EFTs for Bound State Problems

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EFT methods from Bound States to Binary Systems
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30 Oct 2020

Previous Work

Bethe (1947)

Caswell, Lepage

Bodwin, Braaten, Lepage

Labelle

Grinstein, Rothstein

Brambilla, Pineda, Soto, Vairo: pNRQD — First 2 talks

Yndurain, Titard

Beneke, Smirnov: Method of Regions (not an EFT)

Luke, Rothstein, Stewart, Hoang

Applications

Study non-relativistic dynamics in field theory.

- QED: Hydrogen, Positronium and Muonium. Can study energy levels (Lamb shift and hyperfine splitting), and decay widths.
- QCD
  - Υ ($\bar{b}b$)
  - $\bar{t}t$ near threshold
  - non-perturbative effects in a systematic way
- Gravity

Focus on QED since it is Abelian and and closest to gravity.

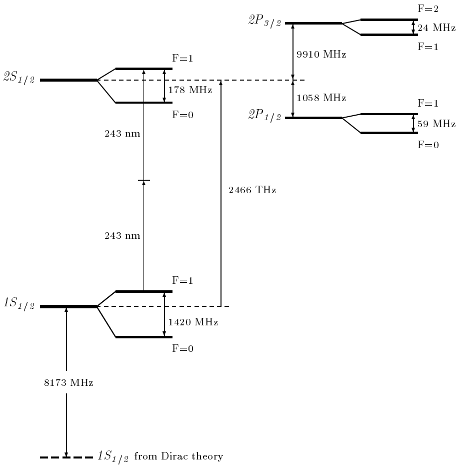
In QED, the expansion parameter is $\alpha \sim 1/137 \ll 1$.
Nevertheless, one cannot always use perturbation theory in α.

Hydrogen Atom: One needs to solve the Schrödinger equation with the potential

$$V = -\frac{4\pi\alpha}{r}$$

The Schrödinger equation sums up multiple iterations of the Coulomb potential.

Hydrogen Atom



Size of Splittings

binding energy:	α^2
fine structure:	α^4
Lamb shift:	$\alpha^5 \ln \alpha \sim 1058\,\text{MHz}$
hyperfine structure:	$\alpha^4 \dfrac{m_e}{m_p}$

$$m_e \sim 7.5 \times 10^{14}\ \text{MHz}$$
$$m_e \alpha^2 \sim 4 \times 10^{10}\ \text{MHz}$$

Classical Limit

H atom is a quantum bound state, and we do not have an $\hbar$ expansion.

For two-body scattering in the CM frame,

$$\mathbf{p} + (-\mathbf{p}) \to \mathbf{p}' + (-\mathbf{p}'),$$

we can use $\mathbf{p}$ and $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ as variables.

Hydrogen	Binary Black Holes
$J \sim \hbar$	$J \sim 10^{77}\hbar$
$v \sim \alpha$	$v \ll 1$ (PN) or PM
$p/m \sim v$	$p/m \sim v \ll 1$ (PN) or γ
$p \sim q$	$q \ll p$
$\left(\frac{\alpha}{mr}\right)^c \left(\frac{1}{mr}\right)^q = \left(\frac{\alpha q}{m}\right)^c \left(\frac{q}{m}\right)^q$	$\left(\frac{GM}{r}\right)^c \left(\frac{1}{mr}\right)^q = \left(\frac{GM}{r}\right)^c \left(\frac{q}{m}\right)^q$

In reality, classical corrections are smaller than quantum ones.

Scales

3 important scales

- m (hard)
- $p \sim mv$ (soft)
- $E \sim mv^2$ (radiation, ultrasoft)

$$J \sim mvr \sim \frac{mv}{q} \gg 1 \qquad \text{Gravity}$$

Region	Hydrogen	Binary Black Holes
hard	(m, m)	(m, m)
soft	(mv, mv)	$(q,\ q) = \frac{1}{J}(mv,\ mv)$
potential	$(mv^2,\ mv)$	$(qv,\ q) = \frac{1}{J}(mv^2,\ mv)$
ultrasoft, radiation	$(mv^2,\ mv^2)$	$(qv,\ qv) = \frac{1}{J}(mv^2,\ mv^2)$

Goal for the Effective Theory

- Have a systematic expansion in some small parameter v or α or R_s/r
- Separate scales consistently
- Sum large logarithms (from ratios of scales) using the renormalization group

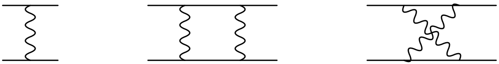
$$\ln \frac{p}{m}, \quad \frac{1}{2} \ln \frac{E}{m}, \quad \ln \frac{E}{p} \to \ln v \to \ln \alpha$$

- Determine scale for α_s:

$$\alpha_s(m), \qquad \alpha_s(mv), \qquad \alpha_s(mv^2)$$

Integrate out Hard Scale *m*

First consider the static case where particles move along fixed worldlines separated by r — get a theory with a potential.



$$e^{-\frac{i}{\hbar}\int_{-T/2}^{T/2} H(t)\mathrm{d}t} = e^{-\frac{i}{\hbar}V(r)T}$$

The diagrams are for the expansion of both sides.

- V is a matrix in the different color sectors (for $q\overline{q}$, **1** and **8.**
- get powers of $1/\hbar^n$ (superclassical)
- get powers of T (pinch singularities)

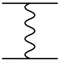
$$\frac{1}{k^0+i\epsilon}\frac{1}{k^0-i\epsilon}$$

Match onto V

Determine V order by order:

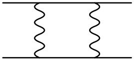
$$V = V_0 + V_1 + \dots$$

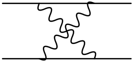
$$-iV_0 T =$$



Integrate over all locations of the two ends. So keep one end fixed and integrate over the other to get V_0.

$$-iTV_1 - \frac{1}{2} T^2 V_0^2 =$$

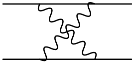




Integrate over all locations of the ends, but there is a time-ordering on each line. So V_0 is equal to the sum of the two diagrams.

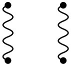
QCD:

$$V_0 : \quad C_F \qquad \text{box} : \quad C_F^2 \qquad \text{crossed-box} : \quad C_F - \frac{1}{2} C_A$$

$$-iTV_1 = -\frac{1}{2} C_A$$ 

The pinch singularities and the superclassical pieces are gone.

Suppose the Wilson lines can be treated as sources $L = J\phi$.

 $+$  $+ \quad \ldots \quad = \exp\Bigg\{$  $\Bigg\}$

Loops are gone because particle line is erased.

This is the exponentiation in QED. In QCD, the sources are correlated, because one has to keep track of color indices, and there are corrections from $\left[T^A, T^B\right] \propto C_A$.

In NRQED, the exponentiation is more complicated because the propagator is no longer static,

$$\frac{1}{k^0 - \mathbf{p}^2/(2m) + i\epsilon}$$

The potentials is a matching calculation, and has an $\hbar$ expansion. $V(r)$ and r is now dynamical. ADM singularity is not present in the matching.

Related: WKB

$$\psi = e^{\frac{i}{\hbar}S} \qquad\qquad S = S_0 + \hbar S_1 + \dots$$

The expansion is in the exponent.

Large N_c QCD

Functional integral

$$e^{\frac{i}{\hbar} N_c L}$$

When you include baryons, the baryon-pion interaction also has the same form with N_c factored out. Baryon behaves like a classical source,

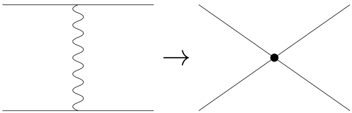
$$m_B = -\frac{9 g_A^2 M_\pi^3}{32\pi f_\pi^2}$$

obtained from solving the classical equations of motion for the pion, and computing the energy in the pion cloud.

AM, PLB 336 (1994) 502

Matching on to the Potential

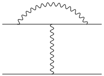
One photon exchange leads to a potential:



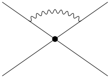
$$V = \frac{U_c}{\mathbf{q}^2} + U_2 + U_s\,\mathbf{S^2} + \frac{U_r(\mathbf{p}^2 + \mathbf{p}'^2)}{2\mathbf{q}^2} - \frac{i\mathbf{U}_\Lambda \cdot (\mathbf{p}' \times \mathbf{p})}{\mathbf{q}^2}$$
$$+U_t\left(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{3\,\mathbf{q}\cdot\boldsymbol{\sigma}_1\,\mathbf{q}\cdot\boldsymbol{\sigma}_2}{\mathbf{q}^2}\right) + \ldots\,,$$

Radiative Corrections

Radiative corrections (e.g. anomalous magnetic moment):

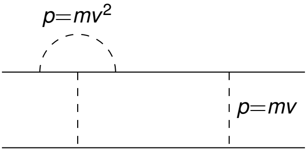


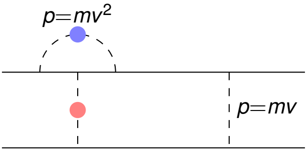
Is this a radiative correction to the coefficients in the potential, such as U_c, U_s, etc. or is this a radiative correction in the effective theory?



EFT sorts out the contributions from different scales.

What is the problem?





Graphs involve $\alpha_s(mv)$ and $\alpha_s(mv^2)$ at the same time. In calculations, one has $\alpha_s(\mu)$.

Renormalization group series

$$\begin{array}{cccc} \text{LL} & \text{NLL} & \text{NNLL} & \\ \end{array}$$

$$V = \begin{pmatrix} 1 & & \\ \alpha \mathsf{L} & \alpha & \\ \alpha^2 \mathsf{L}^2 & \alpha^2 \mathsf{L} & \alpha^2 \\ \alpha^3 \mathsf{L}^3 & \alpha^3 \mathsf{L}^2 & \ldots \\ \vdots & & \end{pmatrix} \begin{array}{l} \text{tree} \\ \text{1-loop} \\ \text{2-loop} \\ \text{3-loop} \\ \\ \end{array}$$

is large log

Multiply by α^2 in QED because binding energy is order α^2.

The Effective Theory

- Expansion in powers of velocity v and $\alpha \sim v$ in power counting.
- Non-relativistic fermions with propagator

$$\frac{1}{E - \mathbf{p}^2/2m}$$

- Potentials $V_{\rm eff}(\mathbf{p}, \mathbf{p}')$ such as the Coulomb potential, hyperfine interaction, etc.
- Radiation (ultrasoft) photons — multipole expanded
- Soft photons
- Multiple modes introduced here were then applied to SCET.
- Static theory not the $v \to 0$ limit or $m \to \infty$ limit

Power Counting

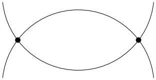
Kinetic Energy:

$$\begin{array}{ccc} E & \frac{\mathbf{p}^2}{2m} & \frac{\mathbf{p}^4}{8m^3} \\ v^2 & v^2 & v^4 \\ 1 & 1 & v^2 \end{array}$$

Potentials:

$$\begin{array}{cccc} \frac{1}{\mathbf{q}^2} \sim \frac{1}{r} & \frac{1}{m|\mathbf{q}|} \sim \frac{1}{mr^2} & \frac{1}{m^2} & \frac{\sigma_1 \cdot \sigma_2}{m^2} \\ \frac{1}{v} & 1 & v & v \end{array}$$

Loops



Loop graph with two insertions of the potential:

$$T\left(V^{(a)}\ V^{(b)}\right) \sim V^{(a+b)}$$

$$T\left(\frac{1}{\mathbf{q}^2}\ \frac{1}{m\,|\mathbf{q}|}\right) \sim \frac{1}{\mathbf{q}^2}$$

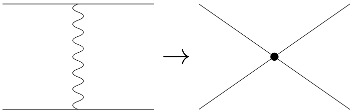
Shows that the static potential is not the one to use for bound states, since the $1/(m\,|\mathbf{q}|)$ potential is not present.

Potentials

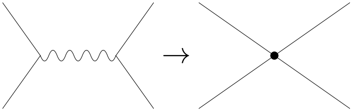
$$V(\mathbf{p}, \mathbf{p}') = V^{(-1)} + V^{(0)} + V^{(1)} + V^{(2)} + \dots$$

$$\begin{aligned}
V^{(-1)} &= \frac{U_c}{\mathbf{q}^2}, \\
V^{(0)} &= \frac{U_k}{|\mathbf{q}|}, \\
V^{(1)} &= U_2 + U_s\,\mathbf{S}^2 + \frac{U_r(\mathbf{p}^2 + \mathbf{p}'^2)}{2\mathbf{q}^2} - \frac{i\mathbf{U}_\Lambda \cdot (\mathbf{p}' \times \mathbf{p})}{\mathbf{q}^2} \\
&\quad + U_t\left(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{3\,\mathbf{q} \cdot \boldsymbol{\sigma}_1\,\mathbf{q} \cdot \boldsymbol{\sigma}_2}{\mathbf{q}^2}\right),
\end{aligned}$$

Matching Conditions



Only difference between Hydrogen and Positronium is annihilation contributions to the potentials, that first start at order $1/m^2$:

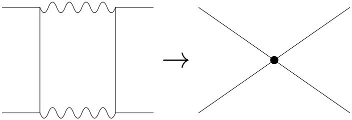


Coefficients

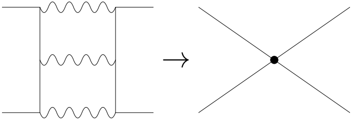
Particles of mass $m_{1,2}$ and charge $-e$, Ze

$$
\begin{aligned}
U_c &= -4\pi Z\alpha \\
U_2 &= \frac{\pi Z\alpha}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)^2 \\
U_s &= \frac{4\pi Z\alpha}{3m_1 m_2}+\frac{\pi\alpha}{m_e^2} \\
U_r &= -\frac{4\pi Z\alpha}{m_1 m_2} \\
U_k &= \frac{\pi^2 Z^2 m_r \alpha^2}{m_1 m_2}
\end{aligned}
$$

Widths



$$U_2 + U_s \mathbf{S}^2 = -i \frac{\pi \alpha^2}{m_e^2} \left(2 - \mathbf{S}^2\right)$$



$$U_2 + U_s \mathbf{S}^2 = -i \frac{4\pi \alpha^3 \left(\pi^2 - 9\right)}{9\pi m_e^2} \mathbf{S}^2$$

Can RG improve the widths.

Renormalization Group Evolution

Scales m, mv and mv^2 (analogous to M_W, m_b and Λ_{QCD}):

Two-stage running: (Conventional method):

- Match to QED/QCD at $\mu = m$.
- Integrate μ from m to mv.
- Integrate out soft (mv) modes
- Integrate μ from mv to mv^2.
- Compute matrix elements

Velocity Renormalization Group

(Luke, Rothstein, A.M.; Stewart, A.M.)

One-stage running:

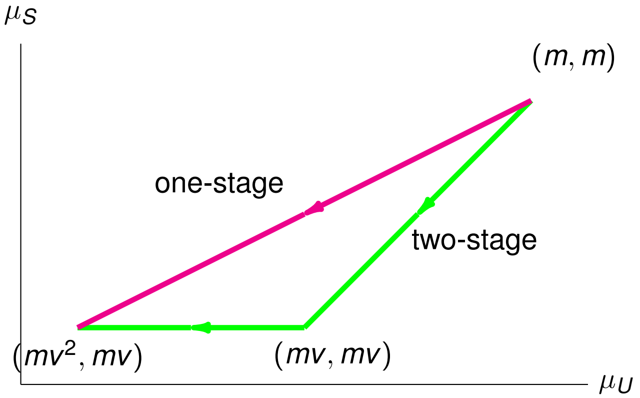
- Use a subtraction velocity ν

$$\begin{aligned} \mu_S &= m\nu \\ \mu_U &= m\nu^2 \end{aligned}$$

- Match to QED/QCD at $\nu = 1$
- Integrate from $\nu = 1$ to $\nu = v$.
- Compute matrix elements

In higher order loops, get $\log E/\mu_U$, $\log p/\mu_S$ and $\log \sqrt{2mE}/\mu_S$, and no logs in the counterterms.

Run in velocity, not momentum



- Two methods give different answers.
- One-stage VRG agrees with explicit QED calculations.

Renormalization Group Evolution

$$\mu \frac{\mathrm{d}U}{\mathrm{d}\mu} = \gamma^S + \gamma^U \equiv \Gamma^S \qquad mv < \mu < m$$

$$\mu \frac{\mathrm{d}U}{\mathrm{d}\mu} = \gamma^U \qquad mv^2 < \mu < mv$$

$$\nu \frac{\mathrm{d}U}{\mathrm{d}\nu} = \gamma^S + 2\gamma^U \qquad v < \nu < 1$$

Structure of QED equations:

$$\gamma^S = \alpha \lambda^S U^2 \qquad \gamma^U = \alpha^2 \lambda^U$$

Renormalization Group Evolution

Two-stage running in μ:

$$U(mv^2) = U(m) + \left[2\alpha^2\lambda^U + \alpha\lambda^S U_2^2(m)\right] \ln v + \left[\alpha^3\lambda^U\lambda^S U(m)\right] \ln^2 v + \left[\frac{1}{3}\alpha^5 \left(\lambda^U\right)^2 \lambda^S\right] \ln^3 v$$

One-stage running in v:

$$U(v) = U(1) + \left[2\alpha^2\lambda^U + \alpha\lambda^S U_2^2(1)\right] \ln v + \left[2\alpha^3\lambda^U\lambda^S U(1)\right] \ln^2 v + \left[\frac{4}{3}\alpha^5 \left(\lambda^U\right)^2 \lambda^S\right] \ln^3 v$$

Energy Series for QED

Compute anomalous dimensions. Coulomb potential (α) does not run, but the other potentials can. LO is the Coulomb biding energy $\sim \alpha^2$

$$\begin{aligned}
\gamma_{\rm LO} &: \alpha^2 \\
\gamma_{\rm NLO} &: \alpha^3 \to 0 \\
\gamma_{\rm NNLO} &: \alpha^4 \quad \left(1 + \alpha \ln \alpha + \alpha^2 \ln^2 \alpha + \alpha^3 \ln^3 \alpha + \ldots\right) \\
\gamma_{\rm N^3LO} &: \alpha^4 \alpha \quad \left(1 + \alpha \ln \alpha + \alpha^2 \ln^2 \alpha + \alpha^3 \ln^3 \alpha + \ldots\right)
\end{aligned}$$

In QCD, the Coulomb potential (α_s) runs.

So one can compute

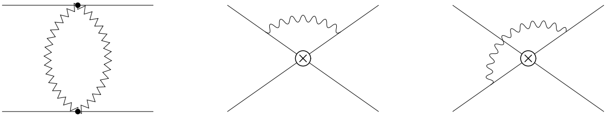
$$\begin{array}{cccc} \alpha^5 \ln \alpha & \alpha^6 \ln^2 \alpha & \alpha^7 \ln^3 \alpha & \ldots \\ \alpha^6 \ln \alpha & \alpha^7 \ln^2 \alpha & \alpha^8 \ln^3 \alpha & \ldots \end{array}$$

energy series in QED by integrating the RG equations.

$\alpha^5 \ln \alpha$ is the Lamb shift

Compute γ

$$V^{(1)} = U_2 + U_s\,\mathbf{S^2} + \frac{U_r(\mathbf{p}^2+\mathbf{p}'^2)}{2\mathbf{k}^2} - \frac{i\mathbf{U}_\Lambda\cdot(\mathbf{p}'\times\mathbf{p})}{\mathbf{k}^2}$$
$$+U_t\left(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \frac{3\,\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2}{\mathbf{k}^2}\right),$$



$+\ldots$

(a) (b) (c)

RG Equation

At NNLO,

$$\nu \frac{dU_k}{d\nu} = 0 \qquad \propto C_A \text{ in QCD}$$

$$\nu \frac{dU_2}{d\nu} = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1}+\frac{Z}{m_2}\right)^2 U_c + \frac{14 Z^2 \alpha^2}{3 m_1 m_2} = \gamma_0 U_c$$

The 2 is from running in velocity.

$$\gamma_0 = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1^2}+\frac{Z}{4 m_1 m_2}+\frac{Z^2}{m_2^2}\right)$$

without the 2:

$$\gamma_0 = \frac{2\alpha}{3\pi}\left(\frac{1}{2m_1^2}-\frac{3Z}{4 m_1 m_2}+\frac{Z^2}{2m_2^2}\right)$$

γ_0

$$\gamma_0 = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1^2} + \frac{Z}{4m_1 m_2} + \frac{Z^2}{m_2^2}\right)$$

γ_0 is a constant in QED, since α does not run

Integrate:

$$U_2(\nu) = U_2(1) + \gamma_0 U_c \ln \nu$$

Only a single term, so the NNLO series terminates for QED

$\alpha^5 \ln \alpha$, $\alpha^6 \ln^2 \alpha$, $\alpha^7 \ln^3 \alpha$, $\alpha^8 \ln^4 \alpha$, $\ldots \alpha^{104} \ln^{100} \alpha \ldots$

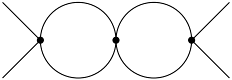
Bethe Lamb Shift

$$\begin{aligned} \Delta E &= \langle U_2 \rangle \\ &= \gamma_0 U_c \ln \nu \, |\psi(0)|^2 \\ &= -\frac{8Z^4\alpha^5 m_R^3}{3\pi n^3}\left(\frac{1}{m_1^2}+\frac{Z}{4m_1 m_2}+\frac{Z^2}{m_2^2}\right)\ln Z\alpha, \end{aligned}$$

H—(Bethe 1947)

$$|\psi(0)|^2 = \frac{(m_R Z\alpha)^3}{\pi n^3} \qquad nS \text{ state}$$

N3LO



$$\nu \frac{dU_{2+s}}{d\nu}\bigg|_{\mathrm{N^3LO}} = \rho_{ccc}\, U_c^3 + \rho_{cc2}\, U_c^2 \left(U_{2+s} + U_r\right)$$
$$+\rho_{c22}\, U_c \left(U_{2+s}^2 + 2U_{2+s}U_r + \frac{3}{4}U_r^2 - 5U_t^2\mathbf{S^2}\right)$$
$$+\rho_{ck}\, U_c U_k + \rho_{k2}\, U_k \left(U_{2+s} + U_r/2\right)$$
$$+\rho_{c3}\, U_c \left(U_3 + U_{3s}S^2 + \frac{1}{2}U_{rk}\right),$$

where $U_{2+s} = U_2 + U_s\mathbf{S}^2$ and $\rho_{c22} = -m_R^2/4\pi^2$.

$$\int \text{const} = \ln \nu$$
$$\int \ln \nu = \frac{1}{2} \ln^2 \nu$$
$$\int \ln^2 \nu = \frac{1}{3} \ln^3 \nu$$

N3LO series terminates after 3 terms

$$\alpha^6 \ln \alpha, \qquad \alpha^7 \ln^2 \alpha, \qquad \alpha^8 \ln^3 \alpha$$

Results for QED

$\alpha^8 \ln^3 \alpha$	Lamb	H	agree/new
		$\mu^+ e^-$, $e^+ e^-$	new
	(no h.f.s.)		agree
$\alpha^4 \ln^3 \alpha$	(no $\Delta\Gamma/\Gamma$)		agree
$\alpha^7 \ln^2 \alpha$	Lamb	H, $\mu^+ e^-$, $e^+ e^-$	agree
	h.f.s.	H, $\mu^+ e^-$, $e^+ e^-$	agree
$\alpha^3 \ln^2 \alpha$	$\Delta\Gamma/\Gamma$	$e^+ e^-$	agree
$\alpha^6 \ln \alpha$	Lamb, h.f.s.	H, $\mu^+ e^-$, $e^+ e^-$	agree
$\alpha^2 \ln \alpha$	$\Delta\Gamma/\Gamma$	$e^+ e^-$	agree

$\alpha^8 \ln^3 \alpha$

AM, Stewart PRL 85 (2000) 2248

$$\frac{1}{3}\gamma_0^2 \rho_{c22} U_c^3(1) \ln^3 \nu,$$

Lamb shift for the nS state (no HFS, Γ at this order)

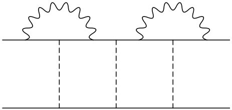
$$\begin{aligned} \Delta E &= \frac{64 m_R^5 \alpha^8 Z^6}{27\pi^2 n^3}\left(\frac{1}{m_1^2} + \frac{Z}{4 m_1 m_2} + \frac{Z^2}{m_2^2}\right)^2 \ln^3(Z\alpha) \\ &= \frac{3 m_e \alpha^8 \ln^3 \alpha}{8\pi^2 n^3} \quad \text{(positronium)} \end{aligned}$$

Have the recoil terms and the Ps result.

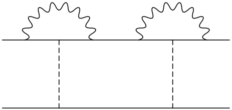
8 kHz for Hydrogen $2S_{1/2} - 2P_{1/2}$

$\alpha^8 \ln^3 \alpha$

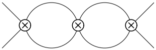
Karshenboim 1993	$a_{63} = -8/27$
Malampalli and Sapirstein PRL 1998	$a_{63} = -0.652$
Goidenko et al. PRL 1999	$a_{63} = -0.296$
Yerokhin hep-ph/0001327	$a_{63} = -0.652$
AM, Stewart (2000)	$a_{63} = -8/27$ (also recoil terms)



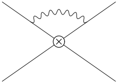
(a)



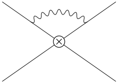
(b)



(a)



(b)



(b)

New Results: QED

1. Find a universal description of $\ln \alpha$ terms.
   A single RG equation gives the Lamb shift, hyperfine splitting and decay widths for Hydrogen, Muonium, and o,p-Positronium
2. Understand the structure of the series and why they terminate
   - LO series: $\alpha^5 \ln \alpha$
   - NLO series: $\alpha^6 \ln \alpha$, $\alpha^7 \ln^2 \alpha$, $\alpha^8 \ln^3 \alpha$
3. some ∞ series: $\alpha^2 \ln \alpha \left(\alpha^3 \ln^2 \alpha\right)^n$.
4. Computation of energy levels to order $\alpha^8 \ln^3 \alpha$

Conclusions

1. Systematic way to separate scales in non-relativistic bound states
2. All large logs summed using the velocity RG
3. Universal description of QED logs.
   Checks: $\alpha^5 \ln \alpha$, $\alpha^6 \ln \alpha$, $\alpha^7 \ln^2 \alpha$, $\alpha^8 \ln^3 \alpha$
4. QCD: can distinguish $\alpha_s(m)$, $\alpha_s(mv)$ and $\alpha_s(mv^2)$. All can appear in the same equatiion — QCD Lamb shift Hoang, AM, Stewart
5. Multiple modes leads to headaches