{512}\pi^2 \\ a_6 &= r_{a_6} + \frac{25911}{256}\pi^2 \end{aligned}}$$

Using classical and/or quantum gravitational scattering

Extracting PN-expanded dynamics from quantum scattering amplitudes:
Corinaldesi '56 '71, Barker-Gupta-Haracz 66, Barker-O'Connell 70, Iwasaki 71, Hiida-Okamura72,Okamura-Ohta-Kimura-Hiida 73,…,Bjerrum-Bohr-Donoghue-Vanhove 2014,….

Extracting PM-expanded dynamics from classical and/or quantum scattering:
TD'16,18,19; CheungRothsteinSolon'18; BCRSSZ'19;….

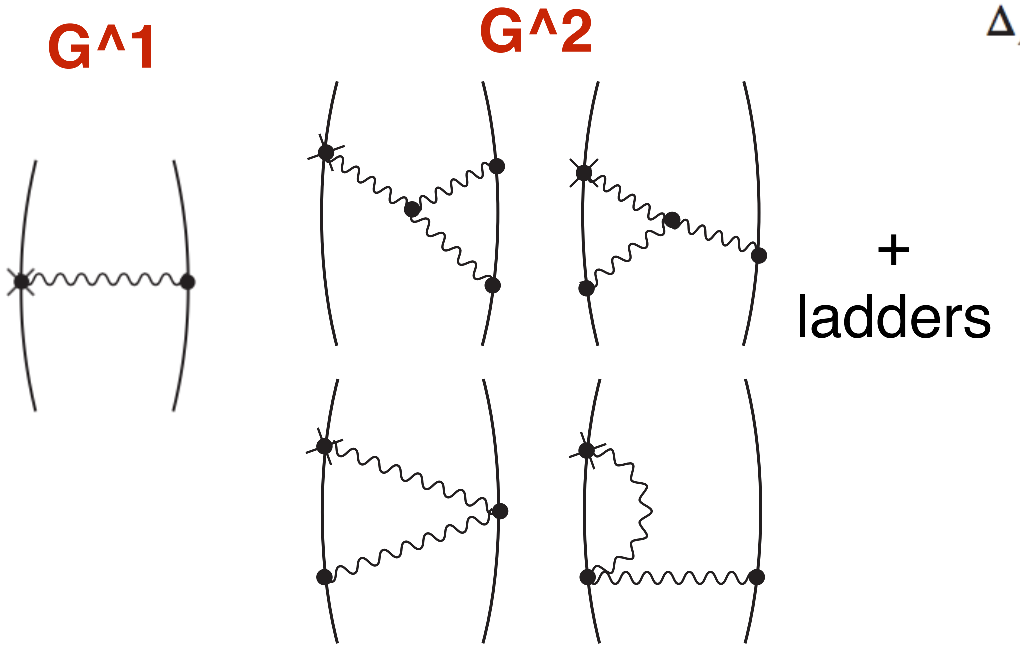
Several aspects:

dictionary classical scattering <—> Hamiltonian
dictionary quantum scattering <—> Hamiltonian

using either PN-expansion or PM-expansion

PM Perturbation Theory for Classical Gravitational Scattering

Bel-Martin ’75-’81, Portilla ’79, Westpfahl ’85, Damour’16’18,…,Kälin-Porto’20



$$\Delta p_{1\mu} = 2G\int d\sigma_1 d\sigma_2 p_{1\alpha}p_{1\beta} \times \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'}(x_1(\sigma_1)-x_2(\sigma_2))p_{2\alpha'}p_{2\beta'} + 2G\int d\sigma_1 d\sigma_1' p_{1\alpha}p_{1\beta}\partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'}(x_1(\sigma_1) - x_1(\sigma_1'))p_{1\alpha'}p_{1\beta'} + O(G^2)$$

$$\Delta p_{1\mu} = 8\pi G\int \frac{d^4k}{(2\pi)^4} ik_\mu p_1^\alpha p_1^\beta \frac{P_{\alpha\beta;\alpha'\beta'}}{k^2} p_2^{\alpha'} p_2^{\beta'} \times \int d\sigma_1 \int d\sigma_2 e^{ik.(x_1(\sigma_1)-x_2(\sigma_2))}.$$

$$\frac{1}{2}\chi_{\rm class}(E,J) = \frac{1}{j}\chi_1(\hat{E}_{\rm eff},\nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\rm eff},\nu) + O(G^3)$$

$$j \equiv \frac{J}{Gm_1m_2}$$

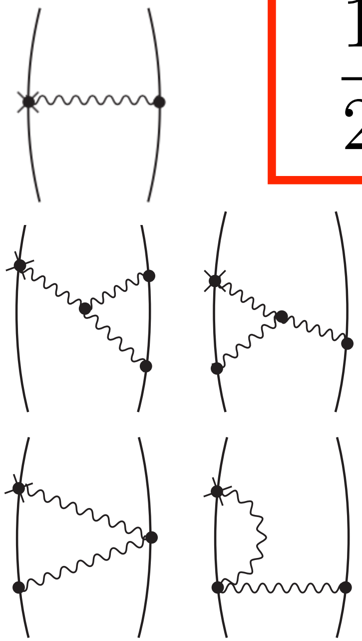
$$\hat{\mathcal{E}}_{\rm eff} \equiv \frac{\mathcal{E}_{\rm eff}}{\mu} \equiv \frac{(E_{\rm real})^2 - m_1^2 - m_2^2}{2m_1m_2} = \frac{s - m_1^2 - m_2^2}{2m_1m_2}.$$

$$\frac{1}{2}\chi^{\rm real}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2p_2^2}}.$$

$$\hat{\mathcal{E}}_{\rm eff} = -\frac{p_1\cdot p_2}{m_1m_2} = \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$\chi_1(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{2\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{\hat{\mathcal{E}}_{\rm eff}^2 - 1}}, \quad \chi_2(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{1+2\nu(\hat{\mathcal{E}}_{\rm eff}-1)}}.$$

Simple Map: Scattering angle <-> EOB dynamics



$$\frac{1}{2}\chi(\gamma, j) = \frac{\chi_1(\gamma)}{j} + \frac{\chi_2(\gamma)}{j^2} + \frac{\chi_3(\gamma)}{j^3} + \cdots$$

TD'16-18
Bini-TD-Geralico'20

$$0 = g_{\rm eff}^{\mu\nu} P_\mu P_\nu + \mu^2 + Q$$

$$g_{\rm eff}^{\mu\nu}$$

Schwarzschild metric M=m1+m2

$$Q = \left(\frac{GM}{R}\right)^2 q_2(E) + \left(\frac{GM}{R}\right)^3 q_3(E) + O(G^4)$$

$$\Delta\chi_n(\gamma) \equiv \chi_n(\gamma) - \chi_n^{\rm Schw}(\gamma),$$

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}.$$

$$\frac{\mathcal{E}_{\rm eff}}{\mu} = \gamma = -\frac{p_1 \cdot p_2}{m_1 m_2}$$

$$q_2(\gamma) = -\frac{4}{\pi}\Delta\chi_2(\gamma)$$

$$q_3(\gamma) = \frac{4}{\pi}\frac{2\gamma^2-1}{\gamma^2-1}\Delta\chi_2(\gamma) - \frac{\Delta\chi_3(\gamma)}{\sqrt{\gamma^2-1}}$$

$$q_2(\gamma,\nu) = \frac{3}{2}(5\gamma^2-1)\left(1 - \frac{1}{h(\gamma,\nu)}\right)$$

$$h(\gamma,\nu) = \sqrt{1+2\nu(\gamma-1)}$$

Linear combinations of the scattering coefficients!

Application to the ACV eikonal scattering phase (massless or ultra-relativistic scattering)

Amati-Ciafaloni-Veneziano'90+ Ciafaloni-Colferai'14+ Bern et al'20+ DiVecchia et al'20

$$\delta^{\text{eikonal}} = \frac{1}{\hbar}(\delta^{\text{R}} + i\delta^{\text{I}}) + \text{quantum corr.}$$

$$\boxed{\frac{1}{2}\chi^{\text{eikonal}} = 2\frac{\gamma}{j} + \frac{16}{3}\frac{\gamma^3}{j^3} + \cdots}$$

valid in the HE limit gamma-> infty

Using the chi—> Q dictionary
this corresponds to the HE limits:

$$\boxed{\begin{aligned} q_2^{\text{HE}} &= \frac{15}{2}\gamma^2 \\ q_3^{\text{HE}} &= \gamma^2 \end{aligned}}$$

i.e. an HE limit for the EOB mass-shell condition (TD'18)

$$0 = g_{\text{eff}}^{\mu\nu}(X)P_\mu P_\nu + \mu^2 + Q(X,P)$$

$$\boxed{0 = g^{\mu\nu}_{\text{Schw}}P_\mu P_\nu + \left(\frac{15}{2}\left(\frac{GM}{R}\right)^2 + \left(\frac{GM}{R}\right)^3\right)P_0^2}$$

3PM computation (Bern-Cheung-Roiban-Shen-Solon-Zeng'19)

using a combination of techniques: generalized unitarity; BCJ double-copy; 2-loop amplitude of quasi-classical diagrams; **EFT transcription** (Cheung-Rothstein-Solon'18); **resummation of PN-expanded integrals for potential-gravitons**

$$\chi_3^{\rm cons} = \chi_3^{\rm Schw} - \frac{2\nu\sqrt{\gamma^2-1}}{h^2(\gamma,\nu)}\bar{C}^{\rm cons}(\gamma)$$

$$q_3^{\rm cons} = \frac{3}{2}\frac{(2\gamma^2-1)(5\gamma^2-1)}{\gamma^2-1}\left(\frac{1}{h(\gamma,\nu)}-1\right) + \frac{2\nu}{h^2(\gamma,\nu)}\bar{C}^{\rm cons}(\gamma)$$

$$\overline{C}^{\rm cons}(\gamma) = \frac{2}{3}\gamma(14\gamma^2+25) + 2(4\gamma^4-12\gamma^2-3)\frac{\mathcal{A}(v)}{\sqrt{\gamma^2-1}}$$

$$h(\gamma,\nu) \equiv \frac{\sqrt{s}}{M} = \sqrt{1+2\nu(\gamma-1)}$$

$$\mathcal{A}(v) \equiv \mathrm{arctanh}(v) = \frac{1}{2}\ln\frac{1+v}{1-v} = 2\,\mathrm{arcsinh}\sqrt{\frac{\gamma-1}{2}}$$

puzzling HE limits when compared to ACV and Akcay et al'12

$$\frac{1}{2}\chi^{\rm cons} = 2\frac{\gamma}{j} + (12 - 8\ln(2\gamma))\frac{\gamma^3}{j^3} + O(G^4)$$

$q_3^{\rm cons} \approx +8\ln(2\gamma)\gamma^2$ instead of $q_3^{\rm ACV} \approx +1\gamma^2$

confirmations: 5PN (Bini-TD-Geralico'19); 6PN (Blümlein-Maier-Marquard-Schäfer'20, Bini-TD-Geralico'20); 3PM (Cheung-Solon'20, Kälin-Porto'20)

Suggested resolutions of these puzzles: non commutative limits?

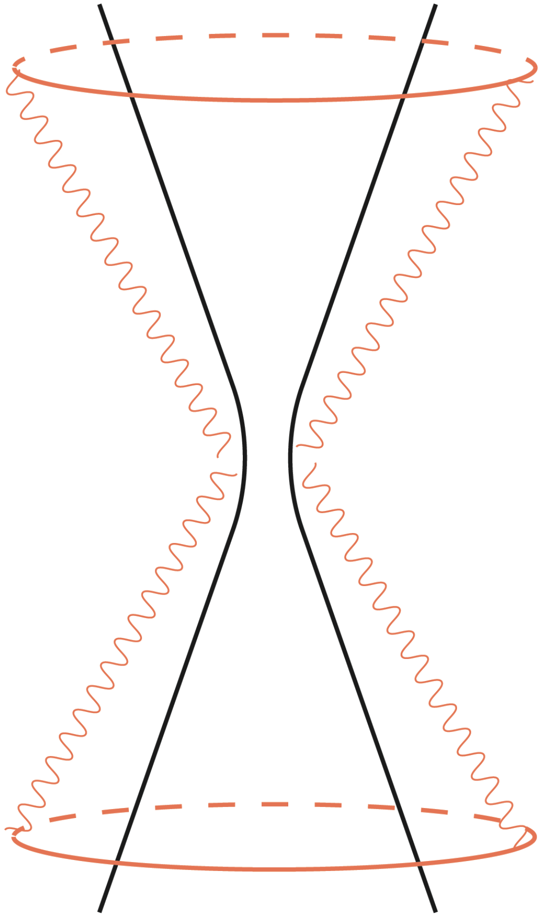
(BCRSSZ'19,TD'19,…)

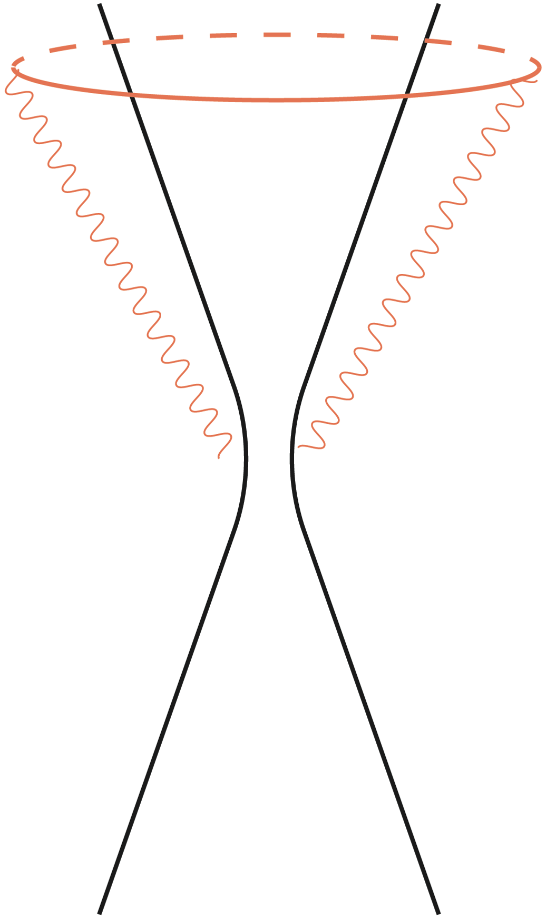
h—> 0
h—> infty
G—>0
[1/c —>0]
gamma—> infty
q=hbar/b—>0
mass—>0
nu—> 0

Bohr 1948:
The domain of validity of the Born-Feynman expansion is GE_1 E_2/(hbar v) << 1, while the domain of validity of the classical scattering is GE_1 E_2/(hbar v) >> 1!

Recently DiVecchia et al. brought a new light on this puzzle by emphasizing (using two different approaches) the crucial role of radiative effects for recovering HE finiteness

Conservative vs Radiation-reacted
Classical Gravitational Scattering





Studied **analytically** in Bini-TD'12;
and **numerically** in TD-Guercilena-Hinder-Hopper-Nagar-Rezzolla'14

Radiation-Reaction Contribution to the Classical Scattering Angle (TD 2010.01641)

$$\boxed{\chi^{\rm tot} = \chi^{\rm cons} + \chi^{\rm rad}}$$

where, to first order in Rad-Reac, one has (Bini-TD'12)

$$\boxed{\chi^{\rm rad}(E,J) = -\frac{1}{2}\frac{\partial \chi^{\rm cons}}{\partial E} E^{\rm rad} - \frac{1}{2}\frac{\partial \chi^{\rm cons}}{\partial J} J^{\rm rad}}$$

chi^cons=O(G^1)

O(G^3)

O(G^4)

O(G^3)

O(G^2) [TD-Deruelle'81]

enough to compute J^rad in PM at O(G^2). Waveform: $h_{ij}^{\rm TT} = \frac{f_{ij}(t-r,\theta,\phi)}{r} + O\left(\frac{1}{r^2}\right)$

DeWitt'71, Thorne'80

$$\boxed{J_k^{\rm rad} = \frac{\epsilon_{kij}}{16\pi G}\int du\, d\Omega \left[f_{ia}\partial_u f_{ja} - \frac{1}{2}x^i \partial_j f_{ab}\partial_u f_{ab}\right],}$$

a priori need the 2PM, O(G^2)-accurate waveform (Kovacs-Thorne'77, Bel et al'81, Westpfahl'85)
[though one can simplify the computation TD'20]

$$v \equiv \frac{v_1+v_2}{1+v_1 v_2} = \sqrt{1-\frac{1}{\gamma^2}}$$

$$\boxed{\mathcal{I}(v) = -\frac{16}{3} + \frac{2}{v^2} + \frac{2(3v^2-1)}{v^3}\mathcal{A}(v) \quad \mathcal{A}(v) \equiv \mathrm{arctanh}(v) = \frac{1}{2}\ln\frac{1+v}{1-v}}$$

$$\boxed{\begin{aligned} &\frac{J^{\mathrm{rad}}}{J} = \frac{J^{\mathrm{rad}}_{yz}}{bP} = \frac{2(2\gamma^2-1)}{\sqrt{\gamma^2-1}}\frac{G^2 m_1 m_2}{b^2}\mathcal{I}(v) + O(G^3) \\ &\frac{1}{2}\chi^{\mathrm{rad}}(\gamma, j, \nu) = +\frac{\nu}{h^2(\gamma,\nu) j^3}(2\gamma^2-1)^2\mathcal{I}(v) + O(G^4) \end{aligned}}$$

contains the same arctanh as ch^{BCRSSZ} but with opposite coefficient when v—>1

Low-velocity limit $\frac{1}{2}\chi^{\mathrm{rad}} = +\frac{8G^3}{5c^5}\frac{m_1^3 m_2^3}{J^3}\nu v^2 + \cdots$ agrees with Bini-TD'12

HE (or massless) limit

$$\boxed{\begin{aligned} &\frac{1}{2}\chi^{\mathrm{rad}} \overset{\mathrm{HE}}{=} +\left(-\frac{20}{3} + 8\ln(2\gamma)\right)\alpha^3 + O(G^4) \\ &\frac{1}{2}\chi^{\mathrm{cons}} = 2\frac{\gamma}{j} + (12 - 8\ln(2\gamma))\frac{\gamma^3}{j^3} + O(G^4) \\ &\frac{1}{2}(\chi^{\mathrm{cons}} + \chi^{\mathrm{rad}}) = 2\frac{\gamma}{j} + \frac{16}{3}\frac{\gamma^3}{j^3} = \chi^{\mathrm{ACV}} \end{aligned}}$$

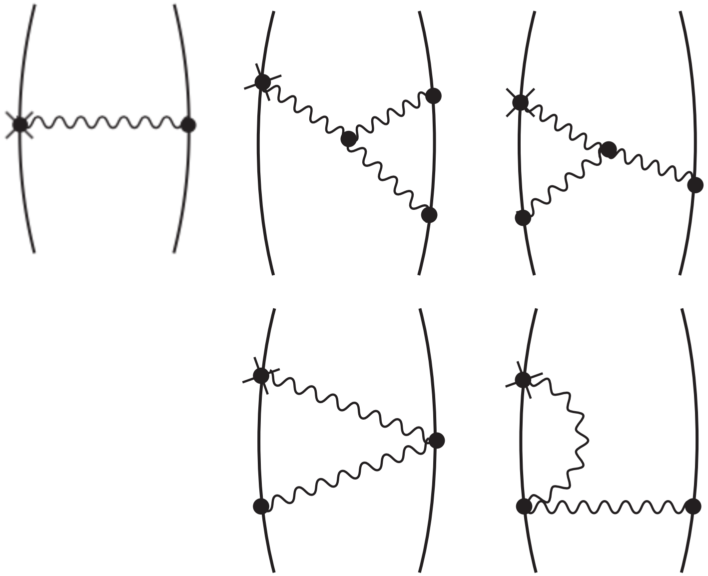
Summary

$$G_{\rm sym} = \mathrm{P}\frac{1}{k^2}$$

$$G_{\rm ret} = \mathrm{P}\frac{1}{k^2} + i\pi \mathrm{sign}(k^0)\delta(k^2$$

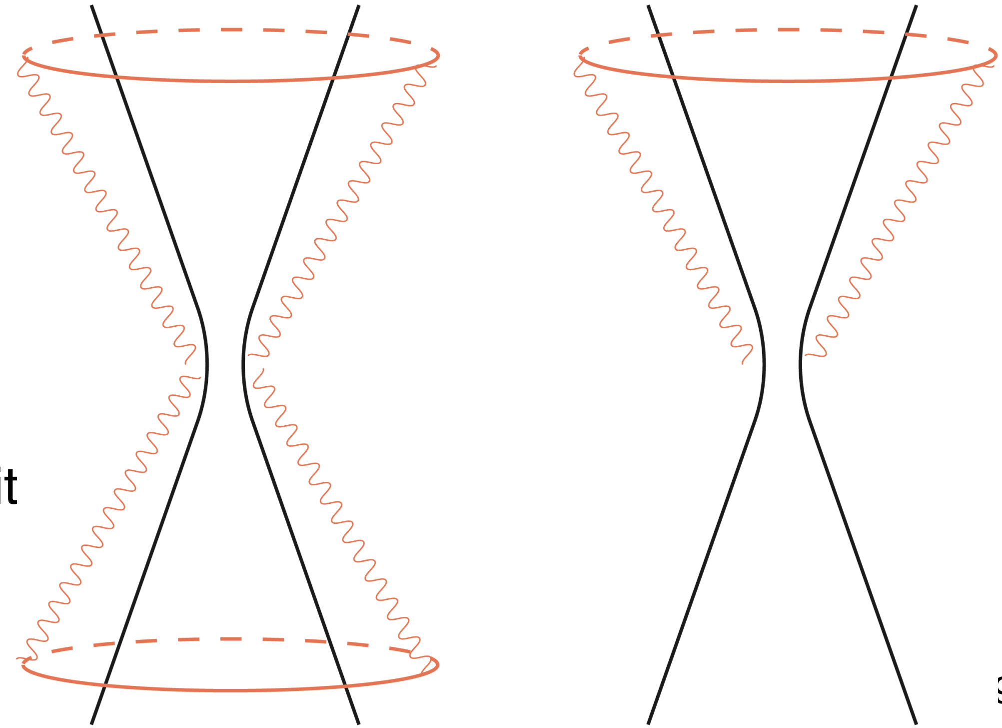
$$G_{\rm F} = \mathrm{P}\frac{1}{k^2} + i\pi\delta(k^2)$$

$$\Delta p^\mu =$$



Quantum vs classical ?
Which action?
FWF, Schw-Keldysh,
log <vac-vac>

$$\dot{p} = -\frac{\partial H(q,p)}{\partial q} + \mathcal{F}^{\rm RR}$$



F^RR is small, O(G^2/c^5),
and << F^cons at low-velocities,
but is crucially important at HE
to ensure the existence of a massless limit

Non-commutativity of
G—>0 and HE limits
must also play a role