Consiglio Nazionale delle Ricerche
Istituto per le Applicazioni del Calcolo "Mauro Picone"

TuttiFrutti (TF) Method: A theoretical journey towards the 6PN two-body Hamiltonian

[Classical GR interaction of binary systems, combining information via PN, PM, MPM, SF, EOB, EFT, Delaunay averaging formalisms]

Donato Bini @ IAC.CNR.IT

[Based on works done in collaboration with T. Damour, A. Geralico, S. Laporta, P. Mastrolia]

EFT Methods from Bound States to Binary Systems
University of Padova, 28-30 October, 2020

PN, PM, MPM, SF, EOB, EFT, Delaunay averaging formalisms

Capturing information from….

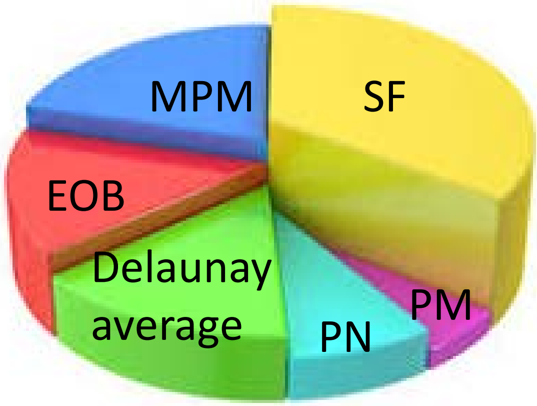
TuttiFrutti

And leading to….

6PN nonlocal part of the two-body Hamiltonian

6PN local part of the two-body Hamiltonian

Main steps of the TF method in a schematic representation



MPM → Separate off the nonlocal part of the Hamiltonian and perform Delaunay averaging (leading to averaged H_{nonloc})

SF→ Compute z_1 @ O(eccentricity^n)

EOB→ Use the first law of the two-body dynamics to translate the redshift information into a $O(p_r^n)$ in the EOB Hamiltonian in DJS gauge

Determine $H_{loc,1SF}$ by «subtracting» the averaged H_{nonloc}

PM→Use EOB-PM theory to determine most of the nonlinear-in- ν dependence of the coefficients in the EOB Hamiltonian.

Nonlocal effects @6PN

Strategy: Computed in h coords and translated in EOB coords

1 Evaluate flux split integrals @ 2PN to have the nonlocal Hamiltonian @6PN since nonlocal effects start @4PN

2 Integrated flux split equals averaged Hamiltonian

3 Compare averaged Hamiltonian in h and EOB coordinates

First accomplishment

4 Fix the nonlocal part of EOB potentials up to 6PN

Passing from nonloc to loc…

Key tool: 1SF

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$ sym mass ratio

4

Fix the nonlocal part of EOB potentials up to 6PN

Second accomplishment

5

Evaluate z_1 from 1SF, expanded in eccentricity, which incorporates both local and nonlocal effects

6

Evaluate z_1 in EOB using the local part of the Hamiltonian in a parametrized form with unknown coefficients

Third accomplishment

7

Compare the two and fix the local EOB potentials at $O(\nu)$ since we can use 1SF only

8

Compute the scattering angle in EOB with a parametrized ν dependence of unknown coefficients

9

Use a special «rule» to fix the allowed dependence on ν of the local EOB potentials

10

Determine all missing coefficients except for 2 @5PN and 4 @6PN

GI quantities involved in the TF approach

→ Delaunay averaged NL Hamiltonian

→ Redshift

→ Scattering angle

Nonlocal action at 6PN order

As a starting point we need an explicit expression for the nonlocal part of the action

Computed in:
D. Bini, T. Damour, A. Geralico, *Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders*, Phys. Rev. D 102, 024062 (2020)

- $$S_{\rm nonloc,f}^{\le 6\rm PN} = S_{\rm nonloc,f}^{4+5+6\rm PN} + S_{\rm nonloc}^{5.5\rm PN}$$

- $$S_{\rm nonloc,f}^{4+5+6\rm PN}[x_1(s_1), x_2(s_2)] = -\int dt\, H_{\rm nonloc,f}^{4+5+6\rm PN}(t)$$

- $$H_{\rm nonloc,f}^{4+5+6\rm PN}(t) = \frac{G^2\mathcal{M}}{c^3}\,{\rm Pf}_{2r_{12}^f(t)/c}\int \frac{dt'}{|t-t'|}\mathcal{F}_{\rm 2PN}^{\rm split}(t,t')$$

Flexed version of the distance among the two bodies

Total ADM conserved mass-energy of the system

At the 6PN accuracy, the nonlocal action can be linearly decomposed into its 4 + 5 + 6PN piece (depending on the choice of a flexibility function f), and a 5.5PN piece (not depending on f).

It is enough to work in h coordinates…

$$r^f(t) = f(t) r^h(t)$$

“Flexed” version of the harmonic coordinate radial distance between the two bodies. f(t) denotes a function of the instantaneous state of the system. Its introduction is convenient for reasons which will be clear later.

To show the essential steps of the TF method it is enough to work in h coordinates. The flexibility function simply generates another piece to be added the h-coordinate nonlocal Hamiltonian which can be cosidered later.

The h-coordinates nonlocal part of the 2-body Hamiltonian is

$$H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}(t) \;=\; \frac{G\mathcal{M}}{c^3}\mathrm{Pf}_{2r^h_{12}(t)/c}\int \frac{dt'}{|t-t'|}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t')\,.$$

What should I do?

1

Evaluate flux split integrals @ 2PN to have the nonlocal Hamiltonian @6PN since nonlocal effects start @4PN

Compute the flux split integral

$$H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}(t) = \frac{G\mathcal{M}}{c^3}\mathrm{Pf}_{2r^h_{12}(t)/c}\int \frac{dt'}{|t-t'|}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t')\,.$$

H nonloc,h is the integral over t' of the flux split F(t,t')/|t-t'|
To have H nonloc,h at 4PN one needs the flux split at the N level
To have H nonloc,h at 5PN one needs the flux split at the 1PN level
To have H nonloc,h at 6PN one needs the flux split at the 2PN level

T. Damour, P. Jaranowski and G. Schafer, Phys. Rev. D 89, 064058 (2014)

Up to 2 PN the flux split follows trivially from the GW flux (function), while beyond 2PN tail contributions appear, mix, and make things not so simple.

I need the flux function 2PN

What is the difference between flux and flux split at 2PN?

t,t

$$\mathcal{F}_{\rm 2PN}(t) = \frac{G}{c^5}\frac{1}{5}I^{(3)}_{ab}(t)I^{(3)}_{ab}(t) + \ldots$$ GW flux

$$\mathcal{F}^{\rm split}_{\rm 2PN}(t,t') = \frac{G}{c^5}\frac{1}{5}I^{(3)}_{ab}(t)I^{(3)}_{ab}(t') + \ldots$$

t,t'

Various multipolar contributions as the building block of the flux function: electric quadrupole, electric octupole, magnetic quadrupole, electric hexadecapole, magnetic octupole.

It is the flux split version of the GW flux which enters the nonlocal part of the two-body Hamiltonian!

I need the h multipole moments @ 2PN

Building blocks of the nonlocal Hamiltonian

See e.g. K.G. Arun, L. Blanchet, B.R. Iyer and M.S.S. Qusailah, Phys. Rev. D 77, 064034 (2008)

$$H^{4+5+6\rm PN}_{\rm nonloc,h}(t) = -\frac{G^2 H}{c^5}{\rm Pf}_{2s/c}\int\frac{d\tau}{|\tau|}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t+\tau) + 2\frac{G^2 H}{c^5}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t)\ln\left(\frac{r^h_{12}(t)}{s}\right)$$

Evaluate the multipole moments along a 2PN ellipticlike orbit, obtaining the integrand as a function of t and t' so to proceed with the t' and t integrations.

I need the parametric equations of the orbit

[The integration is conveniently performed by expanding the integrands in powers of a small eccentricity!]

2PN quasi-Keplerian parametrization of ellipticlike orbits

$$
\begin{aligned}
r &= a_r(1 - e_r \cos u)\,, \\
\ell &= n(t - t_P) = u - e_t \sin u + f_t \sin V + g_t(V - u)\,, \\
\bar{\phi} &= \frac{\phi - \phi_P}{K} = V + f_\phi \sin 2V + g_\phi \sin 3V\,,
\end{aligned}
$$

$$V(u) = 2 \arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan\frac{u}{2}\right]$$

Different types of eccentricity: temporal, radial, azimuthal.
All orbital elements are coordinate-dependent. They can however be expressed in terms of GI variables like E and J.

Source of computational difficulties when evaluating functions of the two-body dynamics along the orbit. Simplifications arise when taking a series expansion in eccentricity

The parametrization depends on the (coordinate dependent) orbital elements! → We need the h coordinate orbital elements at 2PN

And now?

I can finally take the (Delaunay) average of the nonlocal Hamiltonian in h coordinates, leading to a GI invariant quantity.

$$\langle H^h_{\rm nonloc,h}\rangle \equiv \langle H^{4+5+6{\rm PN}}_{\rm nonloc,h}\rangle = F^h(a_r^h, e_t^h)\,,$$

$$\begin{aligned}
F^h(a_r^h, e_t^h) \;&=\; \frac{\nu^2}{(a_r^h)^5}\left[\mathcal{A}^{4{\rm PN}}(e_t^h)+\mathcal{B}^{4{\rm PN}}(e_t^h)\ln a_r^h\right]\\
&+\; \frac{\nu^2}{(a_r^h)^6}\left[\mathcal{A}^{5{\rm PN}}(e_t^h)+\mathcal{B}^{5{\rm PN}}(e_t^h)\ln a_r^h\right]\\
&+\; \frac{\nu^2}{(a_r^h)^7}\left[\mathcal{A}^{6{\rm PN}}(e_t^h)+\mathcal{B}^{6{\rm PN}}(e_t^h)\ln a_r^h\right]
\end{aligned}$$

Explicit expressions in successive papers at 4,5 and 6PN. Polynomial in eccentricity, computed up to a fixed maximum power.

The same average but within EOB framework…

We have computed

$$\langle H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}} \rangle = F^h(a_r^h, e_t^h)$$

in harmonic coordinates. We can do the same in EOB coordinates and the end result is the same due to its GI content

GI

$$\langle H^{\mathrm{eob}}_{\mathrm{nonloc,h}} \rangle = \langle H^{h}_{\mathrm{nonloc,h}} \rangle$$

We need the EOB orbital elements and their relations with the h orbital elements.

If the EOB part (lhs) contains undetermined coefficients these will be fixed by the above equality.

The EOB Hamiltonian (EOB coords)

$$H = M\sqrt{1 + 2\nu(\hat{H}_{\rm eff} - 1)}$$

→Both the local and nonlocal information concerning the Hamiltonian are encoded into the effective Hamiltonian.

→The effective Hamiltonian is written in terms of several potentials characterized by both a local and nonlocal part.

→We are interested in the nonlocal part to compare with our previous computation h-coordinate based.

EOB total effective Hamiltonian

$$\widehat{H}^2_{\rm eff} = A(u;\nu)\Big(1+p_\phi^2u^2+A(u;\nu)\bar{D}(u;\nu)p_r^2+\widehat{Q}(u,p_r;\nu)\Big)$$

$$\widehat{Q}(u,p_r;\nu) = p_r^4q_4(u;\nu)+p_r^6q_6(u;\nu)+p_r^8q_8(u;\nu)+p_r^{10}q_{10}(u;\nu)+\dots$$

DJS gauge: expansion in powers of p_r

$$\begin{aligned}
A(u;\nu) &= 1-2u+\nu a^{\nu^1}(u)+\nu^2a^{\nu^2}(u)+\nu^3a^{\nu^3}(u)+\dots\\
\bar{D}(u;\nu) &= 1+\nu\bar{d}^{\nu^1}(u)+\nu^2\bar{d}^{\nu^2}(u)+\nu^3\bar{d}^{\nu^2}(u)+\dots\\
q_4(u;\nu) &= \nu q_4^{\nu^1}(u)+\nu^2q_4^{\nu^2}(u)+\nu^3q_4^{\nu^3}(u)+\dots\\
q_6(u;\nu) &= \nu q_6^{\nu^1}(u)+\nu^2q_6^{\nu^2}(u)+\nu^3q_6^{\nu^3}(u)+\dots\\
q_8(u;\nu) &= \nu q_8^{\nu^1}(u)+\nu^2q_8^{\nu^2}(u)+\nu^3q_8^{\nu^3}(u)+\dots\,.
\end{aligned}$$

Separate **EOB potentials** into **local** and **nonlocal** parts

$$\begin{aligned}
A &= A^{\rm loc,h}+A^{\rm nonloc,h}=A^{\rm loc,f}+A^{\rm nonloc,f}\,,\\
\bar{D} &= \bar{D}^{\rm loc,h}+\bar{D}^{\rm nonloc,h}=\bar{D}^{\rm loc,f}+\bar{D}^{\rm nonloc,f}\,,\\
\widehat{Q} &= \widehat{Q}^{\rm loc,h}+\widehat{Q}^{\rm nonloc,h}=\widehat{Q}^{\rm loc,f}+\widehat{Q}^{\rm nonloc,f}\,.
\end{aligned}$$

$$\delta A = a_5^{\rm nonloc}u^5+a_6^{\rm nonloc}u^6+a_7^{\rm nonloc}u^7$$

$$\begin{aligned}
\delta\bar{D} &= \bar{d}_4^{\rm nonloc}u^4+\bar{d}_5^{\rm nonloc}u^5+\bar{d}_6^{\rm nonloc}u^6\,,\\
\delta\widehat{Q} &= p_r^4(q_{43}^{\rm nonloc}u^3+q_{44}^{\rm nonloc}u^4+q_{45}^{\rm nonloc}u^5)\\
&+ p_r^6(q_{62}^{\rm nonloc}u^2+q_{63}^{\rm nonloc}u^3+q_{64}^{\rm nonloc}u^4)\\
&+ p_r^8(q_{81}^{\rm nonloc}u+q_{82}^{\rm nonloc}u^2+q_{83}^{\rm nonloc}u^3)\\
&+ p_r^{10}(q_{10,0}^{\rm nonloc}+q_{10,1}^{\rm nonloc}u+q_{10,2}^{\rm nonloc}u^2)
\end{aligned}$$

Use unknown coefficients

Use the GI invariance property to impose

$$\langle H^{\rm eob}_{\rm nonloc,h}\rangle=\langle H^{h}_{\rm nonloc,h}\rangle$$

and fix all EOB nonloc coefficients (including logs).

Intermediate result

$$\langle H^{\rm eob}_{\rm nonloc,h}\rangle = \langle H^{h}_{\rm nonloc,h}\rangle$$

The nonlocal part of all the EOB potentials (a, $\bar{d}$, q_n) needed @ 6PN is fully known.

$$\hat{H}_{\text{eff}}^2 = A(u;\nu)\Big(1 + p_\phi^2 u^2 + A(u;\nu)\bar{D}(u;\nu)p_r^2 + \hat{Q}(u,p_r;\nu)\Big)$$

$$\hat{Q}(u,p_r;\nu) = p_r^4 q_4(u;\nu) + p_r^6 q_6(u;\nu) + p_r^8 q_8(u;\nu) + p_r^{10} q_{10}(u;\nu) + \dots$$

$$A = A^{\text{loc,h}} + A^{\text{nonloc,h}}$$
$$\bar{D} = \bar{D}^{\text{loc,h}} + \bar{D}^{\text{nonloc,h}}$$
$$\hat{Q} = \hat{Q}^{\text{loc,h}} + \hat{Q}^{\text{nonloc,h}}$$

To be determined

Fully determined

TABLE VI: h-route nonlocal EOB coefficients.

$a_7^{\mathrm{nl,c}}$	$\left(\frac{206740}{567}\ln(2)+\frac{12664}{105}-\frac{4617}{14}\ln(3)-\frac{5044}{405}\gamma\right)\nu + \left(-\frac{1139672}{945}\ln(2)+\frac{10132}{105}+\frac{10449}{7}\ln(3)+\frac{101272}{315}\gamma\right)\nu^2 + \left(-\frac{112}{5}+32\gamma+\frac{1214624}{945}\ln(2)-\frac{4860}{7}\ln(3)\right)\nu^3\,,$
$a_7^{\mathrm{nl,log}}$	$-\frac{2522}{405}\nu+\frac{50636}{315}\nu^2+16\nu^3\,,$
$\bar{d}_6^{\mathrm{nl,c}}$	$\left(-\frac{6381680}{189}\ln(2)+\frac{2043541}{2835}+\frac{1765881}{140}\ln(3)-\frac{64096}{45}\gamma+\frac{9765625}{2268}\ln(5)\right)\nu + \left(\frac{28429312}{189}\ln(2)-\frac{3576231}{70}\ln(3)+\frac{167906}{105}+\frac{302752}{105}\gamma-\frac{9765625}{378}\ln(5)\right)\nu^2 + \left(-\frac{9908480}{63}\ln(2)-\frac{744704}{945}+\frac{9765625}{252}\ln(5)+\frac{2944}{3}\gamma+\frac{1275021}{28}\ln(3)\right)\nu^3\,,$
$\bar{d}_6^{\mathrm{nl,log}}$	$-\frac{32048}{45}\nu+\frac{151376}{105}\nu^2+\frac{1472}{3}\nu^3$
$q_{45}^{\mathrm{nl,c}}$	$\left(\frac{70925884}{63}\ln(2)+\frac{13212013}{5670}-\frac{3873663}{16}\ln(3)-\frac{8787109375}{27216}\ln(5)-\frac{617716}{315}\gamma\right)\nu + \left(\frac{92560887}{280}\ln(3)-\frac{12619052648}{2835}\ln(2)-\frac{1437979}{63}+\frac{632344}{315}\gamma+\frac{7755859375}{4536}\ln(5)\right)\nu^2 + \left(-\frac{177316}{35}+\frac{11263031264}{2835}\ln(2)+\frac{16544}{9}\gamma-\frac{4091796875}{2268}\ln(5)+\frac{2908467}{20}\ln(3)\right)\nu^3$
$q_{45}^{\mathrm{nl,log}}$	$-\frac{308858}{315}\nu+\frac{316172}{315}\nu^2+\frac{8272}{9}\nu^3$
$q_{64}^{\mathrm{nl,c}}$	$\left(-\frac{211076833264}{14175}\ln(2)-\frac{137711989}{28350}-\frac{9678652821}{5600}\ln(3)+\frac{447248}{1575}\gamma+\frac{153776136875}{23328}\ln(5)+\frac{96889010407}{116640}\ln(7)\right)\nu + \left(\frac{44592947739}{2800}\ln(3)+\frac{2411178384736}{42525}\ln(2)-\frac{126070663}{4725}-\frac{26848}{175}\gamma-\frac{796015515625}{27216}\ln(5)-\frac{96889010407}{19440}\ln(7)\right)\nu^2 + \left(-\frac{40513708}{4725}-\frac{109566260523}{5600}\ln(3)+\frac{1424826953125}{54432}\ln(5)+\frac{96889010407}{12960}\ln(7)+\frac{2368}{5}\gamma-\frac{431564554688}{8505}\ln(2)\right)\nu^3$
$q_{64}^{\mathrm{nl,log}}$	$\frac{223624}{1575}\nu-\frac{13424}{175}\nu^2+\frac{1184}{5}\nu^3$
$q_{83}^{\mathrm{nl,c}}$	$\left(\frac{5196312336176}{35721}\ln(2)+\frac{17515638027261}{313600}\ln(3)-\frac{63886617280625}{1016064}\ln(5)-\frac{29247366220639}{933120}\ln(7)-\frac{709195549}{132300}\right)\nu + \left(-\frac{177055674739808}{297675}\ln(2)-\frac{43719724468071}{156800}\ln(3)+\frac{366449151015625}{1524096}\ln(5)+\frac{26506549233199}{155520}\ln(7)-\frac{1746293}{70}\right)\nu^2 + \left(\frac{57604236136064}{99225}\ln(2)+\frac{104675833000341}{39200}\ln(3)-\frac{73366198046875}{381024}\ln(5)-\frac{7709596970957}{38880}\ln(7)-\frac{154862}{21}\right)\nu^3$
$q_{83}^{\mathrm{nl,log}}$	0

List of the EOB nonlocal coefficients determined by comparison of Delaunay averaging

How can we determine the local part of the EOB potentials?

→Compute a GI invariant quantity (the redshift) which incorporates both local and nonlocal effects in two different ways:

1) 1SF (computations only linear in ν)
2) Using the full EOB Hamiltonian

$$z_1 = \langle \frac{\partial H}{\partial m_1} \rangle$$

Compare the two («subtract» nonlocal from the full local plus nonlocal objects), and identify the local part of the EOB potentials! But only linear in ν!

Computing z_1 1SF

$$
\begin{aligned}
\delta z_1^{e^8}(u) &= C_3 u^3 + C_4 u^4 + (C_5^{\mathrm{c}} + C_5^{\ln} \ln u) u^5 \\
&+ (C_6^{\mathrm{c}} + C_6^{\ln} \ln u) u^6 + C_{13/2} u^{13/2} \\
&+ (C_7^{\mathrm{c}} + C_7^{\ln} \ln u) u^7 + C_{15/2} u^{15/2} \\
&+ (C_8^{\mathrm{c}} + C_8^{\ln} \ln u + C_8^{\ln^2} \ln^2 u) u^8 + C_{17/2} u^{17/2} \\
&+ (C_9^{\mathrm{c}} + C_9^{\ln} \ln u + C_9^{\ln^2} \ln^2 u) u^9 \\
&+ (C_{19/2}^{\mathrm{c}} + C_{19/2}^{\ln} \ln u) u^{19/2} \\
&+ O_{\ln(u)}(u^{10}),
\end{aligned}
$$

SF people could have computed this before. Unfortunately, this was not the case and we had to do ourselves.

External computation, performed by using standard 1SF technology.

C_3	$\frac{15}{64}$
C_4	$\frac{3001}{384} - \frac{287}{4096}\pi^2$
C_5^{c}	$\frac{4597}{96} - \frac{162109375}{2304}\ln(5) - \frac{11332791}{1280}\ln(3) + \frac{55}{6}\gamma + \frac{15967961}{90}\ln(2) - \frac{474715}{196608}\pi^2$
$C_5^{\ln}$	$\frac{55}{12}$
C_6^{c}	$-\frac{9863051}{40320} + \frac{96889010407}{442368}\ln(7) - \frac{64481546637}{114688}\ln(3) - \frac{5977}{240}\gamma - \frac{16605499789}{5040}\ln(2) + \frac{1466047}{196608}\pi^2 + \frac{4761539921875}{3096576}\ln(5)$
$C_6^{\ln}$	$-\frac{5977}{480}$

Converting z_1 1SF information in EOB potentials (analytic 1SF encodes both local+nonlocal effects)

$$
\begin{aligned}
q_8^{\nu^1}(u) &= B_1 u + B_2 u^2 + B_{5/2} u^{5/2} \\
&+ B_3 u^3 + B_{7/2} u^{7/2} \\
&+ (B_4^{\rm c} + B_4^{\ln} \ln u) u^4 + B_{9/2} u^{9/2} \\
&+ (B_5^{\rm c} + B_5^{\ln} \ln u) u^5 + (B_{11/2}^{\rm c} + B_{11/2}^{\ln} \ln u) u^{11/2} \\
&+ O_{\ln(u)}(u^6)\,,
\end{aligned}
$$

Subtract the (sum of local and nonlocal) EOB potential obtained from 1SF from their nonlocal parts obtained by using Delaunay averaging to obtain the local part of the EOB potentials.
1SF allows to compute only terms linear in ν: all other coefficients should be included in a parametrized form.

B_1	$-\frac{27734375}{126}\ln(5) + \frac{6591861}{350}\ln(3) + \frac{21668992}{45}\ln(2) - \frac{35772}{175}$
B_2	$\frac{13841287201}{17280}\ln(7) - \frac{393786545409}{156800}\ln(3) - \frac{16175693888}{1575}\ln(2) + \frac{875090984375}{169344}\ln(5) + \frac{5790381}{2450}$
$B_{5/2}$	$+\frac{5994461}{12700800}\pi$

Intermediate result

All the EOB potentials have been determined for both their **nonlocal part (completely)** and their **local part (at linear order in ν).** The high-order in ν of the local part of the Hamiltonian is unkown, and written by using parameters!

Determining the local part of the EOB potential @ O(ν^1)

$$
\begin{aligned}
a_{4+5+6\mathrm{PN,loc,f}} &= \left[\left(\frac{2275}{512}\pi^2-\frac{4237}{60}\right)\nu+\left(\frac{41}{32}\pi^2-\frac{221}{6}\right)\nu^2\right]u^5+\left[\left(-\frac{1026301}{1575}+\frac{246367}{3072}\pi^2\right)\nu+a_{6,f}^{(\nu)}\right]u^6\\
&+\left[\left(-\frac{2800873}{262144}\pi^4+\frac{608698367}{1769472}\pi^2-\frac{1469618167}{907200}\right)\nu+a_{7,f}^{(\nu)}\right]u^7,\\
\bar{d}_{4+5+6\mathrm{PN,loc,f}} &= \left[\left(\frac{1679}{9}-\frac{23761}{1536}\pi^2\right)\nu+\left(-260+\frac{123}{16}\pi^2\right)\nu^2\right]u^4+\left(\frac{331054}{175}\nu-\frac{63707}{512}\nu\pi^2+\bar{d}_{5,f}^{(\nu)}\right)u^5\\
&+\left[\left(\frac{229504763}{98304}\pi^2+\frac{135909}{262144}\pi^4-\frac{99741733409}{6350400}\right)\nu+\bar{d}_{6,f}^{(\nu)}\right]u^6,
\end{aligned}
$$

words

$$
\begin{aligned}
q_{4,4+5+6\mathrm{PN,loc,f}} &= \left(20\nu+q_{43}^{(\nu)}\right)u^3+\left[\left(-\frac{93031}{1536}\pi^2+\frac{1580641}{3150}\right)\nu+q_{44,f}^{(\nu)}\right]u^4\\
&+\left[\left(\frac{81030481}{65536}\pi^2-\frac{3492647551}{423360}\right)\nu+q_{45,f}^{(\nu)}\right]u^5,\\
q_{6,4+5+6\mathrm{PN,loc,f}} &= \left(-\frac{9}{5}\nu+q_{62}^{(\nu)}\right)u^2+\left(\frac{123}{10}\nu+q_{63,f}^{(\nu)}\right)u^3\\
&+\left[\left(-\frac{9733841}{327680}\pi^2-\frac{112218283}{294000}\right)\nu+q_{64,f}^{(\nu)}\right]u^4,\\
q_{8,5+6\mathrm{PN,loc,f}} &= q_{82}(\nu)u^2+\left(-\frac{7447}{560}\nu+q_{83}^{(\nu)}\right)u^3,\\
q_{10,6\mathrm{PN,loc,f}} &= q_{10,2}(\nu)u^2.
\end{aligned}
$$

Parametrize the local part of the EOB potential at all orders ν^n (n>1) by using uknown coefficients!

Almost all of them will be determined by using the scattering angle

Using the ν dependence of the scattering angle to determine most of the ν structure of the local Hamiltonian

The scattering angle depend on unknown EOB parameters (at either 5PN and 6PN)

$$\frac{1}{2}(\chi(\gamma,j)+\pi) = -\int_0^{u_{\max}} \frac{\partial}{\partial j} p_r(u;\gamma,j)\frac{du}{u^2}$$

$$\begin{aligned}\gamma^2 &= \hat{\mathcal{E}}^2_{\rm eff} = \hat{H}^{2\,{\rm EG}}_{\rm eff,loc,f}(u,p_r,j;\nu)\\ &= H_S^2 + (1-2u)\hat{Q}^{\rm EG}_{H\,{\rm loc,f}}(u,H_S;\nu)\end{aligned}$$

$$p_r^2(\gamma,j,u) = \frac{\gamma^2-(1-2u)\left(1+j^2u^2+\hat{Q}^{\rm EG}_{E\,{\rm loc,f}}(u,\gamma;\nu)\right)}{(1-2u)^2}$$

$$p_r = p_r^{(0)} + G p_r^{(1)} + G^2 p_r^{(2)} + \dots ,$$

$$\begin{aligned}\chi_5^{\rm loc,f} =\ & \frac{1}{5p_\infty^5} - \frac{2}{p_\infty^3}\eta^2 + \frac{32-8\nu}{p_\infty}\eta^4\\
&+\left[320+\left(-\frac{1168}{3}+\frac{41}{8}\pi^2\right)\nu+24\nu^2\right]p_\infty\eta^6\\
&+\left[640+\left(\frac{5069}{144}\pi^2-\frac{227059}{135}\right)\nu\right.\\
&\left.+\left(-\frac{287}{24}\pi^2+\frac{7342}{9}\right)\nu^2-40\nu^3\right]p_\infty^3\eta^8\\
&+\left[\frac{1792}{5}+\left(-\frac{1460479}{525}+\frac{111049}{960}\pi^2\right)\nu\right.\\
&+\left(\frac{41026}{15}-\frac{40817}{640}\pi^2-\frac{4}{15}\bar{d}_5^{\nu^2}\right)\nu^2\\
&\left.+\left(-\frac{11108}{9}+\frac{451}{24}\pi^2\right)\nu^3+56\nu^4\right]p_\infty^5\eta^{10}\\
&+\left[\left(\frac{93031}{2304}\pi^2-\frac{498343703}{604800}\right)\nu\right.\\
&+\left(\frac{2827607}{1152}-\frac{31633}{768}\pi^2\right)\nu^2\\
&+\left(\frac{205}{16}\pi^2-\frac{253361}{96}\right)\nu^3+\frac{212879}{384}\nu^4+\frac{63}{64}\nu^5\\
&\left.-2q^4_{3\rm EG}-4q^3_{4\rm EG}-\frac{4}{3}q^2_{5\rm EG}\right]p_\infty^7\eta^{12}\,.\end{aligned}$$

The ν-rule

$$\widetilde{\chi}_n^{\mathrm{loc,f}}(\gamma;\nu) \equiv [h(\gamma;\nu)]^{n-1}\,\chi_n^{\mathrm{loc,f}}(\gamma;\nu)\,,$$

$$h(\gamma;\nu) \equiv \sqrt{1+2\nu(\gamma-1)} = \frac{H}{Mc^2}\,.$$

The total (local plus nonlocal) scattering angle satisfies the property shown by Damour

$$C_n^{\mathrm{tot}}:\widetilde{\chi}_n^{\mathrm{tot}}(\gamma;\nu) = P_{d_n}^{\gamma}(\nu)\ ;\ \text{with } d_n \equiv \left[\frac{n-1}{2}\right]$$

The introduction of the flexibility function allows both the local and the nonlocal parts to satisfy this rule!

T. Damour, *Classical and quantum scattering in post-Minkowskian gravity*,
Phys. Rev. D **102** (2), 024060 (2020). [arXiv:1912.02139 [gr-qc]].

Result

We have determined the local 5PN (first) and 6PN (later) EOB Hamiltonian modulo

2 unknown at 5PN and 4 more unknown at 6PN.

Note that thinking of all possible terms entering the 6PN real Hamiltonian, we have determined 147 terms of the 151 total number!

What remains….

$$A(u;\nu) = 1 - 2u + \nu a^{\nu^1}(u) + \nu^2 a^{\nu^2}(u) + \nu^3 a^{\nu^3}(u) + \dots$$
$$\bar{D}(u;\nu) = 1 + \nu \bar{d}^{\nu^1}(u) + \nu^2 \bar{d}^{\nu^2}(u) + \nu^3 \bar{d}^{\nu^2}(u) + \dots$$

Reminder of EOB potentials

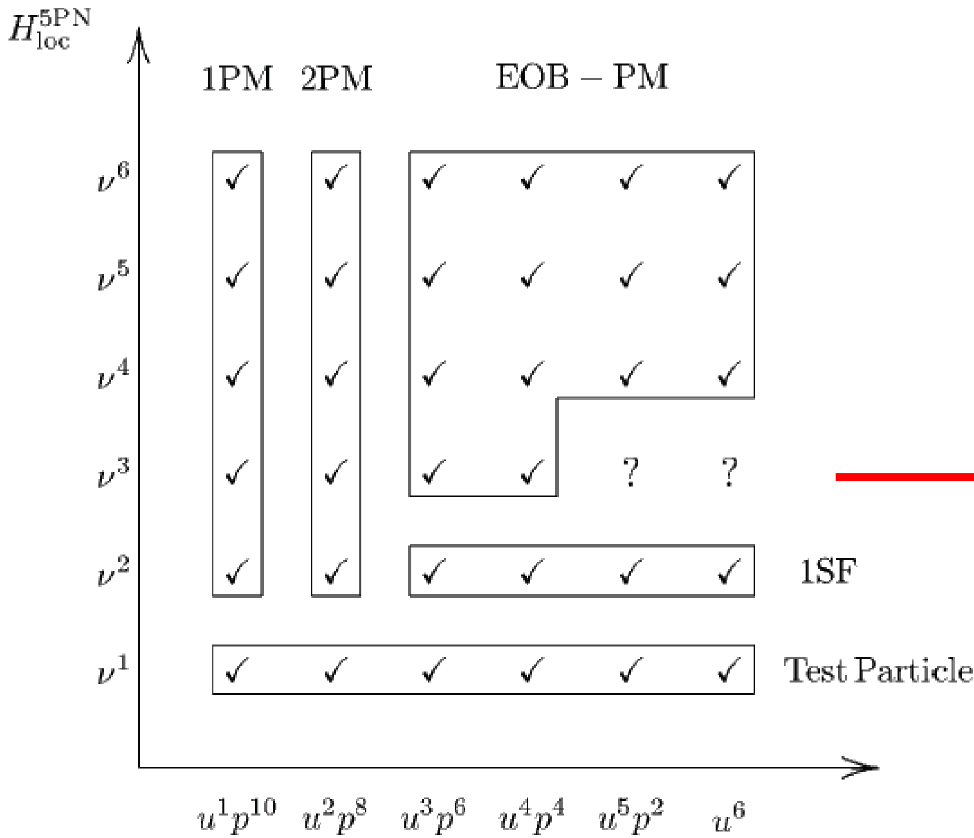
List of the f-route EOB potentials in p_r-gauge.

$a_5^{\rm loc,f}$	$\left(-\frac{4237}{60} + \frac{2275}{512}\pi^2\right)\nu + \left(\frac{41}{32}\pi^2 - \frac{221}{6}\right)\nu^2$
$a_6^{\rm loc,f}$	$\left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_6^{\nu^2}\nu^2 + 4\nu^3$
$a_7^{\rm loc,f}$	$\left(-\frac{2800873}{262144}\pi^4 + \frac{608698367}{1769472}\pi^2 - \frac{1469618167}{907200}\right)\nu + a_7^{\nu^2}\nu^2 + a_7^{\nu^3}\nu^3$
$\bar{d}_4^{\rm loc,f}$	$\left(\frac{1679}{9} - \frac{23761}{1536}\pi^2\right)\nu + \left(\frac{123}{16}\pi^2 - 260\right)\nu^2$
$\bar{d}_5^{\rm loc,f}$	$\left(\frac{331054}{175} - \frac{63707}{512}\pi^2\right)\nu + \bar{d}_5^{\nu^2}\nu^2 + \left(-\frac{205}{16}\pi^2 + \frac{1069}{3}\right)\nu^3$
$\bar{d}_6^{\rm loc,f}$	$\left(\frac{229504763}{98304}\pi^2 + \frac{135909}{262144}\pi^4 - \frac{99741733409}{6350400}\right)\nu + \bar{d}_6^{\nu^2}\nu^2 + \left(\frac{45089}{72} - \frac{44489}{1536}\pi^2 - \bar{d}_5^{\nu^2} - 15a_6^{\nu^2}\right)\nu^3 - 48\nu^4$
$q_{43}^{\rm loc,f}$	$20\nu - 83\nu^2 + 10\nu^3$
$q_{44}^{\rm loc,f}$	$\left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{2075}{3} + \frac{31633}{512}\pi^2\right)\nu^2 + \left(640 - \frac{615}{32}\pi^2\right)\nu^3$
$q_{45}^{\rm loc,f}$	$\left(\frac{81030481}{65536}\pi^2 - \frac{3492647551}{423360}\right)\nu + q_{45}^{\nu^2}\nu^2 + \left(-\frac{14}{3}\bar{d}_5^{\nu^2} + \frac{36677}{1152}\pi^2 - \frac{474899}{216}\right)\nu^3 + \left(\frac{1435}{32}\pi^2 - \frac{7375}{6}\right)\nu^4$
$q_{62}^{\rm loc,f}$	$-\frac{9}{5}\nu - \frac{27}{5}\nu^2 + 6\nu^3$
$q_{63}^{\rm loc,f}$	$\frac{123}{10}\nu - \frac{69}{5}\nu^2 + 116\nu^3 - 14\nu^4$
$q_{64}^{\rm loc,f}$	$\left(-\frac{9733841}{327680}\pi^2 - \frac{112218283}{294000}\right)\nu + \left(\frac{156397}{1280}\pi^2 - \frac{21996581}{21000}\right)\nu^2 + \left(\frac{6977}{6} - \frac{29665}{256}\pi^2\right)\nu^3 + \left(\frac{287}{8}\pi^2 - \frac{3640}{3}\right)\nu^4$
$q_{82}^{\rm loc,f}$	$\frac{6}{7}\nu + \frac{18}{7}\nu^2 + \frac{24}{7}\nu^3 - 6\nu^4$
$q_{83}^{\rm loc,f}$	$-\frac{7447}{560}\nu - \frac{963}{56}\nu^2 - \frac{117}{10}\nu^3 - 147\nu^4 + 18\nu^5$
$q_{10,2}^{\rm loc,f}$	$-\frac{11}{21}\nu - \frac{11}{7}\nu^2 - \frac{20}{7}\nu^3 - \frac{5}{3}\nu^4 + 6\nu^5$

H EOB boxed

Schematic representation of the results obtained at 5PN

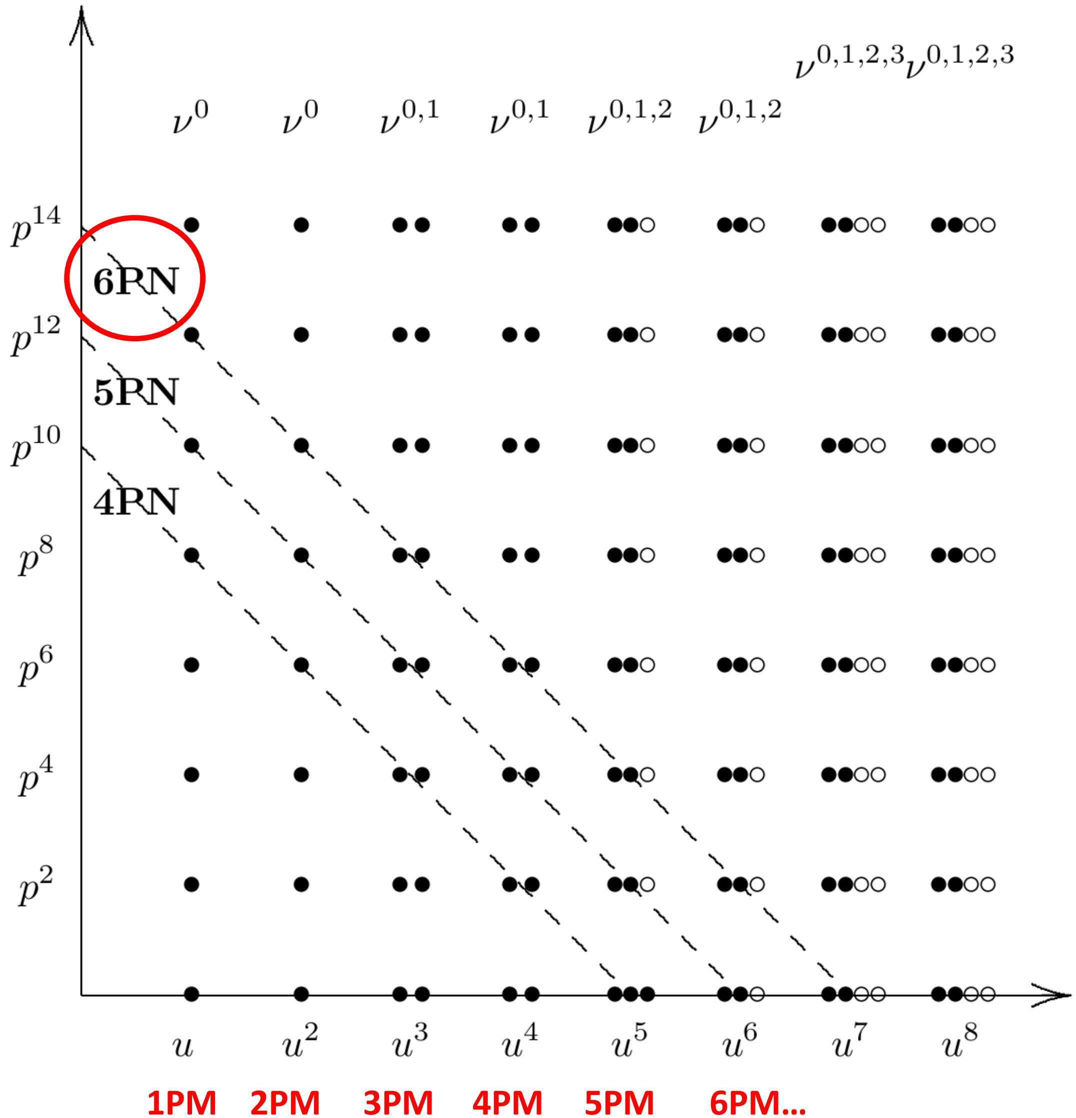
The real Hamiltonian @5PN contains 97 coefficients of which we have determined 95!



D.Bini, T.Damour and A.Geralico,
Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders,
Phys. Rev. D 102, no.2, 024062 (2020)
[arXiv:2003.11891 [gr-qc]].

The missing terms are second order in ν in the EOB potentials: 1SF cannot help!

What is needed to complete the 2 body dynamics at a fixed PM level



Each vertical column of dots describes the PN expansion (keyed by powers of p^2) of an energy-dependent function parametrizing the scattering angle.

The various columns at a given PM level correspond to increasing powers of ν.

D. Bini, T. Damour and A.Geralico, ``Sixth post-Newtonian local-in-time dynamics of binary systems,'' Phys. Rev. D 102, no.2, 024061 (2020) [arXiv:2004.05407 [gr-qc]].

Schematic representation of the irreducible information contained, at each PM level (keyed by a power of u = GM/r), in the local dynamics.

The next challenge…

Can we improve the present knowledge of the nonlocal part of the scattering angle?

Reminder

To be evaluated along hyperboliclike orbits, expanding the flux split (@2 PN) in 1/eccentricity, i.e., computing LO, NLO, NNLO,….

$$W^{\rm nonloc,h}(E,J;\nu) = \frac{GE}{c^5}\times$$
$$\mathrm{Pf}_{2r_{12}^h(t)/c}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dt dt'}{|t-t'|}\mathcal{F}_{\rm GW}^{\rm split}(t,t')$$

$$W^{\rm nonloc,h}(E,J;\nu) = \int_{-\infty}^{+\infty} dt\, H^{\rm nonloc,h}(t)$$

$$\chi^{\rm nonloc,h}(E,J,\nu) = \frac{\partial W^{\rm nonloc,h}(E,J,\nu)}{\partial J}$$

$$
\begin{aligned}
\frac{1}{2}\chi^{\rm tail,h} &= \frac{1}{2M^2\nu}\frac{\partial}{\partial j}W^{\rm tail,h} \\
&= \frac{\chi_4^{\rm tail,h}(\bar{E},\nu)}{j^4} + \frac{\chi_5^{\rm tail,h}(\bar{E},\nu)}{j^5} + \frac{\chi_6^{\rm tail,h}(\bar{E},\nu)}{j^6} \\
&\quad + \frac{\chi_7^{\rm tail,h}(\bar{E},\nu)}{j^7} + O\left(\frac{1}{j^8}\right) \, ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\chi_4^{\rm tail,h}}{j^4} &= \frac{1}{2M^2\nu}\frac{\partial}{\partial j}W^{\rm tail,h\,LO_j}\, , \\
\frac{\chi_5^{\rm tail,h}}{j^5} &= \frac{1}{2M^2\nu}\frac{\partial}{\partial j}W^{\rm tail,h\,NLO_j}\, , \\
\frac{\chi_6^{\rm tail,h}}{j^6} &= \frac{1}{2M^2\nu}\frac{\partial}{\partial j}W^{\rm tail,h\,NNLO_j} \\
\frac{\chi_6^{\rm tail,h}}{j^7} &= \frac{1}{2M^2\nu}\frac{\partial}{\partial j}W^{\rm tail,h\,N^3LO_j}\, .
\end{aligned}
$$

$O(G^4)$

$O(G^7)$

Notation used in our paper(s)

$$\frac{1}{2}\chi^{\mathrm{tail,h}}(p_\infty, j;\nu) \;=\; \nu\frac{p_\infty^4}{j^4}\left(A_0^{\mathrm{tail,h}}(p_\infty;\nu) + \frac{A_1^{\mathrm{tail,h}}(p_\infty;\nu)}{p_\infty j} + \frac{A_2^{\mathrm{tail,h}}(p_\infty;\nu)}{(p_\infty j)^2} + \frac{A_3^{\mathrm{tail,h}}(p_\infty;\nu)}{(p_\infty j)^3} + \cdots\right)$$

NNLO NNNLO

$$\begin{aligned}
A_0(p_\infty;\nu) &= A_0^{\mathrm{tail,h,N}} + A_0^{\mathrm{tail,h,1PN}} + A_0^{\mathrm{tail,h,2PN}} + \cdots \\
A_1(p_\infty;\nu) &= A_0^{\mathrm{tail,h,N}} + A_1^{\mathrm{tail,h,1PN}} + A_1^{\mathrm{tail,h,2PN}} + \cdots \\
A_2(p_\infty;\nu) &= A_2^{\mathrm{tail,h,N}} + A_2^{\mathrm{tail,h,1PN}} + A_2^{\mathrm{tail,h,2PN}} + \cdots \\
A_3(p_\infty;\nu) &= A_3^{\mathrm{tail,h,N}} + A_3^{\mathrm{tail,h,1PN}} + A_3^{\mathrm{tail,h,2PN}} + \cdots
\end{aligned}$$

NNLO $O(G^6)$

At NNLO we found large integrands which made difficult even their numerical integration.

The collaboration with **S. Laporta** and **P. Mastrolia** has allowed to obtain **high-precision numerical values first**, e.g.,

Q_{20}	524.767292180212584342735955703101758476141999557369011937728711238498839830097712093907037158196060831706238995205677052067946783744966475134730111010455883184170170829347212071124106113165 8613485679

Successively, we could fit these results with **PSLQ** algorithm

Q_{20}	$\frac{25883}{1800}+\frac{22333}{140}\mathrm{K}-\frac{625463}{3360}\pi-\frac{361911}{560}\pi\ln 2+\frac{99837}{160}\pi\zeta(3)$

And finally we could analytically check the results by using HPLs of w=4

Noticeably, in the intermediate results the Catalan constant and $\zeta(3)$ enter. However, the first cancels out in the scattering angle while BOTH cancel out in the periastron advance, leading to a rational coefficient only (at NNLO)

The NNLO accomplishment

Integral entering our computation (in its reduced, final form)

$$J(x) \equiv \int_0^1 dT\,(1-x^2) \times$$
$$\times \frac{16\,\mathrm{arctanh}^3(T) - 3\,\mathrm{Li}_3\left[\left(\frac{(1-T)(1-x)}{(1+T)(1+x)}\right)^2\right]}{2i(T+x)(T+1/x)}$$

Expression in terms of HPLs w=4

sought for result

$$J(i) = -\frac{1}{2}\pi^2\,\mathrm{K} + \frac{9}{2}\pi\zeta(3)$$

NNLO: Further developments and recent accomplishments

→ Nonlocal part of the scattering angle @N^3LO (still unpublished)

→ Interplay of works in the ordinary space and in the Fourier space (better understanding of these two complementary povs)

Bini, Damour, Geralico, Laporta, Mastrolia, in its final phase….

Computations in the Fourier space are apparently simpler than the corresponding in the ordinary space

Working in the ordinary space is more immediate but produces very large integrals to be computed

$$A_0^{\rm tail,h,N} = \pi\left[-\frac{37}{5}\ln\left(\frac{p_\infty}{2}\right)-\frac{63}{4}\right],$$

LO

$$A_0^{\rm tail,h,1PN} = \pi\left[\left(-\frac{1357}{280}+\frac{111}{10}\nu\right)\ln\left(\frac{p_\infty}{2}\right)-\frac{2753}{1120}+\frac{1071}{40}\nu\right]p_\infty^2,$$

$$A_0^{\rm tail,h,2PN} = \pi\left[\left(-\frac{27953}{3360}+\frac{2517}{560}\nu-\frac{111}{8}\nu^2\right)\ln\left(\frac{p_\infty}{2}\right)-\frac{155473}{8960}+\frac{109559}{40320}\nu-\frac{186317}{5040}\nu^2\right]p_\infty^4$$

$$A_1^{\rm tail,h,N} = -\frac{6656}{45}-\frac{6272}{45}\ln\left(4\frac{p_\infty}{2}\right),$$

NLO

$$A_1^{\rm tail,h,1PN} = \left[\left(-\frac{74432}{525}+\frac{13952}{45}\nu\right)\ln\left(4\frac{p_\infty}{2}\right)+\frac{114368}{1125}+\frac{221504}{525}\nu\right]p_\infty^2,$$

$$A_1^{\rm tail,h,2PN} = \left[\left(-\frac{881392}{11025}+\frac{288224}{1575}\nu-\frac{21632}{45}\nu^2\right)\ln\left(4\frac{p_\infty}{2}\right)+\frac{48497312}{231525}-\frac{5134816}{23625}\nu-\frac{25465952}{33075}\nu^2\right]p_\infty^4$$

$$A_2^{\rm tail,h,N} = \pi\left[-122\ln\left(\frac{p_\infty}{2}\right)-\frac{99}{4}-\frac{2079}{8}\zeta(3)\right],$$

NNLO

$$A_2^{\rm tail,h,1PN} = \pi\left[\left(\frac{811}{2}\nu-\frac{13831}{56}\right)\ln\left(\frac{p_\infty}{2}\right)-\frac{41297}{112}-\frac{9216}{7}\ln(2)+\frac{49941}{64}\zeta(3)+\left(\frac{3303}{4}\zeta(3)+\frac{1937}{8}\right)\nu\right]p_\infty^2,$$

$$\begin{aligned}A_2^{\rm tail,h,2PN} = \pi\Bigg[&\left(\frac{64579}{1008}-785\nu^2+\frac{75595}{168}\nu\right)\ln\left(\frac{p_\infty}{2}\right)-\frac{40711}{128}\zeta(3)+\frac{1033549}{4536}+\frac{10704}{7}\ln(2)\\&+\left(\frac{75520}{21}\ln(2)+\frac{8008171}{8064}-\frac{660675}{256}\zeta(3)\right)\nu+\left(-\frac{100935}{64}\zeta(3)-\frac{583751}{864}\right)\nu^2\Bigg]p_\infty^4,\end{aligned}$$

At NNLO?

One of these integrands

«About 2-300 times this block»

```
(-86222/315*(-1+T)*
   (-((Tp^12+567504/43111*Tp^10+181497/43111*Tp^8-6429152/
        105476/43111*Tp*(Tp^12-1627314/26369*Tp^10-5379177
        524393/43111*(Tp^12-4160988/524393*Tp^10-24738249/
        6509256/43111*Tp*(-3308411/271219*Tp^8+1662010/271
        386007/43111*(1+Tp)*(-1+Tp)*(Tp^12+6987756/128669*
        21516708/43111*Tp*(-6616822/1793059*Tp^10-6616822/
```

A good job!

Q40num = 2324.\
2945518755079244796761032384673115986096012600915287113669334105968155\
9651694952785135498410096031353631295560553413266360924201075481405964\
2335939609295101495919259244145297218612521402336653114469276802586585\
5818941914970953912386638933986533600312540324370182179717623944381629\
7914887792750942760325300191696425379993558158933940567719749455512295\
0654024975716659024974807825147802357227383231

Q40PSLQ = -(21644756692/7640325) + (13986515191 Pi)/
  19051200 - (5616 Pi^2)/25 + (122517 Pi^4)/
  2240 - (11235344 Zeta[3])/51975

A comment on our fruitful collaboration with HEP-EFT people

Bini, Damour, Geralico, Laporta, Mastrolia

Fitting is easy only once you know the structure of the expected result!

Producing numerical results for the sought for integrals with high numerical precision (about 500 estimated correct digits).
Fitting of results with available algorithms (PSLQ) and having an «experimental math» solution.
Obtaining the complete analytical solution by using HLP.

Complete agreement with the recent work by J. Blumlein, A. Maier, P. Marquard, G. Schafer, arXiv: 2010.13672 @5PN.

See Eqs. 63-64 there with the determination of the Pi^2 part of the 5PN missing coefficients in TF

$$\bar{d}_5 = r_{d_5} + \frac{306545}{512}\pi^2$$
$$a_6 = r_{a_6} + \frac{25911}{256}\pi^2$$

Typo in our 6PN nonloc paper arXiv:2007.11239 [gr-qc]

Warning

Eq. (8.27), first line $-\frac{155}{12}\ln 2 \quad \rightarrow \quad -\frac{155}{112}\ln 2$

Conclusions and plans for future works

- Can TF method be applied to reach 7PN? ⇒ In principle yes…for both local and nonlocal H!
- Can one push forward our knowledge of the nonlocal in time dynamics and obtain more information about the nonlocal part of the scattering angle?

⇓

- Large-eccentricity expansion of flux split integrals computed along hyperboliclike motion. [Fruitful collaboration with S. Laporta and P. Mastrolia…We went to N3LO]
- Forthcoming 5PN h-coordinate-based complete Hamiltonian? [See a recent work by J. Blumlein, A. Maier, P. Marquard, G. Schafer, arXiv: 2010.13672]
- Scattering angle from 1SF? ⇒ [Maybe, hopefully we will see first numerical 1SF results before their analytical counterparts]
- Scattering angle @ 4PM from amplitudes? [Maybe, these are all new and promising avenues…See e.g., Z. Bern, et al. Phys. Rev. Lett. 122, no.20, 201603 (2019)]
- Other recent EFT approaches/results [See S. Foffa , R. Sturani, W. J. Torres Bobadilla, arXiv: 2010.13730; G.Kalin and R.A. Porto, JHEP 01, 072 (2020); L.Blanchet, S.Foffa, F.Larrouturou and R.Sturani, Phys. Rev. D 101, no.8, 084045 (2020); …]

Thanks for your kind attention!

I would like to thank the organizers of this meeting especially for stimulating interactions among different communities of theoretical physicists, with the aim of making immediate progress from mutual collaborations.