Gravitational Two-Body Hamiltonians from Scattering Amplitudes

October 29, 2020

EFT Methods from Bound States to Binary Systems

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ZB, C. Cheung, R. Roiban, C. H. Shen, M. Solon, M. Zeng,
arXiv:1901.04424 and arXiv:1908.01493.
ZB, A. Luna, R. Roiban, C. H. Shen, M. Zeng,
arXiv:2005.03071
ZB, J. Parra-Martinez, R. Roiban, E. Sawyer, C.-H. Shen,
arXiv 2010.08559

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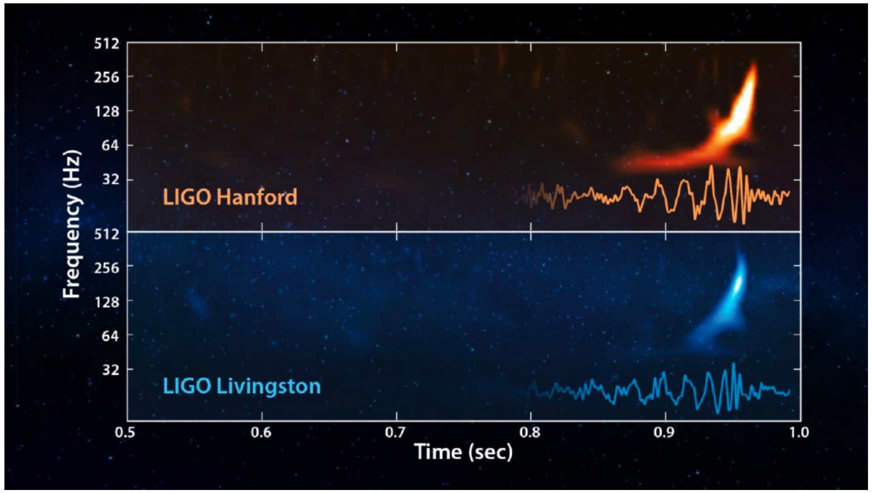
Overview

Era of gravitational wave astronomy has begun.



We in the scattering amplitudes community are eager to help contribute to problems of importance for gravitational wave physics.

Outline

1. Some basic ideas from scattering amplitudes.
2. Search for useful structures.
3. EFT approach.
4. Applications
   - — $O(G^3)$ two-body Hamiltonian.
   - — $S_1 S_2$ $O(G^2)$ two-body Hamiltonian.
   - — Eikonal formula for spinning case.
   - — New results for tidal operators.
5. Outlook.

Earlier amplitudes based talks from Bjerrum-Bohr, Kosower, Parra-Martinez.

Some Basic Principles

General Relativity is a mature subject. Surprising that there can be a radically different way of looking at old problems.

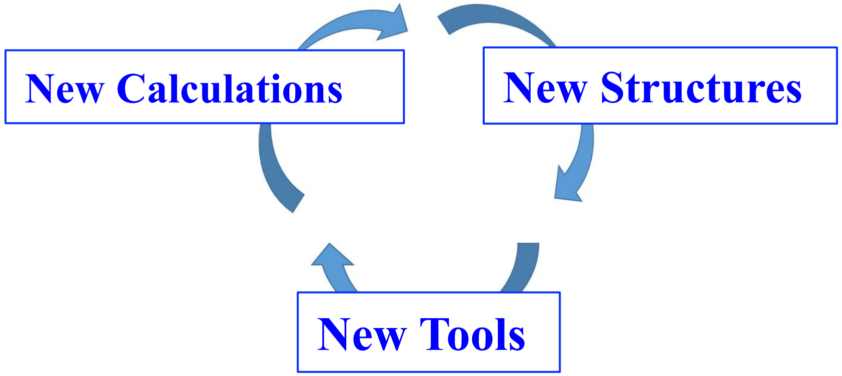
Some of the ideas from scattering amplitudes

- On-shell. No gauge fixing, coordinate, or field variable dependence.
- Generalized unitarity.
- Double copy and color-kinematics duality. (gravity) ~ (gauge theory)^2
- Hidden simplicity.
- Advanced integration methods

These ideas have already proven useful: Here will discuss applications for bound-state case:

1. high orders
2. spin
3. tidal effects

The search for new structures.



New Calculations

New Structures

New Tools

A virtuous cycle.

$$A_n(1^-, 2^-, 3^+, \dots n^+) = i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \cdots \langle n1 \rangle}$$

Parke Taylor;
Mangano, Parke, Xu

- **Key priority for new calculations is to uncover new and useful structures.**
- **Simultaneously push state of the art for quantities of interest to LIGO/Virgo**

Examples of identified relevant structures in gravitational 2 body problem:

1. **Double copy.**
2. **Direct amplitude relations to scattering angles (which feed into EOB).**
3. **Eikonal phase capturing the physics.**
4. **Infinite sequences of tidal operators.**

See Porto’s and
Bjerrum Bohr’s talks

From Tree to Loops: Generalized Unitarity Method

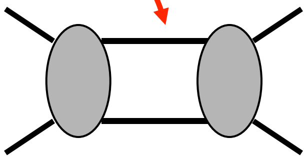
Use tree amplitudes to build higher-order (loop) amplitudes.

$$E^2 = \vec{p}^{\,2} + m^2$$

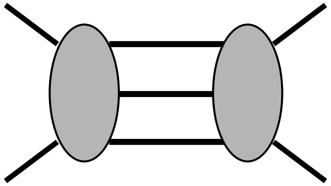
on-shell

ZB, Dixon, Dunbar and Kosower (1994)

Two-particle cut:

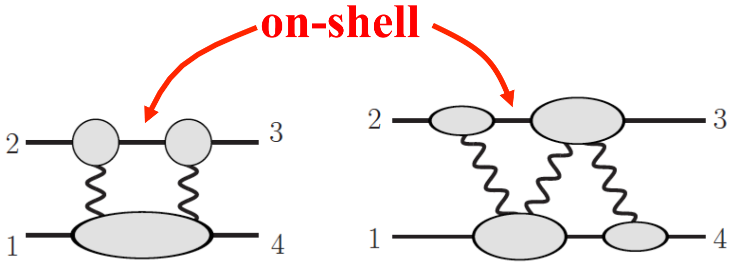


Three-particle cut:



- **Systematic assembly of complete loop amplitudes from tree amplitudes.**
- **Works for any number of particles or loops.**

Generalized unitarity as a practical tool for loops.

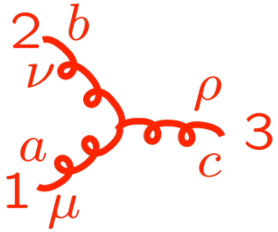


ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger;
ZB, Carrasco, Johansson, Kosower
and many others

Idea used in the "NLO revolution" in QCD collider physics.
No gauge fixing in the formalism.

Three Vertices

Standard perturbative approach:



Three-gluon vertex:

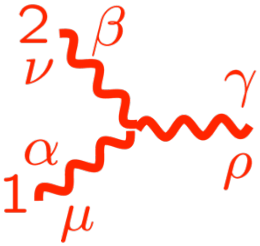
$$V_{3\,\mu\nu\sigma}^{abc} = -g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three-graviton vertex:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) =$$

$$\begin{aligned}&\mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})\\ &\quad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})\\ &\quad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})\\ &\quad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]\end{aligned}$$



About 100 terms in three vertex

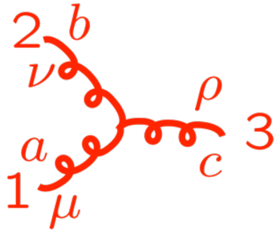
Naïve conclusion: Gravity is a nasty mess.

Simplicity of Gravity Amplitudes

People were looking at gravity amplitudes the wrong way. On-shell viewpoint overwhelmingly more powerful.

On-shell **three vertices contains all information:** $E_i^2 - \vec{k}_i^2 = 0$

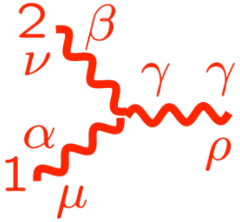
Yang-Mills (QCD) gauge theory:



"color" factor

$$-g f^{abc} (\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

Einstein gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

Starting from this on-shell vertex any multi-loop amplitude can be constructed via modern methods

Gravitons are like two gluons!

KLT Relation Between Gravity and Gauge Theory

See also Bjerrum-Bohr's talk

KLT (1985)

Kawai-Lewellen-Tye string relations in low-energy limit:

gravity

gauge-theory color ordered

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12} A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)\,,$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34} A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(2,1,4,3,5)$$

$$+ is_{13}s_{24} A_5^{\text{tree}}(1,3,2,4,5)\, A_5^{\text{tree}}(3,1,4,2,5)$$

Inherently gauge invariant!

Generalizes to explicit all-leg form.

ZB, Dixon, Perelstein, Rozowsky

1. **Gravity is derivable from gauge theory.**
2. **Once gauge-theory amplitude is simplified, so is gravity.**
3. **Standard Lagrangian methods offer no hint why this is possible.**
4. **It is very general property of gravity.**

Duality Between Color and Kinematics

Consider five-point tree amplitude:

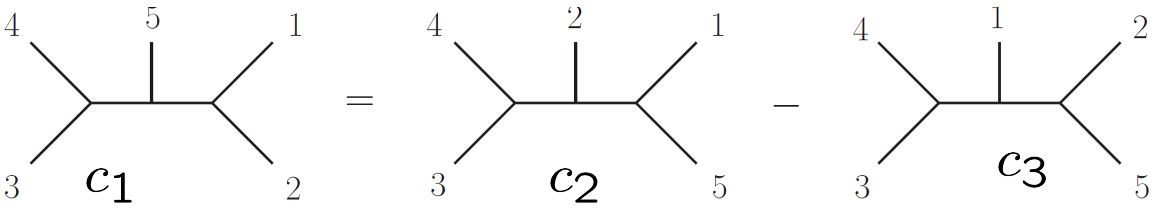
ZB, Carrasco, Johansson (BCJ)

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

color factor

kinematic numerator factor

Feynman propagators



$$c_1 = f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} \qquad c_2 = f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5} \qquad c_3 = f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 + c_2 + c_3 = 0 \;\; \Leftrightarrow \; n_1 + n_2 + n_3 = 0}$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* algebraic constraint equations.

Gravity from Gauge Theory

ZB, Carrasco, Johansson

gauge theory (QCD):

$$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i\, n_i}{D_i}$$

color factor

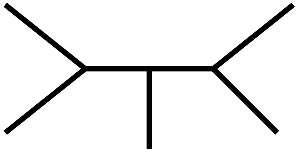
kinematic numerator factor

Feynman propagators

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

$$\boxed{c_i \to n_i}$$



sum over diagrams with only 3 vertices

Einstein gravity:

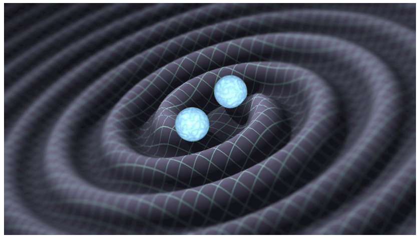
$$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

Gravity and gauge theory kinematic numerators are the same!

Cries out for a unified description of gravity with gauge theory, presumably along the lines of string theory.

Scattering Amplitudes and Gravitational Radiation



A small industry had developed to study this.

- **Connection to scattering amplitudes.**

  Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Luna, Nicholson, O'Connell, White; Guevara; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon; Damour; Bautista, Guevara; Kosower, Maybee, O'Connell; Plefka, Steinhoff, Wormsbecher; Foffa, Mastrolia, Sturani, Sturm; Guevara, Ochirov, Vines; Chung, Huang, Kim, Lee; etc.

- **Worldline approach for radiation and double copy.**

  Goldberger and Ridgway; Goldberger, Li, Prabhu, Thompson; Chester; Shen; Almeida, Foffa, Sturani

- **Technical issues having to do with keeping right physical states.**

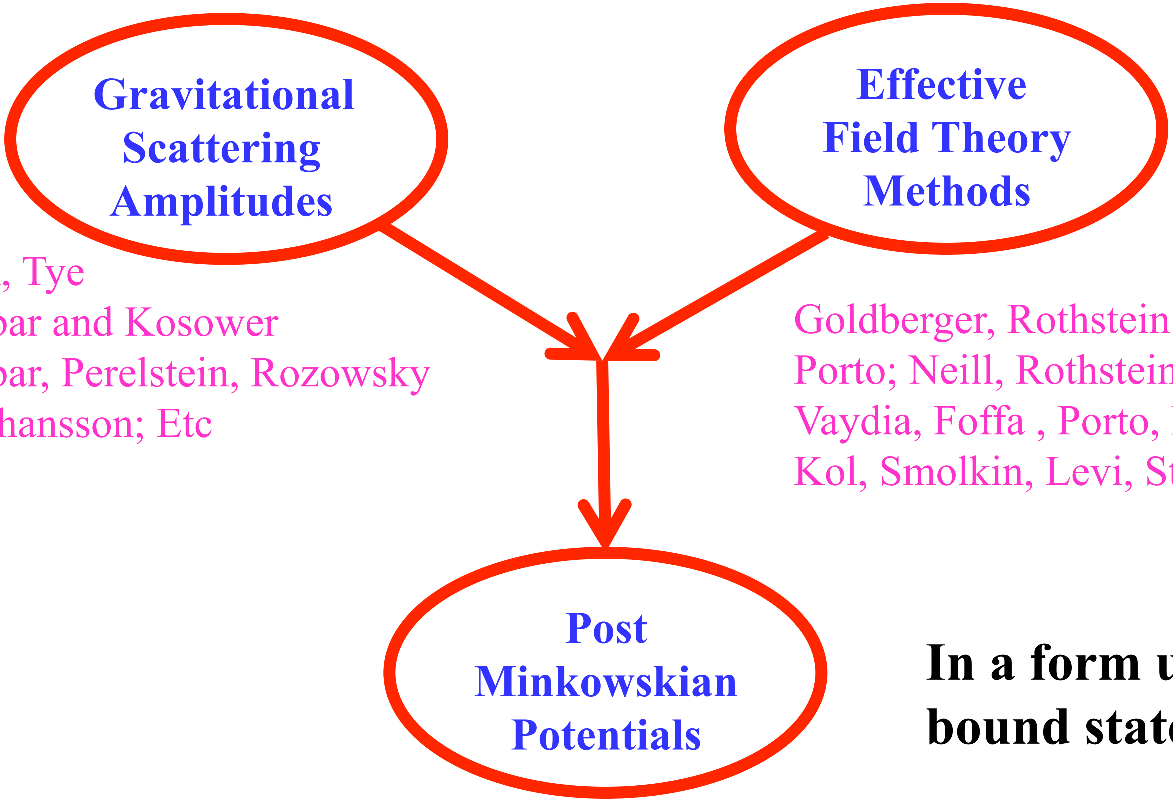
  Luna, Nicholson, O'Connell, White; Johansson, Ochirov; Johansson, Kalin; Henrik Johansson, Gregor Kälin, Mogull.

Key question: Can we do something of direct interest for LIGO/Virgo theorists?

Effective Field Theory Approach

ZB, Cheung, Roiban, Shen, Solon, Zeng

Cheung, Rothstein, Solon (2018)



Amplitudes community

Gravitational Scattering Amplitudes

Effective Field Theory Methods

EFT community

Kawai, Lewellen, Tye
ZB, Dixon, Dunbar and Kosower
ZB, Dixon, Dunbar, Perelstein, Rozowsky
ZB, Carrasco, Johansson; Etc

Goldberger, Rothstein;
Porto; Neill, Rothstein;
Vaydia, Foffa , Porto, Rothstein, Sturani;
Kol, Smolkin, Levi, Steinhoff, etc.

Post Minkowskian Potentials

In a form useful for bound state problem

The EFT directly gives us a two-body Hamiltonian of a form appropriate to enter LIGO analysis pipeline (after importing into EOB or pheno models).

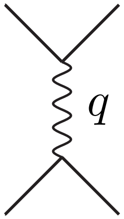
We prefer the EFT matching when we need to guarantee correctness.

Potentials and Amplitudes

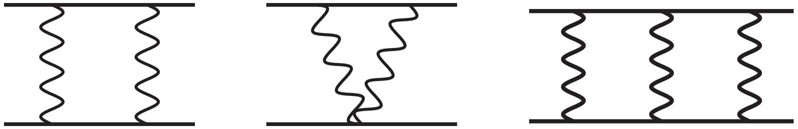
Iwasaki; Bjerrum-Bohr, Donoghue, Vanhove; Neill, Rothstein
Bjerrum-Bohr, Damgaard, Festuccia, Planté. Vanhove

Tree-level: Fourier transform gives classical potential.

$$V(r) \sim \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} A^{\text{tree}}(\mathbf{q})$$



At higher orders things quickly become less obvious:



- **What we learned in grad school on $\hbar$ counting is wrong. Loops have classical pieces.**
- **Double counting and iteration.**
- **$1/\hbar$ scaling of loop amplitudes.**
- **Non-uniqueness of potential.**
- **Cross terms between $1/\hbar$ and $\hbar$**

Talks from Donoghue and Bjerrum-Bohr

$$e^{iS_{\text{classical}}/\hbar}$$

$1/\hbar^L$ at L loops

Piece of loops are classical: Our task is to extract these pieces.

We harness EFT to clean up confusion

EFT is a Clean Approach

No need to re-invent the wheel.
Build EFT from which we can read off potential.

Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

$$L_{\text{kin}} = \int_{\boldsymbol{k}} A^{\dagger}(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_A^2} \right) A(\boldsymbol{k}) + \int_{\boldsymbol{k}} B^{\dagger}(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_B^2} \right) B(\boldsymbol{k})$$

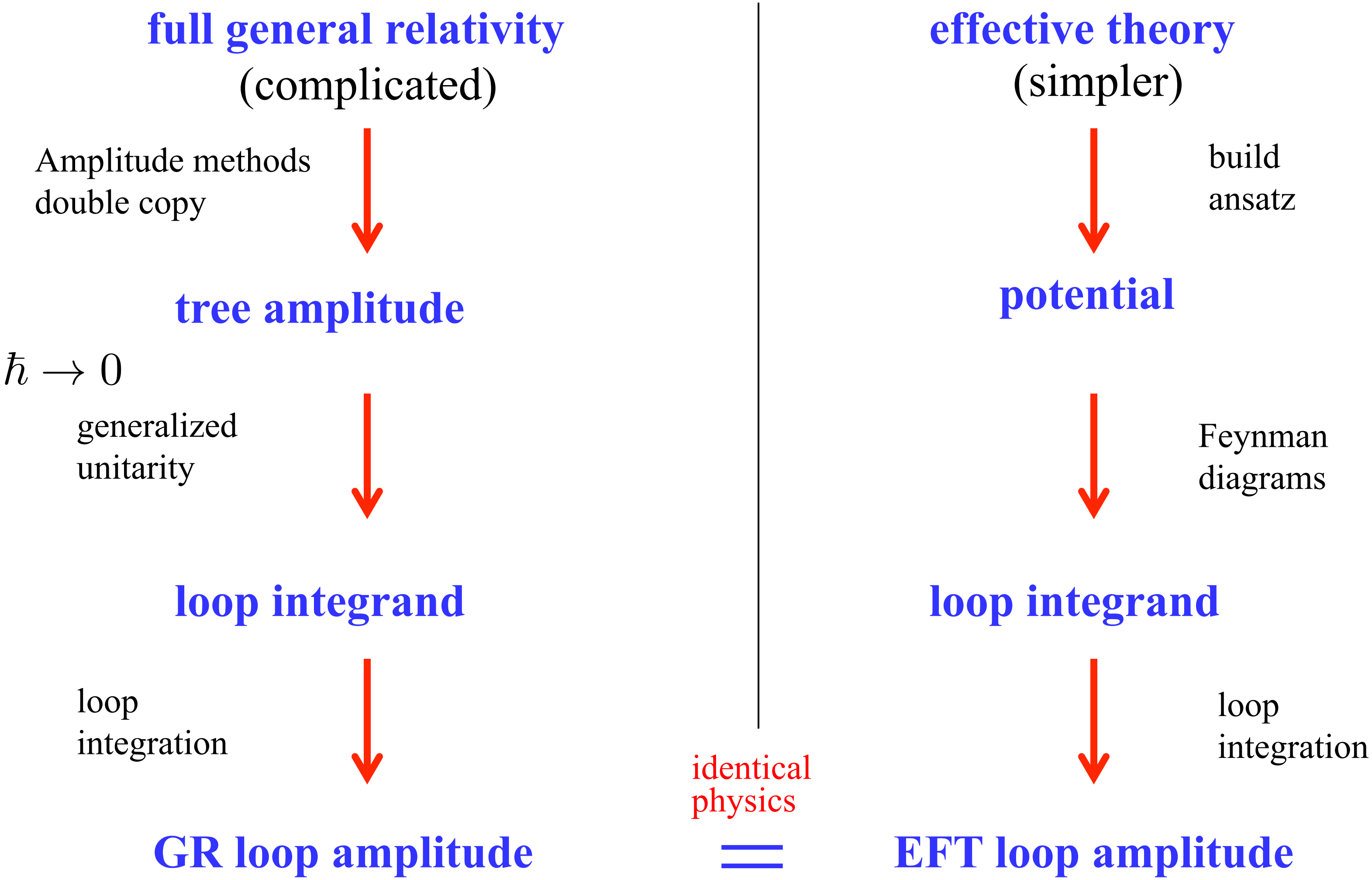
***A*, *B* scalars represents spinless black holes**

$$L_{\text{int}} = -\int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}')\, A^{\dagger}(\boldsymbol{k}')A(\boldsymbol{k})B^{\dagger}(-\boldsymbol{k}')B(-\boldsymbol{k})$$

Match amplitudes of this theory to the full theory in classical limit to extract a potential.

We prefer the EFT approach whenever we need to guarantee correctness.

EFT Matching



full general relativity
(complicated)

Amplitude methods
double copy

tree amplitude

$\hbar \to 0$

generalized
unitarity

loop integrand

loop
integration

GR loop amplitude

effective theory
(simpler)

build
ansatz

potential

Feynman
diagrams

loop integrand

loop
integration

EFT loop amplitude

identical
physics

$=$

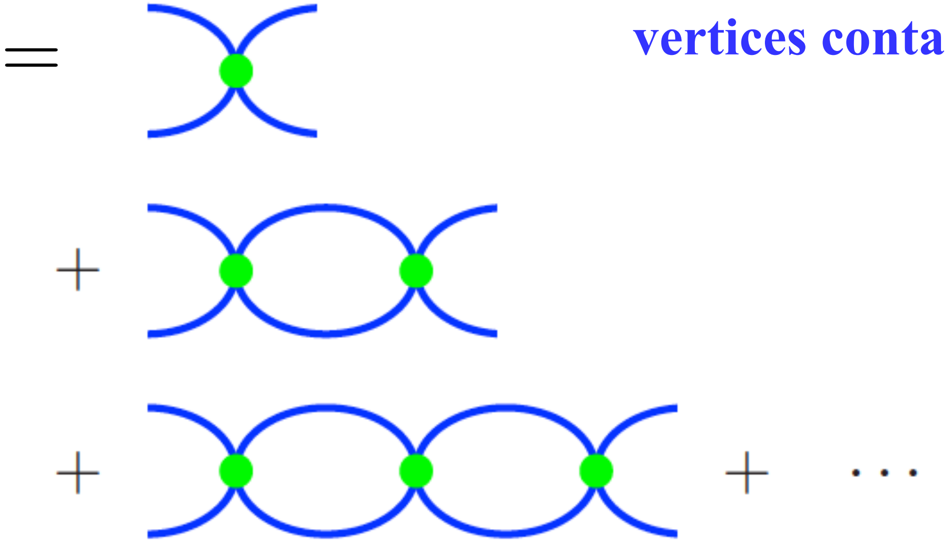
Roundabout but efficiently determines potential

Feynman diagrams for EFT

- **EFT scattering amplitudes easy to compute using Feynman diagrams.**
- **No need for advanced methods.**

Newton's constant

$$A_{\rm EFT} = \sum_{i=1}^{\infty} G^i A^{(i)}_{\rm EFT}$$



vertices contain all powers of *G*

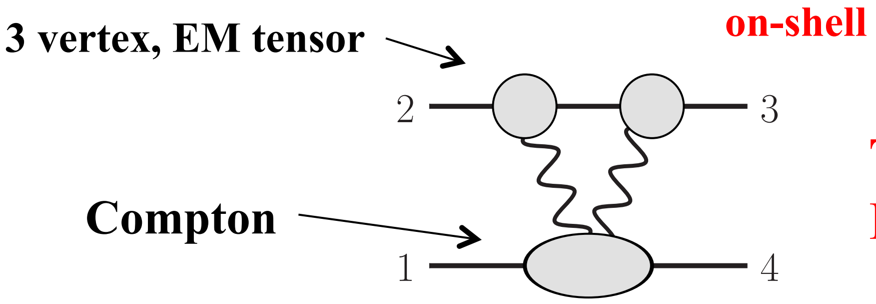
Match to Full Theory

General Relativity: Unitarity + Double Copy

- Long-range force: Two matter lines must be separated by on-shell propagators.
- Classical potential: 1 matter line per loop is cut (on-shell).

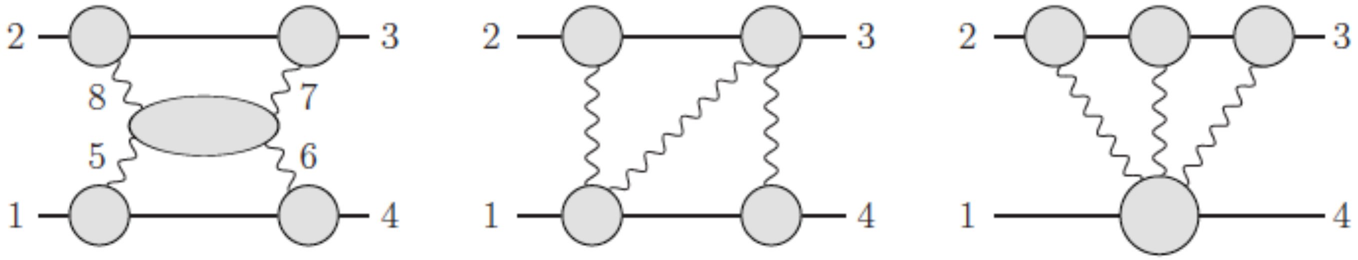
Neill and Rothstein ; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

Only independent unitarity cut for 2 PM 2 body Hamiltonian.



Treat exposed lines on-shell (long range).
Pieces we want are simple!

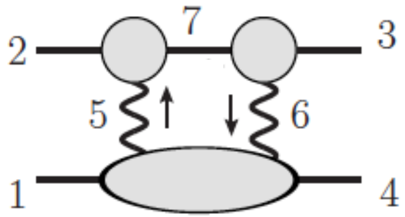
Independent generalized unitarity cuts for 3 PM.



Our amplitude tools fit perfectly with extracting pieces we want.

gravity **loops**

Generalized Unitarity Cuts



2^{nd} post-Minkowkian order

KLT relations

$$\begin{aligned} C_{\text{GR}} &= \sum_{h_5,h_6=\pm} M_3^{\text{tree}}(3^s, 6^{h_6}, -7^s)\, M_3^{\text{tree}}(7^s, -5^{h_5}, 2^s)\, M_4^{\text{tree}}(1^s, 5^{-h_5}, -6^{-h_6}, 4^s) \\ &= \sum_{h_5,h_6=\pm} it[A_3^{\text{tree}}(3^s, 6^{h_6}, -7^s)\, A_3^{\text{tree}}(7^s, -5^{h_5}, 2^s)\, A_4^{\text{tree}}(1^s, 5^{-h_5}, -6^{-h_6}, 4^s)] \\ &\qquad\qquad \times [A_3^{\text{tree}}(3^s, 6^{h_6}, -7^s)\, A_3^{\text{tree}}(7^s, -5^{h_5}, 2^s)\, A_4^{\text{tree}}(4^s, 5^{-h_5}, -6^{-h_6}, 1^s)] \end{aligned}$$

Problem of computing the generalized cuts in gravity is reduced to multiplying and summing gauge-theory tree amplitudes.

$$A_4^{\text{tree}}(1^s, 2^+, 3^+, 4^s) = i\frac{m^2\,[2\,3]}{\langle 2\,3\rangle\,\tau_{12}} \qquad A_4^{\text{tree}}(1^s, 2^+, 3^-, 4^s) = i\frac{\langle 3|\,1\,|2]^2}{s_{23}\tau_{12}}$$

$$\tau_{12} = 2p_1 \cdot p_2$$

$$s_{23} = (p_1 + p_2)^2$$

- **For spinless case, same logic works to all orders: KLT and BCJ work for massless *n*-point in *D*-dimension. Dimensional reduction gives massive case**
- **Unwanted states (dilaton) easy to remove with physical state projectors.**

Amplitude in Conservative Classical Potential Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (BCRSSZ)

To make longer story short. The O(G^3) or 3PM conservative terms are:

$$
\begin{aligned}
\mathcal{M}_3 = &\frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2 \xi}\left[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right. \\
&\left. - \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\right] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi}\left[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)F_1 - 32 m^2 \nu^2 \left(1-2\sigma^2\right)^3 F_2\right]
\end{aligned}
$$

$$
m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,
$$

$$
\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,
$$

- **Amplitude remarkably compact.**
- **Arcsinh and the appearance of a mass singularity is new and robust feature. Cancels mass singularity of real radiation, as expected from KLN theorem.**

  Di Vecchia, Heissenberg, Russo, Veneziano; Damour
- **IR finite parts of amplitude directly connected to scattering angle.**

  Expanded on by Kälin, Porto; Bjerrum-Bohr, Cristofoli, Damgaard
- **Derived conservative scattering angle has simple mass dependence.**

  Observed by Antonelli, Buonanno, Steinhoff, van de Meent, Vines (1901.07102)
  Comprehensive understanding: Damour

Conservative 3PM Hamiltonian

BCRSSZ

The O(G^3) 3PM Hamiltonian:

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2+m_1^2} + \sqrt{\boldsymbol{p}^2+m_2^2} + V(\boldsymbol{p},\boldsymbol{r})$$

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{3} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i,$$

Newton in here

$$c_1 = \frac{\nu^2 m^2}{\gamma^2\xi}\left(1-2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2\xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$c_3 = \frac{\nu^2 m^4}{\gamma^2\xi}\left[\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right.$$
$$- \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2}$$
$$\left.+ \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\right],$$

$$m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$
$$\xi = E_1E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1\cdot p_2/m_1m_2,$$

- **Expanding in velocity gives infinite sequence of terms in PN expansion.**
- **Can be put into EOB form.** Antonelli, Buonannom Steinhoff,van de Meent, Vines

How do we know it is right?

Original check:

Compared to 4PN Hamiltonians after canonical transformation

Damour, Jaranowski, Schäfer; Bernard, Blanchet, Bohé, Faye, Marsat

Thibault Damour seriously questioned correctness.
Specific corrections proposed. Damour, arXiv:1912.02139v1

New calculations confirm our 3PM result:

1. **Recent papers confirm our result in 6PN overlap.** Blümlein, Maier, Marquard, Schäfer; Bini, Damour, Geralico
2. **New calculations reproducing our 3PM result.**
3. **Scattering angle check.** Cheung and Solon; Kälin, Liu, Porto
   ZB, Ita, Parra-Martinez, Ruf
4. **Adding real radiation removes mass singularity.** Di Vecchia, Heissenberg, Russo,Veneziano; Damour

See Damour's, Parra-Martinez, Heissenberg's talk

3PM results have passed highly nontrivial checks and careful scrutiny.

Eikonal-Phase Insight

Eikonal discussed in talks from
Bjerrum-Bohr and Heissenberg

Recall eikonal phase

slowly varying

$$\mathcal{M}(\boldsymbol{q}) \sim \int d^{2-2\epsilon}\boldsymbol{b}\, C(s,\boldsymbol{b},\ldots)e^{i\chi(\boldsymbol{b},s,\ldots)}e^{i\boldsymbol{b}\cdot\boldsymbol{q}}$$

eikonal phase

Dominant part: stationary phase approximation:

$$\Delta p = q = -\nabla_b\chi(b) \qquad 2\sin\frac{\theta}{2} = \frac{\sqrt{s}\,|\nabla_{\boldsymbol{b}_e}\chi(\boldsymbol{b}_e)|}{m_1m_2\sqrt{\sigma^2-1}} \qquad \sigma = \frac{p_1\cdot p_2}{m_1m_2}$$

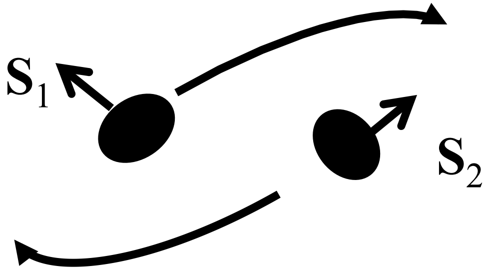
- **Directly extracted from gauge-invariant amplitude.**
- **No need to solve Hamilton's equation.**
- **Angle feeds into EOB.**

Expect there to be a scalar function—the eikonal phase—from which all classical scattering information can be extracted.

May seem like a bold claim, but we can explicitly check.

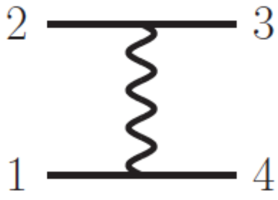
Kerr Black Hole Case

ZB, Luna, Roiban Shen, Zeng, 2005.03071



- **Orbital angular momentum not conserved.**
- **Orbital or motion complicated, not in a plane.**

Is there a notion of eikonal phase for spinning particles?



Yes, from scattering amplitude perspective it should be there:

At O(G):

$$\chi_1 = \frac{\xi E}{|\boldsymbol{p}|}\left[-a_1^{(0)} \ln \boldsymbol{b}^2 - \frac{2a_1^{(1,i)}}{\boldsymbol{b}^2}(\boldsymbol{p} \times \boldsymbol{S}_i) \cdot \boldsymbol{b} \cdot + \cdots\right]$$

coefficients in paper

Impulse and spin kick is indeed given by simple formula:

$$\Delta \boldsymbol{p}_\perp = [\boldsymbol{P}_\perp, i\chi_1] = \nabla_{\boldsymbol{b}_\perp} \chi_1$$

Poisson bracket

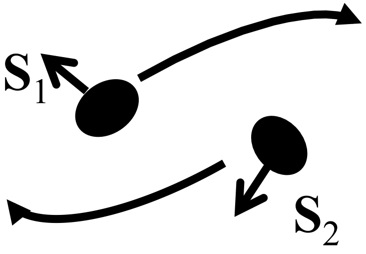
$$\Delta \boldsymbol{S}_i = [\boldsymbol{S}, i\chi_1] = -\epsilon^{ijk} \frac{\partial \chi_1}{\partial S_a^j} S_a^k$$

Here we use rest frame spins;
Maybee, O'Connell, Vines; Guevara, Ochirov, Vines had earlier form with covariant spin.

Kerr Black Hole Case

ZB, Luna, Roiban, Shen, Zeng

Does this continue to higher order?



Conjecture:

$$\boxed{\Delta \mathcal{O} = e^{-i\chi^{\mathcal{D}}}[\mathcal{O}, e^{i\chi^{\mathcal{D}}}]}$$

$$[\boldsymbol{P}_\perp, i\chi_1] = \nabla_{\boldsymbol{b}_\perp}\chi_1 \qquad [\boldsymbol{S}, i\chi_1] = -\epsilon^{ijk}\frac{\partial \chi}{\partial S_a^j}S_a^k \qquad \chi^{\mathcal{D}} g \equiv \chi g + iD_{SL}(\chi, g)$$

$$\mathcal{D}_{SL}(f,g) \equiv -\sum_{a=1,2}\epsilon^{ijk}S_a^k\frac{\partial f}{\partial S_a^i}\frac{\partial g}{\partial L^j}$$

As part of obtaining this formula generated new state of the art O(G^2) S_1 S_2 two-body Hamiltonian.

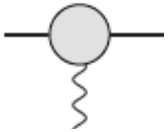
- **Conjecture that the eikonal phase indeed determines physical observables to all orders.**
- **Possible analytic continuations to bound state observables or that it can be imported to EOB.** **See talks from Damour and Porto**
- **Similar formulas might exist for covariant instead of rest-frame spin.**

Guevara, Ochirov, Vines

While still conjectural, example of structures we are seeking.

Nontrivial Double Copy and Spin

ZB, Luna Roiban, Shen, Zeng



Consider generic O(*G*) energy momentum tensor for spin:

$$T^{\mu\nu} = \frac{p_1^\mu p_1^\nu}{m} \sum_{n=0}^{\infty} \frac{C_{ES^{2n}}}{(2n)!} \left(\frac{q\cdot S(p_1)}{m}\right)^{2n} - \frac{i}{m} q_\rho p_1^{(\mu} S(p_1)^{\nu)\rho} \sum_{n=1}^{\infty} \frac{C_{BS^{2n+1}}}{(2n+1)!} \left(\frac{q\cdot S(p_1)}{m}\right)^{2n}$$

Closely related to worldline Lagrangian. Porto, Rothstein; Levi, Steinhoff

For $C_{ES^{2n}} = 1, \ C_{BS^{2n}} = 1$ **matches Vines' Kerr black hole tensor**

Does this have a double-copy construction? Yes!

On-shell, can factorize the energy momentum tensor:

$$T^{\mu\nu} = -i\varepsilon(s,p_2)V_{3,\mathrm{GR}}^{\mu\nu}\varepsilon(s,p_1)$$
$$= [-i\varepsilon(s_L,p_2)V_3^{\mu}\varepsilon(s_L,p_1)][-i\varepsilon(s_R,p_2)V_3^{\nu}\varepsilon(s_R,p_1)]$$

gauge theory

$$\varepsilon(s,p) = \varepsilon(s_L,p)\otimes\varepsilon(s_R,p)$$

At O(*G*) energy momentum tensor factorizes to all orders in spin! Only works on shell.

More Double Copy

Consider higher-spin electrodynamics

ZB, Luna. Roiban, Shen, Zeng

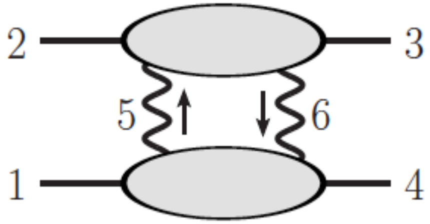
$$\mathcal{L}_{s,\mathrm{EM}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + D^{\dagger}_{\mu}\bar{\phi}_s D^{\mu}\phi_s - m^2\bar{\phi}_s\phi_s + e(g-1)F_{\mu\nu}\bar{\phi}_s M^{\mu\nu}\phi_s$$

Compare to minimally coupled gravity:

$$\mathcal{L}_{\mathrm{min}} = -R(e,\omega) + \frac{1}{2}g^{\mu\nu}\nabla(\omega)_\mu\phi_s\nabla(\omega)_\nu\phi_s - \frac{1}{2}m^2\phi_s\phi_s$$

$$\nabla(\omega)_\mu\phi_s \equiv \partial_\mu\phi_s + \frac{i}{2}\omega_{\mu ef}M^{ef}\phi_s$$

Two theories seem pretty different.



However, simple KLT-like formula holds for Compton amplitudes used for constructing O(*G*²) S₁ S₂ potentials.

$$i\mathcal{M}(1^s,2^s,3^h,4^h) = -4\pi i G\frac{p_1\cdot p_3\, p_1\cdot p_4}{p_3\cdot p_4}A(1^0,2^0,3^A,4^A)\,A(1^s,2^s,3^A,4^A)$$

electrodynamics special to 4 points

Used in extraction of two body S_1 S_2 Hamiltonian at O(G^2)

Remarkable idea: gauge theory likely encodes all information for spin interaction in General Relativity to all orders in *G* and in spin.

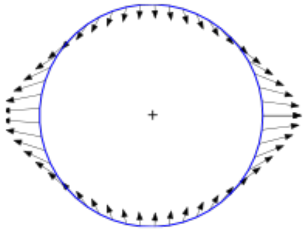
Our paper demonstrates this through O(G^2) and S_1 S_2

Field Theory Approach to Tidal Effects

ZB, Luna, Roiban, Shen, Zeng (2 weeks ago)

Large literature on tidal effects. Worldline is popular.

Flanagan, Hinderer; Damour, Nagar; Carney, Wade and Irwin; Bini and Damour; Steinhoff, Hinderer, Buonanno,Taracchini; Henry, Faye, Blanchet; Goldberger, Rothstein; Bini, Damour, Geralico; etc



analogous to EM susceptibililty

We use a field theory approach to tidal effects.

Cheung, Solon; Haddad and Helset

Systematically add higher-derivative operators corresponding to tidal effects

$$S^{\rm QFT}_{\rm tidal}\Big|_{\rm linear} = m\int d^4x\sqrt{-g}\ \sum_{n=2}^{\infty}\sum_{l=0}^{\infty}(\mu^{(n,l)}\,\phi\hat{E}^{(l)}_{\mu_1\cdots\mu_n}\hat{E}^{(l)\,\mu_1\cdots\mu_n}\phi+\sigma^{(n,l)}\,\phi\hat{B}^{(l)}_{\mu_1\cdots\mu_n}\hat{B}^{(l)\,\mu_1\cdots\mu_n}\phi)$$

Electric and magnetic combinations of Weyl

$$S^{\rm QFT}_{\rm tidal}\Big|_{\rm non\text{-}linear} = m\int d^4x\sqrt{-g}\ \sum_{n=2}^{\infty}(\rho_e^{(n)}\,\phi\hat{E}_{\mu_1}{}^{\mu_2}\hat{E}_{\mu_2}{}^{\mu_3}\cdots\hat{E}_{\mu_n}{}^{\mu_1}\phi+\rho_m^{(n)}\,\phi\hat{B}_{\mu_1}{}^{\mu_2}\hat{B}_{\mu_2}{}^{\mu_3}\cdots\hat{B}_{\mu_n}{}^{\mu_1}\phi)+\cdots$$

Weyl tensor

$$\hat{P}^{\nu}_{\mu} = \frac{1}{m^2}(\partial_\mu\partial^\nu-\delta^\nu_\mu\partial^2)$$

$$\hat{E}^{(l)}_{\mu_1\mu_2\ldots\mu_n} = \frac{i^{m+2}}{m^{m+2}}\mathrm{Sym}_{\mu_1\ldots\mu_n}[\nabla_{\nu_n}\ldots\nabla_{\nu_3}\nabla^{\rho_1}\ldots\nabla^{\rho_l}C_{\mu_1\alpha\mu_2\beta}\hat{P}^{\nu_n}_{\mu_n}\ldots\hat{P}^{\nu_3}_{\mu_3}\nabla_{(\rho_1}\ldots\nabla_{\rho_l)}\nabla^\alpha\nabla^\beta\ ]$$

Similar for magnetic type

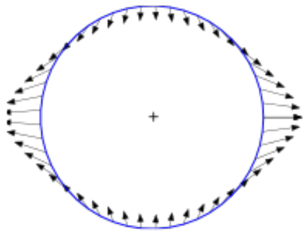
By design we match standard worldline tidal operators at leading order in *G*

$$\int d\tau E^{(l)}_{\mu_1\ldots\mu_n}E^{(l)\mu_1\ldots\mu_n}\longleftrightarrow m_i\int d^4x\sqrt{-g}\phi_i\hat{E}^{(l)}_{\mu_1\ldots\mu_n}\hat{E}^{(l)\mu_1\ldots\mu_n}\phi_i$$

$$\int d\tau B^{(l)}_{\mu_1\ldots\mu_n}B^{(l)\mu_1\ldots\mu_n}\longleftrightarrow m_i\int d^4x\sqrt{-g}\phi_i\hat{B}^{(l)}_{\mu_1\ldots\mu_n}\hat{B}^{(l)\mu_1\ldots\mu_n}\phi_i$$

Structures in Tidal Effects

ZB, Parra-Martinez, Roiban, Sawyer, Shen



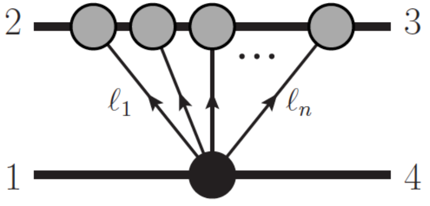
Leading order effects from tidal operators relatively simple.

What structures can we uncover here?

Bini, Damor, Geralico

$$V(\boldsymbol{p},\boldsymbol{r}) \sim c_{i,k}(\boldsymbol{p})\, m \left(\frac{Gm}{|\boldsymbol{r}|}\right)^i \left(\frac{R}{|\boldsymbol{r}|}\right)^k$$

$$\ell_i \sim \hbar$$



$\phi\mathcal{O}\phi$ **tidal operator**

Leading order contributions from each operator simple

- **At leading order easy to align operator basis with worldline.**
- **Leading in ℓ_i is always classical.**
- **Keep only leading in ℓ_i**
  - **— No expansion of operator.**
  - **— No expansion of energy momentum tensor. All are identical!**
- **Double copy is very simple. Weyl tensor is product of two gauge theory $F_{\mu\nu}$**
- **No iteration, EFT matching is trivial.**

Related recent papers: Haddad, Helset; Cheung, Shah, Solon; Mougiakakos, Vanhove

Structures in Tidal Effects

ZB, Parra-Martinez, Roiban, Sawyer, Shen

***E* and *B* tensors have no more than rank 3. Also parity.**

Simple relations between operators

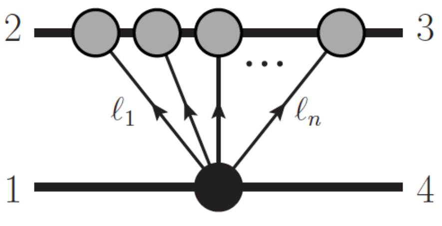
Similar results on infinite sequences of tidal operators:

Cheung, Shah, Solon.

$$(E^n) = n \sum_{2p+3q=n} \frac{1}{2^p 3^q} \frac{\Gamma(p+q)}{\Gamma(p+1)\Gamma(q+1)} (E^2)^p\, (E^3)^q$$

$$(B^{2k}) = \frac{1}{2^{k-1}} (B^2)^k, \qquad (\mathcal{O}) \equiv \mathrm{Tr}[\mathcal{O}]$$

$$\mathcal{O} = \{E^2, B^2, E^3\} \qquad \mathcal{N}_{(\mathrm{E}^2)^n} = \mathcal{N}_{(\mathrm{B}^2)^n} = 2^{2n+2} G^{2n} \pi^{2n} m_1 m_2^{2n+1}$$



$\phi \mathcal{O}^n \phi$

$$\widetilde{\mathcal{M}}_{(\mathcal{O})^n} = \mathcal{N}_{(\mathcal{O})^n} \left[\frac{1}{|\boldsymbol{r}|^h} \left(a_{(\mathcal{O})} + b_{(\mathcal{O})} \frac{(\boldsymbol{r}\cdot\boldsymbol{u}_1)^2}{\boldsymbol{r}^2} + c_{(\mathcal{O})} \frac{(\boldsymbol{r}\cdot\boldsymbol{u}_1)^4}{\boldsymbol{r}^4} \right) \right]^n$$

position space factorization

$$E^2\!: \quad a_{(\mathrm{E}^2)} = \frac{3(1-3\sigma^2+3\sigma^4)}{2\pi^2}, \qquad b_{(\mathrm{E}^2)} = \frac{9(1-2\sigma^2)}{2\pi^2}, \qquad c_{(\mathrm{E}^2)} = \frac{9}{2\pi^2}, \qquad h = 6$$

$$B^2\!: \quad a_{(\mathrm{B}^2)} = \frac{9\sigma^2(\sigma^2-1)}{2\pi^2}, \qquad b_{(\mathrm{B}^2)} = \frac{9(1-2\sigma^2)}{2\pi^2}, \qquad c_{(\mathrm{B}^2)} = \frac{9}{2\pi^2}, \qquad h = 6$$

$$E^3\!: \quad a_{(\mathrm{E}^3)} = -\frac{3(2-9\sigma^2+9\sigma^4)}{8\pi^3}, \qquad b_{(\mathrm{E}^3)} = -\frac{27(1-2\sigma^2)}{8\pi^3}, \qquad c_{(\mathrm{E}^3)} = -\frac{27}{8\pi^3}. \qquad h = 9$$

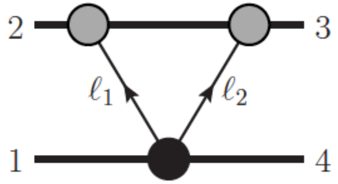
$$\boxed{V_{(\mathcal{O})^n}(\boldsymbol{p},\boldsymbol{r}) = -\frac{\mathcal{N}_{(\mathcal{O})^n}}{4E_1E_2\,|\boldsymbol{r}|^{nh}} \sum_{k=0}^{n}\sum_{l=0}^{k} \binom{n}{k}\binom{k}{l} a_{\mathcal{O}}^{n-k}\, b_{\mathcal{O}}^{l}\, c_{\mathcal{O}}^{k-l}\, (\sigma^2-1)^{2k-l} \frac{\Gamma(\frac{1}{2}+2k-l)\Gamma(\frac{1}{2}hn)}{\sqrt{\pi}\,\Gamma(2k-l+\frac{1}{2}hn)}}$$

Binomial expansion gives exact formulas for potential for any *n*

Obtained leading terms in 2-body potential for infinite sequences of tidal operators

Additional Comments

ZB, Parra-Martinez, Roiban, Sawyer, Shen



- **Evaluated all D^nE^2 and D^nB^2 tidal operators.** **All one loop integrals can be evaluated.** Haddad and Helset

- **Example of interaction between spin and D^nE^2 tidal worked out.**

- **Generating higher G corrections well understood in momentum space, though integrals can be nontrivial.**

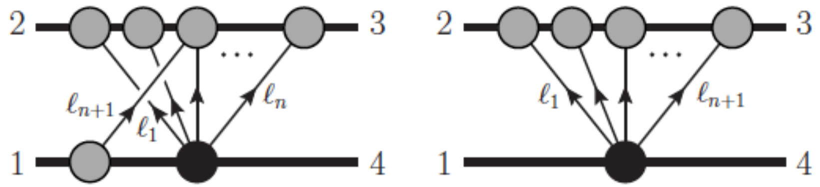
  Cheung and Solon; Porto

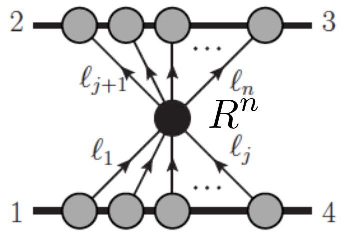
- **Higher-dimension modifications of GR can also be done with same methods. Beyond simplest cases, UV divergences and renorm of tidal.**

  Brandhuber, Travaglini; Edmond and Moynahan; Cristofoli

- **Test mass case can be done exactly, compact formulas. Also nice results on all orders in Schwarzschild background. Used geodesics.**

  Cheung, Shah and Solon





Outlook

Amplitude methods have a lot of promise and their use has already been tested for a variety of problems.

- **State of the art for higher orders.** ZB, Cheung, Roiban, Shen, Solon, Zeng
- **Radiation.** Cristofoli, Gonzo, Kosower, O'Connell; Parra-Martinez, Herrmann, Ruf, Zeng
- **Finite size effects.** Cheung and Solon; Haddad and Helset; Kälin, Liu, Porto; Cheung, Shah, Solon; ZB, Parra-Martinez, Roiban, Sawyer, Shen.
- **Spin.** Vaidya; Geuvara, O'Connell, Vines; Chung, Huang, Kim, Lee; ZB, Luna, Roiban, Shen, Zeng; etc

See talks from Kosower and Parra-Martinez.

Much of the future progress will be driven by our ability to evaluate integrals.

Our current preferred methods are IBP and differential equations.
Only single-scale integrals to all orders in PM expansion.

Parra-Martinez, Ruf and Zeng

Summary

- Amplitudes provide a new and useful way to think about problems of direct interest to gravitational-wave community.
- Amplitudes are independent of gauges, coordinates and field variables, making it simpler to identify useful new structures.
- Examples described: pushed state of the art for $O(G^3)$, eikonal encoding of spin observables, explicit results for infinite sequences of tidal operators. Methods work well for a variety of topics.

In the coming years we can expect new advances, not only in gravitational-wave physics, but also in understanding gravity and its relation to gauge theory.