Eikonal Exponentiation & Universality in GRAVITATIONAL SCATTERING

EFT Methods
from Bound States
to Binary Systems

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+ in preparation

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Outline

- **Eikonal Exponentiation** $S \sim e^{2i\delta}$ $\to \delta = \delta_0 + \delta_1 + \delta_2 + \dots$
  1PM 2PM 3PM
  - CLASSICAL LIMIT OF THE AMPLITUDE
  - RESUMMATION IN IMPACT-PARAMETER SPACE

- **Universality at High Energy**
  - ANALYTICITY + CROSSING SYMMETRY
    $$\Longrightarrow \quad \mathrm{Re}\,\delta_2 \underset{s \gg m^2}{\sim} f(\delta_0, \mathrm{Im}\,\delta_2)$$
  - $\mathrm{Im}\,\delta_2$ FROM UNITARITY AT HIGH ENERGY

- **Massive $\mathcal{N}=4$ SUGRA at Any Energy**
  - AMPLITUDE IN THE SOFT REGION $\Longrightarrow$ δ and χ
  - χ_2 AT HIGH AND LOW ENERGY
    3PM

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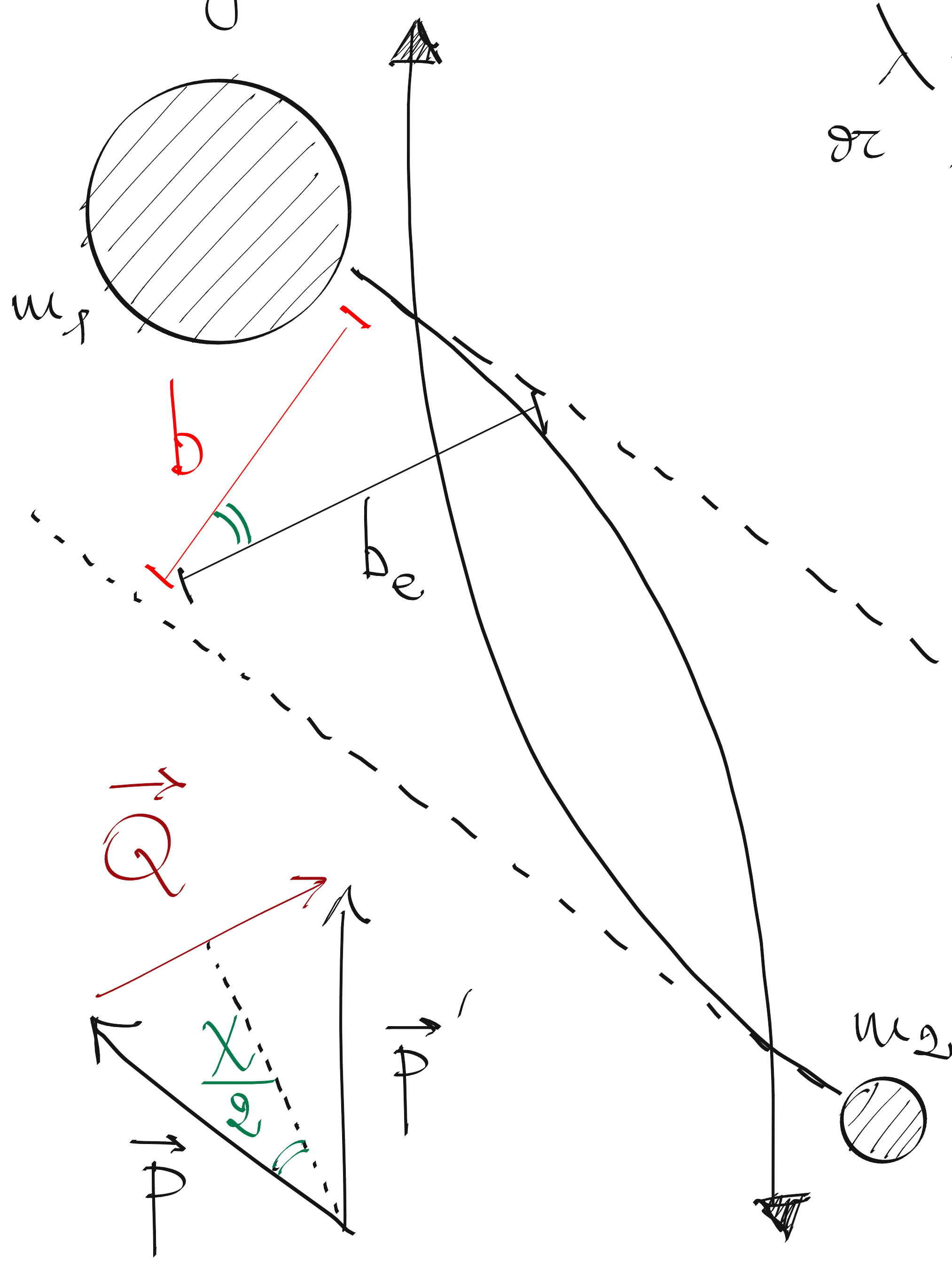
3PM

Classical Regime of Gravitational Scattering

- Length scales:



QUANTUM WAVELENGTH

$\lambda \sim 1/m$

or $\lambda \sim 1/\sqrt{s}$

SCHWARZSCHILD RADIUS

$R_S \sim G_N m$

or $R_S \sim G_N \sqrt{s}$

IMPACT PARAMETER

$$\boxed{\lambda \ll R_S \ll b}$$

CLASSICAL REGIME

WEAK-COUPLING REGIME

total momentum transfer $\quad |\vec{Q}| = 2|\vec{p}|\sin\frac{\chi}{2}$ $\quad$ SCATTERING ANGLE

$b\,|\vec{p}| = L$ $\quad$ angular momentum

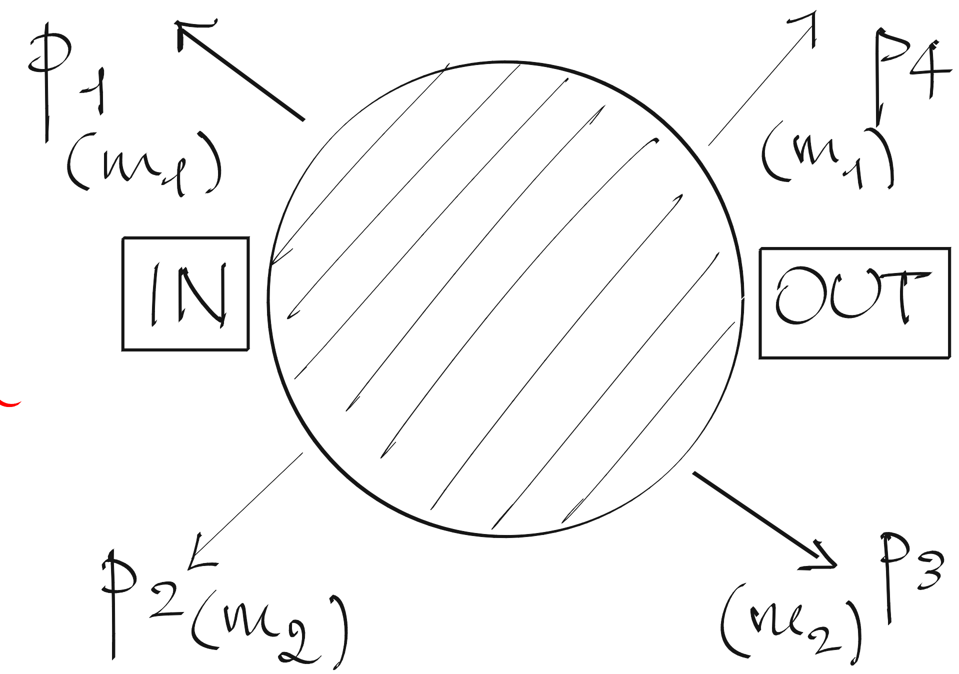
$$b = b_e \cos\frac{\chi}{2}$$

Elastic 2-to-2 Amplitude, $s=0$

- Momentum Space:

$$s = -(p_1+p_2)^2 = m_1^2 + m_2^2 + 2m_1m_2\sigma$$

$$q^2 = (p_1+p_4)^2 \quad \text{perturbative momentum transfer}$$



- Perturbative expansion:

$$\mathcal{A}(s,q^2) = \mathcal{A}_0(s,q^2) + \mathcal{A}_1(s,q^2) + \mathcal{A}_2(s,q^2) + \dots$$

TREE $O(G_N)$ 1 LOOP $O(G_N^2)$ 2 LOOPS $O(G_N^3)$

- Impact Parameter Space

$$\tilde{\mathcal{A}}(s,b) = \int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}} \frac{\mathcal{A}(s,\vec{q})}{4E_{CM}p_{CM}} e^{i\vec{b}\cdot\vec{q}}$$

$\boxed{\text{LARGE-}b \text{ asymptotics of } \tilde{\mathcal{A}}(s,b)}$

$\Longleftrightarrow$

$\boxed{\text{SMALL-}q^2 \text{ asymptotics of } \mathcal{A}(s,q^2)}$

up to $(q^2)^n$ $(n=0,1,2,\dots)$

Classical Limit of the Amplitude

Near-forward limit of $A(s,q^2)$

$q^2 \ll m_1^2, m_2^2, s$

$$A_0(s,q^2) = \frac{c_0^{CL}}{q^2} + (\ldots)$$

$$A_1(s,q^2) = \frac{c_1^{SCL}}{(q^2)^{1+\epsilon}} + \frac{c_1^{CL}}{(q^2)^{\frac{1}{2}+\epsilon}} + \frac{c_1^{Q}}{(q^2)^{\epsilon}} + (\ldots)$$

$$A_2(s,q^2) = \frac{c_2^{SSCL}}{(q^2)^{1+2\epsilon}} + \frac{c_1^{SCL}}{(q^2)^{\frac{1}{2}+2\epsilon}} + \frac{c_1^{CL}}{(q^2)^{2\epsilon}} + (\ldots)$$

Long-range behavior of $\tilde{A}(s,b)$

$$\tilde{A}_0(s,b) = \frac{f_0^{CL}}{(b^2)^{-\epsilon}} + (\ldots)$$

$$\tilde{A}_1(s,b) = \frac{f_1^{SCL}}{(b^2)^{-2\epsilon}} + \frac{f_1^{CL}}{(b^2)^{\frac{1}{2}-2\epsilon}} + \frac{f_1^{Q}}{(b^2)^{1-2\epsilon}} + (\ldots)$$

$$\tilde{A}_2(s,b) = \frac{f_2^{SSCL}}{(b^2)^{-3\epsilon}} + \frac{f_2^{SCL}}{(b^2)^{\frac{1}{2}-3\epsilon}} + \frac{f_2^{CL}}{(b^2)^{1-3\epsilon}} + (\ldots)$$

CL $\rightarrow \mathcal{O}\left(\frac{1}{\hbar}\right)$ classical

SCL $\rightarrow \mathcal{O}\left(\frac{1}{\hbar^2}\right)$ superclassical

SSCL $\rightarrow \mathcal{O}\left(\frac{1}{\hbar^3}\right)$ supersuperclassical

Q $\rightarrow \mathcal{O}(\hbar^0)$ 1st quantum

$(\ldots)$ = higher-order in q^2 or in $1/b$

Exponentiation in Impact Parameter Space

- The b-space amplitude $\tilde{A}(s,b)$ resums:

$$\boxed{1 + i\tilde{A}(s,b) = \left[1 + 2i\Delta(s,b)\right] e^{2i\delta(s,b)}}$$

LARGE EIKONAL PHASE

- IDENTIFICATIONS:

classical & quantum pieces determine $\delta = \delta_0 + \delta_1 + \delta_1'$, $\Delta = \Delta_1 + \cdots$

$$\begin{cases} 2\delta_0 = \dfrac{f_0^{CL}}{(b^2)^{-\epsilon}} \\[2ex] 2\delta_1 = \dfrac{f_1^{CL}}{(b^2)^{\frac{1}{2}-2\epsilon}}, \quad 2\Delta_1 = \dfrac{f_1^{Q}}{(b^2)^{1-2\epsilon}} \\[2ex] 2\delta_2 = \dfrac{f_2^{CL} - i f_0 f_1^{Q}}{(b^2)^{1-3\epsilon}} \end{cases}$$

- CONSTRAINTS:

the superclassical pieces are redundant

$$\begin{cases} f_1^{SCL} = \dfrac{i}{2}\left(f_0^{CL}\right)^2 \\[2ex] f_2^{SSCL} = -\dfrac{1}{3!}\left(f_0^{CL}\right)^3 \\[2ex] f_2^{SCL} = i\, f_0^{CL} f_1^{CL} \end{cases}$$

|| useful cross-checks for loop calculations ||

Eikonal Amplitude, Scattering Angle

- $$1 + i\tilde{A}(s,b) = \left[1 + 2i\Delta(s,b)\right] e^{2i\delta(s,b)} \simeq e^{2i\delta(s,b)}$$

  LARGE EIKONAL PHASE

  $$\Rightarrow \quad iA(s,Q^2) = 4E_{CM}|\vec{p}_{CM}| \int d^{D-2}\vec{b}_e \; e^{-i\vec{Q}\cdot\vec{b}_e}\left(e^{2i\delta(s,b_e)} - 1\right)$$

- Saddle point: $$\boxed{\vec{Q} = 2\,\frac{\partial\delta(s,\vec{b}_e)}{\partial\vec{b}_e}}$$ total momentum transfer

- $|\vec{Q}| = 2|\vec{p}|\sin\frac{\chi}{2}$, $\quad L = b|\vec{p}|$, $\quad b = b_e\cos\frac{\chi}{2}$

  $\Rightarrow \quad \chi(s,L)$ scattering angle

- The eikonal method applies both to massive and massless scattering!! in the CLASSICAL REGIME.

Example: $m=0$ tree level

- Relevant scales: $\lambda \ll R_S \ll b$ or $\frac{1}{\sqrt{s}} \ll G_N\sqrt{s} \ll b$
- Tree-level amplitude:

$$\mathcal{A}_0(s,q^2) = \frac{1}{q^2}\, 8\pi G_N s \qquad \tilde{\mathcal{A}}_0(s,b) = \frac{1}{(b^2)^{-\epsilon}}\, \frac{G_N s\,\Gamma(-\epsilon)}{\pi^{-\epsilon}}$$

- Eikonal: $2\delta_0(s,b) = G_N s \dfrac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}}$, indeed $G_N s \gg 1$.

$$1 + i\tilde{\mathcal{A}}_0(s,b) \simeq e^{2i\delta_0(s,b)}$$

- Scattering angle: $\vec{Q} = -\dfrac{2G_N s\,\Gamma(1-\epsilon)}{\pi^{-\epsilon}\, b^{1-2\epsilon}}\,\hat{b} \xrightarrow[\epsilon\to 0]{} -G_N s\,\dfrac{\hat{b}}{b}$

$$\Rightarrow \boxed{\chi_0 = \frac{2G_N\sqrt{s}}{b} = 2\left(\frac{R_S}{b}\right) \ll 1}$$

Imaginary and Real Part of δ

- A_0 is real $\Longrightarrow$ δ_0 is real.
- $\mathrm{Im}(A_1) \sim \frac{i}{2!}(\delta_0)^2 \Longrightarrow \delta_1$ is real.
- $\mathrm{Im}(A_2) \neq 0$ and $\boxed{\mathrm{Im}\,\delta_2 \neq 0}$

  $\mathrm{Im}\,\delta_2$ contains infrared divergent contributions $\frac{1}{\epsilon}$ due to intermediate virtual gravitons.

Such divergences must cancel against those arising from soft Bremsstrahlung in observables.

- $\mathrm{Re}\,\delta_2$ is infrared finite $\Longrightarrow$

$$\boxed{\begin{array}{l}\vec{Q} = 2\dfrac{\partial}{\partial \vec{b}_e}\left[\delta_0(s,b_e) + \delta_1(s,b_e) + \mathrm{Re}\,\delta_2(s,b_e)\right] \\ \text{IR (and UV) finite 3PM momentum} \\ \text{transfer} \Longrightarrow \text{scattering angle } \chi.\end{array}}$$

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Analyticity, Crossing Symmetry & Universality

For theories where the GRAVITON is the highest-spin $m=0$ state, (INCLUDING GR!) assuming:

- ANALITICITY in s: $A(s,q^2)$ is the boundary value of a real analytic function $\mathcal{M}(\tilde{s},q^2)$ with a branch cut for $\tilde{s} \geq (m_1+m_2)^2$

$$A(s,q^2) = \mathcal{M}(s+i0,q^2) \qquad \mathcal{M}^*(\tilde{s},q^2) = \mathcal{M}(\tilde{s}^*,q^2)$$

- CROSSING SYMMETRY: $\mathcal{M}(\tilde{s},q^2) = \mathcal{M}(\tilde{u},q^2)$

$$\tilde{u} = 2(m_1^2+m_2^2) - \tilde{s} + q^2$$

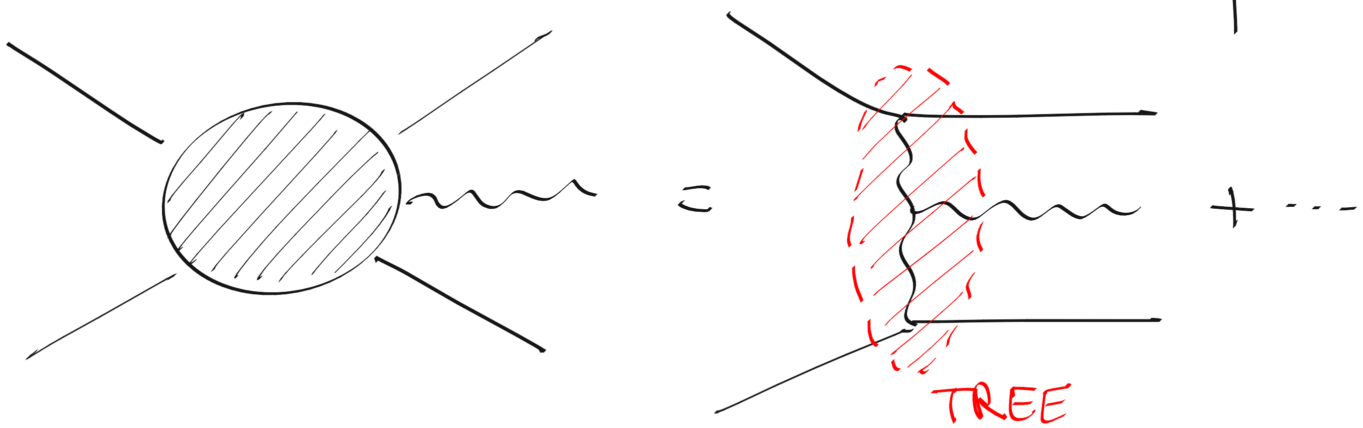
- HIGH-ENERGY asymptotics: $A_2(s,q^2) \sim s^3 \log s \; G(q^2, m_{1,2})$ after eikonal subtraction

$$\Rightarrow \boxed{\underbrace{\mathrm{Re}(2\delta_2)}_{\text{IR finite (?)}} \underset{s \gg m_{1,2}^2}{\sim} \frac{\pi}{2\log s} \underbrace{\mathrm{Im}(2\delta_2)}_{\text{IR div.}} - \underbrace{\frac{\partial\delta_0}{\partial s}\left(2\frac{\partial\delta_0}{\partial \vec{b}}\right)^2}_{\text{IR div.}} + O\left(\frac{1}{\log s}\right)}$$

$\mathrm{Im}\,\delta_2$ from the 3-Particle Cut

Estimate $\mathrm{Im}\,\delta_2$ in the high-energy limit:

- Construct the 2-to-3 amplitude in a specific theory



- Bosonic string
  - $\rightarrow$ KLT relations
  - $\rightarrow$ Field Theory limit
- Focus on the high-energy limit (DOUBLE REGGE).

- Im 

unitarity

$\Longrightarrow$ $\mathrm{Im}\,\delta_2$ at high energy

- $$\boxed{\mathrm{Re}\,(2\delta_2) \underset{s \gg m_1^2, m_2^2}{\sim} \frac{4 G_N^3 s^2}{b^2}}$$

✓ IR finite

✓ agrees with ACV90 $\quad \mathcal{O}\left(\delta_0 \left(\frac{R_s}{b}\right)^2\right)$

✓ dominated by graviton exchange

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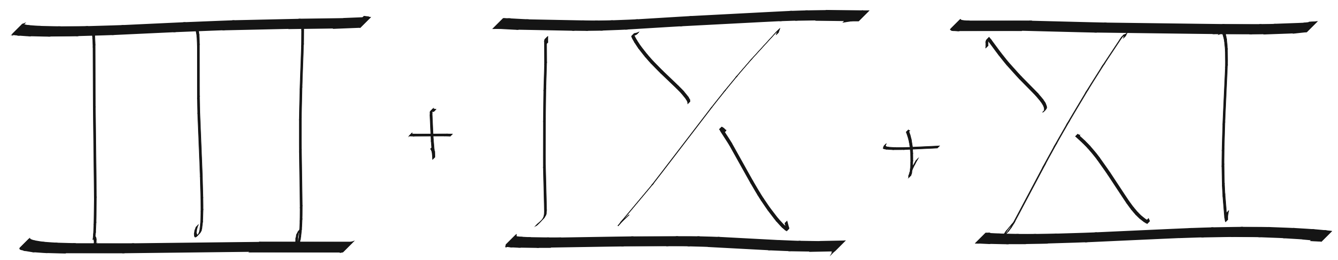
    ↳ 3PM

$\mathcal{N}=8$ SUGRA

- Calculate the amplitude in the near-forward limit for a specific theory, for generic $s \geq (m_1+m_2)^2$.
- s-u symmetric amplitude (dilaton-axion)

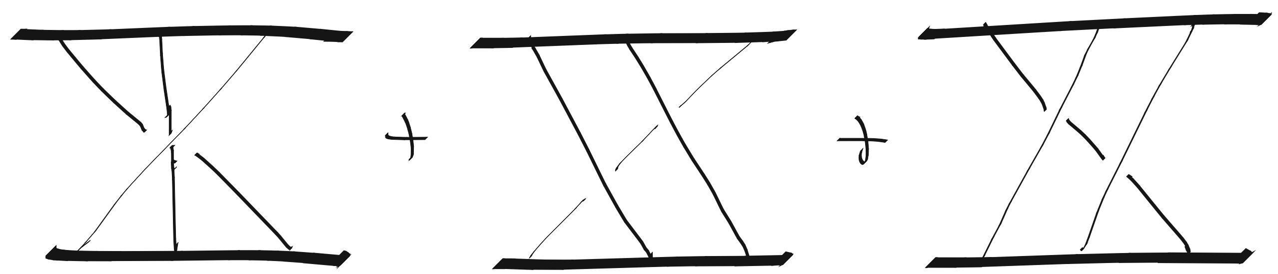
$$A_2(s,q^2) = \frac{(8\pi G_N)^3}{2}\left[(s-m_1^2-m_2^2)^4+(u-m_1^2-m_2^2)^4-(q^2)^4\right]$$

$$\times\Big[(s-m_1^2-m_2^2)^2\Big($$



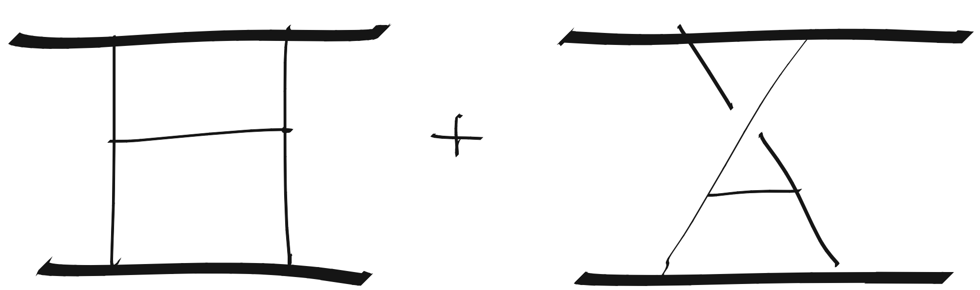
$$\Big)$$

$$+(u-m_1^2-m_2^2)^2\Big($$



$$\Big)$$

$$+(q^2)^2\Big($$



$$+ \cdots \Big)\Big]$$

[2005.04236 Parra-Martinez, Ruf, Zeng] adapted to the int. of real scalars.

Evaluation of the Loop Integrals

$$\text{III} = \iint \frac{d^D\ell_1 \, d^D\ell_2}{\left[2\bar{p}_1\cdot\ell_1 - i0 + (\ell_1^2 - \ell_1 q)\right]\left[-2\bar{p}_2\cdot\ell_1 - i0 + (\ell_1^2 - \ell_1 q)\right]\left[-2\bar{p}_1\cdot\ell_2 - i0 + (\ell_2^2 - \ell_2\cdot q)\right]\left[2\bar{p}_2\cdot\ell_2 - i0 + (\ell_2^2 - \ell_2 q)\right]}$$

$$\times \frac{1}{(\ell_1^2 - i0)(\ell_2^2 - i0)\left((\ell_1 + \ell_2 - q)^2 - i0\right)}$$

$$p_1 = -\bar{p}_1 + q/2$$
$$p_2 = -\bar{p}_2 - q/2$$
$$p_3 = \bar{p}_2 - q/2$$
$$p_4 = \bar{p}_1 + q/2$$

- Hard region: $q^2 \to 0, \quad \ell_1, \ell_2 \sim \mathcal{O}(m_{1,2})$

$$\text{III}^{(h)} = c_0 (q^2)^0 + c_1 (q^2)^1 + \cdots$$

irrelevant for the long range behavior of $\delta(s,b)$

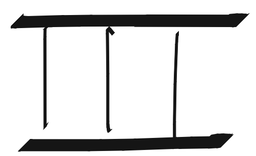
- Soft region: $q^2 \to 0, \quad \ell_1, \ell_2 \sim \mathcal{O}(q)$

$$\text{III}^{(s)} = \frac{C_{\text{III}}^{(SSCL)}}{(q^2)^{1+2\epsilon}} + \frac{C_{\text{III}}^{(SCL)}}{(q^2)^{\frac{1}{2}+2\epsilon}} + \frac{C_{\text{III}}^{(CL)}}{(q^2)^{2\epsilon}} + \cdots$$

relevant region!

Evaluation of the Soft Integrals

- Basis of soft master integrals [2005.04236]

 $\overset{(s)}{=} \sum_j c_j \, f_{\mathrm{III},j}(x)$

$$x = y - \sqrt{y^2 - 1}$$
$$y \simeq \sigma + \mathcal{O}(q^2)$$
$$s = m_1^2 + m_2^2 + 2 m_1 m_2 \sigma$$

- Differential equations from IBP reduction (LiteRed, FIRE6)

$$d \vec{f}_{\mathrm{III}}(x) = \epsilon \left[A_0 \, d\log x + A_{+1} \, d\log(x+1) + A_{-1} \, d\log(x-1) \right] \vec{f}_{\mathrm{III}}(x)$$

- Boundary Conditions evaluated in the STATIC LIMIT

$$x \to 1$$

without restricting to the potential region

Re δ_2 and χ_2

- Exponentiation works as expected. 3PM eikonal:

$$\mathrm{Re}\,\delta_2 = \frac{8 m_1^2 m_2^2 G_N^3 \sigma^6}{b^2(\sigma^2-1)^2} - \frac{8 m_1^2 m_2^2 G_N^3 \sigma^4}{b^2(\sigma^2-1)} \cosh^{-1}(\sigma)\left[1 - \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\right]$$

IR finite, ϵ has been sent to zero.

- 3PM scattering angle:

$$\chi_2 = -\frac{m_1^3 m_2^3 \sigma^6 G_N^3}{L^3(\sigma^2-1)}\left[\boxed{\frac{16}{3}\frac{1}{\sqrt{\sigma^2-1}}} - \boxed{\frac{32 m_1 m_2}{s}}\right] - \frac{32\, m_1^4 m_2^4 \sigma^4 G_N^3}{L^3 s}\left[\boxed{\cosh^{-1}(\sigma)} - \boxed{\frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\cosh^{-1}(\sigma)}\right]$$

terms also appearing in the potential region

New terms appearing in the full soft region

- **High-energy limit:** $s = m_1^2 + m_2^2 + 2m_1 m_2 \sigma \gg m_{1,2}^2$

$$\chi_2 = -\frac{m_1^3 m_2^3 \sigma^6 G_N^3}{L^3(\sigma^2-1)}\left[\frac{16}{3}\frac{1}{\sqrt{\sigma^2-1}} - \frac{32 m_1 m_2}{s}\right] \to \underset{\text{ACV90}}{\frac{4}{3}\left(\frac{G_N s}{L}\right)^3} \sim \mathcal{O}\left(\left(\frac{R_s}{b}\right)^3\right)$$

$$-\underbrace{\frac{32\, m_1^4 m_2^4 \sigma^4 G_N^3}{L^3 s}}_{\mathcal{O}\left(\left(\frac{R_s}{b}\right)^3\right)} \underbrace{\cosh^{-1}(\sigma)}_{\mathcal{O}(\text{logs})} \underbrace{\left[1 - \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\right]}_{\mathcal{O}(s^{-1})}$$

$(R_s \sim G_N \sqrt{s})$

- **Near-static limit:** $\sqrt{\sigma^2-1} = \frac{E_{CM}\, P}{m_1 m_2}$ $\quad \sigma - 1 = \mathcal{O}(v^2) \ll 1$

$$\chi_2 = -\frac{32 m_1^3 m_2^3 \sigma^6 G_N^3}{L^3}\left[\underset{\text{0PN}}{\boxed{\frac{1}{6}\frac{1}{(\sigma^2-1)^{\frac{3}{2}}}}} - \boxed{\frac{m_1 m_2}{s(\sigma^2-1)}}\right]$$

$$-\frac{32\, m_1^4 m_2^4 \sigma^4 G_N^3}{L^3 s}\left[\underset{\text{2PN}}{\boxed{\cosh^{-1}(\sigma)}} - \boxed{\frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\cosh^{-1}(\sigma)}\right]$$

1.5 PN

remnant of radiation (?)

$\left(\cosh^{-1}(\sigma) \sim \sqrt{\sigma^2-1}\right)$

cf. [2010.01641 T. Damour]

Conclusions

- Analyticity + Crossing Symmetry
  $\Longrightarrow$ Universal relation $\boxed{\mathrm{Re}\,\delta_2 \underset{s \gg m_{1,2}^2}{\sim} f(\delta_0, \mathrm{Im}\,\delta_2)}$
- Explicit ev. of $\mathrm{Im}\,\delta_2$ from unitarity at high energy
  $\Longrightarrow$ $\mathrm{Re}\,\delta_2$ in agreement with ACV90
- δ_2 and χ_2 for $\boxed{\mathcal{N}=8 \text{ SUGRA}}$ at any energy in the full soft region
  $\Longrightarrow$ smooth high-energy limit
  $\Longrightarrow$ new HALF-ODD PN terms at low energy

Outlook:
- Radiation/spin in the eikonal (operator)
- GR radiation effects ([2010.01641 T. Damour])
- Comparison w/ EOB, effective field theory, direct evaluation of $\langle \vec{Q} \rangle$

Thank you
for your
attention !