Nonrelativistic EFTs for bound states from a QCD perspective

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Outline

- Non-Relativistic EFTs
- pNRQCD
- Poincaré invariance
- Van der Waals EFT

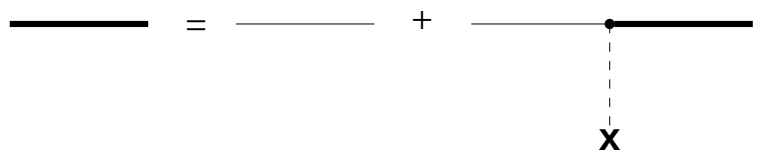
Non-Relativistic EFTs

Non-relativistic quantum theory of bound states

Non-relativistic bound states accompanied the history of the quantum theory from its inception to the establishing of the quantum theory of fields:

- 1926 Schrödinger equation: $\left(\frac{\mathbf{p}^2}{2m}+V\right)\phi = E\phi$

$$\begin{cases} g = g_0 + g_0(-iV)g \\ g_0 = \frac{i}{E-\mathbf{p}^2/(2m)} \end{cases}$$



- 1927 Pauli equation: $\left(\frac{(\mathbf{p}-e\mathbf{A})^2}{2m}+V-\frac{\boldsymbol{\sigma}\cdot e\mathbf{B}}{2m}\right)\phi = E\phi$

The relevant scales of the non-relativistic bound state dynamics are

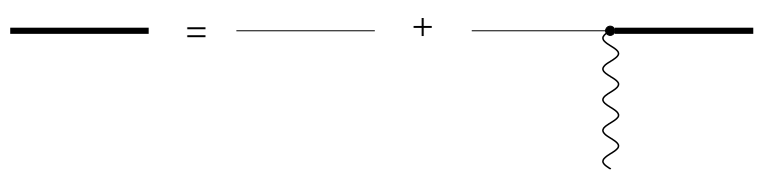
- $E \sim \frac{\mathbf{p}^2}{2m} \sim V \sim mv^2$,
- $p \sim 1/r \sim mv$;

a crucial observation: if v(elocity) $\ll 1$, then $m \gg mv \gg mv^2$.

Relativistic quantum theory of bound states

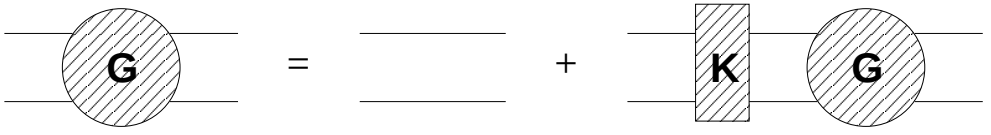
- 1928 Dirac equation: $(i\not{D} - m)\,\psi = 0$

$$\begin{cases} g^D = g_0^D + g_0^D(-ie\not{A})g^D \\ g_0^D = \frac{i}{\not{p}-m} \end{cases}$$

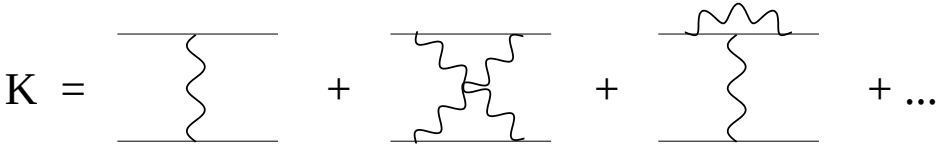


- 1951 Bethe–Salpeter equation:

$$\begin{cases} G = G_0 + G_0 K G \\ G_0 = g_0^D \otimes g_0^D \end{cases}$$



All the complexity of the field theory is in the kernel

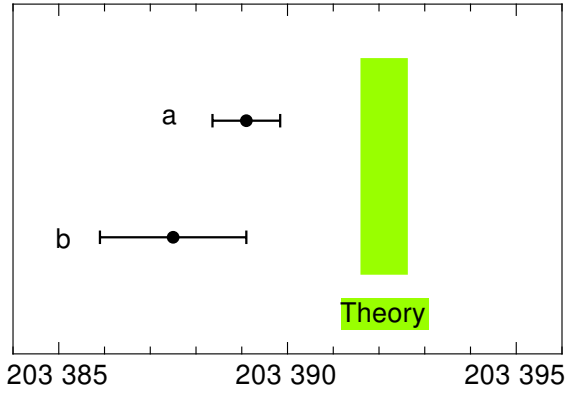


which only in the non-relativistic limit reduces to the Coulomb potential, but, in general, keeps entangled all bound-state scales.

Disentangling the bound-state scales at the Lagrangian level has advantages.

(I) It facilitates higher-order perturbative calculations.

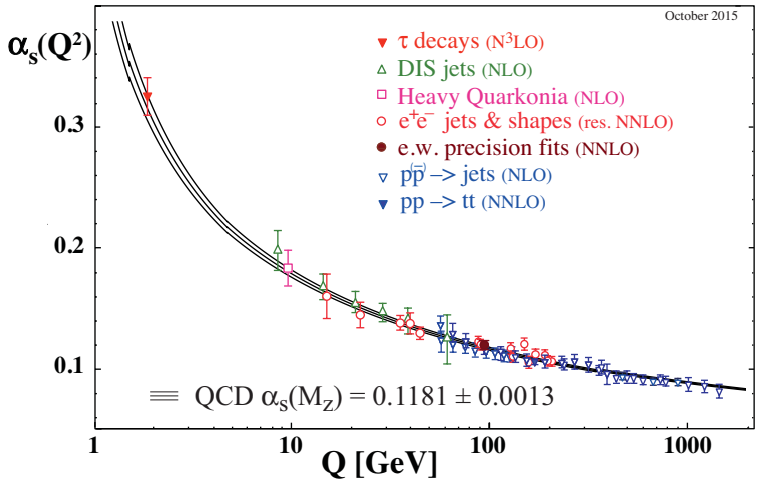
E.g. it took twenty-five years to go from the calculation of the $m\alpha^5$ correction in the hyperfine splitting of the positronium ground state to the $m\alpha^6 \ln \alpha$ term!

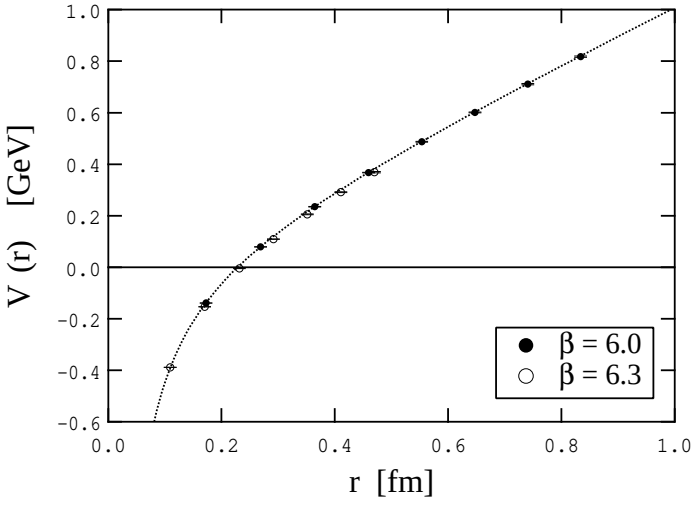


Relevant for

- atomic physics: Hydrogen atom (e.g. proton radius), positronium (e.g. width, hfs), ...
- $t\bar{t}$ threshold production, ...
- ...

(II) In QCD, it factorizes automatically high-energy (perturbative) contributions from low-energy (non-perturbative, thermal, ...) ones.





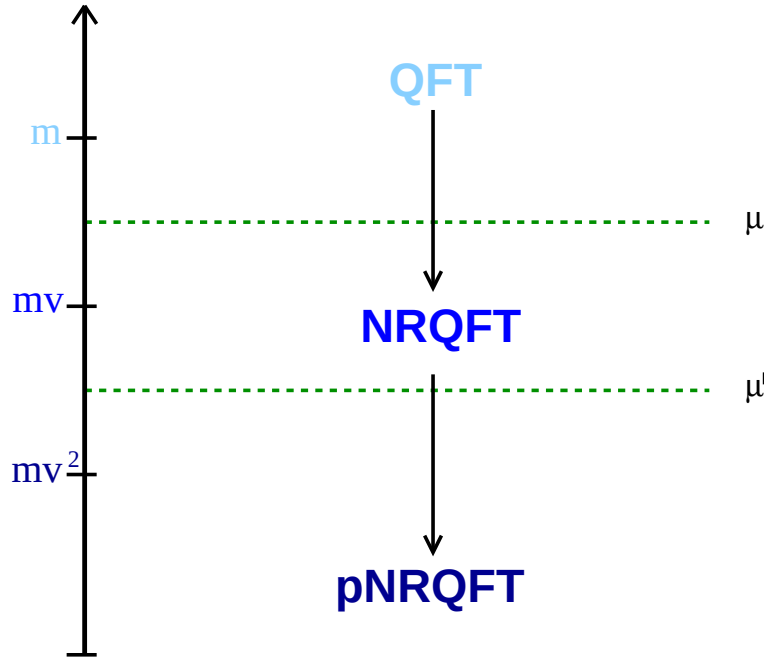
Relevant for

- pionium and precision chiral dynamics, ...
- nucleon-nucleon systems, ...
- quarkonia and new quarkonium states
- confinement and lattice calculations, ...
- quarkonium in heavy ion collisions: factorization of thermal contributions.

(III) More conceptually:
it provides a field theoretical foundation of the Schrödinger equation:

$$\mathcal{L}_{\rm pNRQFT} = \int d^3r\, \phi^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V \right) \phi + \Delta\mathcal{L}$$

The Lagrangian $\mathcal{L}_{\rm pNRQFT}$, which separates the Schrödinger dynamics of the two-particle field ϕ from the low-energy dynamics encoded in $\Delta\mathcal{L}$, defines an effective field theory.



pNRQCD

pNRQCD: EFT for $Q\bar{Q}$

pNRQCD is the EFT for nonrelativistic quark-antiquark pairs ($Q\bar{Q}$) near threshold.

- QFT = QCD
- It is obtained by integrating out hard and soft gluons with p or E scaling like m, mv.
- The d.o.f. are $Q\bar{Q}$ pairs (sometimes cast in color singlet S and color octet O) and ultrasoft modes (e.g. light quarks, low-energy gluons):
  $\phi = S$
- The Lagrangian is organized as an expansion in $1/m$ and r.
- The form of $\Delta\mathcal{L}$ and of the ultrasoft modes depends on the low energy dynamics.
- The power counting is
  $\rightarrow p \sim 1/r \sim mv$ (soft scale),
  $\rightarrow E \sim \mathbf{p}^2/2m \sim V^{(0)} \sim \mathbf{P}_{\rm cm} \sim 1/\mathbf{R}_{\rm cm} \sim mv^2$ (ultrasoft scale),
  $\rightarrow$ operators in $\Delta\mathcal{L}$ scale like $(mv^2)^{\rm dimension}$.

○ Brambilla Pineda Soto Vairo RMP 77 (2005) 1423

Matching the potential

Although the equation of motion of pNRQCD is at leading order a Schrödinger equation, as an EFT pNRQCD provides a description of the bound state that goes beyond quantum mechanics.

- The potential is a matching coefficient of the EFT that may be computed from first principle by matching Green's functions in QCD with Green's function in pNRQCD, it is scheme and scale dependent, and undergoes renormalization.
  It may be organized as an expansion in $1/m$:

$$V = V^{(0)} + \frac{V^{(1)}}{m} + \frac{V^{(2)}}{m^2} + ...$$

- The interaction terms contained in $\Delta\mathcal{L}$ provide corrections to the quantum mechanical picture.

Matching the potential

- The static potential:

$$V^{(0)}(r) \;=\; \lim_{T\to\infty}\frac{i}{T}\ln\langle\,\square\,\rangle - \Delta\mathcal{L}\,\text{effects};\qquad \square = \exp\left\{ig\oint_{r\times T} dz^\mu A_\mu\right\}$$

Wilson loops (as matching Green's functions) guarantee gauge invariance.

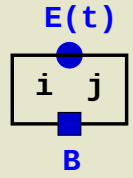
- The $1/m$ potential:

$$V^{(1)} = -\frac{1}{2}\int_0^\infty dt\, t\;\left\langle\;\begin{array}{c}\text{E(t)}\\ \square\end{array}\;\right\rangle \quad -\Delta\mathcal{L}\,\text{effects}$$

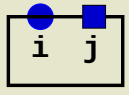
- Brambilla Pineda Soto Vairo PRD 63 (2001) 014023

Matching the potential

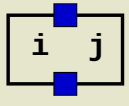
- The spin dependent (SD) $1/m^2$ potential:

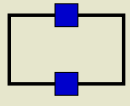
$$V_{\rm SD}^{(2)} = -\frac{r^k}{r^2} c_F\, \epsilon^{kij}\, i \int_0^\infty dt\, t \langle$$ 

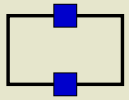
$$\rangle\, \mathbf{L}_1 \cdot \mathbf{S}_2 + (1 \leftrightarrow 2)\, |V_{LS}^{(2)}$$

$$-\frac{r^k}{r^2}\left(c_F\, \epsilon^{kij}\, i \int_0^\infty dt\, t \,\langle$$ 

$$\rangle - \frac{2c_F - 1}{2} \nabla^k V^{(0)} \right) \mathbf{L}_1 \cdot \mathbf{S}_1 + (1 \leftrightarrow 2)\, |V_{LS}^{(1)}$$

$$-c_F^2 \hat{r}_i \hat{r}_j i \int_0^\infty dt \left(\langle$$ 

$$\rangle - \frac{\delta_{ij}}{3} \langle$$ 
$$\rangle \right) \left(\mathbf{S}_1 \cdot \mathbf{S_2} - 3(\mathbf{S}_1 \cdot \hat{\mathbf{r}})(\mathbf{S_2} \cdot \hat{\mathbf{r}}) \right) |V_T$$

$$+\left(\frac{2}{3} c_F^2 i \int_0^\infty dt \langle$$ 
$$\rangle - 4 \left(d_{sv} + \frac{4}{3} d_{vv} \right) \delta^{(3)}(\mathbf{r}) \right) \mathbf{S}_1 \cdot \mathbf{S_2}\, |V_S - \Delta\mathcal{L}\ \text{effects}$$

$c_F = 1 + \alpha_{\rm s}/\pi(13/6 + 3/2 \ln m/\mu) + ...)$, $d_{sv,vv} = O(\alpha_{\rm s}^2)$ from NRQCD.

- Pineda Vairo PRD 63 (2001) 054007

Matching the potential

- The spin independent (SI) $1/m^2$ potential:

$$V_{\rm SI}^{(2)} = \frac{1}{2}\left\{\mathbf{p}_1^2, \hat{r}_i\hat{r}_j\,\frac{i}{2}\int_0^\infty dt\,t^2\langle\ \boxed{\mathtt{i}\quad\mathtt{j}}\ \rangle\right\} + (1\leftrightarrow 2)\ \Big|V_{p^2}^{(1)}$$

$$+\frac{1}{r^2}\left(\delta^{ij}-3\hat{r}_i\hat{r}_j\right)\frac{i}{4}\int_0^\infty dt\,t^2\langle\ \boxed{\mathtt{i}\quad\mathtt{j}}\ \rangle\,\mathbf{L}_1^2 + (1\leftrightarrow 2)\ \Big|V_{L^2}^{(1)}$$

$$-\frac{1}{2}\left\{\mathbf{p}_1\cdot\mathbf{p}_2, \hat{r}_i\hat{r}_j\,i\int_0^\infty dt\,t^2\langle\ \boxed{\mathtt{i}\quad\mathtt{j}}\ \rangle\right\}\ \Big|V_{p^2}^{(2)}$$

$$-\frac{1}{2r^2}\left(\delta^{ij}-3\hat{r}_i\hat{r}_j\right)\frac{i}{2}\int_0^\infty dt\,t^2\langle\ \boxed{\mathtt{i}\quad\mathtt{j}}\ \rangle\,\mathbf{L}_1\cdot\mathbf{L}_2 + (1\leftrightarrow 2)\ \Big|V_{L^2}^{(2)}$$

$+$ momentum independent terms $-$ $\Delta\mathcal{L}$ effects

- Pineda Vairo PRD 63 (2001) 054007

- The expressions of the potentials are suitable for lattice QCD calculations.
  - ...

    M.Koma Y.Koma Wittig PRL 97 (2006) 122003

    M.Koma Y.Koma Wittig PoS LATTICE2007 (2007) 111

    M.Koma Y.Koma NPB 769 (2007) 79

- They are also suitable to perturbative QCD calculations.
  - ...

    Peset Pineda Stahlhofer EPJC 77 (2017) 681

Weak coupling pNRQCD

$$\Delta\mathcal{L} = \int d^3r\ \mathrm{Tr}\left\{O^\dagger\left(iD_0 - \frac{\mathbf{p}^2}{m} + \cdots - V_o\right)O \right.$$
$$\left. V_A O^\dagger \mathbf{r}\cdot g\mathbf{E}\, S + \mathrm{H.c.} + \frac{V_B}{2} O^\dagger \mathbf{r}\cdot g\mathbf{E}\, O + \mathrm{c.c.}\right\} + \cdots$$
$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + \sum_{i=1}^{n_f}\bar{q}_i\, i\not{D} q_i$$

The (weak coupling) matching coefficients are the Coulomb potential:

$$V(r) = -C_F\frac{\alpha_{\mathrm{s}}}{r} + \ldots, \qquad V_o(r) = \frac{1}{2N}\frac{\alpha_{\mathrm{s}}}{r} + \ldots, \qquad N = 3,\ C_F = \frac{4}{3}$$

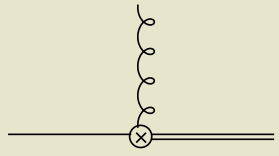
and $V_A = 1 + \mathcal{O}(\alpha_{\mathrm{s}}^2)$, $V_B = 1 + \mathcal{O}(\alpha_{\mathrm{s}}^2)$.

- Pineda Soto NP PS 64 (1998) 428
  Brambilla Pineda Soto Vairo NPB 566 (2000) 275

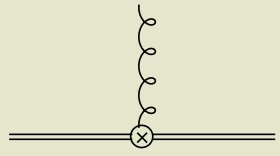
Feynman rules

$$= \theta(t)\, e^{-it(\mathbf{p}^2/m+V)}$$

$$= \theta(t)\, e^{-it(\mathbf{p}^2/m+V_o)} \left(e^{-i\int dt\, A^{\rm adj}}\right)$$



$$= \mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\, \mathrm{S}$$

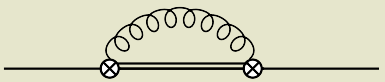


$$= \mathrm{O}^\dagger \{\mathbf{r}\cdot g\mathbf{E},\, \mathrm{O}\}$$

- Brambilla Pineda Soto Vairo NPB 566 (2000) 275, RMP 77 (2005) 1423

Weak coupling static potential

$$V^{(0)}(r,\mu') = \lim_{T\to\infty}\frac{i}{T}\ln\langle\ \square\ \rangle - \quad + \cdots$$



$$= E_0(r) + \frac{i}{N}\int_0^\infty dt\, e^{-it(V_o-V)}\langle \mathrm{Tr}\,\mathbf{r}\cdot g\mathbf{E}(t)\,\mathbf{r}\cdot g\mathbf{E}(0)\rangle(\mu') + \cdots$$

- Brambilla Pineda Soto Vairo PRD 60 (1999) 091502

The static energy $E_0(r)$ is known at three loops:

$$E_0(r) = \Lambda_s - \frac{C_F\alpha_{\rm s}}{r}(1+\#\alpha_{\rm s}+\#\alpha_{\rm s}^2+\#\alpha_{\rm s}^3+\#\alpha_{\rm s}^3\ln\alpha_{\rm s}+\#\alpha_{\rm s}^4\ln^2\alpha_{\rm s}+\#\alpha_{\rm s}^4\ln\alpha_{\rm s}+\dots)$$

- Anzai Kiyo Sumino PRL 104 (2010) 112003
  A.Smirnov V.Smirnov Steinhauser PRL 104 (2010) 112002

Note on exponentation

The computation is simplified through the logarithm of the Wilson loop (exponentiation):

$$\ln\left\langle ... \right\rangle = g^2 C_F$$ $$+ g^4 C_F \left[-\tfrac{1}{2} C_A$$ $$+ \tfrac{1}{2} C_A$$ $$+$$ $$\right]$$

$$+ g^6 C_F \left[\tfrac{1}{4} C_A^2$$  $$+ \tfrac{1}{2} C_A^2$$  $$- \tfrac{1}{4} C_A^2$$  $$+ \tfrac{1}{4} C_A^2$$  $$+ \tfrac{1}{4} C_A^2$$ 

$$- \tfrac{1}{2} C_A$$  $$+ \tfrac{1}{2} C_A$$  $$+ \tfrac{1}{2} C_A$$  $$+$$  $$+$$ $$\Big] + \mathcal{O}(g^8)$$

- Schröder PLB 447 (1999) 321

An efficient way to calculate the exponent of Wilson loops is the so-called replica trick:

$$\langle W_1 \cdot W_2 \cdots W_N \rangle = 1 + N \ln\langle W \rangle + \mathcal{O}(N^2)$$

$W_i =$ ith copy of W in a replicated theory of QCD not interacting with the others.

- Gardi Laenen Stavenga White JHEP 1011 (2010) 155
  Gardi Smillie White JHEP 1306 (2013) 088

Infrared logarithms

$\ln \alpha_s$ in E_0 signals the cancellation of contributions coming from soft and ultrasoft gluons:

$$\ln \alpha_s = \ln \frac{\mu'}{1/r} + \ln \frac{\alpha_s/r}{\mu'}$$

Infrared logarithms in the potential may be computed in the EFT solving the ADM problem.

- Appelquist Dine Muzinich PRD 17 (1978) 2074

$$\begin{aligned} V^{(0)}(r,\mu') &= -C_F \frac{\alpha_s(1/r)}{r} \left\{ 1 + \frac{\alpha_s(1/r)}{4\pi} a_1 + \left(\frac{\alpha_s(1/r)}{4\pi}\right)^2 a_2 \right. \\ &+ \left(\frac{\alpha_s(1/r)}{4\pi}\right)^3 \left[\frac{16\,\pi^2}{3} C_A^3 \ln r\mu' + a_3 \right] \\ &\left. + \left(\frac{\alpha_s(1/r)}{4\pi}\right)^4 \left[a_4^{L2} \ln^2 r\mu' + \left(a_4^L + \frac{16}{9}\pi^2 C_A^3 \beta_0 (-5 + 6\ln 2) \right) \ln r\mu' + \cdots \right] \right\} \end{aligned}$$

- Brambilla Pineda Soto Vairo PRD 60 (1999) 091502
  Brambilla Garcia Soto Vairo PLB 647 (2007) 185

Resummation of logarithms

The potential satisfies renormalization group equations

$$\begin{cases} \mu' \dfrac{d}{d\mu'} V^{(0)} = -\dfrac{2}{3} C_F \dfrac{\alpha_{\rm s}}{\pi} r^2 \left[V_o^{(0)} - V^{(0)} \right]^3 \left(1 + \dfrac{\alpha_{\rm s}}{\pi}\, c \right) \\ \mu' \dfrac{d}{d\mu'} V_o^{(0)} = \dfrac{1}{N} \dfrac{\alpha_{\rm s}}{\pi} r^2 \left[V_o^{(0)} - V^{(0)} \right]^3 \left(1 + \dfrac{\alpha_{\rm s}}{\pi}\, c \right) \\ \mu' \dfrac{d}{d\mu'} \alpha_{\rm s} = \alpha_{\rm s} \beta(\alpha_{\rm s}); \qquad\qquad c = \dfrac{-5 n_f + C_A (6\pi^2 + 47)}{108} \end{cases}$$

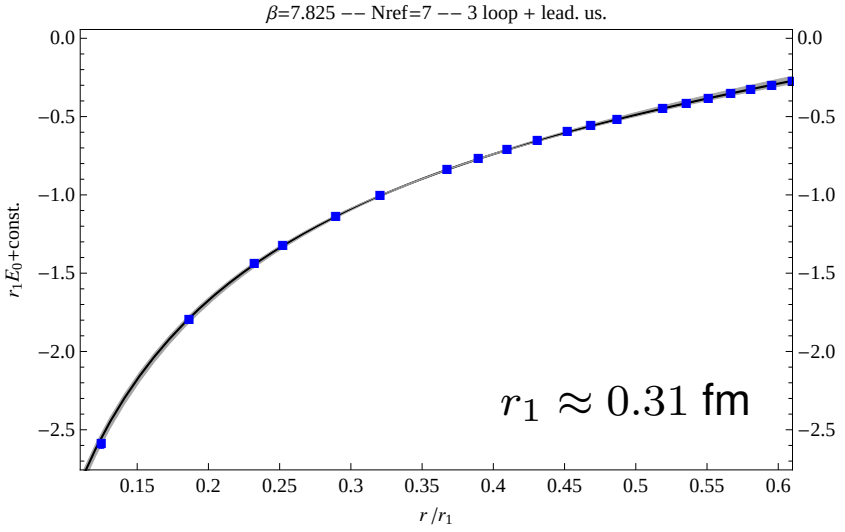
whose solution provides $V^{(0)}$ with N^{3}LL accuracy:

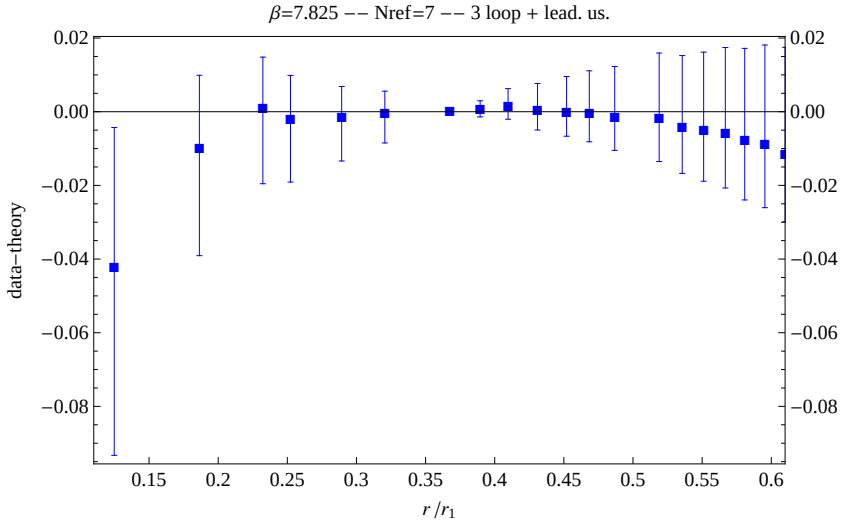
$$V^{(0)}(r,\mu') = V^{(0)}(r,1/r) - \frac{C_F C_A^3}{6\beta_0} \frac{\alpha_{\rm s}^3(1/r)}{r} \left\{ \left(1 + \frac{3}{4} \frac{\alpha_{\rm s}(1/r)}{\pi} a_1 \right) \ln \frac{\alpha_{\rm s}(1/r)}{\alpha_{\rm s}(\mu')} \right. \\ \left. \left(\frac{\beta_1}{4\beta_0} - 6c \right) \left[\frac{\alpha_{\rm s}(\mu')}{\pi} - \frac{\alpha_{\rm s}(1/r)}{\pi} \right] \right\}$$

- Pineda Soto PLB 495 (2000) 323
  Brambilla Garcia Soto Vairo PRD 80 (2009) 034016

Static energy vs lattice data

A typical application is the comparison of the perturbative expression of the QCD static energy with lattice data at short distances and the extraction of $\Lambda_{\overline{\rm MS}}$ or equivalently $\alpha_{\rm s}$:





$$\Lambda_{\overline{\rm MS}} = 315^{+18}_{-12}\ {\rm MeV} \quad \text{or} \quad \alpha_{\rm s}(M_Z) = 0.1166^{+0.0012}_{-0.0008}$$

- Bazavov Brambilla Garcia Petreczky Soto Vairo PRD 90 (2014) 074038

Poincaré invariance

EFTs preserve all the invariances of the fundamental QFT.
Therefore NREFTs are constrained by the Poincaré invariance of the fundamental QFT, although Lorentz invariance is apparently broken by the nonrelativistic expansion.

It has been suggested, even before the establishing of EFTs, that Poincaré invariance provides non trivial constraints on the form of the potentials.

- Dirac RMP 21 (1949) 302

Within NREFTs these constraints may be implemented in a rigorous setting. They allow to fix some of the matching coefficients/potentials of the NREFT to all orders and nonperturbatively without computing them. In QCD, these constraints can be tested against lattice determinations.

- Brambilla Gromes Vairo PRD 64 (2001) 076010, PLB 576 (2003) 314
  Berwein Brambilla Hwang Vairo PRD 99 (2019) 094008

Poincaré Algebra

For any Poincaré invariant theory the generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ of time translation, space translations, rotations, and Lorentz boosts satisfy the Poincaré algebra:

$$
\begin{aligned}
[\mathbf{P}^i, \mathbf{P}^j] &= 0 \\
[\mathbf{P}^i, H] &= 0 \\
[\mathbf{J}^i, \mathbf{P}^j] &= i\epsilon_{ijk}\mathbf{P}^k \\
[\mathbf{J}^i, H] &= 0 \\
[\mathbf{J}^i, \mathbf{J}^j] &= i\epsilon_{ijk}\mathbf{J}^k \\
[\mathbf{P}^i, \mathbf{K}^j] &= -i\delta_{ij}H \\
[H, \mathbf{K}^i] &= -i\mathbf{P}^i \\
[\mathbf{J}^i, \mathbf{K}^j] &= i\epsilon_{ijk}\mathbf{K}^k \\
[\mathbf{K}^i, \mathbf{K}^j] &= -i\epsilon_{ijk}\mathbf{J}^k
\end{aligned}
$$

Once $\mathbf{P}$, $\mathbf{J}$ are written in terms of the EFT fields and H, $\mathbf{K}$ have been matched, the algebra constraints the matching coefficients of H, which include the potentials, and $\mathbf{K}$.

The Poincaré algebra imposes the following constraints on the potentials:

$$V_{LS}^{(1)}(r) - V_{LS}^{(2)}(r) + \frac{1}{2r}V^{(0)\prime}(r) = 0$$

$$4V_{L^2}^{(1)}(r) - 2V_{L^2}^{(2)}(r) + rV^{(0)\prime}(r) = 0$$

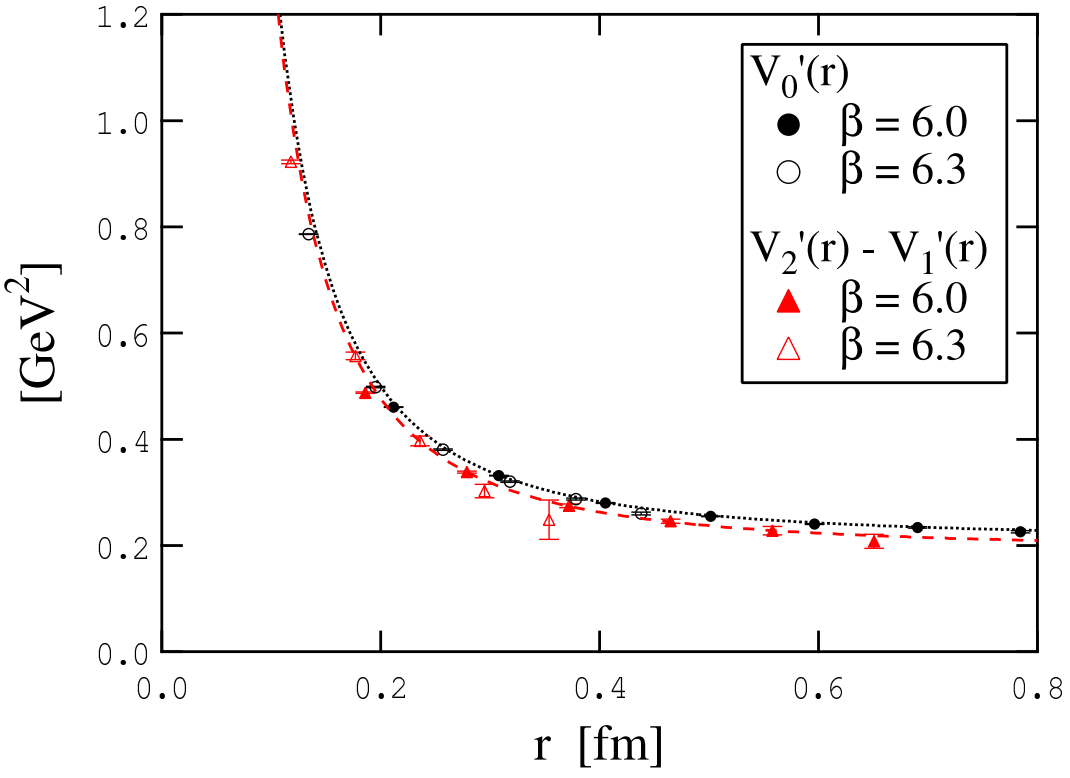
$$4V_{p^2}^{(1)}(r) - 2V_{p^2}^{(2)}(r) + V^{(0)}(r) - rV^{(0)\prime}(r) = 0$$

The constraints are generic and do not depend on the dynamical content of the EFT.
They are satisfied by any potential defined in an EFT and derived from a relativistic QFT.

- Gromes ZPC 26 (1984) 401
  Barchielli Brambilla Prosperi NCA 103 (1990) 59
  Brambilla Gromes Vairo PRD 64 (2001) 076010, PLB 576 (2003) 314

Lattice verification of the Poincaré constraints

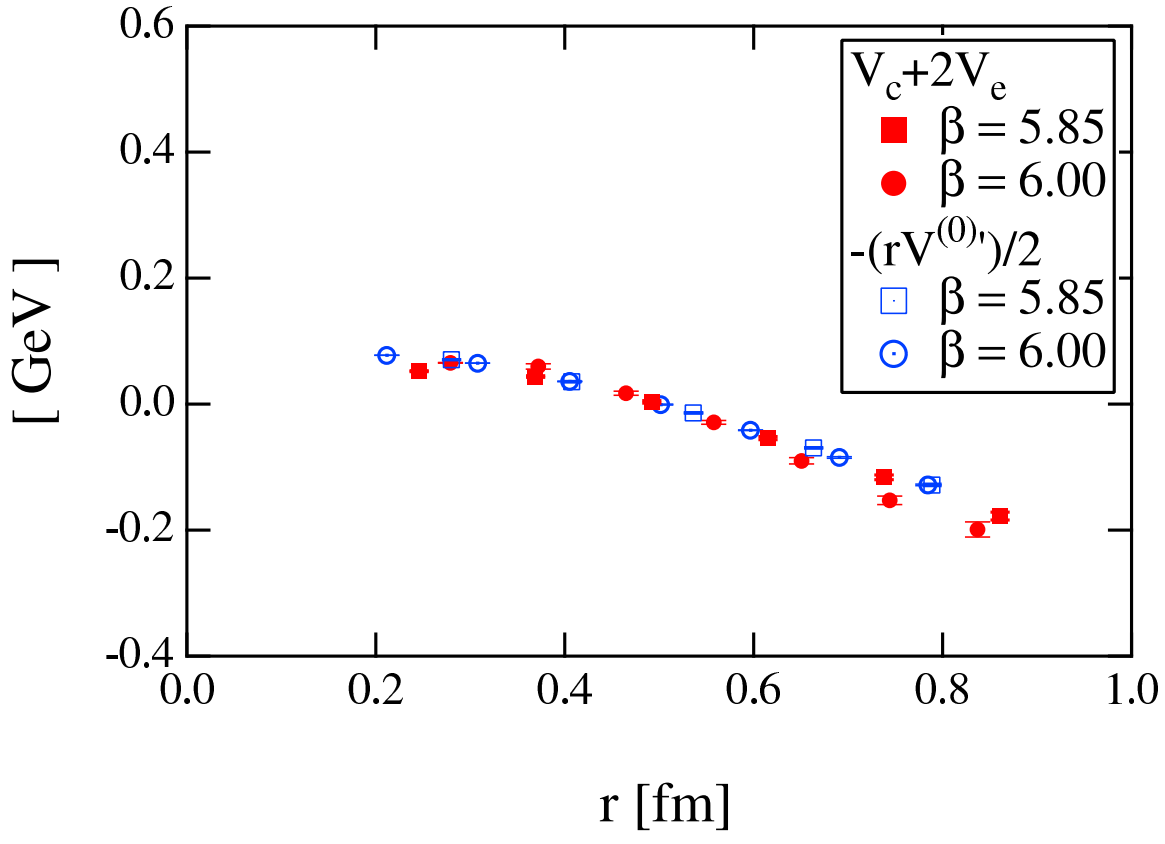
A lattice verification of $V_{LS}^{(1)}(r) - V_{LS}^{(2)}(r) + \frac{1}{2r}V^{(0)\prime}(r) = 0$:



○ M.Koma Y.Koma NPB 769 (2007) 79

Lattice verification of the Poincaré constraints

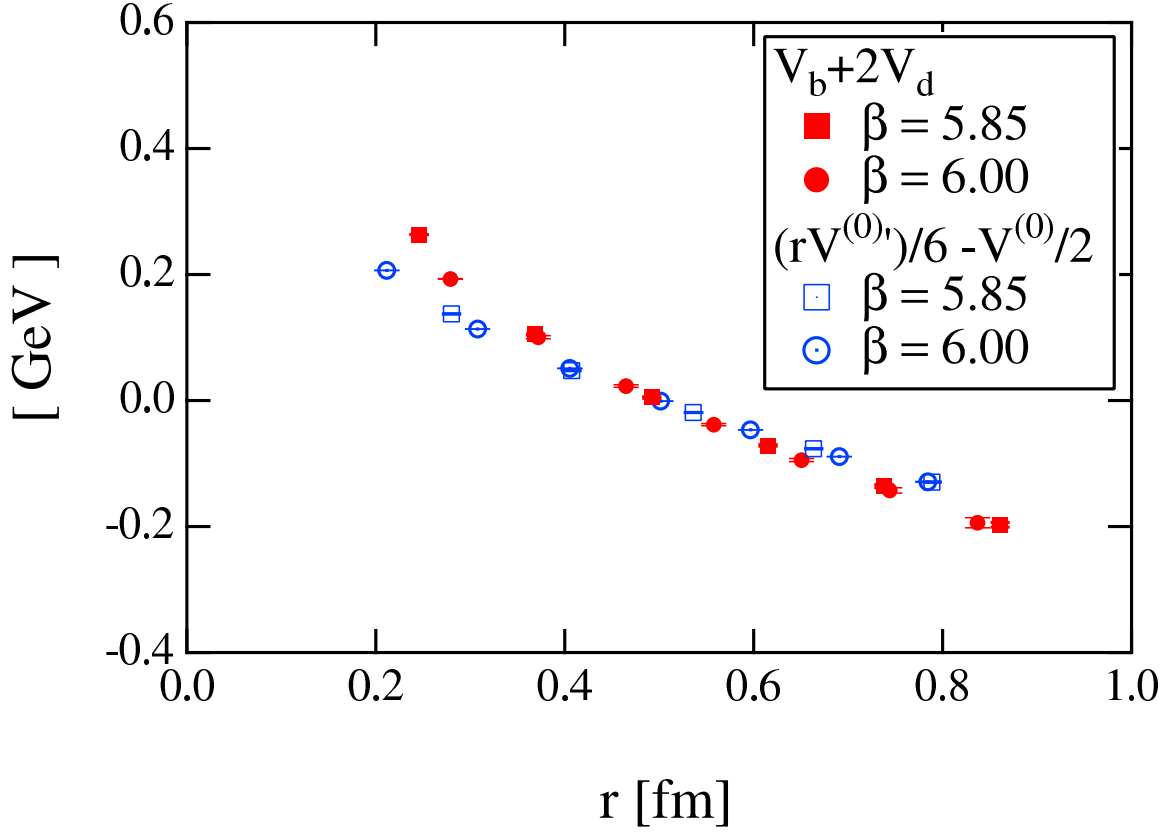
A lattice verification of $4V^{(1)}_{L^2}(r) - 2V^{(2)}_{L^2}(r) + rV^{(0)\prime}(r) = 0$:



○ M.Koma Y.Koma Wittig PoS LATTICE2007 (2007) 111

Lattice verification of the Poincaré constraints

A lattice verification of $4V^{(1)}_{p^2}(r) - 2V^{(2)}_{p^2}(r) + V^{(0)}(r) - rV^{(0)\prime}(r) = 0$:



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Note on PN gravity

As a general observation, if we call $\mathbf{P}$ the center of mass momentum of a 2 particle system, then the $1/m^2$ center of mass momentum dependent potentials have the form

$$\frac{\mathbf{P}^2}{8m^2}\left(4V_{p^2}^{(1)} - 2V_{p^2}^{(2)}\right) + \frac{\mathbf{P}^2 - (\hat{\mathbf{r}}\cdot\mathbf{P})^2}{8m^2}\left(4V_{L^2}^{(1)} - 2V_{L^2}^{(2)}\right)$$

which, using the constraints induced by Poincaré invariance, can be written as

$$-\frac{1}{2}\left(\mathbf{v}_{\rm avg}^2 V^{(0)} - (\hat{\mathbf{r}}\cdot\mathbf{v}_{\rm avg})^2 r V^{(0)\prime}\right)$$

where $\mathbf{v}_{\rm avg} = \mathbf{P}/2m$ is the average velocity of the 2 particles.
Poincaré invariance does not constrain terms that depend on the relative momentum $\mathbf{p}$, although the dynamics of the underlaying QFT may relate terms that depend on $\mathbf{P}$ with terms that depend on $\mathbf{p}$, leading, eventually, to constrain also the latter.

This kind of consideration have been used recently to constrain the form of the potential in 5PN and 6PN expansion of general relativity.

- Nicolis arXix:2010.04743

Van der Waals EFT

WEFT: EFT for bound state-bound state interaction

WEFT is the EFT for the large distance interaction of two bound states:
R (distance between the bound states) $\gg r$ (size of the bound state).

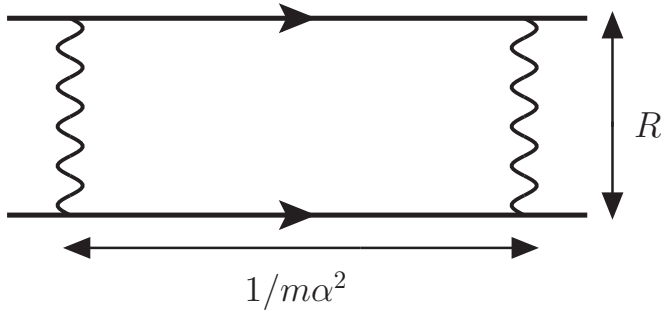
- WEFT may be matched to pNRQFT.
- It is obtained by integrating out modes associated with the energy scale $1/R$ and mv^2 from pNRQFT.
- $\phi_n = S_n S_n$, calling S_n the field associated to the bound state in pNRQFT with quantum number n.
- The (van der Waals) potential depends therefore on two quantum numbers n_1 and n_2.
- We may distinguish two situations:
  $1/R \gg mv^2$: short distance van der Waals interaction
  $1/R \ll mv^2$: long distance van der Waals interaction

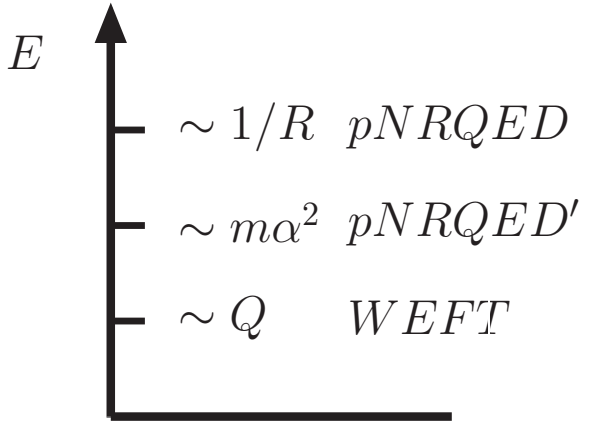
- Holstein PRD 78 (2008) 013001
  Brambilla Shtabovenko Tarrus Vairo PRD 95 (2017) 116004

Short distance van der Waals interaction in QED

In QED $v \sim \alpha$.

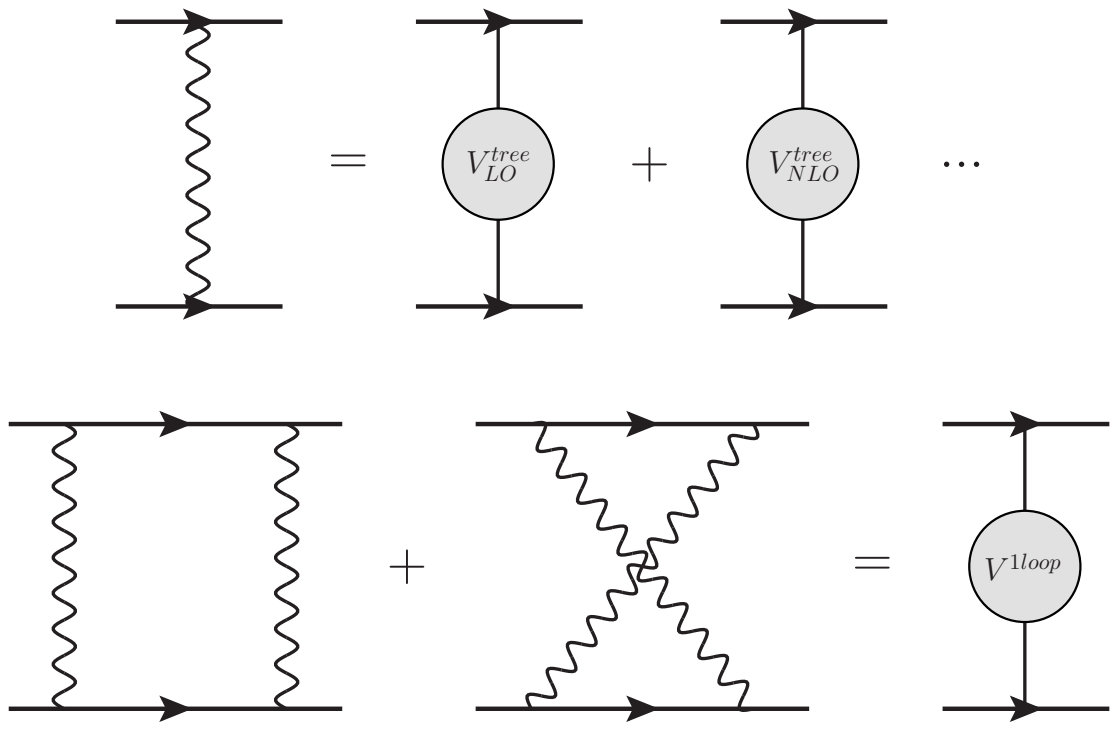
The short distance hierarchy corresponds to a hierarchy of EFTs:





WEFT follows from pNRQED by integrating out subsequently the scales:
$1/R$ (pNRQED′) and $m\alpha^2$.

Short distance van der Waals interaction: matching pNRQED′

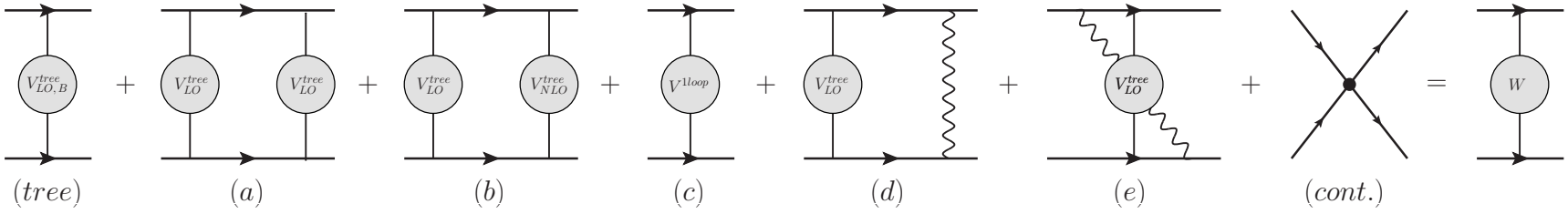


+ contact terms inherited from NRQED.

In the lhs of the matching (pNRQED), the photons couple through electric ad magnetic dipoles; in the rhs of the matching, there appear the effective potentials of pNRQED′.

Short distance van der Waals interaction in WEFT

The potential in WEFT matches the potentials in pNRQED′, their iterations, loops involving low energy photons (energy or momentum of order $m\alpha^2$) and contact terms:



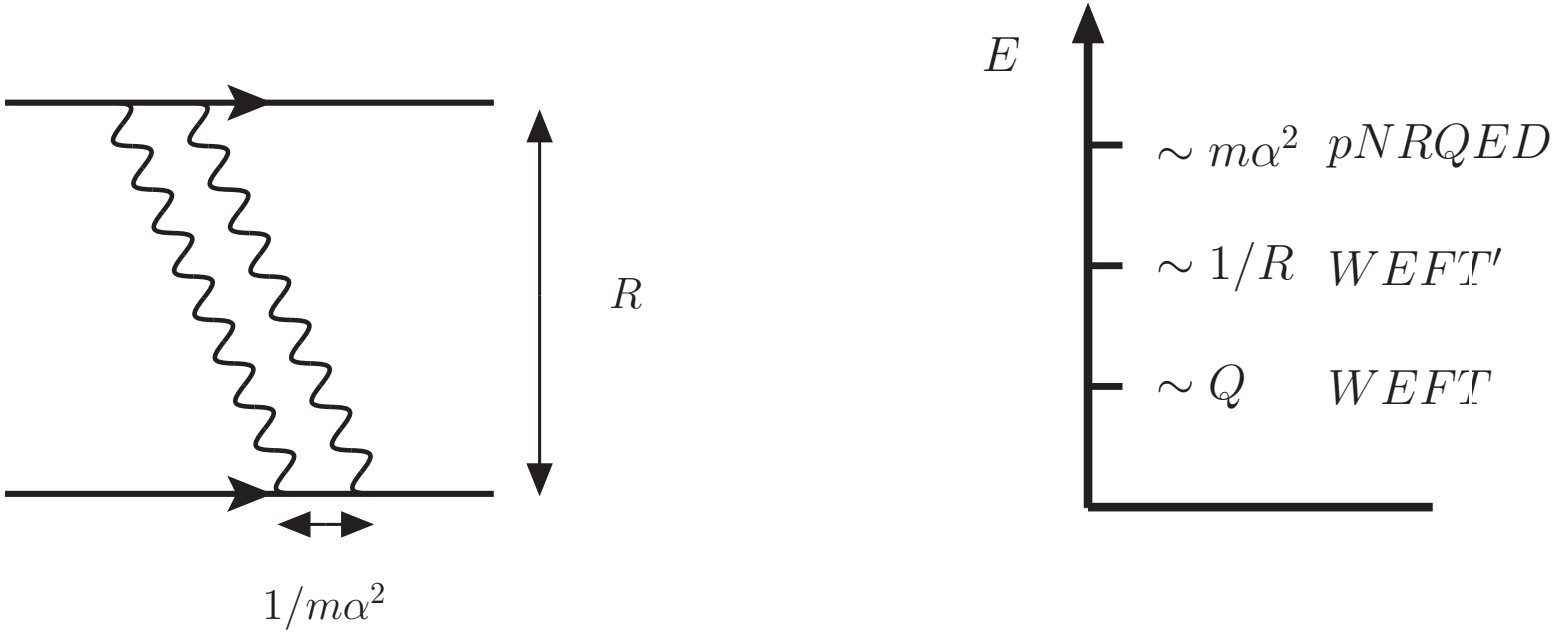
At leading order, one obtains the London potential:

$$W(R) = -\frac{C_6}{R^6}, \qquad C_6 = -\frac{3}{8\pi^2}\sum_{m_1,\,m_2}{}' \frac{p_E(n_1,m_1)p_E(n_2,m_2)}{\Delta E_{n_1 m_1} + \Delta E_{n_2 m_2}}\,.$$

$$p_E(n,m) = \frac{e^2}{3}\langle n|\boldsymbol{x}|m\rangle\cdot\langle m|\boldsymbol{x}|n\rangle, \qquad \Delta E_{nm} = E_n - E_m$$

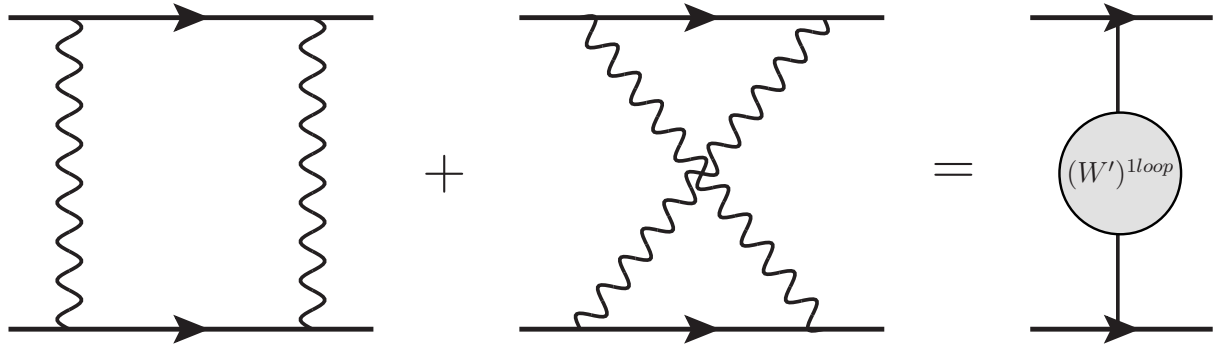
Long distance van der Waals interaction in QED

The long distance hierarchy corresponds to a hierarchy of EFTs:



WEFT follows from pNRQED by integrating out subsequently the scales: $m\alpha^2$ (WEFT′) and $1/R$.

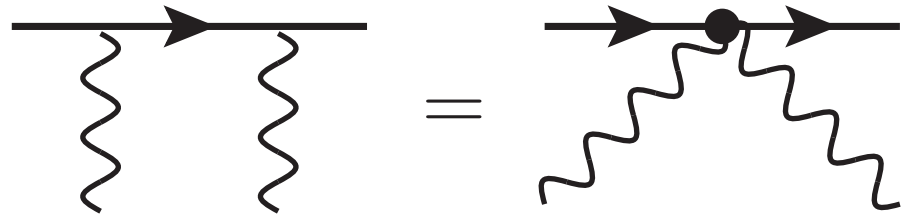
Long distance van der Waals interaction: matching WEFT′



+ contact terms inherited from NRQED.

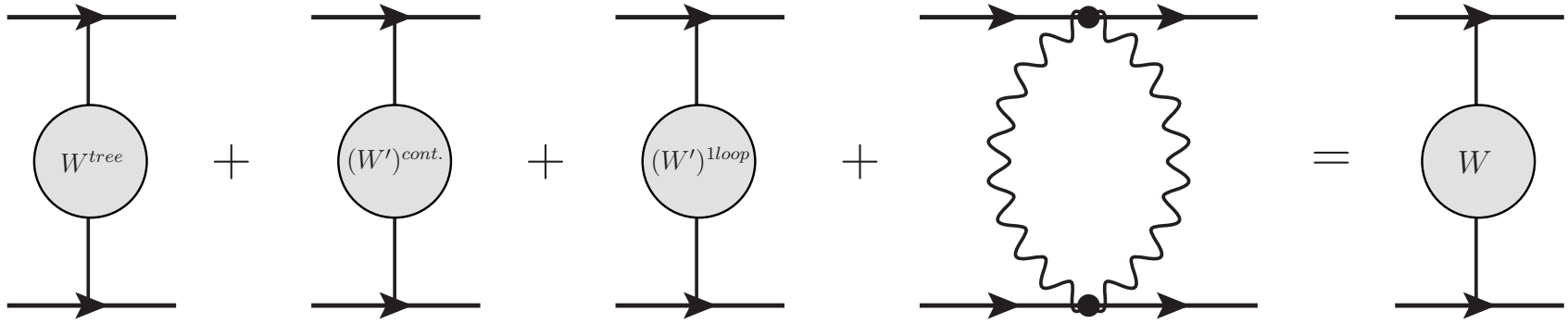
In the lhs of the matching (pNRQED), the photons couple through electric ad magnetic dipoles; in the rhs of the matching, there appear the effective potentials of WEFT′.

In the 1 particle sector an effective vertex with 2 electric fields is generated with strength proportional to the electric polarizability α_n:



Long distance van der Waals interaction in WEFT

The potential in WEFT matches the potentials in WEFT′, loops involving low energy photons (energy or momentum of order $1/R$) and contact terms:



The tree level exchange involves 2 magnetic dipole vertices.

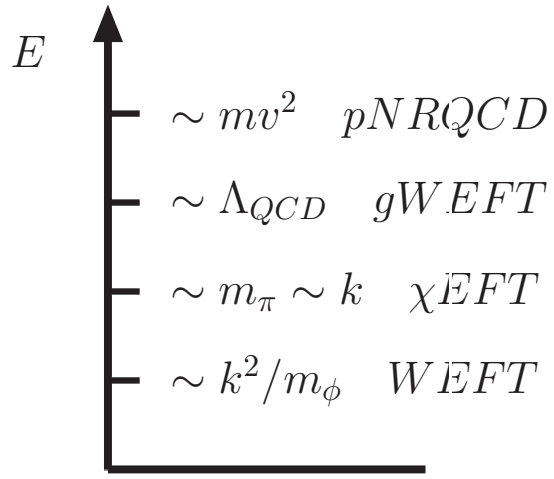
At leading order, one obtains the Casimir–Polder potential:

$$W(R) = -\frac{C_7}{R^7}, \qquad C_7 = \frac{23}{4\pi}\alpha_{n1}\alpha_{n2}$$

$$\alpha_n = \frac{1}{2\pi}\sum_m{}' \frac{p_E(n,m)}{\Delta E_{nm}} \quad \text{(electric polarizability)}$$

Long distance van der Waals interaction in QCD

In QCD, more energy scales and intermediate EFTs are involved to match with WEFT:

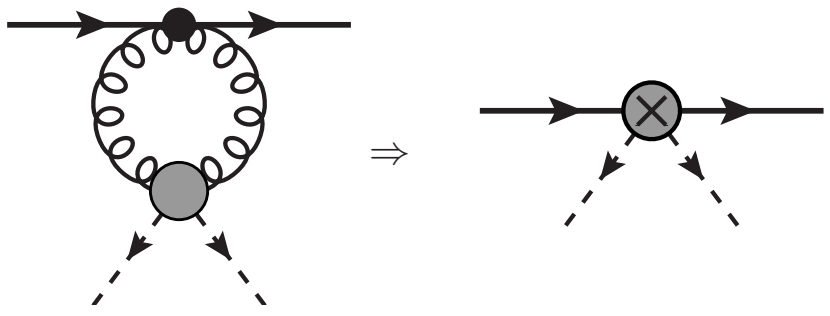


The matching to WEFT requires coupling the QCD bound state (quarkonium) with chiral perturbation theory, the EFT for the would-be Goldstone bosons of QCD (pions).
The inverse of the pion mass sets the range of the QCD van der Waals interaction at long distances ($R \sim 1/m_\pi$) and m_π^2/m_ϕ sets the scale of the WEFT.

- Brambilla Krein Tarrus Vairo PRD 93 (2016) 054002

Matching gWEFT to χEFT

The matching from pNRQCD to gWEFT is similar to the one from pNRQED to WEFT′. Under the assumed hierarchy the matching from gWEFT to χEFT is nonperturbative and leads to an EFT where nonrelativistic quarkonia are coupled with pions:



This coupling may be computed from the trace anomaly:

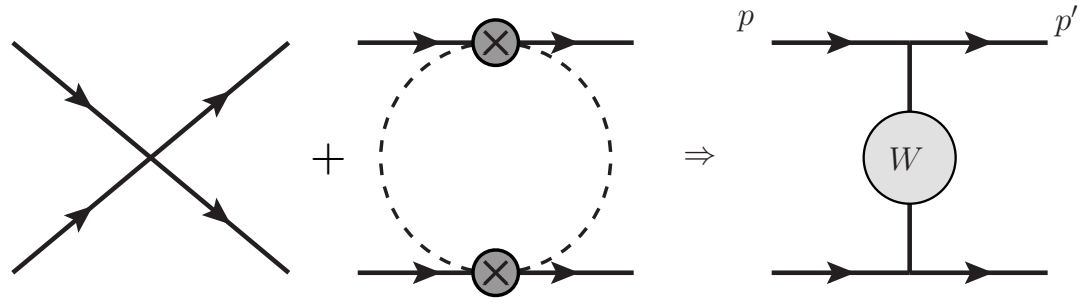
$$g^2\langle\pi^+(p_1)\pi^-(p_2)|\boldsymbol{E}_a^2|0\rangle = \frac{8\pi^2}{\beta_0}\left(\kappa_1 p_1^0 p_2^0 - \kappa_2 p_1^i p_2^i + 3m_\pi^2\right)$$

where κ_1 and κ_2 can be obtained from pionic transitions of quarkonium states.

- Voloshin PPNP 61 (2008) 455

Long distance van der Waals interaction in WEFT

The potential in WEFT matches the two pion exchange in χEFT and contact terms:

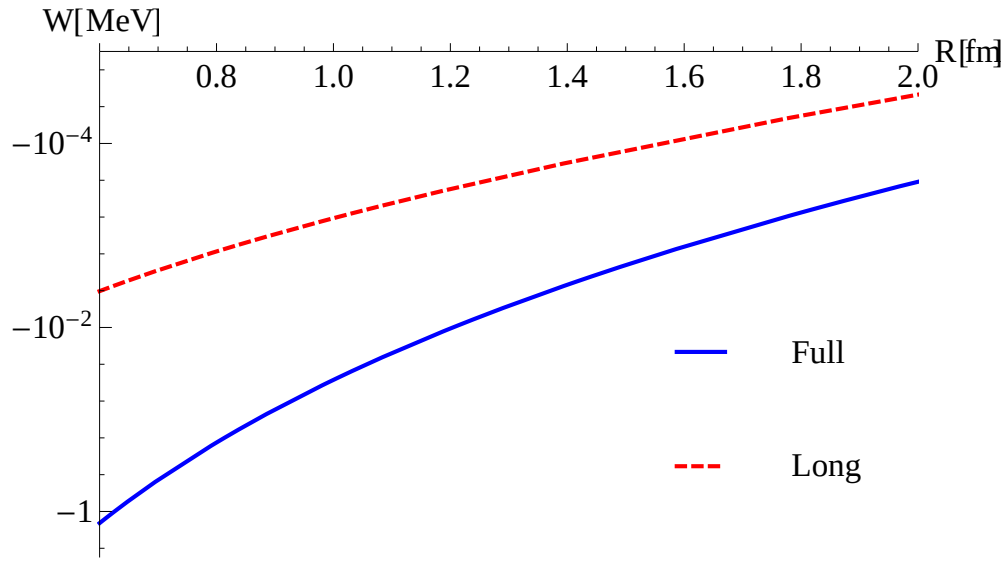


In the long range one obtains (K_n are modified Bessel functions of the 2nd kind):

$$W(R) = -\frac{3\pi\beta^2 m_\pi^2}{8\beta_0^2 R^5}\left[\left(4\left(\kappa_2+3\right)^2(m_\pi R)^3+\left(3\kappa_1^2+43\kappa_2^2+14\kappa_1\kappa_2\right)m_\pi R\right)K_1(2m_\pi R)\right.$$
$$\left.+2\left(2\left(\kappa_2+3\right)\left(\kappa_1+5\kappa_2\right)(m_\pi R)^2+\left(3\kappa_1^2+43\kappa_2^2+14\kappa_1\kappa_2\right)\right)K_2(2m_\pi R)\right]$$
$$\underset{R\gg 1/(2m_\pi)}{\approx} -\frac{3(3+\kappa_2)^2\pi^{3/2}\beta^2}{4\beta_0^2}\frac{m_\pi^{9/2}}{R^{5/2}}e^{-2m_\pi R}$$

where $\beta = -\frac{1}{3N}\langle n|\mathbf{r}\cdot\frac{1}{E_n-\mathbf{p}^2/m-V_o}\mathbf{r}|n\rangle$ is the polarizability in pNRQCD.

Long distance van der Waals potential



- Brambilla Krein Tarrus Vairo PRD 93 (2016) 054002

Note on the gravitational force between polarizable objects

One can compute at low energy quantum gravitational corrections to the force between a pair of localized polarizable objects. The force is induced by gravitational quadrupole moments (rather than dipole moments as in QED) due to quantum fluctuations in the metric associated with the exchange of two gravitons.

The equivalent of the London limit (R shorter than the inverse of some characteristic frequency of response in the system but larger than the size of the gravitational source) is

$$W(R) = -\frac{C_{10}}{R^{10}}, \qquad C_{10} = \frac{315}{\pi} G^2 \int_0^\infty d\omega\, \alpha_1(i\omega)\alpha_2(i\omega)$$

The equivalent of the Casimir–Polder limit (R larger than the inverse of some characteristic frequency of response in the system) is

$$W(R) = -\frac{C_{11}}{R^{11}}, \qquad C_{11} = \frac{3987}{4\pi} G^2 \alpha_1(0)\alpha_2(0)$$

where $\alpha_i(\omega)$ are the gravitational quadrupole polarizabilities.

- Ford Hertzberg Karouby PRL 116 (2016) 151301

Conclusions

Conclusions

NREFTs for bound states of different kind have been developed and extensively used in QCD and QED. They all provide Schrödinger-like equations at leading order.

At higher order the nonrelativistic particles interact with the other degrees of freedom. This interaction modifies the leading Schrödinger-like picture. The power counting of the EFT quantifies the size of this modification.

The potentials in the Schrödinger equation can be computed systematically by matching either perturbatively or nonperturbatively to the fundamental theory. They are scheme and scale dependent and undergo renormalization.

The EFT realizes manifestly or in a more subtle way all the symmetries of the fundamental theory. Noteworthy this is also the case for the Lorentz symmetry, which is manifestly broken in NREFTs.

Many results are generic and may serve as a guide also in the construction of other NREFTs for instance of general relativity.