Gravity from Amplitudes

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SAGEX

Scattering Amplitudes: from Geometry to Experiment

Niels Bohr Institutet

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Outline

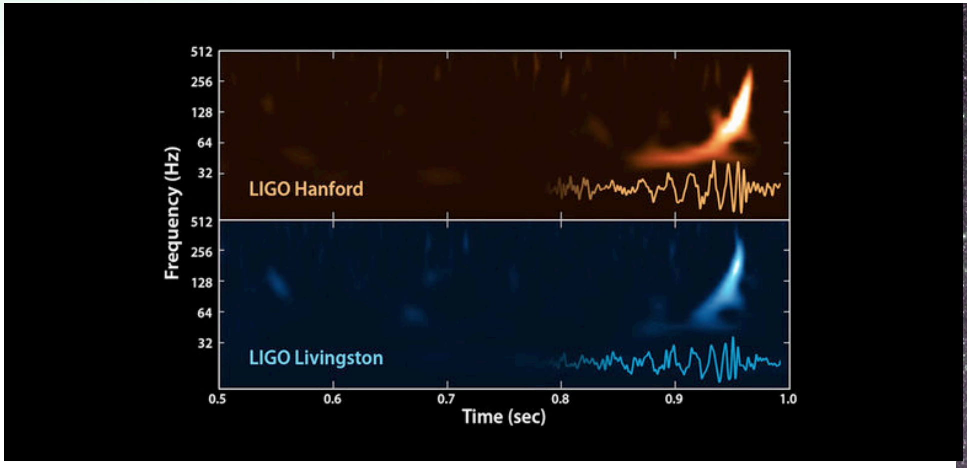
Will discuss the framework of computations of effects of gravity treating low energy quantum general relativity as a perturbative effective field theory (see John's talk)

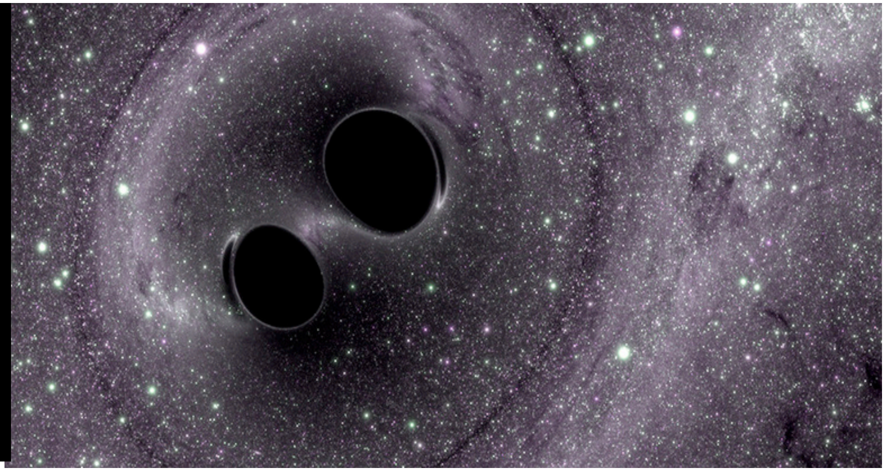
Discuss applications of such formalism for computation of observables in classical gravity and for effects of perturbative quantum gravity.

Discuss modern amplitudes techniques and on-shell toolboxes for computations
(see also David's, Zvi's, and Julio's talks)

Outlook

Exciting motivations to study gravity





- First direct observation of a binary merger of black holes -> access to gravitational interactions in extreme regimes!
  - Tasks: Supplement conventional analysis

    Match observational precision

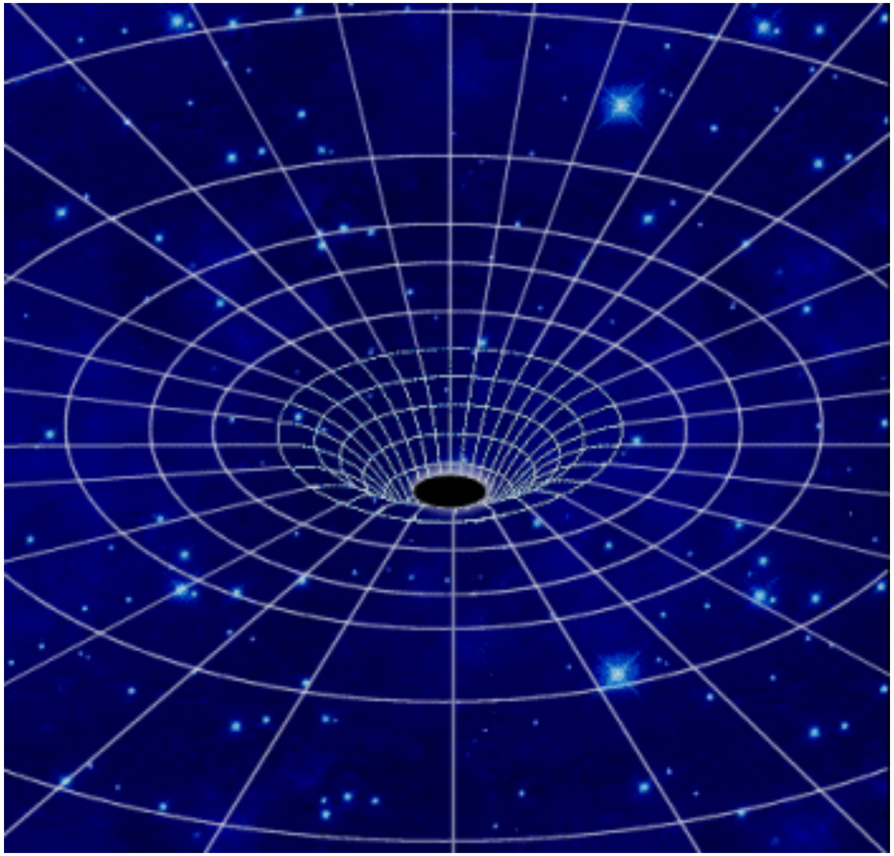
    Develop gravity phenomenology

    Hope for new efficiency from amplitude methods

Fundamental paradox

- Einstein's theory presents us with a beautiful theory for gravity
- However geometrical description that does not fit well with notion of quantum mechanics
- Quantum mechanical extension of General Relativity?

A fundamental question of theoretical physics!



A new hope?

Perturbative quantization of gravity

- Known since the 1960ties that a particle version of General Relativity can be derived from the Einstein Hilbert Lagrangian (Feynman, DeWitt)
- Expand Einstein-Hilbert Lagrangian :

$$\mathcal{L}_{\mathrm{E}H} = \int d^4x \left[\sqrt{-g}R\right] \qquad g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

- Derive vertices as in a particle theory - compute amplitudes as Feynman diagrams!

Quantum theory for gravity

- Gravity as a theory with self-interactions

- Non-renormalisable theory! ('t Hooft and Veltman)

Dimensionful coupling:

$G_N = 1/M^2_{planck}$

- Traditional belief : – no known symmetry can remove all UV-divergences

Except string theory..

Quantum gravity as an effective field theory

- (Weinberg) proposed to view the quantization of general relativity from the viewpoint of effective field theory.

$$\mathcal{L} = \sqrt{-g}\left[\frac{2R}{\kappa^2} + \mathcal{L}_{\text{matter}}\right]$$

$$\Downarrow$$

$$\mathcal{L} = \sqrt{-g}\left\{\frac{2R}{\kappa^2} + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \ldots\right\}$$

- (Donoghue) and (NEJBB, Donoghue, Holstein) did the first one-loop concrete computation in such a framework

Off-shell gravity amplitudes

- Vertices: 3pt, 4pt, 5pt,..n-pt
- Complicated expressions
- Expand Lagrangian, tedious process....

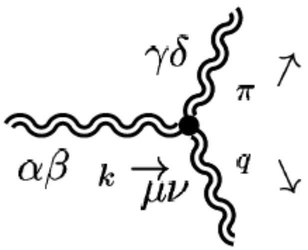
$$\begin{aligned} V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\,\mathrm{sym}\Big[& -\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\ & + \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \\ & - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\ & + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big], \end{aligned}$$

45 terms + sym

(DeWitt;Sannan)

Off-shell gravity amplitudes

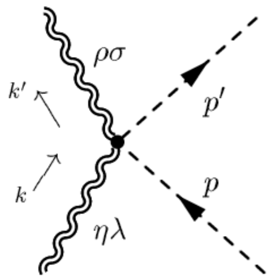
(Donoghue) and (NEJBB, Donoghue, Holstein)



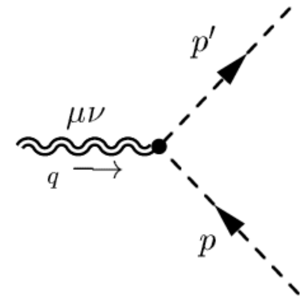
$$
\begin{aligned}
\tau_{10}{}^{\mu\nu}_{\alpha\beta\gamma\delta}(k,q) = -\frac{i\kappa}{2} \times \Bigg(& \mathcal{P}_{\alpha\beta\gamma\delta}\Big[k^\mu k^\nu + (k-q)^\mu (k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu} q^2\Big] \\
& + 2q_\lambda q_\sigma \Big[I_{\alpha\beta}{}^{\sigma\lambda} I_{\gamma\delta}{}^{\mu\nu} + I_{\gamma\delta}{}^{\sigma\lambda} I_{\alpha\beta}{}^{\mu\nu} - I_{\alpha\beta}{}^{\mu\sigma} I_{\gamma\delta}{}^{\nu\lambda} - I_{\gamma\delta}{}^{\mu\sigma} I_{\alpha\beta}{}^{\nu\lambda}\Big] \\
& + \Big[q_\lambda q^\mu \Big(\eta_{\alpha\beta} I_{\gamma\delta}{}^{\nu\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}{}^{\nu\lambda}\Big) + q_\lambda q^\nu \left(\eta_{\alpha\beta} I_{\gamma\delta}{}^{\mu\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}{}^{\mu\lambda}\right) \\
& - q^2 \left(\eta_{\alpha\beta} I_{\gamma\delta}{}^{\mu\nu} - \eta_{\gamma\delta} I_{\alpha\beta}{}^{\mu\nu}\right) - \eta^{\mu\nu} q_\sigma q_\lambda \left(\eta_{\alpha\beta} I_{\gamma\delta}{}^{\sigma\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}{}^{\sigma\lambda}\right)\Big] \\
& + \Big[2q_\lambda \big(I_{\alpha\beta}{}^{\lambda\sigma} I_{\gamma\delta\sigma}{}^{\nu} (k-q)^\mu + I_{\alpha\beta}{}^{\lambda\sigma} I_{\gamma\delta\sigma}{}^{\mu} (k-q)^\nu + I_{\gamma\delta}{}^{\lambda\sigma} I_{\alpha\beta\sigma}{}^{\nu} k^\mu + I_{\gamma\delta}{}^{\lambda\sigma} I_{\alpha\beta\sigma}{}^{\mu} k^\nu\big) \\
& + q^2 \left(I_{\alpha\beta\sigma}{}^{\mu} I_{\gamma\delta}{}^{\nu\sigma} + I_{\alpha\beta}{}^{\nu\sigma} I_{\gamma\delta\sigma}{}^{\mu}\right) + \eta^{\mu\nu} q_\sigma q_\lambda \left(I_{\alpha\beta}{}^{\lambda\rho} I_{\gamma\delta\rho}{}^{\sigma} + I_{\gamma\delta}{}^{\lambda\rho} I_{\alpha\beta\rho}{}^{\sigma}\right)\Big] \\
& + \Big\{(k^2 + (k-q)^2)\left[I_{\alpha\beta}{}^{\mu\sigma} I_{\gamma\delta\sigma}{}^{\nu} + I_{\gamma\delta}{}^{\mu\sigma} I_{\alpha\beta\sigma}{}^{\nu} - \frac{1}{2}\eta^{\mu\nu}\mathcal{P}_{\alpha\beta\gamma\delta}\right] \\
& - \left(I_{\gamma\delta}{}^{\mu\nu} \eta_{\alpha\beta} k^2 + I_{\alpha\beta}{}^{\mu\nu} \eta_{\gamma\delta} (k-q)^2\right)\Big\}\Bigg).
\end{aligned}
$$

Off-shell gravity amplitudes

2-scalar-1-graviton

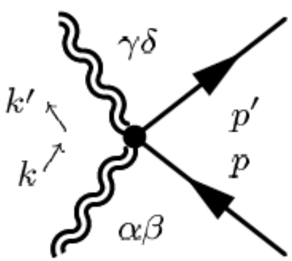


2-scalar-2-graviton



$$V_{\eta\lambda\rho\sigma}(p,p') = \frac{i\kappa^2}{2}\Bigg[\Big\{I_{\eta\lambda\alpha\delta}I_{\rho\sigma\beta}{}^{\delta} - \frac{1}{4}\{\eta_{\eta\lambda}I_{\rho\sigma\alpha\beta}+\eta_{\rho\sigma}I_{\eta\lambda\alpha\beta}\}\Big\}\left(p^{\alpha}p'^{\beta} + p'^{\alpha}p^{\beta}\right) - \frac{1}{2}\left\{I_{\eta\lambda\rho\sigma} - \frac{1}{2}\eta_{\eta\lambda}\eta_{\rho\sigma}\right\}[(p\cdot p') - m^2]\Bigg]$$

$$\tau_{\mu\nu}(p,p') = -\frac{i\kappa}{2}\left[p_\mu p'_\nu + p_\nu p'_\mu - \eta_{\mu\nu}\{(p\cdot p') - m^2\}\right]$$

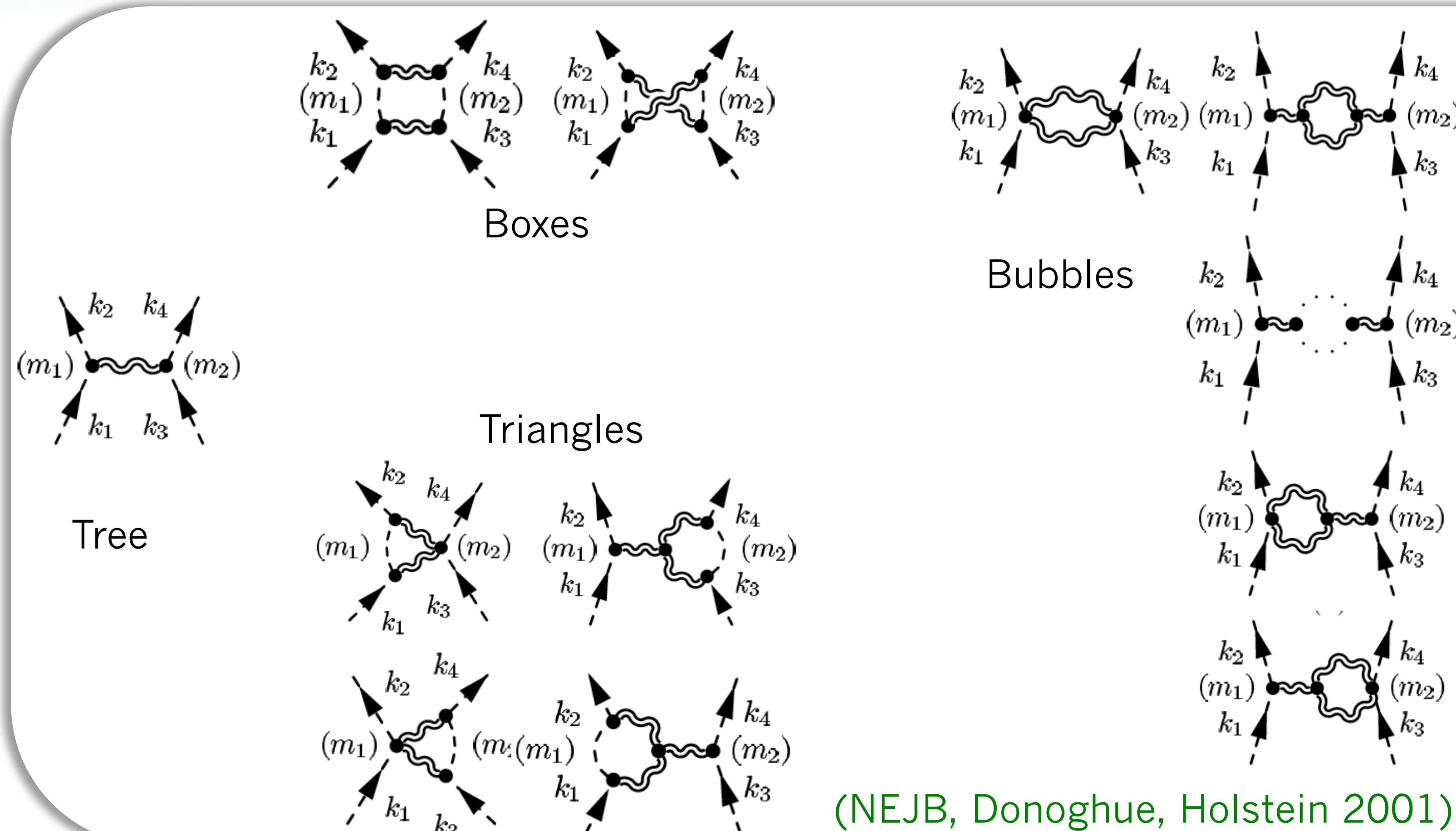


$$= i\kappa^2\Bigg\{-\frac{1}{2}\Big(\frac{1}{2}(\not{p}+\not{p}') - m\Big)\mathcal{P}^{\alpha\beta,\gamma\delta} - \frac{1}{16}[\eta^{\alpha\beta}(\gamma^{\gamma}(p+p')^{\delta} + \gamma^{\delta}(p+p')^{\gamma})$$
$$+\ \eta^{\gamma\delta}(\gamma^{\alpha}(p+p')^{\beta} + \gamma^{\beta}(p+p')^{\alpha})] + \frac{3}{16}(p+p')_{\epsilon}\gamma^{\xi}(I^{\xi\phi,\alpha\beta}I_{\phi}{}^{\epsilon,\gamma\delta} + I^{\xi\phi,\gamma\delta}I_{\phi}{}^{\epsilon,\alpha\beta})$$
$$+\ \frac{i}{16}\epsilon^{\rho\sigma\eta\lambda}\gamma_{\lambda}\gamma_{5}(I^{\alpha\beta}{}_{,\eta\nu}I^{\gamma\delta}{}_{,\sigma\nu}k'_{\rho} - I^{\gamma\delta}{}_{,\eta\nu}I^{\alpha\beta}{}_{,\sigma\nu}k_{\rho})\Bigg\}.$$

vierbein

2-fermion-2-graviton

One-loop (off-shell) gravity computation



(NEJB, Donoghue, Holstein 2001)

One-loop result for gravity

- Four point one-loop amplitude can be deduced to take the form

$$\mathcal{M} \sim \left(A + Bq^2 + \ldots + \alpha\kappa^4 \frac{1}{q^2} + \beta_1 \kappa^4 \ln(-q^2) + \beta_2 \kappa^4 \frac{m}{\sqrt{-q^2}} + \ldots\right)$$

Focus on deriving these ~>

Long-range behavior

(no higher derivative contributions)

Short range behavior

Important differences compared to QCD computations

One-loop result for gravity

- The result for the amplitude (in coordinate space) after summing all diagrams is:

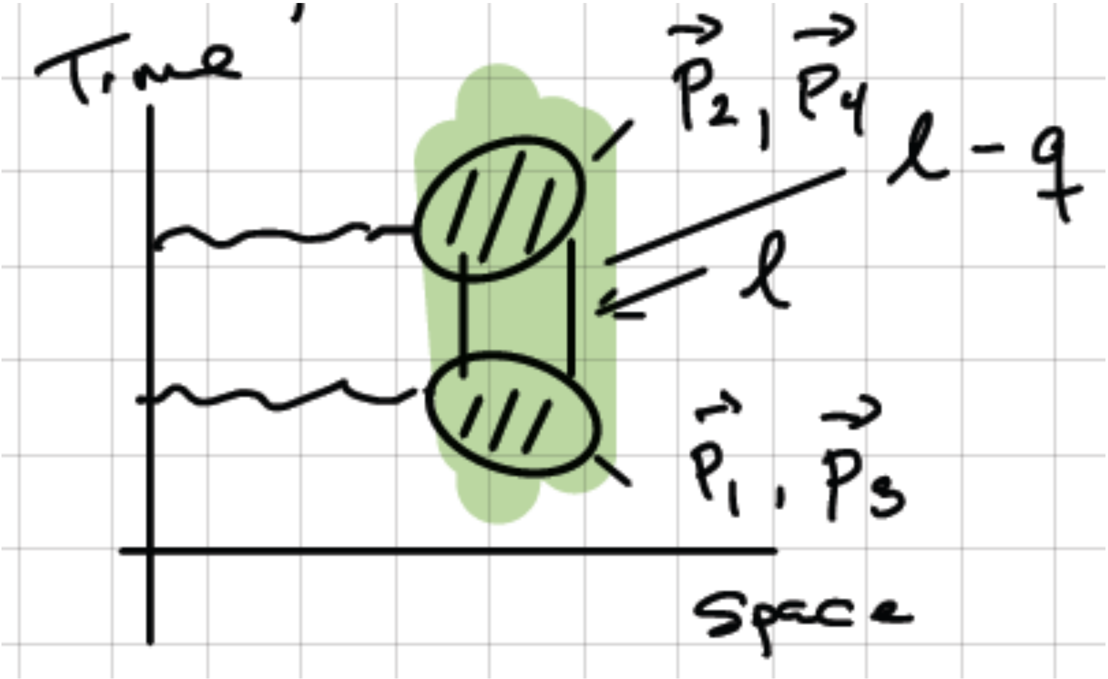
(NEJB, Donoghue, Holstein)

$$-\frac{Gm_1m_2}{r}\left[1+3\underbrace{\frac{G(m_1+m_2)}{r}}_{\text{Post-Newtonian term}}+\underbrace{\frac{41}{10\pi}\frac{G\hbar}{r^2}}_{\text{New quantum term}}\right]$$

Post-Newtonian term in complete accordance with general relativity*: (Iwasaki; Holstein and Ross; Neill and Rothstein, NEJB, Damgaard, Festuccia, Plante, Vanhove)

Born subtraction

- Born subtraction important to make contact with Post-Newtonian limit.



Einstein-Infeld-Hoffman Potential

- *In order to see this. Solve for potential in non-relativistic limit, (Born subtraction)

$$
\begin{aligned}
i\langle f|T|i\rangle &= -2\pi i\delta(E-E') \\
&\times \left[\langle f|\tilde{V}_{bs}(\mathbf{q})|i\rangle + \sum_n \frac{\langle f|\tilde{V}_{bs}(\mathbf{q})|n\rangle\langle n|\tilde{V}_{bs}(\mathbf{q})|i\rangle}{E-E_n+i\epsilon} + \ldots\right]
\end{aligned}
$$

$$
\langle f|\tilde{V}_{bs}(\mathbf{q})|i\rangle = -\frac{Gm_1m_2}{r}\left[1+3\frac{G(m_1+m_2)}{r}\right]
$$

- Contact with Einstein-Infeld-Hoffmann Hamiltonian

$$
\tilde{V}_{bs}(r) = V(r) + \frac{7Gm_1m_2(m_1+m_2)}{2c^2r^2}
$$

Post-Newtonian interaction potentials

$$H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_4^2}{2m_2} - \frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_4^4}{8m_2^3} - \frac{Gm_1m_2}{r} - \frac{G^2m_1m_2(m_1+m_2)}{2r^2} - \frac{Gm_1m_2}{2r}\left(\frac{3\vec{p}_1^2}{m_1^2} + \frac{3\vec{p}_4^2}{m_2^2} - \frac{7\vec{p}_1\cdot\vec{p}_4}{m_1m_2} - \frac{(\vec{p}_1\cdot\vec{r})(\vec{p}_4\cdot\vec{r})}{m_1m_2r^2}\right)$$

(Einstein-Infeld-Hoffman, Iwasaki)

Crucial subtraction of Born term to in order to get the correct PN potential

(3 – 7/2 -> -1/2)

Laboratory for quantum gravity

Consistent quantization

- Working low energy version of quantum gravity
- Possible to derive exact low energy corrections and start thinking about concrete phenomenology in quantum gravity.

(see John's talk)

Important classical application:

- General relativity: hbar-> 0 limit of path-integral
  - "Surprising" feature: Classical physics from loop diagrams!

(Iwasaki; Donoghue, Holstein; Kosower, Maybee, O'Connell)

Explanation: contributions appear in loop diagrams feature a cancellation of the loop diagram hbar factor

(mass/hbar) expansion.

Gravity as an EFT

- General relativity augmented by higher derivative operators –most general modified theory
  - Tiny consequences for most observables – since curvature is really small. Interesting connection between observed bounds and theory
- Quantum theory -> classical limit general relativity
  - Hamiltonians for gravitational systems and post-Newtonian and post-Minkowskian observables
- Unique quantum effects
  - Measurable consequences? Interpretation?

Spin

- Easy to incorporate effects of fundamental spin in amplitude computations. E.g. vierbein formalism.
- Spin-1/2 and spin-1 post-Newtonian computations similar to the scalar case (Holstein and Ross)
  - Confirms universality of (NEJB, Donoghue, Holstein)
- Similar off-shell constructions of metric was done for spin-0, spin-1/2 confirming universality and leading order classical metric + new quantum effects. (NEJB, Donoghue, Holstein)
- Generic classical "infinite" spin effects still complicated beyond one-loop..

Feynman diagrams in gravity theories

Number of unpleasant features

- Complicated
  - infinitely many vertices..
- Numerous double (mixed) contractions
- Factorial growth in number of legs (like Yang-Mills)
- Loop diagrams: no ordering / no planarity!
- Loop integrations: tensor integrals / difficult to find a good basis..

String theory

String theory given us lots of ideas..

Fact: Using (weak) string theory as a way to learn more about field theory is extremely useful..

A RELATION BETWEEN TREE AMPLITUDES OF CLOSED AND OPEN STRINGS*

H. KAWAI, D.C. LEWELLEN and S.-H.H. TYE

Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853, USA

Received 11 October 1985

We derive a formula which expresses any closed string tree amplitude in terms of a sum of the products of appropriate open string tree amplitudes. This formula is applicable to the heterotic string as well as to the closed bosonic string and type II superstrings. In particular, we demonstrate its use by showing how to write down, without any direct calculation, all four-point heterotic string tree amplitudes with massless external particles.

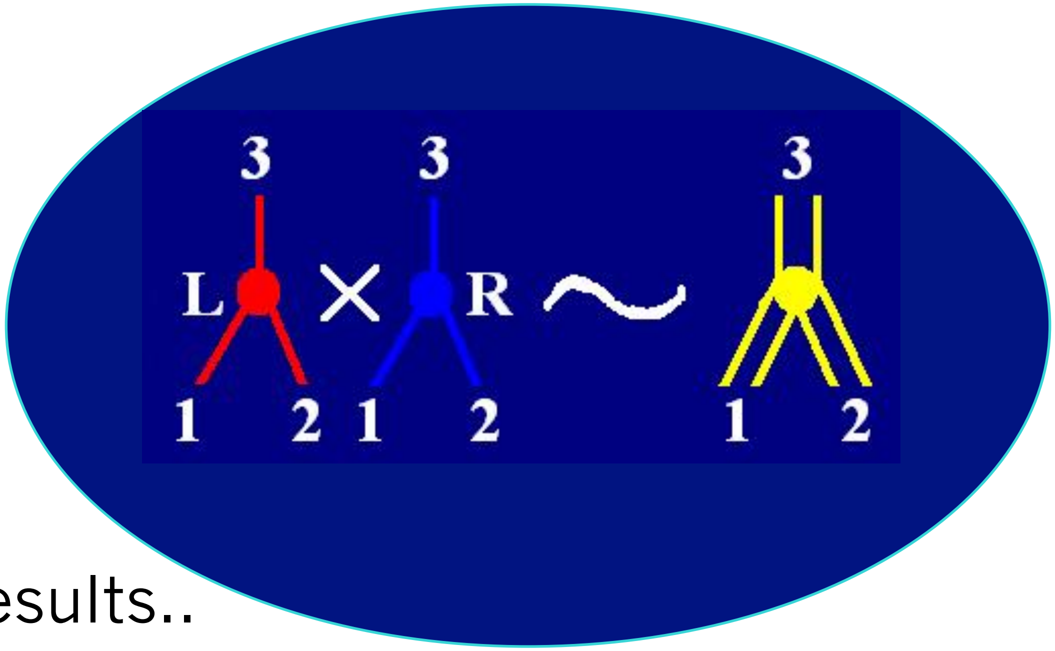
...a more efficient way

Squaring relation for gravity

Gravity from (Yang-Mills)^2 (Kawai, Lewellen, Tye)

Natural from the decomposition of closed strings into open.

Gives a smart way to recycle Yang-Mills results into gravity results.. (Bern et. al.)



Gravity Amplitudes

KLT explicit representation:

$$M_3^{\rm tree}(1,2,3) = -i A_3^{\rm tree}(1,2,3) A_3^{\rm tree}(1,2,3)$$
$$M_4^{\rm tree}(1,2,3,4) = -i s_{12} A_4^{\rm tree}(1,2,3,4) A_4^{\rm tree}(1,2,4,3)$$

$\alpha' \to 0$

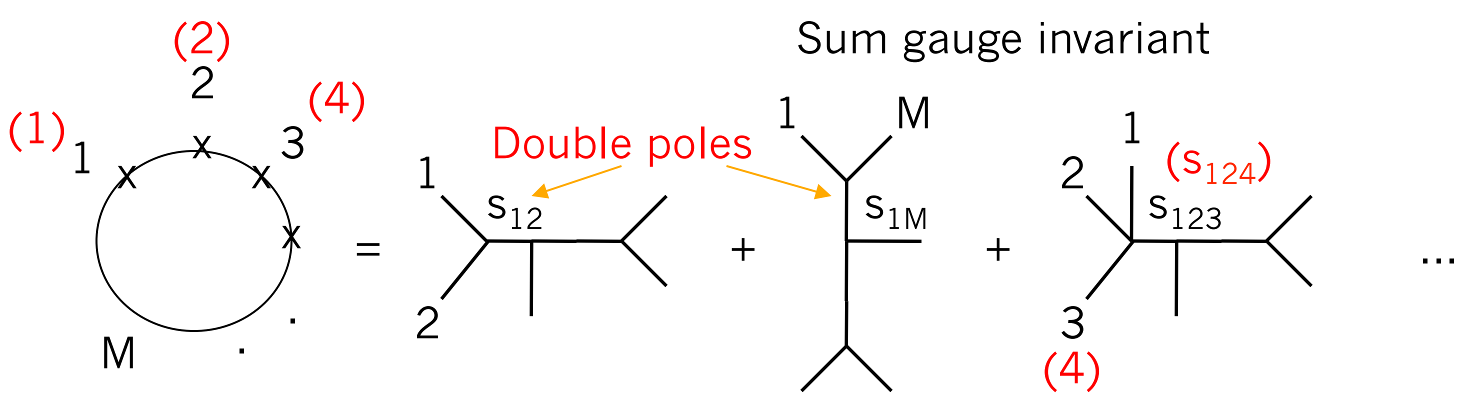
$e^{i\pi\phi} \to \sum_{\pi,\pi'} \prod_{(n-3,\, ij)} s_{ij}$
$= \text{Polynomial}\ (s_{ij})$

No manifest crossing symmetry

(Bern et al)

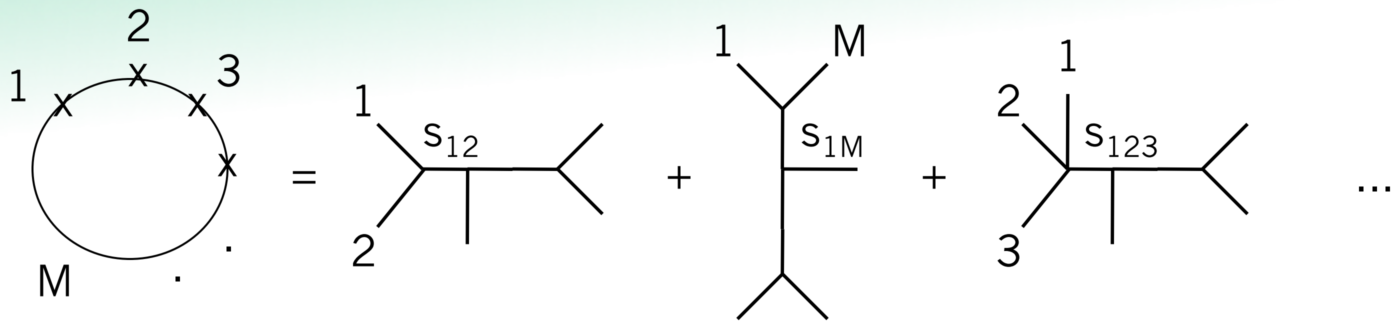
$$M_5^{\rm tree}(1,2,3,4,5) = i s_{12} s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + i s_{13} s_{24} A_5^{\rm tree}(1,3,2,4,5) A_5^{\rm tree}(3,1,4,2,5)$$

Higher point expressions quite bulky ..



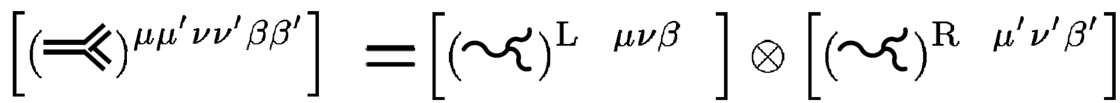
KLT relations work <u>independently</u> of external polarisations

Gravity Amplitudes



(Link to individual Feynman diagrams lost..)

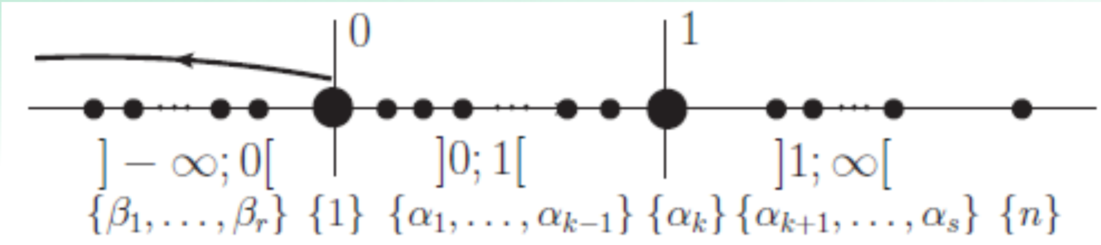
Certain vertex relations possible



Concrete off-shell Lagrangian formulation possible? (double-copy?)

(Bern and Grant; Ananth and Theisen; Hohm)

Monodromy relations



$$\mathcal{A}_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n)$$

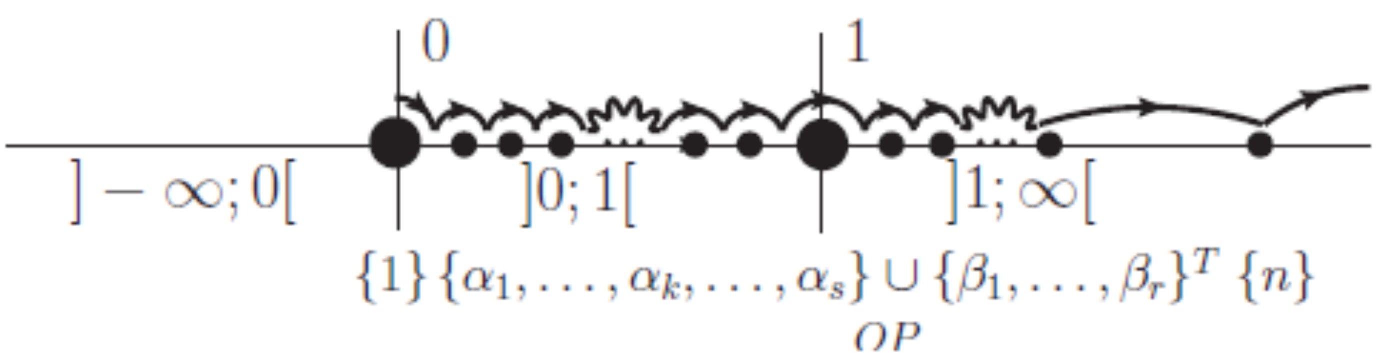
(n-2)! functions in basis

(Kleiss – Kuijf) relations

Monodromy related

(n-3)! functions in basis

(Bern-Carrasco-Johansson) relations



Massive KLT squaring

In all generality we have

$$iM^{\text{tree}} = \sum_{\sigma,\gamma\in\mathfrak{S}_{n-3}} \mathcal{S}[\sigma(2,\cdots,n-2)|\gamma(2,\cdots,n-2)]|_{k_1} \times$$
$$\times A^{\text{tree}}(1,\sigma(2,\cdots,n-2),n-1,n)A^{\text{tree}}(n,n-1,\gamma(2,\cdots,n-2),1)$$

Where

$$\boxed{S[i_1,\ldots,i_r|j_1,\ldots,j_r]|_p = \prod_{t=1}^{r}(p\cdot k_{i_r} + \sum_{s>t}^{r}\theta(i_r,i_s)\,k_{i_r}\cdot k_{i_s})}$$

(NEJB, Damgaard, Feng, Søndergaard; NEJB, Damgaard, Sondergaard, Vanhove)

Key: D=4, spinor-helicity

Spinor products :

$$\langle i\ j\rangle = \epsilon^{mn}\lambda^i_m\lambda^j_n \quad [i\ j] = \epsilon^{\dot{m}\dot{n}}\tilde{\lambda}^i_{\dot{m}}\tilde{\lambda}^j_{\dot{n}}$$

Different representations of the Lorentz group

$$p_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu$$

$$p^\mu p_\mu = 0 \qquad p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

Momentum parts of amplitudes:

$$q_{a\dot{a}} = \mu_a \tilde{\mu}_{\dot{a}} \quad p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \quad 2(p\cdot q) = s_{ij} = -\langle \lambda, \mu\rangle [\tilde{\lambda}, \tilde{\mu}]$$

Spin-2 polarisation tensors in terms of helicities, (squares of those of YM):

(Xu, Zhang, Chang)

$$\varepsilon^-_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]} \qquad \tilde{\varepsilon}^+_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda\rangle} \qquad \begin{matrix} \varepsilon^- & \varepsilon^- \\ \tilde{\varepsilon}^+ & \tilde{\varepsilon}^+ \end{matrix}$$

Simplifications from Spinor-Helicity

$$s_{ij} = -\langle\lambda,\mu\rangle[\tilde{\lambda},\tilde{\mu}]$$

Huge simplifications

$$V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\,\mathrm{sym}\Big[-\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma})$$
$$+\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma})$$
$$-P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta})$$
$$+2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big],$$

45 terms + sym

Vanish in spinor helicity formalism

Contractions

$$\varepsilon^-_{a\dot{a}} = \frac{\lambda_a\tilde{\mu}_{\dot{a}}}{[\tilde{\lambda},\tilde{\mu}]} \qquad \tilde{\varepsilon}^+_{a\dot{a}} = \frac{\mu_a\tilde{\lambda}_{\dot{a}}}{\langle\mu,\lambda\rangle}$$

Gravity: $A_3(1^-,2^-,3^+)$

$$\varepsilon^-\ \varepsilon^-$$
$$\tilde{\varepsilon}^+\ \tilde{\varepsilon}^+$$

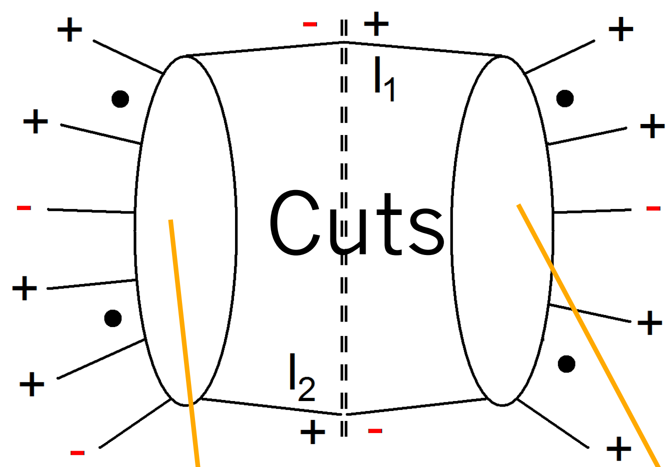
$$= -i\frac{\langle 1\,2\rangle^6}{\langle 2\,3\rangle^2\langle 3\,1\rangle^2}$$

Key: Unitarity

Loop amplitudes

Simpler expressions
for amplitudes

Unitarity



Amplitudes N=4,
N=1, QCD at NLO,
Gravity..

Key: Simple trees

Hidden structure!

New possibilities and matter fields

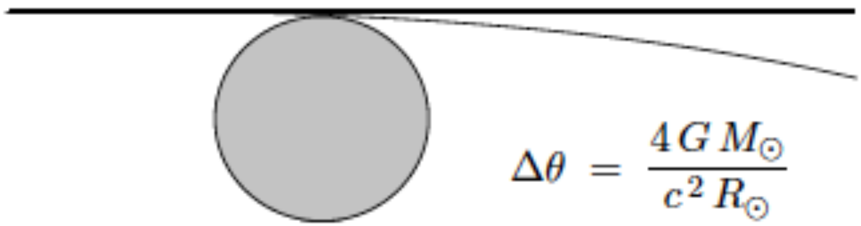
- Unitarity offers many advantages
  - On-shell tree, recursive methods can be used to compute trees.
  - It is easy to consider other types of matter fields just by making the cut with other external particles.
  - Immediate extension to higher loop integrands once trees are known.
    - Extensions to any loop integrand possible (or less impossible than off-shell approaches, current bottle-neck is integrations...)

Helicity method vs. covariant

- The cut is written down in terms of helicity variables (*i.e.* a physical transverse polarisations), has the advantage that 'ghost' contributions are avoided.
- For a covariant cut which is also possible, 'ghosts' would have to be taken into account.
- All symmetry factors plus the various Feynman channels that would normally have be mapped out before the computation are automatically included when calculating the loop amplitude from the cut.

Example: massless matter

- Scattering of massless matter



$$\Delta\theta = \frac{4\,G\,M_\odot}{c^2\,R_\odot}$$

- Bending of light/massless matter around the Sun
- New features: mass-less external fields ~> IR singularities
- New test of universality of matter

Trees and the cut

- We have the Lagrangian

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right]$$

where

$$\mathcal{S}_{\text{scalar}} = \int d^4x \sqrt{-g} \left(-\frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} \Big((\partial_\mu \Phi)^2 - M^2 \Phi^2 \Big) \right)$$

$$\mathcal{S}_{\text{fermion}} = \frac{i}{2} \int d^4x \sqrt{-g}\, \bar{\chi} \not{D} \chi \,,$$

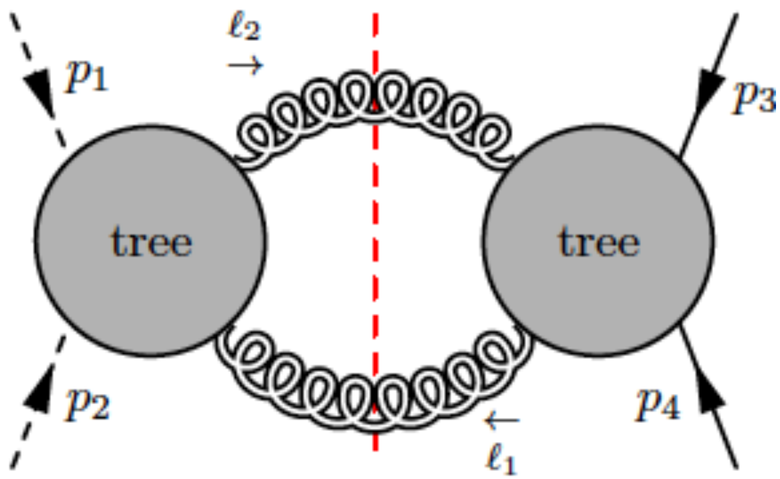
$$\mathcal{S}_{\text{QED}} = -\frac{1}{4} \int d^4x \sqrt{-g} \, (\nabla_\mu A_\nu - \nabla_\nu A_\mu)^2$$

Trees and the cut

- We have the Lagrangian

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right]$$

We want to compute the cut



Trees and the cut

- We have the Lagrangian

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right]$$

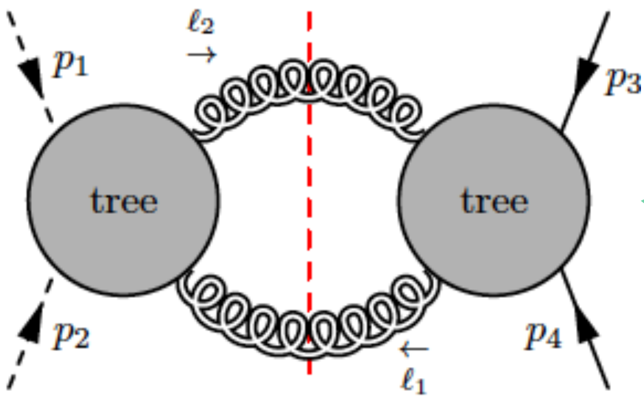
We want to compute the cut

$$\mathcal{M}_X^{(2)}(p_1, p_2, p_3, p_4)\Big|_{\text{disc}} := \frac{1}{2!\, i} \mu^{2\epsilon} \int d\text{LIPS}(\ell_1, -\ell_2)\, (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)$$
$$\times \sum_{\lambda_1, \lambda_2} \mathcal{M}^{(1)}_{X^2 G^2}(p_1, \ell_1, p_2 - \ell_2) \times \mathcal{M}^{(1)}_{\phi^2 G^2}(p_3, \ell_2, p_4, -\ell_1)^\dagger$$

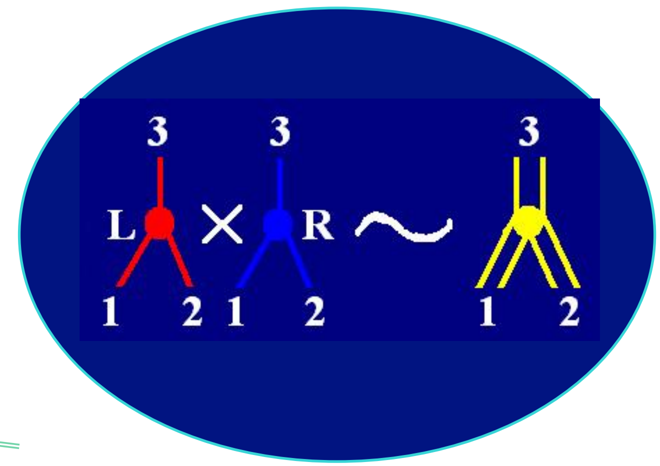
One-loop and the cut

- It is in fact much simpler to capture the long-range behavior from unitarity and on-shell tree amplitudes

$$C_{i,\ldots,j} = \mathrm{Im}_{K_{i,\ldots,j}>0} M^{\text{1-loop}}$$



- No ghosts!



KLT + on-shell 4D input trees recycled from Yang-Mills

(Badger et al; Forde Kosower)

in D-dimensions (from CHY) (NEJB, Cristofoli, Damgaard, Gomez)

(Neill,Rothstein; NEJB, Donoghue, Vanhove)

Photons and scalars

For photons we have

$$\boxed{i\overset{0}{\mathcal{M}}{}^{[h^+(k_1)h^-(k_2)]}_{[\gamma^+(p_1)\gamma^-(p_2)]} = \frac{\kappa^2}{4}\frac{[p_1\,k_1]^2\,\langle p_2\,k_2\rangle^2\,\langle k_2|p_1|k_1]^2}{(p_1\cdot p_2)(p_1\cdot k_1)(p_1\cdot k_2)}}$$

While for scalars

$$\boxed{\begin{aligned} i\overset{0}{\mathcal{M}}{}^{[h^+(k_1)h^+(k_2)]}_{[\phi(p_1)\phi(p_2)]} &= \frac{\kappa^2}{4}\frac{M^4\,[k_1\,k_2]^4}{(k_1\cdot k_2)(k_1\cdot p_1)(k_1\cdot p_2)} \\ i\overset{0}{\mathcal{M}}{}^{[h^-(k_1)h^+(k_2)]}_{[\phi(p_1)\phi(p_2)]} &= \frac{\kappa^2}{4}\frac{\langle k_1|p_1|k_2]^2\,\langle k_1|p_2|k_2]^2}{(k_1\cdot k_2)(k_1\cdot p_1)(k_1\cdot p_2)} \end{aligned}}$$

Result for the amplitude

We can rewrite

$$\mathcal{M}_{\varphi}^{(2)}(p_1,p_2,p_3,p_4) = -\frac{\kappa^4}{32t^2\, i}\sum_{i=1}^{2}\sum_{j=3}^{4}\int \frac{d^D\ell\,\mu^{2\epsilon}}{(2\pi)^D}\,\frac{\mathcal{N}^S}{\ell_1^2\ell_2^2(p_i\cdot\ell_1)(p_j\cdot\ell_1)}$$

where

$$\mathcal{N}^0_{\text{non-singlet}} = \frac{1}{2}\left[\left(\mathrm{tr}_-(\ell_1 p_1 \ell_2 p_3)\right)^4 + \left(\mathrm{tr}_-(\ell_2 p_1 \ell_1 p_3)\right)^4\right]$$

Scalar case

$$\mathcal{N}^{\frac{1}{2}\,+-}_{\text{non-singlet}} = \frac{\left(\mathrm{tr}_-(\ell_1 p_1 \ell_2 p_3)^3\mathrm{tr}_+(p_1 p_3 p_2 \ell_1 p_3 \ell_2)\right) - (\ell_1 \leftrightarrow \ell_2)}{\langle p_2|p_3|p_1]}$$

Fermion case

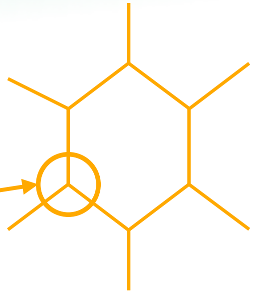
$$\mathcal{N}^{1\,+-}_{\text{non-singlet}} = \frac{\left(\mathrm{tr}_-(\ell_2 p_1 \ell_1 p_3)\mathrm{tr}_+(\ell_2 p_3 \ell_1 p_1 p_3 p_2)\right)^2 + (\ell_1 \leftrightarrow \ell_2)}{\langle p_1|p_3|p_2]^2}$$

Photon case

General 1-loop amplitudes

n-pt amplitude

Vertices carry factors of loop momentum

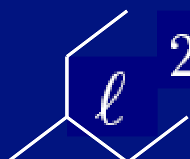


$$\int d^4\ell \frac{F^p(\ell, k, \epsilon)}{\Pi_i p_i^2}$$

p = 2n for gravity
p=n for YM

Propagators

(Passarino-Veltman) reduction

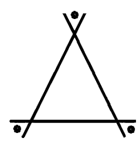


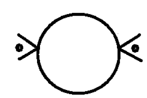
$$2(k \cdot \ell) = (k-\ell)^2 - \ell^2$$

Collapse of a propagator

$$I_r[P^m(l)] \longrightarrow \sum_i I^i_{r-1}[P^{m-1}(l)]$$

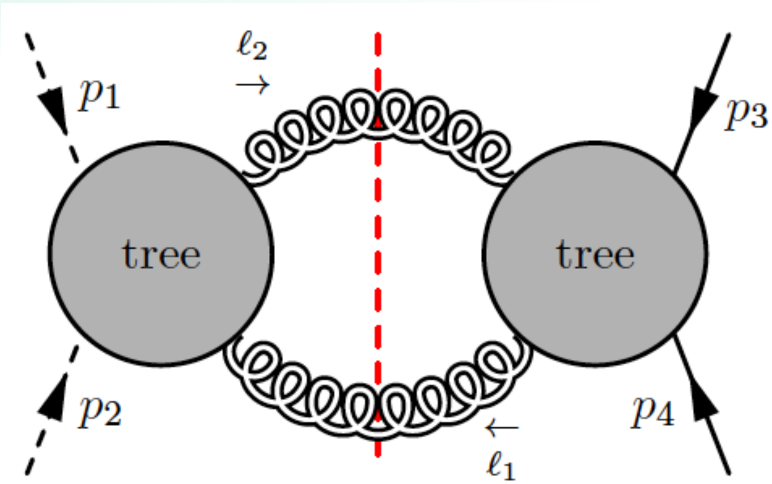
$$I_4^i[P^{m'}(l)] \longrightarrow c_i\, I_4^i[1] + \sum_j I_3^j[P^{m'-1}(l)]$$





$$M^{1-\mathrm{loop}} = \sum_a c_a I_4^a + \sum_a d_a I_3^a + \sum_a e_a I_2^a + R$$

Result for the amplitude

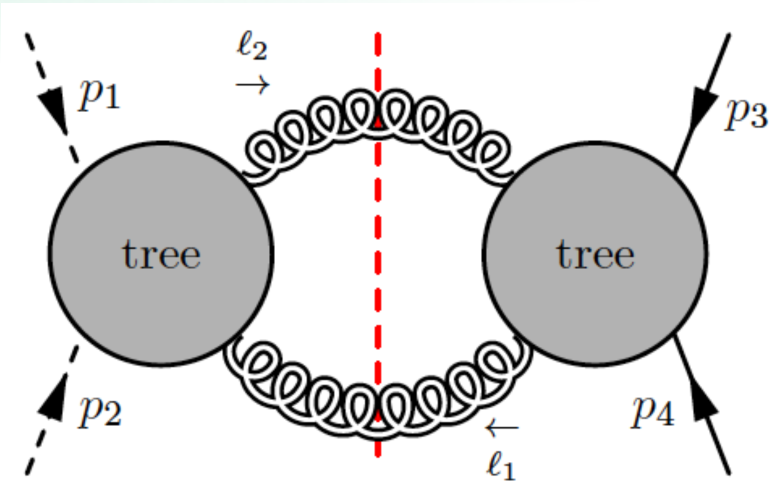


$$bu^{\varphi} = \frac{3}{40} \quad bu^{\gamma} = -\frac{161}{120}$$

$$bu^{X} = -\frac{31}{30}$$

$$-\frac{\mathcal{N}^X}{\hbar}\Big[\hbar\frac{\kappa^4}{4}\Big(4(M\omega)^4(I_4(t,u)+I_4(t,s))+3(M\omega)^2 t I_3(t)$$
$$-15(M^2\omega)^2 I_3(t,M)+bu^X(M\omega)^2 I_2(t)\Big)\Big]$$

Result for the amplitude



$$bu^{\varphi} = \frac{3}{40} \quad bu^{\gamma} = -\frac{161}{120}$$

$$bu^{X} = -\frac{31}{30}$$

Taking the Non-Relativistic low energy limit

$$\frac{\mathcal{N}^X}{\hbar}(M\omega)^2\Big[-\kappa^4\frac{15}{512}\frac{M}{|\boldsymbol{q}|} - \hbar\kappa^4\frac{15}{512\pi^2}\log\left(\frac{\boldsymbol{q}^2}{M^2}\right) + \hbar\kappa^4\frac{bu^X}{(8\pi)^2}\log\left(\frac{\boldsymbol{q}^2}{\mu^2}\right)$$
$$-\hbar\kappa^4\frac{3}{128\pi^2}\log^2\left(\frac{\boldsymbol{q}^2}{\mu^2}\right) + \kappa^4\frac{M\omega}{8\pi}\frac{i}{\boldsymbol{q}^2}\log\left(\frac{\boldsymbol{q}^2}{M^2}\right)\Big]$$

(NEJBB, Donoghue, Holstein, Plante, Vanhove)

Making connection to general relativity

General metric

$$ds^2 = A(r)dt^2 - B(r)^2 dr^2 - r^2 d\Omega^2$$

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2G_N M}{r}$$ Schwarzschild

$$\theta = \frac{4G_N M}{R} + \frac{4G_N^2 M^2}{R^2}\left(\frac{15\pi}{16} - 1\right) + \ldots$$

Can we reproduce?

Bending of light

The simple assumption that we can equate

$$d\sigma = \pi d\rho^2$$

and treat the impact parameter classically is too simple and will <u>only</u> yield the leading contribution.

Thus we have to treat the problem in a more quantum mechanically correct way...

Eikonal Approximation

A partial resolution comes from applying an Eikonal Approximation

Fourier transformation to impact parameter space exponentiates into an eikonal phase, so that a stationary phase method can be applied.
(See e.g. Akhoury, Saotome and Sterman)

$$\mathcal{M}(\boldsymbol{q}) = \mathcal{M}_1^{(1)}(\boldsymbol{q}) + \mathcal{M}^{(2)}(\boldsymbol{q})$$

$$\begin{aligned}\mathcal{M}(\boldsymbol{b}) &= 2(s - M^2)\left[(1 + i\chi_2)e^{i\chi_1} - 1\right] \\ &\simeq 2(s - M^2)\left[e^{i(\chi_1+\chi_2)} - 1\right]\end{aligned}$$

Bending of light via Eikonal Approximation

Now we can compute

$$\chi_1(\boldsymbol{b}) = \frac{\kappa^2 M E}{4} \int \frac{d^2\boldsymbol{q}}{(2\pi)^2}\; e^{-i\boldsymbol{q}\cdot\boldsymbol{b}} \frac{1}{\boldsymbol{q}^2}$$
$$\simeq 4 G_N M E \left[\frac{1}{d-2} - \log(b/2) - \gamma_E \right]$$

$$\chi_2(\boldsymbol{b}) = G_N^2 M^2 E \frac{15\pi}{4b} + \frac{G_N^2 M^2 E}{2\pi b^2} \left(8 b u^\eta - 15 + 48 \log \frac{2b_0}{b} \right)$$

Bending of light via Eikonal Approximation

Leading to static phase when:

$$\frac{\partial}{\partial b}\left(q\,b + \chi_1(b) + \chi_2(b) + \cdots\right) = 0$$

Using that $q = 2E\sin(\theta/2)$

We arrive at:

$$2\sin\frac{\theta}{2} \simeq \theta = -\frac{1}{E}\frac{\partial}{\partial b}\left(\chi_1(b) + \chi_2(b)\right)$$

Bending of light via Eikonal Approximation

Leading to static phase when:

$$\frac{\partial}{\partial b}\left(q\, b + \chi_1(b) + \chi_2(b) + \cdots\right) = 0$$

Using that $q = 2E\sin(\theta/2)$

Or:

$$\theta \simeq \frac{4G_N M}{b} + \frac{15}{4}\frac{G_N^2 M^2 \pi}{b^2} + \left(8bu^S + 9 - 48\log\frac{b}{2b_0}\right)\frac{\hbar G_N^2 M}{\pi b^3} + \ldots$$

Bending of light

Interpreted as a bending angle (eikonal approximation) we have:

$$\theta_\eta \simeq \frac{4GM}{b} + \frac{15}{4}\frac{G^2M^2\pi}{b^2}$$

plus a quantum effect of the order of magnitude:

$$+\frac{8bu^\eta + 9 + 48\log\frac{b}{2r_o}}{\pi}\frac{G^2\hbar M}{b^3}$$

We see that we have universality between scalars, fermions and photons only for the ‘Newton’ and ‘post-Newtonian’ contributions

Bending angle for quantum effects is likely too naïve!

- Should really be treated by quantum means like in QCD… likely to give a diffraction effect as a wave packet treatment.

Classical contributions from perturbative computations

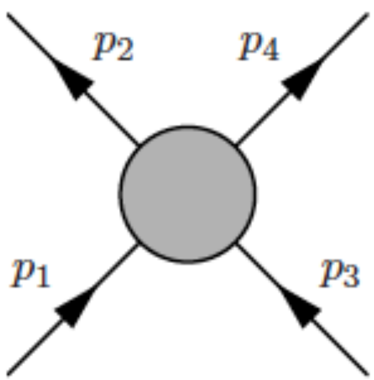
- Use of perturbative framework to compute observables in general relativity
- Truncation to only classical terms
- Only non-analytical piece corresponding to long-distance interactions -> Unitarity cuts useful
- Applications:
  - Computations of post-Newtonian potentials
  - Scattering angle in post-Minkowskian formalism
  - **New urgency**: binary mergers observed by LIGO

NB: Contact with General Relativity require care..!

Example: Massive scalar-scalar scattering

- Will consider scalar-scalar scattering amplitudes mediated through graviton field theory interaction

$$\mathcal{S} = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G_N} + \frac{1}{2}\sum_a \left(g^{\mu\nu}\partial_\mu\phi_a\,\partial_\nu\phi_a - m_a^2\phi_a^2\right)\right]$$

$\mathcal{M} =$ 

$= \sum_{L=0}^{+\infty} \mathcal{M}^{\mathrm{L-loop}}$

$$\mathcal{M}^{\mathrm{L-loop}} \sim O(G_N^{L+1})$$

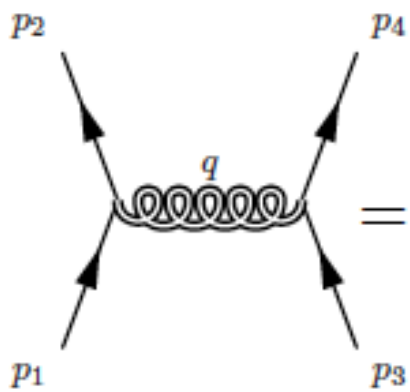
$$p_1^\mu = (E_a, \vec{p})\,, \quad p_2^\mu = (E_a, \vec{p}\,')\,,$$

$$p_3^\mu = (E_b, -\vec{p})\,,\ p_4^\mu = (E_b, -\vec{p}\,')$$

$$|\vec{p}| = |\vec{p}\,'| \qquad q^\mu = p_1^\mu - p_2^\mu$$

Tree level

$$\mathcal{M}^{\text{tree}} =$$



$$= -\frac{4\pi G_N}{E_a E_b}\frac{[2(p_1\cdot p_3)^2 - m_a^2 m_b^2 - |\vec{q}\,|^2(p_1\cdot p_3)]}{|\vec{q}\,|^2}$$

$$p_1\cdot p_3 = E_a(p)E_b(p) + |\vec{p}\,|^2$$

Newton's law through Fourier transform

$$V(r) = -\frac{G m_1 m_2}{r}$$

Unitarity method trees

- Starting from Yang-Mills trees we have $\kappa_{(4)}^2 = 32\pi G_N$

$$iM_s^{\text{tree}}(p_1,p_2,k_1,k_2) = \kappa_{(4)}^2\,(p_1\cdot k_1)\,A_s^{\text{tree}}(p_1,p_2,k_2,k_1)A_0^{\text{tree}}(p_1,k_2,p_2,k_1)$$

s = 0, ½, 1,

- The color striped YM amplitude satisfies

$$A_s^{\text{tree}}(p_1,p_2,k_2,k_1) = \frac{p_1\cdot k_2}{k_1\cdot k_2}\,A_s^{\text{tree}}(p_1,k_2,p_2,k_1)$$

$$\boxed{iM_s^{\text{tree}}(p_1,p_2,k_1,k_2) = \frac{\kappa_{(4)}^2}{e^2}\frac{(p_1\cdot k_1)\,p_1\cdot k_2}{k_1\cdot k_2}\,A_s^{\text{tree}}(p_1,k_2,p_2,k_1)A_0^{\text{tree}}(p_1,k_2,p_2,k_1)}$$

(NEJB, Donoghue, Vanhove)

Unitary cut

- One has

$$A_0^{\text{tree}}(p_1, k_2^+, p_2, k_1^+) = -\frac{m^2\,[k_1\,k_2]^2}{4(p_1\cdot k_1)\,(p_1\cdot k_2)}$$

$$A_0^{\text{tree}}(p_1, k_2^-, p_2, k_1^+) = \frac{\langle k_2|p_1|k_1]^2}{4\,(k_1\cdot p_1)(p_1\cdot k_2)}$$

- This yields

$$\boxed{iM_0^{\text{tree}}(p_1, p_2, k_1^+, k_2^+) = \frac{\kappa_{(4)}^2}{16}\,\frac{1}{(k_1\cdot k_2)}\,\frac{m^4\,[k_1\,k_2]^4}{(k_1\cdot p_1)(k_1\cdot p_2)}}$$

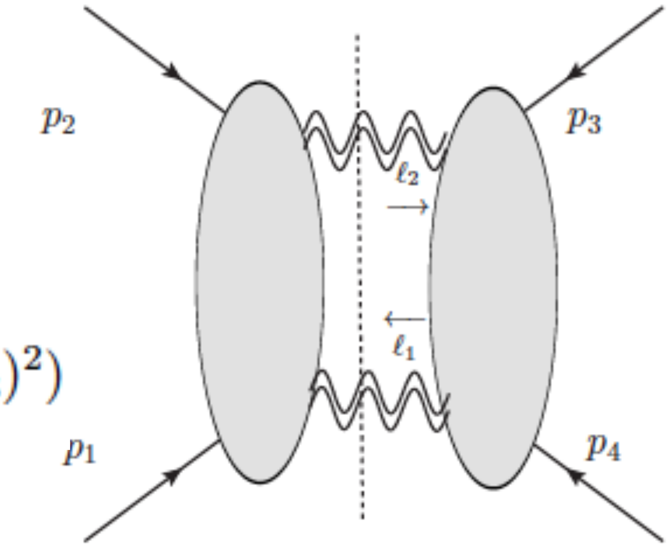
$$\boxed{iM_0^{\text{tree}}(p_1, p_2, k_1^-, k_2^+) = \frac{\kappa_{(4)}^2}{16}\,\frac{1}{(k_1\cdot k_2)}\,\frac{\langle k_1|p_1|k_2]^2\,\langle k_1|p_2|k_2]^2}{(k_1\cdot p_1)(k_1\cdot p_2)}}$$

Compact double cut

- In the cut we have

$$\mathcal{N}^{\text{singlet}} = m_1^2 m_2^2 s^2$$

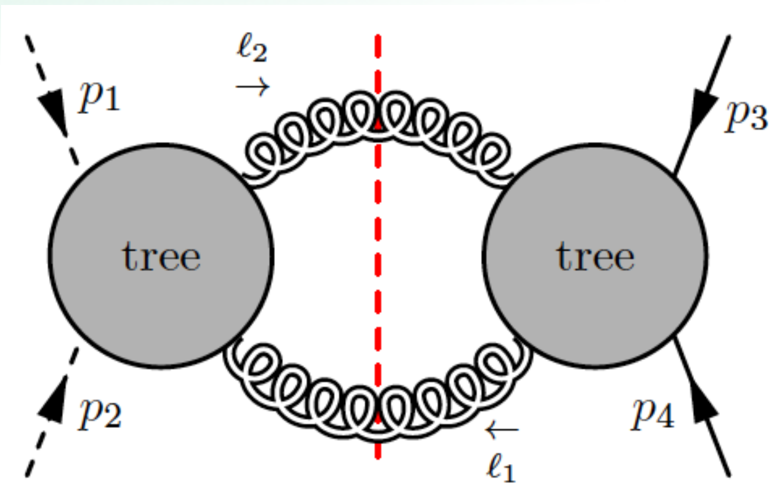
$$\mathcal{N}^{\text{non-singlet}} = \frac{1}{2}(\text{tr}_-(\ell_2 p_1 \ell_1 p_3)^2 + \text{tr}_+(\ell_2 p_1 \ell_1 p_3)^2)$$



$$iM^{1-\text{loop}}\Big|_{disc} = \int \frac{d^D\ell}{(2\pi)^D} \frac{\sum_{\lambda_1,\lambda_2} M^{\text{tree}}_{\lambda_1\lambda_2}(p_1, p_2, -\ell_2^{\lambda_2}, \ell_1^{\lambda_1})(M^{\text{tree}}_{\lambda_1\lambda_2}(p_3, p_4, \ell_2^{\lambda_2}, -\ell_1^{\lambda_1}))^*}{\ell_1^2\ell_2^2}\Big|_{cut}$$

$$\boxed{iM^{1-\text{loop}}\Big|_{disc} = \frac{e^4}{16}\int \frac{d^D\ell}{(2\pi)^D} \frac{\mathcal{N}}{\ell_1^2\ell_2^2\prod_{i=1}^4(p_i\cdot\ell_1)}}$$

Result for the one-loop amplitude



1) Expand out traces
2) Reduce to scalar basis of integrals
3) Isolate coefficients

(NEJB, Donoghue, Vanhove)

(See also Cachazo and Guevara;
(Bern, Cheung, Roiban, Shen, Solon, Zeng)

$$\mathcal{M}^{1-\text{loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b}\left(c_\Box \mathcal{I}_\Box + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots\right)$$

Integrals in the one-loop amplitude

$$\mathcal{M}^{1-\text{loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b}\left(c_\Box \mathcal{I}_\Box + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots\right)$$

$$\mathcal{I}_\Box = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+p_1)^2 - m_a^2 + i\varepsilon)((\ell-p_3)^2 - m_b^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+q)^2 + i\varepsilon)}$$

$$\mathcal{I}_{\bowtie} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+p_1)^2 - m_a^2 + i\varepsilon)((\ell+p_4)^2 - m_b^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+q)^2 + i\varepsilon)}$$

$$\mathcal{I}_{\triangleright} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+p_1)^2 - m_a^2 + i\varepsilon)}$$

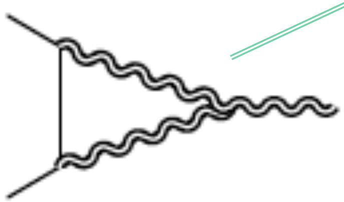
$$\mathcal{I}_{\triangleleft} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell-q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell-p_3)^2 - m_b^2 + i\varepsilon)}$$

Classical pieces in loops

Classical physics from loop diagrams

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} \frac{1}{(\ell+q)^2 + i\epsilon} \frac{1}{(\ell+p_1)^2 - m_1^2 + i\epsilon}$$

$$(\ell + p_1)^2 - m_1^2 = \ell^2 + 2\ell \cdot p_1 \simeq 2m_1\ell_0$$



(NEJB, Damgaard, Festuccia, Plante, Vanhove)

$$\frac{1}{2m_1} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} \frac{1}{(\ell+q)^2 + i\epsilon} \frac{1}{\ell_0 + i\epsilon}$$

Classical pieces in loops

$$\frac{1}{2m_1}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2+i\epsilon}\frac{1}{(\ell+q)^2+i\epsilon}\frac{1}{\ell_0+i\epsilon}$$

Close contour

$$\int_{|\vec{\ell}|\ll m}\frac{d^3\vec{\ell}}{(2\pi)^3}\frac{i}{4m}\frac{1}{\vec{\ell}^2}\frac{1}{(\vec{\ell}+q)^2} = -\frac{i}{32m|\vec{q}|}$$

Putting it all together

Imaginary phase

Ignore quantum pieces

$$\mathcal{M}^{1-\text{loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b}\left(c_\Box \mathcal{I}_\Box + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots\right)$$

$$\mathcal{I}_\Box = -\frac{i}{16\pi^2|\vec{q}|^2}\left(-\frac{1}{m_a m_b} + \frac{m_a(m_a - m_b)}{3m_a^2 m_b^2} + \frac{i\pi}{|p|E_p}\right)\left(\frac{2}{3-d} - \log|\vec{q}|^2\right) + \cdots$$

$$\mathcal{I}_{\bowtie} = -\frac{i}{16\pi^2|\vec{q}|^2}\left(\frac{1}{m_a m_b} - \frac{m_a(m_a - m_b)}{3m_a^2 m_b^2}\right)\left(\frac{2}{3-d} - \log|\vec{q}|^2\right) + \cdots$$

$$\mathcal{I}_{\triangleright} = -\frac{i}{32 m_a}\frac{1}{|\vec{q}|} + \cdots$$

$$\mathcal{I}_{\triangleleft} = -\frac{i}{32 m_b}\frac{1}{|\vec{q}|} + \cdots$$

$$c_\Box = c_{\bowtie} = 4\left(m_a^2 m_b^2 - 2(p_1\cdot p_3)^2\right)^2$$

$$c_{\triangleright} = 3m_a^2\left(m_a^2 m_b^2 - 5(p_1\cdot p_3)^2\right)$$

$$c_{\triangleleft} = 3m_b^2\left(m_a^2 m_b^2 - 5(p_1\cdot p_3)^2\right)$$

Relation to a PM potential

- We use the language of old-fashioned time-ordered perturbation theory
- In particular we eliminate by hand
  - Annihilation channels
  - Back-tracking diagrams (no intermediate multiparticle states)
  - Anti-particle intermediate states

We will also assume (classical) long-distance scattering

(Cristofoli, Bjerrum-Bohr, Damgaard, Vanhove)

Relation to a PM potential

- One-loop amplitude after summing all contributions

$$\mathcal{M}^{1-\text{loop}} = \frac{\pi^2 G_N^2}{E_p^2 \xi}\left[\frac{1}{2|\vec{q}|}\left(\frac{c_{\triangleright}}{m_a} + \frac{c_{\triangleleft}}{m_b}\right) + \frac{i}{E_p}\frac{c_{\Box}}{|\vec{p}|}\frac{\left(\frac{2}{3-d} - \log|\vec{q}|^2\right)}{\pi|\vec{q}|^2}\right]$$

Imaginary
super-classical/
singular

- How to relate to a classical potential?
  - Choice of coordinates
  - Born subtraction/Lippmann-Schwinger

Relation to a relativistic PM potential

- Amplitude defined via perturbative expansion around a flat Minkowskian metric

- Now we need to relate the Scattering Amplitude to the potential for a bound state problem – alternative to matching (Cheung, Solon, Rothstein; Bern, Cheung, Roiban, Shen, Solon, Zeng)

- Starting point: the Hamiltonian of the relativistic Salpeter equation

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V}\,, \qquad \hat{\mathcal{H}}_0 = \sqrt{\hat{k}^2 + m_a^2} + \sqrt{\hat{k}^2 + m_b^2}$$

Relation to a potential

- Analysis involves solution of the Lippmann-Schwinger recursive equation:

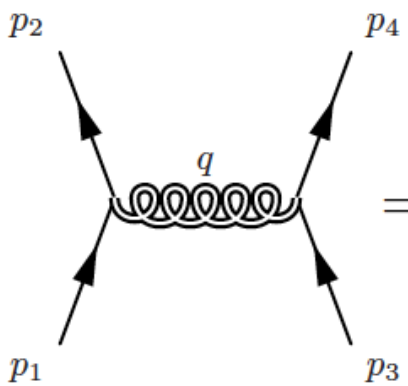
$$\mathcal{M}(p,p') = \langle p|V|p'\rangle + \int \frac{d^3k}{(2\pi)^3} \frac{\langle p|V|k\rangle \mathcal{M}(k,p')}{E_p - E_k + i\epsilon}$$

$$\langle p|V|p'\rangle = \mathcal{M}(p,p') - \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{M}(p,k)\mathcal{M}(k,p')}{E_p - E_k + i\epsilon} + \cdots$$

$$V(p,r) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} V(p,q)$$

Tree level

$\mathcal{M}^{\text{tree}} =$



$$= -\frac{4\pi G_N}{E_a E_b} \frac{\left[2(p_1 \cdot p_3)^2 - m_a^2 m_b^2 - |\vec{q}\,|^2 (p_1 \cdot p_3)\right]}{|\vec{q}\,|^2}$$

$$p_1 \cdot p_3 = E_a(p) E_b(p) + |\vec{p}\,|^2$$

Same result as from matching (Cheung, Solon, Rothstein; Bern, Cheung, Roiban, Shen, Solon, Zeng)

$$V_{1PM}(p,r) = \frac{1}{E_p^2 \xi} \frac{G_N c_1(p^2)}{r} + \cdots$$

$$c_1(p^2) \equiv m_a^2 m_b^2 - 2(p_1 \cdot p_3)^2 \quad , \quad \xi \equiv \frac{E_a E_b}{E_p^2}$$

One-loop

$$\mathcal{M}^{\text{Iterated}} = -\frac{16\pi^2 G_N^2}{E_a(p^2)E_b(p^2)} \int \frac{d^dk}{(2\pi)^d} \frac{A(\vec{p},\vec{k})}{|\vec{p}-\vec{k}|^2} \frac{A(\vec{k},\vec{p}')}{|\vec{p}'-\vec{k}|^2} \frac{\mathcal{G}(p^2,k^2)}{E_a(k^2)E_b(k^2)}$$

$$\mathcal{G}(p^2,k^2) = \frac{1}{E_p - E_k + i\epsilon}$$

Expanded

$$\mathcal{M}^{\text{Iterated}} = \frac{32\pi^2 G_N^2}{E_p^3 \xi} c_1^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{|\vec{p}-\vec{k}|^2 |\vec{p}'-\vec{k}|^2 (k^2-p^2)}$$

$$- \frac{16\pi^2 G_N^2}{E_p^3 \xi^2} \left(\frac{c_1^2(1-\xi)}{2E_p^2 \xi} + 4c_1 p_1 \cdot p_3 \right) \int \frac{d^dk}{(2\pi)^d} \frac{1}{|\vec{p}-\vec{k}|^2 |\vec{p}'-\vec{k}|^2} + \cdots$$

One-loop

$$\mathcal{M}^{\text{Iterated}} = \frac{i\pi G_N^2}{E_p^3\xi}\frac{4c_1^2}{|\vec{p}|}\frac{(\log|\vec{q}|^2 - \frac{2}{3-d})}{|\vec{q}|^2} + \frac{2\pi^2 G_N^2}{E_p^3\xi^2|\vec{q}|}\left(\frac{c_1^2(\xi-1)}{2E_p^2\xi} - 4c_1 p_1\cdot p_3\right)$$

$$\mathcal{M}^{1-\text{loop}} = \frac{\pi^2 G_N^2}{E_p^2\xi}\left[\frac{1}{2|\vec{q}|}\left(\frac{c_\triangleright}{m_a} + \frac{c_\triangleleft}{m_b}\right) + \frac{i}{E_p}\frac{c_\Box}{|\vec{p}|}\frac{(\frac{2}{3-d} - \log|\vec{q}|^2)}{\pi|\vec{q}|^2}\right]$$

$$V_{2\text{PM}}(p,q) = \mathcal{M}^{1-\text{loop}} + \mathcal{M}^{\text{Iterated}}$$

$$V_{2\text{PM}}(p,q) = \frac{\pi^2 G_N^2}{E_p^2\xi|\vec{q}|}\left[\frac{1}{2}\left(\frac{c_\triangleright}{m_a} + \frac{c_\triangleleft}{m_b}\right) + \frac{2}{E_p\xi}\left(\frac{c_1^2(\xi-1)}{2E_p^2\xi} - 4c_1 p_1\cdot p_3\right)\right]$$

Again same result as from matching, no singular term

Effective potential

In fact we do not have to go through either matching procedure or solving Lippmann-Schwinger to derive observables such as the scattering angle

Energy relation makes everything simple:

$$p^2 \; = \; p_\infty^2 - 2E\xi\left[\widetilde{\mathcal{M}}^{cl.}_{tree}(p_\infty^2, r) + \widetilde{\mathcal{M}}^{cl.}_{1-loop}(p_\infty^2, r)\right]$$

(Damour; Bern, Cheung, Roiban, Shen, Solon, Zeng; Kalin, Porto; NEJB, Cristofoli, Damgaard; Cristofoli, Damgaard, Di Vecchia, Heissenberg)

Effective potential

Thus given the classical amplitude

$$\widetilde{\mathcal{M}}^{cl.}(p,r) \equiv -\frac{1}{2E\xi}\sum_{n=1}^{\infty}\frac{G_N^n \widetilde{c}_{(n-1)-loop}(p)}{r^n}$$

$$f_n(E) = \widetilde{c}_{(n-1)-loop}(p_\infty) \quad V_{\text{eff}}(r) \equiv -\sum_{n=1}^{\infty}\frac{G_N^n f_n(E)}{r^n}$$

Non-relativistic Hamiltonian with effective potential

$$\hat{\mathcal{H}} = \hat{p}^2 + V_{\text{eff}}(r)$$

Scattering angle all orders

$$\chi = \sum_{k=1}^{\infty} \widetilde{\chi}_k(b) \,, \quad \widetilde{\chi}(b) \equiv \frac{2b}{k!} \int_0^{+\infty} du \left(\frac{d}{du^2} \right)^k \frac{V_{\text{eff}}^k(r)\, r^{2(k-1)}}{p_\infty^{2k}}$$

(Kalin, Porto; NEJB, Cristofoli,Damgaard)

$$p_r = \sqrt{p_\infty^2 - \frac{L^2}{r^2} - V_{\text{eff}}(r)}$$

$$\frac{\chi}{2} = -\int_{r_m}^{+\infty} dr \, \frac{\partial p_r}{\partial L} - \frac{\pi}{2}$$

Corrects naïve
light-bending
'Bohm's formula'
+ no reference
minimal distance

Any PM order given amplitude...

PM	$\chi^{\rm PM}/\left(\frac{G_N}{p_\infty L}\right)^{\rm PM}$
1	f_1
2	$\frac{1}{2}\pi p_\infty^2 f_2$
3	$2f_3p_\infty^4 + f_1f_2p_\infty^2 - \frac{f_1^3}{12}$
4	$\frac{3}{8}\pi p_\infty^4\left(2f_4p_\infty^2 + f_2^2 + 2f_1f_3\right)$
5	$\frac{8}{3}f_5p_\infty^8 + 4\left(f_2f_3 + f_1f_4\right)p_\infty^6 + f_1\left(f_2^2 + f_1f_3\right)p_\infty^4 - \frac{1}{6}f_1^3f_2p_\infty^2 + \frac{f_1^5}{80}$
6	$\frac{5}{16}\pi p_\infty^6\left(3f_6p_\infty^4 + 3\left(f_3^2 + 2f_2f_4 + 2f_1f_5\right)p_\infty^2 + f_2^3 + 6f_1f_2f_3 + 3f_1^2f_4\right)$
7	$\frac{16}{5}f_7p_\infty^{12} + 8\left(f_3f_4 + f_2f_5 + f_1f_6\right)p_\infty^{10} + 6\left(f_3f_2^2 + 2f_1f_4f_2 + f_1\left(f_3^2 + f_1f_5\right)\right)p_\infty^8$ $+f_1\left(f_2^3 + 3f_1f_3f_2 + f_1^2f_4\right)p_\infty^6 - \frac{1}{8}f_1^3\left(2f_2^2 + f_1f_3\right)p_\infty^4 + \frac{3}{80}f_1^5f_2p_\infty^2 - \frac{f_1^7}{448}$
8	$\frac{35}{128}\pi p_\infty^8\left(4f_8p_\infty^6 + 6\left(f_4^2 + 2\left(f_3f_5 + f_2f_6 + f_1f_7\right)\right)p_\infty^4 + 12\left(f_4f_2^2 + \left(f_3^2 + 2f_1f_5\right)f_2\right.\right.$ $\left.\left.+f_1\left(2f_3f_4 + f_1f_6\right)\right)p_\infty^2 + f_2^4 + 6f_1^2f_3^2 + 12f_1f_2^2f_3 + 12f_1^2f_2f_4 + 4f_1^3f_5\right)$

Confirmation of 3PM & 4PM

Bern, Cheung, Roiban, Shen, Solon, Zeng))

Open question: how to include quantum effects?

Outlook

- Amplitude toolbox for computations already provided many new efficient methods for computation

  - Amplitude tools very useful for computations:
    - Double-copy and KLT
    - Recursion;
    - Unitarity
    - Spinor-helicity
    - CHY formalism
    - Low energy limits of string theory

Outlook

Already **extensive work** on QFT approaches to gravity. **(Many talks)**

Forms the theoretical backbone of current investigations using templates at LIGO/Virgo**.**

Observation: Amplitude methods provide *new efficiency* to calculations.

1) Double-copy allows recycling of Yang-Mills results in gravity.

2) Off-shell -> On-shell (removes clutter from computations, while essence contributions remains). *However important considerations when throwing away off-shell information.* (coordinate dependence etc.)

3) Classical parts are possible to identify and target independently of quantum contributions. (non-analytic pieces, have unique cuts)

4) New technology for integrations is helpful.

5) Possible to address quantum gravity effects.

Outlook

- A number of very impressive 3PM amplitude computations. (Bern et al; Cheung et al, Parra-Martinez et al, Kalin et al) + many more. (see Zvi's, Rafael's, Carlos's and Damour's and Julio's talks)

- Endless additional tasks ahead
  - Improved ways to include effects of spin (Guevara et al; Bautista et al; Johansson at al; Ochirov ; Arkani-Hamed; Chung et al; Huang at al, O'Connell et al; Bern et al + many more)
  - Improved take on radiation effects / new understanding? (Kosower et al; Kalin et al many more)
  - Inclusion of high order curvature terms, finite size effects, tidal effects, double-copy etc (Brandhuber et al; Helset and Haddad; Cheung et al, Di Vecchia et all; Damour)
  - Better understanding of quantum effects and their implications

- Clearly much more physics to learn....

Discussion / Conclusion

- Treating general relativity as an effective field is a smart way to avoid the usual complications and confusions in quantizing gravity
- The new results are unique consequences of an underlying more fundamental theory
  - Effects are tiny but this is a consequence of gravity being a very weak force
- Classical GR has a huge validity
- GR-EFT provides a natural laboratory for investgating low energy quantum corrections

(Weinberg; Donoghue; NEJBB, Donoghue, Holstein)

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SAGEX
Scattering Amplitudes:
from Geometry to Experiment