Towards the 5PN gravitational potential of binary systems

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in collaboration with

J. Blümlein, A. Maier, G. Schäfer [arXiv:2010.13672]

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Outline

1. Asymptotic expansions
2. Contributions to the dynamics of a binary system @ 5PN from potential modes

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Expansion by regions based on [M. Beneke, V.A.Smirnov'98]

Used for expanding a given Feynman integral in a given limit.

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- Divide the space of loop momenta into regions and in every region expand the integrand accordingly

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- Divide the space of loop momenta into regions and in every region expand the integrand accordingly
- Integrate the expanded integrand over the whole integration domain of the loop momenta

Asymptotic Expansions

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In a nutshell:

- Divide the space of loop momenta into regions and in every region expand the integrand accordingly
- Integrate the expanded integrand over the whole integration domain of the loop momenta
- Set scaleless integrals to zero

Asymptotic Expansions

- Worked out in detail and for many examples in
  B. Jantzen, JHEP 12 (2011) 076
- There are programs available to help finding all regions
  - `asy.m` [Pak,Smirnov '10]
  - `asy2.m` [Jantzen,Smirnov,Smirnov '12]
    `asy.m` + Glauber and potential regions

Asymptotic Expansions

In the problem at hand we are interested in the regions

potential:

$$|k_0| \sim \frac{v}{R}, \qquad |k_i| \sim \frac{1}{R} \,,$$

radiation (ultrasoft):

$$|k_0| \sim \frac{v}{R}, \qquad |k_i| \sim \frac{v}{R} \,.$$

Asymptotic Expansions

Our initial domains of integration are thus given by

$$k_i \in D_{\text{pot}} = \left[-\infty, -\tfrac{1}{R}\right] \cup \left[\tfrac{1}{R}, \infty\right],$$

and

$$k_i \in D_{\text{us}} = \left[-\tfrac{1}{R}, \tfrac{1}{R}\right].$$

and we have to apply our expansions accordingly using the Taylor expansion operator

$$T_i^{\text{N}} I_0(v) := \theta(N) \sum_{k=0}^{\infty} T_{i,k} v^k,$$

Asymptotic Expansions

$$I_1 = \int_{-\infty}^{\infty} dk_i I = \int_{D_{\text{pot}}} dk_i T^N_{\text{pot}} I + \int_{D_{\text{us}}} dk_i T^N_{\text{us}} I$$

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$$I_1 = \int_{-\infty}^{+\infty} dk_i T^N_{\rm pot} I - \int_{D_{\rm us}} dk_i T^N_{\rm pot} I + \int_{-\infty}^{+\infty} dk_i T^N_{\rm us} I - \int_{D_{\rm pot}} dk_i T^N_{\rm us} I$$

Asymptotic Expansions

$$I_1 = \int_{-\infty}^{\infty} dk_i \, I = \int_{D_{\text{pot}}} dk_i \, T_{\text{pot}}^N I + \int_{D_{\text{us}}} dk_i \, T_{\text{us}}^N I$$

$$I_1 = \int_{-\infty}^{+\infty} dk_i \, T_{\text{pot}}^N I - \int_{D_{\text{us}}} dk_i \, T_{\text{pot}}^N I + \int_{-\infty}^{+\infty} dk_i \, T_{\text{us}}^N I - \int_{D_{\text{pot}}} dk_i \, T_{\text{us}}^N I$$

$$I_1 = \int_{-\infty}^{+\infty} dk_i \, T_{\text{pot}}^N I - \int_{D_{\text{us}}} dk_i \, T_{\text{us}} \, T_{\text{pot}}^N I + \int_{-\infty}^{+\infty} dk_i \, T_{\text{us}}^N I - \int_{D_{\text{pot}}} dk_i \, T_{\text{pot}} \, T_{\text{us}}^N .$$

Asymptotic Expansions

$$I_1 = \int_{-\infty}^{+\infty} dk_i \, T_{\mathrm{pot}}^{N} I - \int_{D_{\mathrm{us}}} dk_i \, T_{\mathrm{us}} \, T_{\mathrm{pot}}^{N} I + \int_{-\infty}^{+\infty} dk_i \, T_{\mathrm{us}}^{N} I - \int_{D_{\mathrm{pot}}} dk_i \, T_{\mathrm{pot}} \, T_{\mathrm{us}}^{N} I$$

Asymptotic Expansions

$$I_1 = \int_{-\infty}^{+\infty} dk_i \, T_{\text{pot}}^N I - \int_{D_{\text{us}}} dk_i \, T_{\text{us}} \, T_{\text{pot}}^N I + \int_{-\infty}^{+\infty} dk_i \, T_{\text{us}}^N I - \int_{D_{\text{pot}}} dk_i \, T_{\text{pot}} \, T_{\text{us}}^N I$$

$$I_1 = \int_{-\infty}^{+\infty} dk_i \, T_{\text{pot}}^N I + \int_{-\infty}^{+\infty} dk_i \, T_{\text{us}}^N I - \int_{-\infty}^{+\infty} dk_i \, T_{\text{pot}}^N \, T_{\text{us}}^N I$$

if

$$\left[T_{\text{pot}}^N, \, T_{\text{us}}^N \right] = 0$$

Furthermore, overlap term vanishes

$$0 = \int_{-\infty}^{+\infty} dk_i \, T_{\text{pot}}^N \, T_{\text{us}}^N I$$

since $T_{\text{pot}}^N \, T_{\text{us}}^N$ leads to scaleless integrals.

Outline

Literature

Many authors have contributed to calculations for the potential part in the Post-Newtonian setting.

T. Damour, P. Jaranowski and G. Schäfer, [1401.4548]
P. Jaranowski and G. Schäfer, [1508.01016]
L. Bernard, L. Blanchet, A. Bohé, G. Faye and S. Marsat [1512.02876]
T. Marchand, L. Bernard, L. Blanchet, G. Faye, [1707.09289]
L. Bernard, L. Blanchet, G. Faye and T. Marchand, [1711.00283]
S. Foffa and R. Sturani, [1903.05113]
S. Foffa, R.A. Porto, I. Rothstein and R. Sturani, [1903.05118]
T. Damour, P. Jaranowski and G. Schäfer, [1601.01283]
L. Bernard, L. Blanchet, A. Bohé, G. Faye and S. Marsat, [1610.07934]
T. Damour and P. Jaranowski, [1701.02645]
T. Damour and P. Jaranowski, [1701.02645]
S. Foffa, P. Mastrolia, R. Sturani and C. Sturm, [1612.00482]
J. Blümlein, A. Maier, P. Marquard and G. Schäfer, [2003.01692]
S. Foffa, P. Mastrolia, R. Sturani, C. Sturm and W.J. Torres Bobadilla, [1902.10571]
J. Blümlein, A. Maier and P. Marquard, [1902.11180]
D. Bini, T. Damour and A. Geralico, [1909.02375]
D. Bini, T. Damour and A. Geralico, [2003.11891]
J. Blümlein, A. Maier, P. Marquard and G. Schäfer, [2003.07145]
D. Bini, T. Damour and A. Geralico, [2004.05407]
D. Bini, T. Damour and A. Geralico, [2007.11239]

Non-relativistic effective theory

[Goldberger, Rothstein 2004]

Similar to non-relativistic QCD

[Caswell, Lepage 1985; Pineda, Soto 1997; Luke, Manohar, Rothstein 2000; ...]

Full theory: General relativity		Effective theory: NRGR
$S_{\text{GR}} = S_{\text{EH}} + S_{\text{GF}} + S_{pp}$	$\longrightarrow$	$S_{\text{NRGR}} = \int dt \frac{1}{2} m_i v_i^2 + \frac{G m_1 m_2}{r} + \dots$
potential gravitons: $k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{1}{r}$		classical potentials
radiation gravitons: $k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{v}{r}$		radiation gravitons

Potential matching

Expansion of action

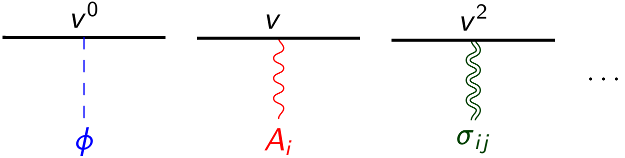
Expand $S_{\rm GR}$ in $v \sim \sqrt{Gm/r} \ll 1$, e.g.

$$S_{\rm pp} = -\sum_i m_i \int dt \sqrt{-g_{\mu\nu}\frac{\partial x_i^\mu}{\partial t}\frac{\partial x_i^\nu}{\partial t}} = -\sum_i m_i \int dt \sqrt{-g_{00}} + \mathcal{O}(v_i)$$

Coupling to spatial components of metric suppressed

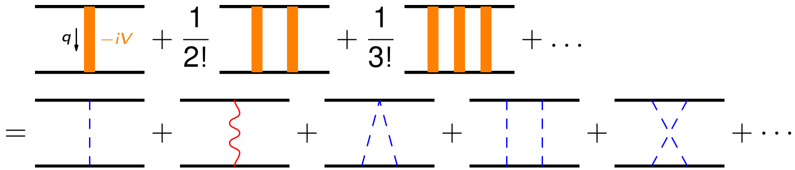
Temporal Kaluza-Klein decomposition [Kol, Smolkin 2010]

$$g^{\mu\nu} = e^{2\phi}\begin{pmatrix} -1 & A_j \\ A_i & e^{-2\frac{d-1}{d-2}\phi}(\delta_{ij}+\sigma_{ij}) - A_iA_j \end{pmatrix}$$



Potential matching
Diagrammatic expansion

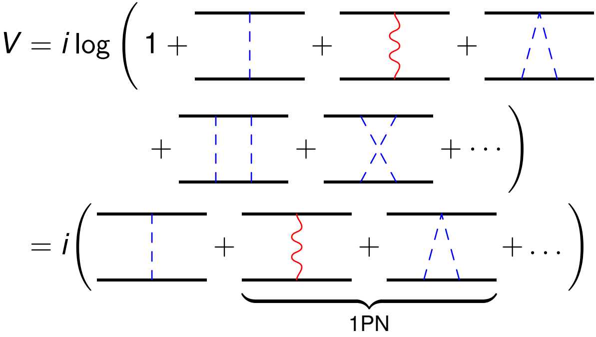
Equate amplitude in effective and full theory:



All momenta potential, $p_0 \sim \frac{v}{r} \ll p_i \sim \frac{1}{r}$
$\hookrightarrow$ expand propagators:

$$\frac{1}{\vec{p}^{\,2} - p_0^2} = \frac{1}{\vec{p}^{\,2}} + \underbrace{\frac{p_0^2}{\vec{p}^{\,4}}}_{\propto \partial_{t_1} \partial_{t_2}} + \mathcal{O}(v^4)$$

Potential matching
Diagrammatic expansion



Potential matching

Diagram selection

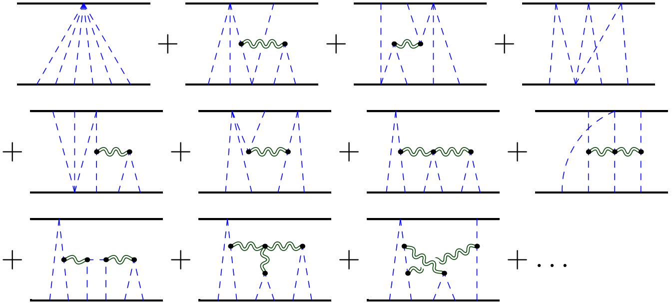
- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams

[Fischler 1977]

Potential matching

Static 5PN calculation

$$-iV^S_{5\text{PN}} =$$



Setup

- (Almost) completely automated setup for
  - Feynman rule extraction from action
  - Diagram generation using `qgraf` [Nogueira]
  - Integral family mapping (`autopsy`)
  - Integral reduction (`crusher`)

- Algebra done with `FORM` [Vermaseren]

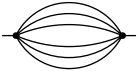
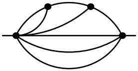
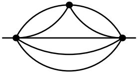
Statistics

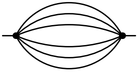
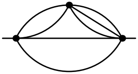
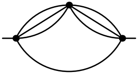
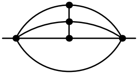
#loops	QGRAF	source irred.	no source loops	no tadpoles	masters
0	3	3	3	3	0
1	72	72	72	72	1
2	3286	3286	3286	2702	1
3	81526	62246	60998	41676	1
4	545812	264354	234934	116498	7
5	332020	128080	101570	27582	4

Potential matching

Reduction to master integrals

Apply integration-by-parts relations (`crusher`): [Chetyrkin, Tkachov 1981; Laporta 2000]

$V_{5\text{PN}}^{S} = \tilde{c}_0$  $+\ \tilde{c}_1$  $+\ \tilde{c}_2$  $+ \dots$

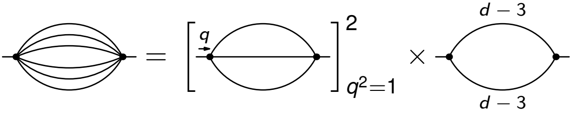
$\overset{\text{IBP}}{=} c_0$  $+\ c_1$  $+\ c_2$  $+\ c_3$ 

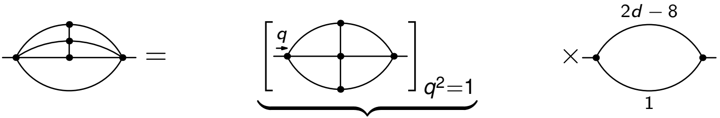
$+\,\mathcal{O}(\epsilon)(2 \times [7\ \text{lines}] + 2 \times [8\ \text{lines}])$

$\tilde{c}_i, c_j$: Laurent series in $\epsilon = \frac{3-d}{2}$, polynomials in m_1, m_2, r^{-1}, G^{-1}

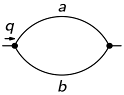
Potential matching
Calculation of master integrals

Master integrals factorise, e.g.





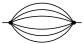
[Lee, Mingulov '15; Foffa,Mastrolia,Sturani,Sturm '16]



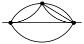
$$= \int \frac{d^d l}{\pi^{d/2}} \frac{1}{((q-l)^2)^a} \frac{1}{(l^2)^b}$$

$$= \frac{1}{(q^2)^{a+b-d/2}} \frac{\Gamma\left(\frac{d}{2}-a\right)\Gamma\left(\frac{d}{2}-b\right)\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}$$

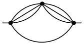
Potential matching
Results for master integrals



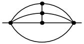
$$= e^{5\epsilon\gamma_E} \frac{\Gamma\left(6-\frac{5d}{2}\right)\Gamma^6\left(-1+\frac{d}{2}\right)}{\Gamma(-6+3d)}$$



$$= e^{5\epsilon\gamma_E} \frac{\Gamma\left(7-\frac{5d}{2}\right)\Gamma\left(3-d\right)\Gamma\left(2-\frac{d}{2}\right)\Gamma^7\left(-1+\frac{d}{2}\right)\Gamma(5-2d)}{\Gamma\left(5-\frac{3}{2}d\right)\Gamma(-2+d)\Gamma\left(-3+\frac{3}{2}d\right)\Gamma\left(-7+3d\right)}$$



$$= e^{5\epsilon\gamma_E} \frac{\Gamma\left(7-\frac{5d}{2}\right)\Gamma^2(3-d)\Gamma^7\left(-1+\frac{d}{2}\right)\Gamma\left(-6+\frac{5d}{2}\right)}{\Gamma\left(6-2d\right)\Gamma^2\left(-3+\frac{3d}{2}\right)\Gamma\left(-7+3d\right)}$$



$$= 6\pi^{7/2}\left[\frac{2}{\epsilon} - 4 - 4\ln(2) - \left(48 + 8\ln(2) - 4\ln^2(2) - 105\zeta_2\right)\epsilon + \mathcal{O}(\epsilon^2)\right].$$

Lagrangian → Hamiltonian

- Calculate potential contribution to the action at Lagrangian level
  $\hookrightarrow \mathcal{L}(r, \dot{r}, \ddot{r}, \dddot{r}, \ldots)$

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full (potential) Hamiltonian @ 5PN

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full (potential) Hamiltonian @ 5PN

- still contains poles in $d - 3$

Result 5PN $\mathcal{O}(\nu)$

$$\hat{H}^{\rm pot}_{\rm 5PN} = +\frac{1}{\varepsilon}\Bigg\{+\nu\Bigg[-\frac{272309}{12600r^6}+\frac{22439p^2}{12600r^5}-\frac{49023p^4}{560r^4}+\frac{1173p^6}{80r^3}-\frac{210947(p.n)^2}{2520r^5}$$
$$+\frac{25169p^2(p.n)^2}{105r^4}-\frac{2271p^4(p.n)^2}{80r^3}-\frac{13059(p.n)^4}{70r^4}-\frac{81p^2(p.n)^4}{r^3}+\frac{77(p.n)^6}{r^3}\Bigg]+\dots\Bigg\}$$
$$+\nu\Bigg[\frac{231p^{12}}{1024}-\frac{253555919}{529200r^6}-\frac{1457872519p^2}{2116800r^5}+\frac{2128837091p^4}{1411200r^4}+\frac{11206267p^6}{141120r^3}+\frac{937p^8}{32r^2}+\frac{805p^{10}}{256r}$$
$$+\pi^2\Bigg(\frac{70399}{1152r^6}+\frac{65291p^2}{1152r^5}-\frac{1328147p^4}{12288r^4}-\frac{7719p^6}{4096r^3}+\frac{6649(p.n)^2}{576r^5}+\frac{5042575p^2(p.n)^2}{6144r^4}$$
$$+\frac{58887p^4(p.n)^2}{4096r^3}-\frac{3293913(p.n)^4}{4096r^4}-\frac{89625p^2(p.n)^4}{4096r^3}+\frac{42105(p.n)^6}{4096r^3}\Bigg)$$
$$+\ln\left(\frac{r}{r_0}\right)\Bigg(-\frac{272309}{1050r^6}+\frac{22439p^2}{1260r^5}-\frac{49023p^4}{70r^4}+\frac{3519p^6}{40r^3}-\frac{210947(p.n)^2}{252r^5}+\frac{201352p^2(p.n)^2}{105r^4}$$
$$-\frac{6813p^4(p.n)^2}{40r^3}-\frac{52236(p.n)^4}{35r^4}-\frac{486p^2(p.n)^4}{r^3}+\frac{462(p.n)^6}{r^3}\Bigg)+\frac{467022407(p.n)^2}{2116800r^5}-\frac{2385014243p^2(p.n)^2}{282240r^4}$$
$$-\frac{162949463p^4(p.n)^2}{235200r^3}-\frac{589p^6(p.n)^2}{16r^2}-\frac{35p^8(p.n)^2}{256r}+\frac{1895797259(p.n)^4}{235200r^4}+\frac{31715507p^2(p.n)^4}{23520r^3}$$
$$+\frac{8951p^4(p.n)^4}{384r^2}-\frac{627281(p.n)^6}{960r^3}-\frac{5117p^2(p.n)^6}{320r^2}+\frac{159(p.n)^8}{28r^2}\Bigg]+\dots$$

Towards full Hamiltonian including tail terms

Add divergent and (local) logarithmic terms from the tail term (far zone)

$$\hat{H}^{\text{tail,sing,log}}_{\text{5PN}} = \frac{1}{\varepsilon}\left\{\left(\frac{16\nu}{105}-\frac{332\nu^2}{105}\right)\frac{1}{r^6}+\left[\left(\frac{236\nu}{35}-\frac{212\nu^2}{35}\right)p^2-\left(\frac{684\nu}{35}+\frac{1264\nu^2}{105}\right)(p.n)^2\right]\frac{1}{r^5}\right.$$
$$+\left[\left(\frac{533\nu}{21}+\frac{706\nu^2}{21}\right)p^4-\left(\frac{7732\nu}{35}+\frac{10936\nu^2}{105}\right)p^2(p.n)^2+\left(\frac{6197\nu}{35}+\frac{2656\nu^2}{35}\right)\times\right.$$
$$\left.(p.n)^4\right]\frac{1}{r^4}+\left[\left(\frac{94\nu}{15}-\frac{94\nu^2}{5}\right)p^6+\left(-\frac{172\nu}{5}+\frac{516\nu^2}{5}\right)p^4(p.n)^2+\left(26\nu-78\nu^2\right)\right.$$
$$\left.\left.\times p^2(p.n)^4\right]\frac{1}{r^3}\right\}$$
$$+\left\{\left[\frac{128\nu}{105}-\frac{2656\nu^2}{105}\right]\frac{1}{r^6}+\left[\left(\frac{1416\nu}{35}-\frac{1272\nu^2}{35}\right)p^2-\left(\frac{4104\nu}{35}+\frac{2528\nu^2}{35}\right)\right.\right.$$
$$\left.\times(p.n)^2\right]\frac{1}{r^5}+\left[\left(\frac{2132\nu}{21}+\frac{2824\nu^2}{21}\right)p^4-\left(\frac{30928\nu}{35}+\frac{43744\nu^2}{105}\right)p^2(p.n)^2\right.$$
$$\left.+\left(\frac{24788\nu}{35}+\frac{10624\nu^2}{35}\right)(p.n)^4\right]\frac{1}{r^4}+\left[\left(\frac{188\nu}{15}-\frac{188\nu^2}{5}\right)p^6\right.$$
$$\left.\left.+\left(-\frac{344\nu}{5}+\frac{1032\nu^2}{5}\right)p^4(p.n)^2+\left(52\nu-156\nu^2\right)p^2(p.n)^4\right]\frac{1}{r^3}\right\}\ln\left(\frac{r}{r_0}\right)$$

[Foffa,Sturani '19]

Towards full Hamiltonian including tail terms

After combining these contributions a finite Hamiltonian can be obtained using a canonical transformation

$$\hat{H}_{\text{5PN}} = \hat{H}^{\text{pot}}_{\text{5PN}} + \hat{H}^{\text{tail,sing,log}}_{\text{5PN}} \overset{\text{can.trafo.}}{\longrightarrow} \hat{H}^{\text{finite}}_{\text{5PN}} = \hat{H}_{\text{5PN}} + \left\{ \hat{H}_{\text{5PN}}, g \right\}$$

$$\begin{aligned}
\hat{H}^{\text{finite}}_{\text{5PN}} = {} & +\nu \Bigg[\frac{231 p^{12}}{1024} - \frac{279775133}{529200 r^6} - \frac{1450584679 p^2}{2116800 r^5} + \frac{2010713771 p^4}{1411200 r^4} + \frac{11206267 p^6}{141120 r^3} + \frac{937 p^8}{32 r^2} \\
& + \frac{805 p^{10}}{256 r} + \pi^2 \Bigg(\frac{70399}{1152 r^6} + \frac{65291 p^2}{1152 r^5} - \frac{1328147 p^4}{12288 r^4} - \frac{7719 p^6}{4096 r^3} + \frac{6649 (p.n)^2}{576 r^5} \\
& + \frac{5042575 p^2 (p.n)^2}{6144 r^4} + \frac{58887 p^4 (p.n)^2}{4096 r^3} - \frac{3293913 (p.n)^4}{4096 r^4} - \frac{89625 p^2 (p.n)^4}{4096 r^3} \\
& + \frac{42105 (p.n)^6}{4096 r^3} \Bigg) - \frac{34541593 (p.n)^2}{2116800 r^5} - \frac{2395722563 p^2 (p.n)^2}{282240 r^4} - \frac{62196341 p^4 (p.n)^2}{78400 r^3} \\
& - \frac{589 p^6 (p.n)^2}{16 r^2} - \frac{35 p^8 (p.n)^2}{256 r} + \frac{631107353 (p.n)^4}{78400 r^4} + \frac{31226291 p^2 (p.n)^4}{23520 r^3} \\
& + \frac{8951 p^4 (p.n)^4}{384 r^2} - \frac{563921 (p.n)^6}{960 r^3} - \frac{5117 p^2 (p.n)^6}{320 r^2} + \frac{159 (p.n)^8}{28 r^2} \\
& + \ln\left(\frac{r}{r_0}\right) \left(\frac{64}{105 r^6} - \frac{18944 p^2}{105 r^5} + \frac{1796 p^4}{105 r^4} + \frac{19136 (p.n)^2}{105 r^5} - \frac{10664 p^2 (p.n)^2}{105 r^4} + \frac{2748 (p.n)^4}{35 r^4} \right) \Bigg] + \cdots
\end{aligned}$$

Towards full Hamiltonian including tail terms

After combining these contributions a finite Hamiltonian can be obtained using a canonical transformation

$$\hat{H}_{5\mathrm{PN}} = \hat{H}^{\mathrm{pot}}_{5\mathrm{PN}} + \hat{H}^{\mathrm{tail,sing,log}}_{5\mathrm{PN}} \overset{\mathrm{can.trafo.}}{\longrightarrow} \hat{H}^{\mathrm{finite}}_{5\mathrm{PN}} = \hat{H}_{5\mathrm{PN}} + \left\{ \hat{H}_{5\mathrm{PN}}, g \right\}$$

$$\begin{aligned}
\hat{H}^{\mathrm{finite}}_{5\mathrm{PN}} \quad = \quad & +\nu\Bigg[\frac{231p^{12}}{1024} - \frac{279775133}{529200r^6} - \frac{1450584679p^2}{2116800r^5} + \frac{2010713771p^4}{1411200r^4} + \frac{11206267p^6}{141120r^3} + \frac{937p^8}{32r^2} \\
& + \frac{805p^{10}}{256r} + \pi^2\Bigg(\frac{70399}{1152r^6} + \frac{65291p^2}{1152r^5} - \frac{1328147p^4}{12288r^4} - \frac{7719p^6}{4096r^3} + \frac{6649(p.n)^2}{576r^5} \\
& + \frac{5042575p^2(p.n)^2}{6144r^4} + \frac{58887p^4(p.n)^2}{4096r^3} - \frac{3293913(p.n)^4}{4096r^4} - \frac{89625p^2(p.n)^4}{4096r^3} \\
& + \frac{42105(p.n)^6}{4096r^3}\Bigg) - \frac{34541593(p.n)^2}{2116800r^5} - \frac{2395722563p^2(p.n)^2}{282240r^4} - \frac{62196341p^4(p.n)^2}{78400r^3} \\
& - \frac{589p^6(p.n)^2}{16r^2} - \frac{35p^8(p.n)^2}{256r} + \frac{631107353(p.n)^4}{78400r^4} + \frac{31226291p^2(p.n)^4}{23520r^3} \\
& + \frac{8951p^4(p.n)^4}{384r^2} - \frac{563921(p.n)^6}{960r^3} - \frac{5117p^2(p.n)^6}{320r^2} + \frac{159(p.n)^8}{28r^2} \\
& + 2\frac{G_N^3 E}{c^{10}}\left(\frac{1}{5}I^{(3)}(t)^2 + \frac{1}{189c^2}O^{(4)}(t)^2 + \frac{16}{45c^2}J^{(3)}(t)^2\right)\ln\left(\frac{r}{r_0}\right)\Bigg] + \cdots
\end{aligned}$$

Checks & Comparison with Literature

- finite Hamiltonian & correct form of the log. terms ✓

Checks & Comparison with Literature

- finite Hamiltonian & correct form of the log. terms ✓
- Schwarzschild limit ✓

$$\hat{H}^{\text{Schw}}_{\text{5PN}} = -\frac{21p^{12}}{1024} - \frac{77p^{10}}{256r} - \frac{445p^{8}}{256r^2} - \frac{161p^{6}}{32r^3} - \frac{499p^{4}}{64r^4} - \frac{125p^{2}}{16r^5} + \frac{5p^{6}(p.n)^2}{32r^2} + \frac{21p^{4}(p.n)^2}{16r^3} + \frac{29p^{2}(p.n)^2}{8r^4} - \frac{(p.n)^4}{8r^4} + \frac{17(p.n)^2}{4r^5} + \frac{5}{16r^6}$$

Checks & Comparison with Literature

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- Agreement with expansion of 3PM results (1 & 2 loops) ✓

[Bern,Cheung,Roiban,Shen,Solon,Zeng '19,Kälin,Liu,Porto '20]

Checks & Comparison with Literature

- finite Hamiltonian & correct form of the log. terms ✓
- Schwarzschild limit ✓

$$\hat{H}^{\mathrm{Schw}}_{\mathrm{5PN}} = -\frac{21p^{12}}{1024} - \frac{77p^{10}}{256r} - \frac{445p^{8}}{256r^2} - \frac{161p^{6}}{32r^3} - \frac{499p^{4}}{64r^4} - \frac{125p^{2}}{16r^5} + \frac{5p^{6}(p.n)^2}{32r^2} + \frac{21p^{4}(p.n)^2}{16r^3} + \frac{29p^{2}(p.n)^2}{8r^4} - \frac{(p.n)^4}{8r^4} + \frac{17(p.n)^2}{4r^5} + \frac{5}{16r^6}$$

- Agreement with expansion of 3PM results (1 & 2 loops) ✓
- Extension of 4PN calculation ✓

Checks & Comparison with Literature

- finite Hamiltonian & correct form of the log. terms ✓
- Schwarzschild limit ✓

$$\hat{H}^{\text{Schw}}_{5\text{PN}} = -\frac{21p^{12}}{1024} - \frac{77p^{10}}{256r} - \frac{445p^8}{256r^2} - \frac{161p^6}{32r^3} - \frac{499p^4}{64r^4} - \frac{125p^2}{16r^5} + \frac{5p^6(p.n)^2}{32r^2} + \frac{21p^4(p.n)^2}{16r^3} + \frac{29p^2(p.n)^2}{8r^4} - \frac{(p.n)^4}{8r^4} + \frac{17(p.n)^2}{4r^5} + \frac{5}{16r^6}$$

- Agreement with expansion of 3PM results (1 & 2 loops) ✓
- Extension of 4PN calculation ✓
- Agreement in static potential @ 5PN ✓

[Foffa,Mastrolia,Sturani,Sturm,Torres Bobadilla '19,Blümlein,Maier,Marquard '19]

Checks & Comparison with Literature

- finite Hamiltonian & correct form of the log. terms ✓
- Schwarzschild limit ✓

$$\hat{H}^{\text{Schw}}_{5\text{PN}} = -\frac{21p^{12}}{1024} - \frac{77p^{10}}{256r} - \frac{445p^{8}}{256r^2} - \frac{161p^{6}}{32r^3} - \frac{499p^{4}}{64r^4} - \frac{125p^{2}}{16r^5} + \frac{5p^6(p.n)^2}{32r^2} + \frac{21p^4(p.n)^2}{16r^3} + \frac{29p^2(p.n)^2}{8r^4} - \frac{(p.n)^4}{8r^4} + \frac{17(p.n)^2}{4r^5} + \frac{5}{16r^6}$$

- Agreement with expansion of 3PM results (1 & 2 loops) ✓
- Extension of 4PN calculation ✓
- Agreement in static potential @ 5PN ✓
- Via can. transformation agreement of $\nu^0, \nu^3, \nu^4, \nu^5$ terms with results from EOB 'Tutti Frutti' ✓

  [Bini,Damour,Geralico '20]

Checks & Comparison with Literature

- finite Hamiltonian & correct form of the log. terms ✓
- Schwarzschild limit ✓

$$\hat{H}^{\rm Schw}_{\rm 5PN} = -\frac{21p^{12}}{1024} - \frac{77p^{10}}{256r} - \frac{445p^{8}}{256r^2} - \frac{161p^{6}}{32r^3} - \frac{499p^{4}}{64r^4} - \frac{125p^{2}}{16r^5} + \frac{5p^6(p.n)^2}{32r^2} + \frac{21p^4(p.n)^2}{16r^3} + \frac{29p^2(p.n)^2}{8r^4} - \frac{(p.n)^4}{8r^4} + \frac{17(p.n)^2}{4r^5} + \frac{5}{16r^6}$$

- Agreement with expansion of 3PM results (1 & 2 loops) ✓
- Extension of 4PN calculation ✓
- Agreement in static potential @ 5PN ✓
- Via can. transformation agreement of $\nu^0, \nu^3, \nu^4, \nu^5$ terms with results from EOB 'Tutti Frutti' ✓
  This partially checks all diagrams $\leq$ 4 loops

Observables – energy E & periastron advance K

- non-local $E_{\rm nl}$ and $K^{\rm nl}$ successfully reproduced
- local E and $K(E,j) \to K^{\rm circ}(j)$

$$
\begin{aligned}
\frac{E^{\rm circ}(j)}{\mu c^2} &= -\frac{1}{2j^2} + \cdots + \left[\left(r^{\rm E}_{\nu 2} + \frac{132979\pi^2}{2048}\right)\nu^2 - \frac{21\nu^5}{1024} + \frac{5\nu^4}{1024}\right. \\
&\quad \left. + \left(\frac{41\pi^2}{512} - \frac{3769}{3072}\right)\nu^3 + \left(r^{\rm E}_{\nu} - \frac{31547\pi^2}{1536}\right)\nu - \frac{1648269}{1024}\right]\frac{1}{j^{12}}\eta^{10} + \frac{E^{\rm circ}_{\rm nl}}{\mu c^2} + O\left(\eta^{12}\right), \\
K^{\rm circ}(j) &= 1 + 3\frac{1}{j^2}\eta^2 + \cdots + \left[\frac{161109}{8} + \left(r^{\rm K}_{\nu} + \frac{488373}{2048}\pi^2\right)\nu + \left(r^{\rm K}_{\nu 2} - \frac{1379075}{1024}\pi^2\right)\nu^2\right. \\
&\quad \left. + \left(-\frac{1627}{6} + \frac{205}{32}\pi^2\right)\nu^3\right]\frac{1}{j^{10}}\eta^{10} + K^{\rm nl}_{4+5\rm PN}(j) + O\left(\eta^{12}\right)
\end{aligned}
$$

- From this one can conclude, cf [Bini,Damour,Geralico [2003.11891]]

$$
\begin{aligned}
\bar{d}_5 &= r_{\bar{d}_5} + \frac{306545}{512}\pi^2 \\
a_6 &= r_{a_6} + \frac{25911}{256}\pi^2
\end{aligned}
$$

Conclusions

- Calculated the potential contributions to the Hamiltonian @ 5PN
- Results agree with literature where available
- Still missing: Addition of finite contributions from the tail
- ToDo: Check of factorising contributions against

[Foffa,Sturani,Torres Bobadilla '20]