# On Gauge Sector of F-theory: Bounds and Limits


**Based mainly on:** **Phys. Rev. D 108 (2023) 086021** w/ P. Oehlmann

**JHEP 06 (2022) 042 + 09 (2022) 143** w/ W. Lerche, T. Weigand

**2407.abcde** w/ R. Alvarez-Garcia, T. Weigand



**Seung-Joo Lee (IBS)**

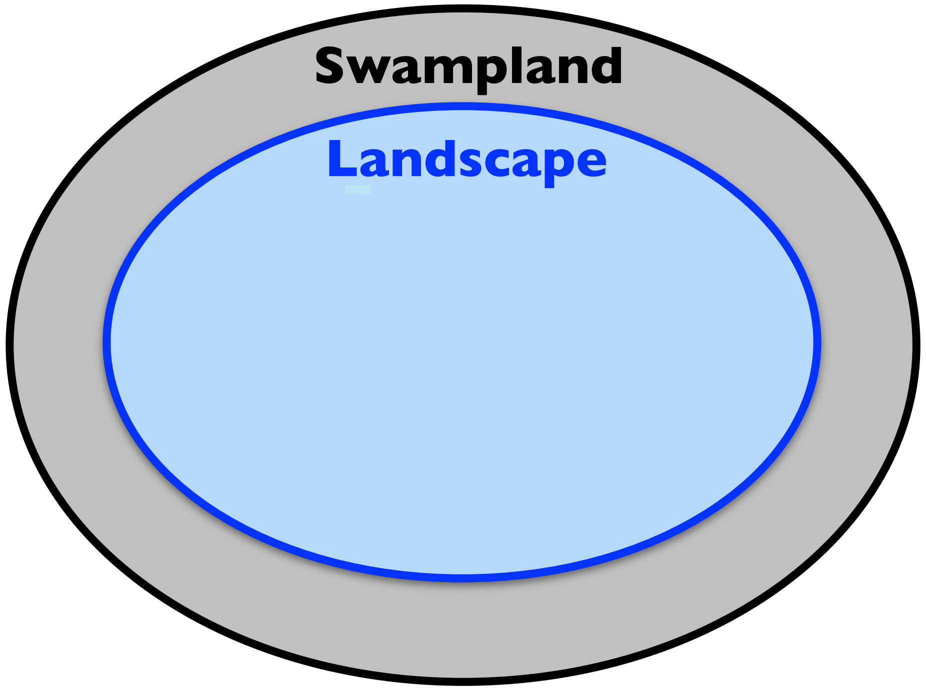

String Phenomenology@Padova

27-June-2024

# The Swampland Program

In Light of the String Landscape

- **Grouping the QFT Models**
  - **Landscape** (EFTs w/ a UV completion into QG)
    - contains the **String Landscape** (String EFTs)
  - **Swampland** (EFTs w/o a UV completion)

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  - Goal: distinguish EFTs in the **Landscape** from those in the **Swampland** [Vafa '05]
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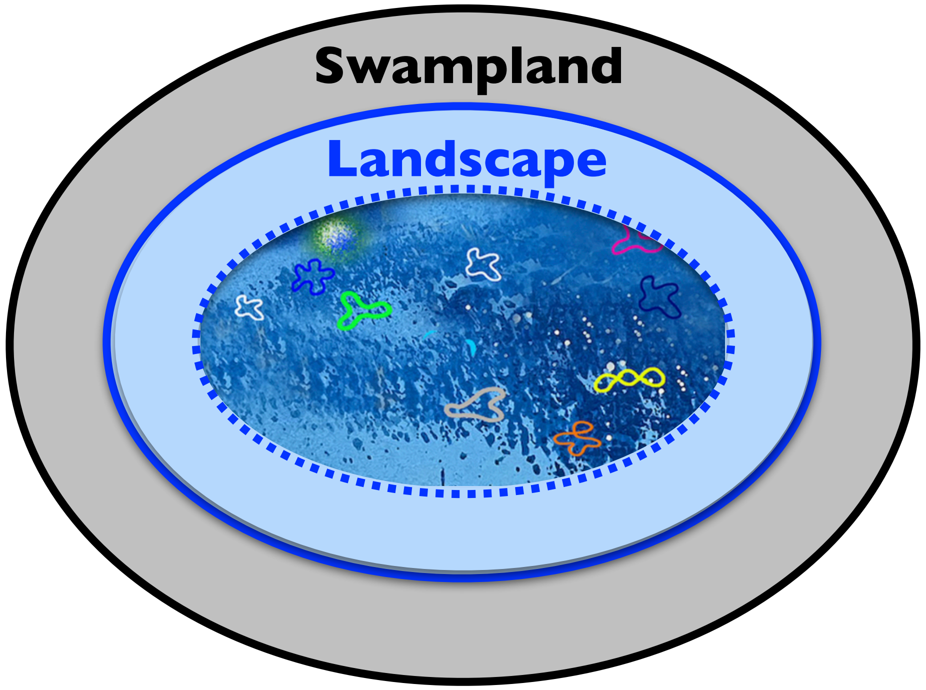


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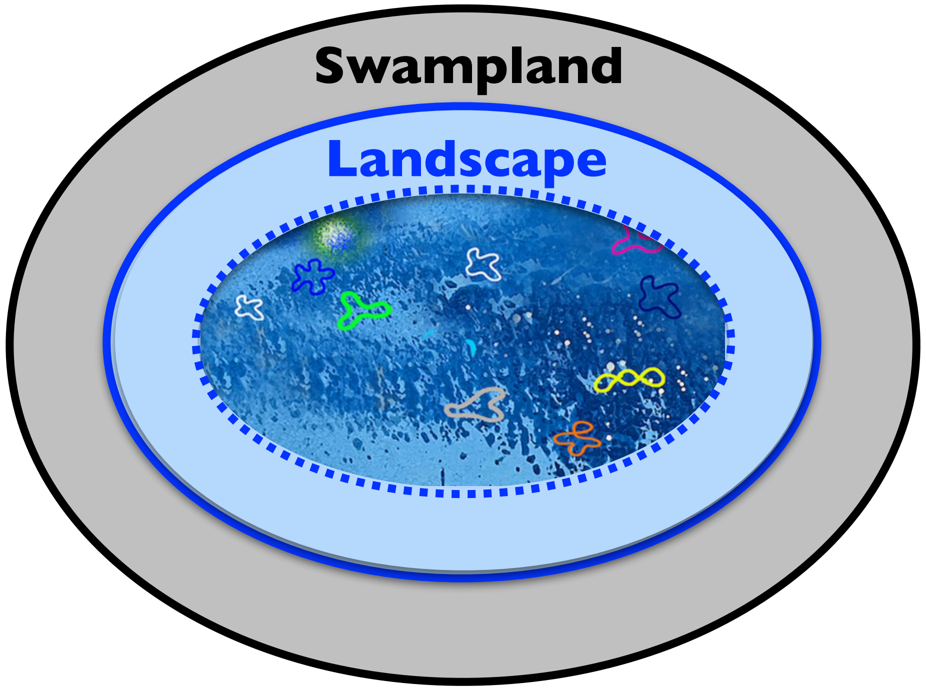


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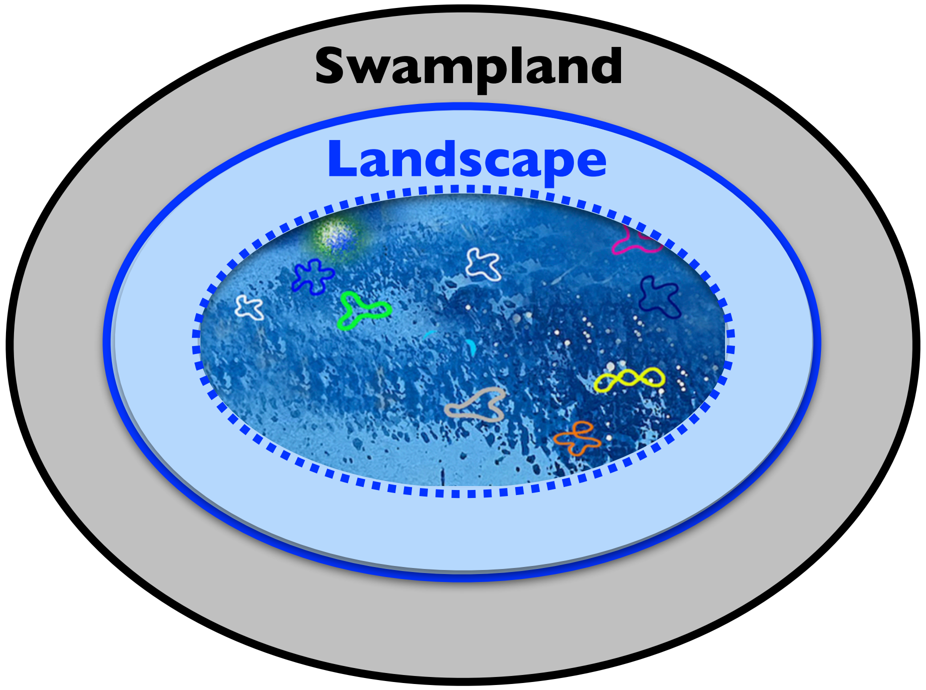


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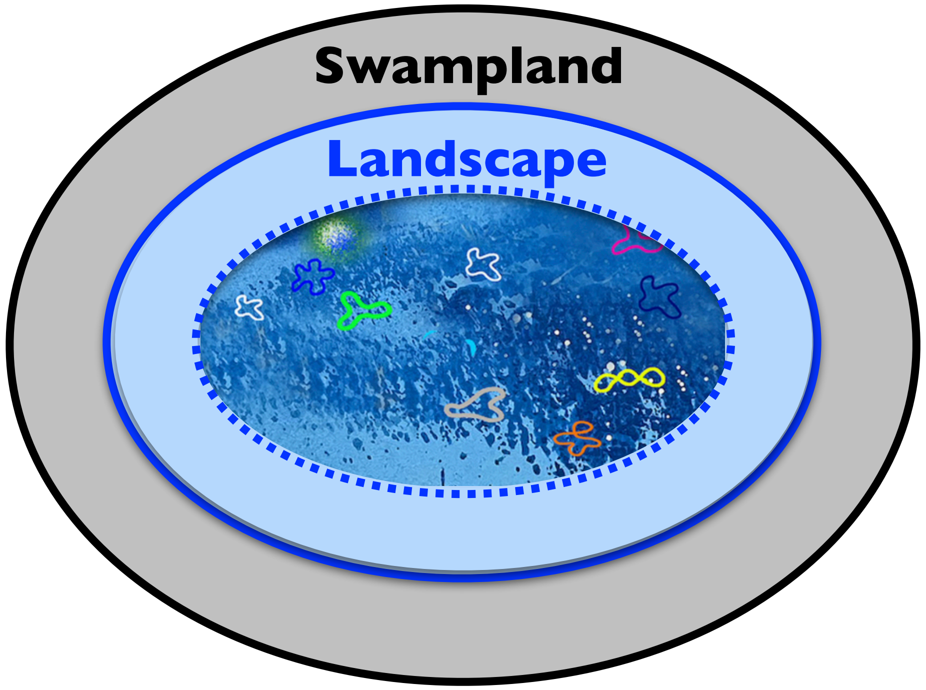


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  → insights on quantum gravity conjectures!

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Limits at Infinite Distance vs. Limitations at Finite Distance

- **Universal Features of EFTs at Infinite Distance**

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| general constraints from the same string? | decompactification beyond expected bounds? |
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# Main Results on Gauge Sector: Bounds & Limits

What this talk will be about

- **EFTs in Scrutiny**
  - Supergravity EFTs w/ minimal SUSY (main focus: EFTs of F-theory, mostly in 6d)

- **Bounds on the Gauge Sector (6d)** as seen by a solitonic string
  - Goal: constrain the global structure of the non-abelian gauge sector
    - order of each cyclic quotient: $m \leq 6$
  - Bonus: find a connection to the rank bound on the abelian gauge sector
    - number of U(1) factors: $N_{U(1)} \leq 18$

- **Limits of the Gauge Sector (6d/8d)** as what sets the bound
  - Goal: classify unconventional "non-minimal" stacks of coalescing 7-branes
    - non-minimal stacks at *infinite* distance: decompactifications
  - Alert: distinguish finite distance limits in disguise from the classification
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Motivation and Main Results

## Rudiments of 6d F-theory

Gauge Sector: Non-abelian and Abelian Sectors

Solitonic Strings: Heterotic String

## Part I.  Bounds

Global Structure of the Non-abelian Sector

Rank of the Abelian Sector

## Part II.  Limits

Non-minimal Brane Stack at Infinite vs. Finite Distance

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Summary and Discussion

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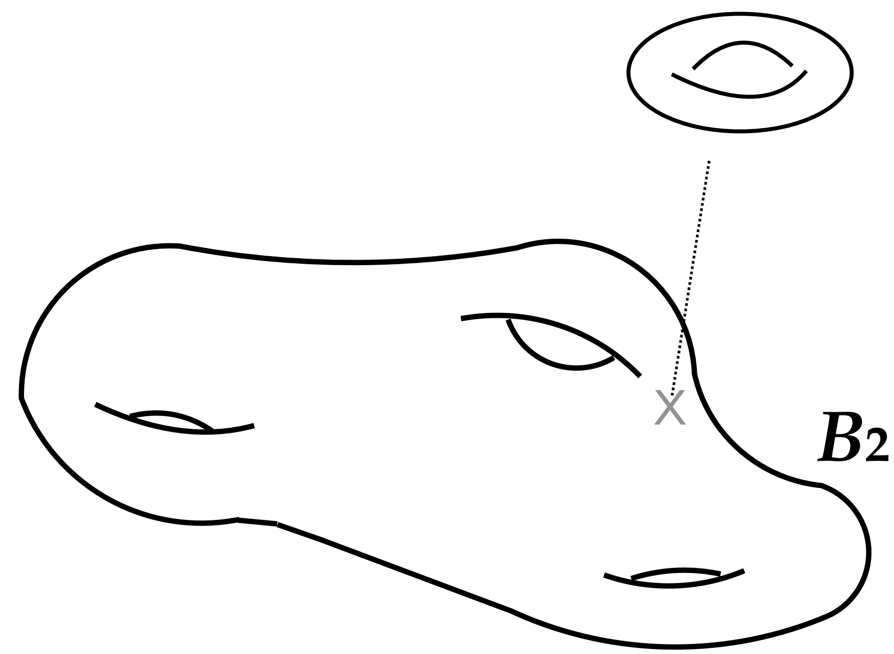
- **6d F-theory**
  - IIB string on compact 2-fold $B_2$ w/ 7-branes on complex curves (varying axio-dilaton)
  - 7-brane configuration encoded in an elliptic Calabi-Yau 3-fold $\pi : Y_3 \to B_2$

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Non-abelian $G_i$ vectors

  - Codim-1 singular fibers over divisors $b_i$

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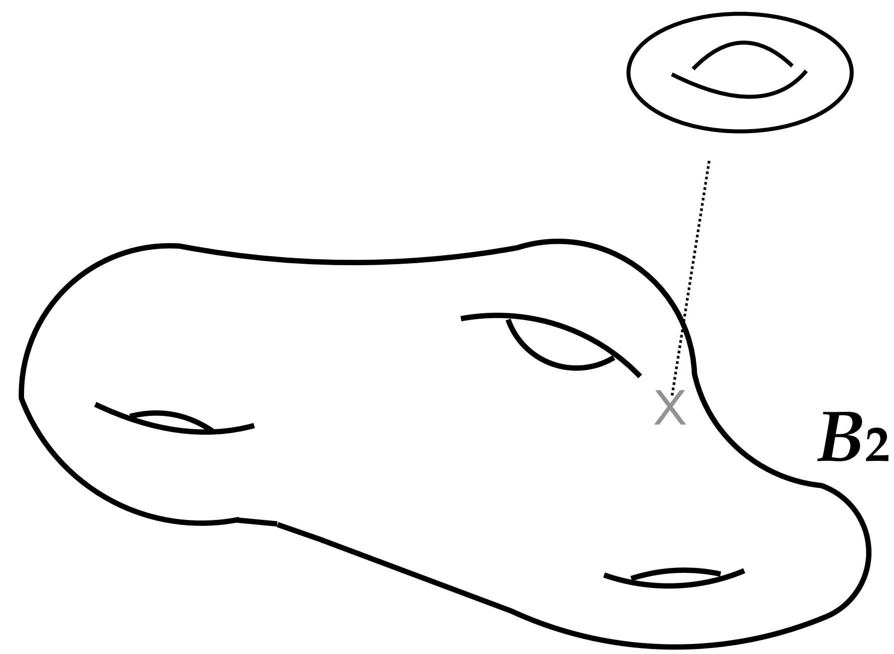


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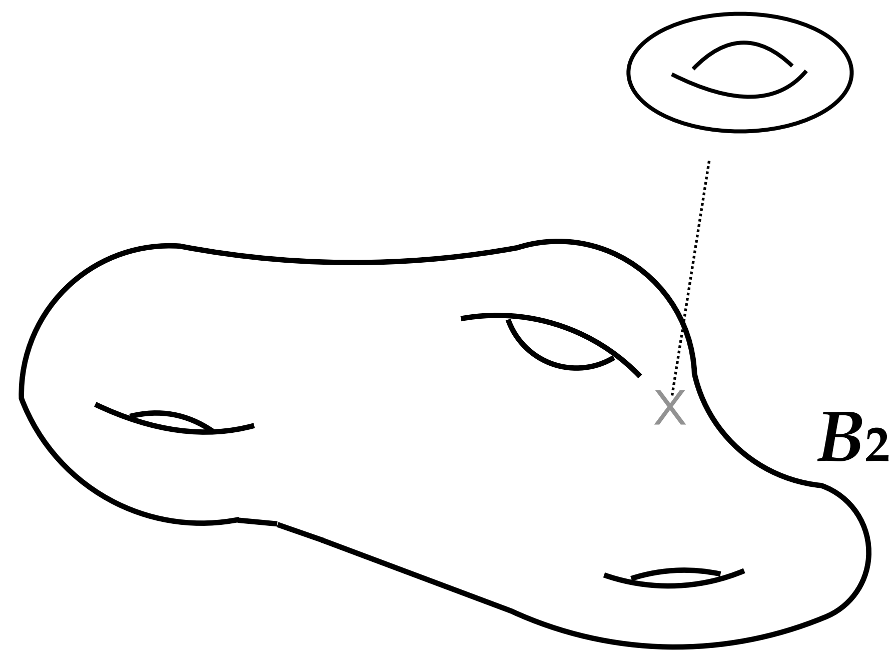

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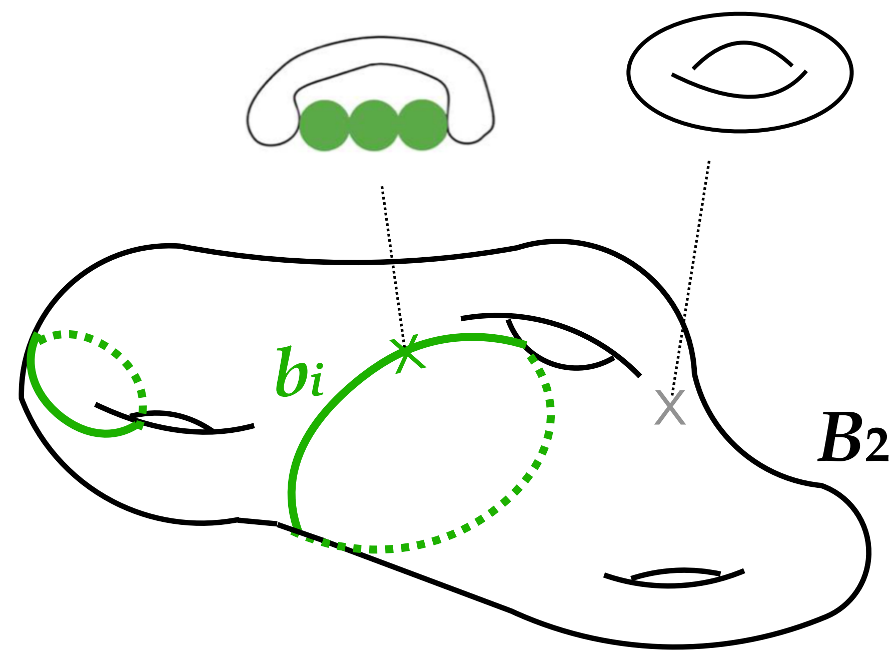


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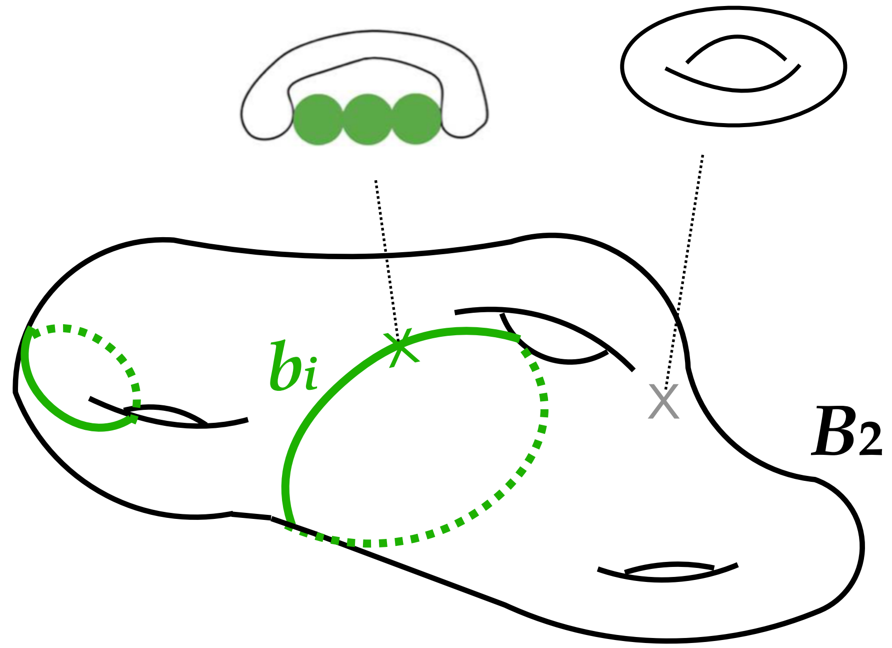

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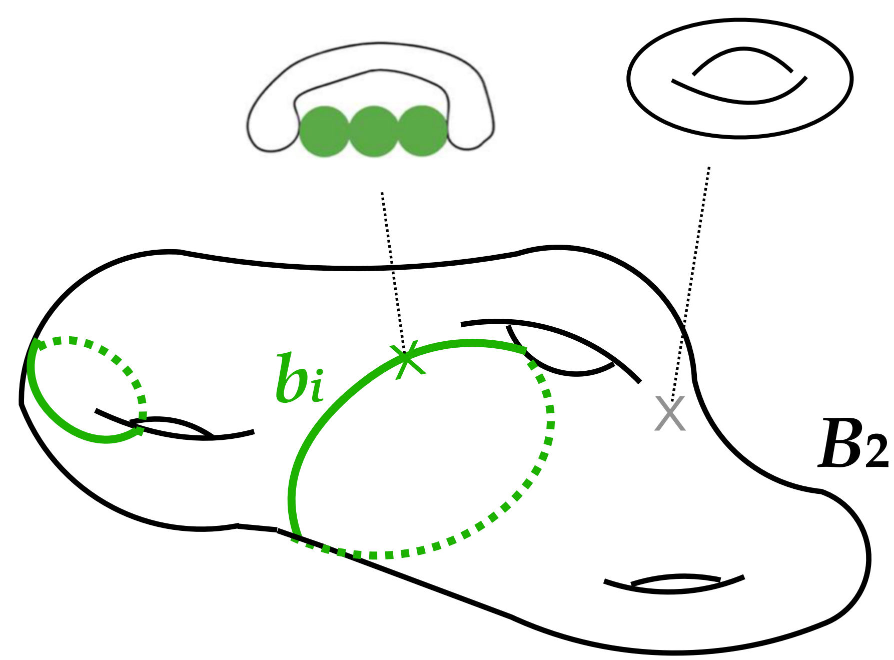


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**Abelian $U(1)_A$ vectors**

- Sections $s_A : B_2 \dashrightarrow Y_3$ to the elliptic fibration

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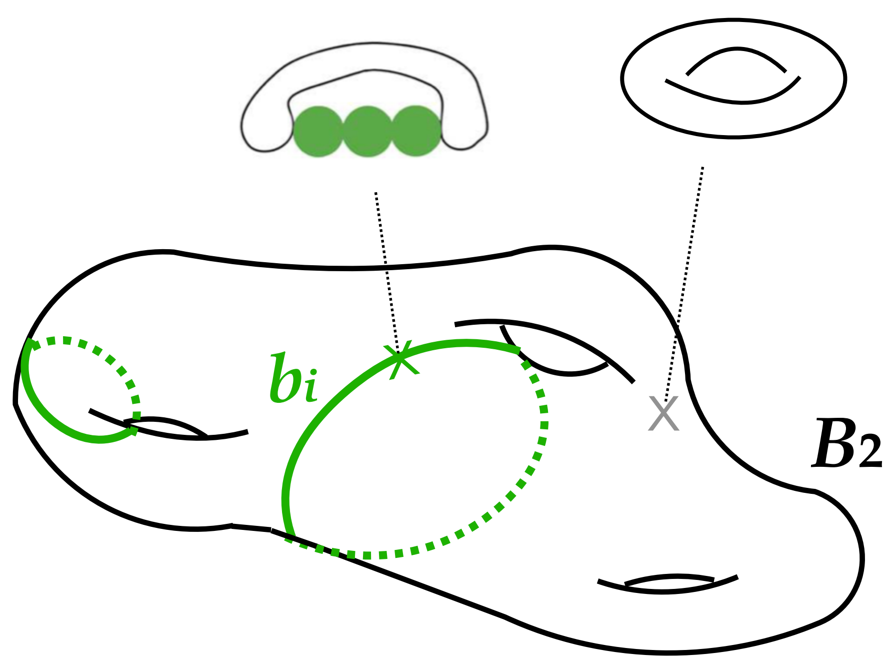


**Non-abelian $G_i$ vectors**

- Codim-1 singular fibers over divisors $b_i$

| Algebra $G$ | ord($f$) | ord($g$) | ord($\Delta$) |
|---|---|---|---|
| $A_N$ | 0 | 0 | $N+1$ |
| $D_N$ | 2 | 3 | $N+2$ |
| $E_6$ | $\geq 3$ | 4 | 8 |
| $E_7$ | 3 | $\geq 5$ | 9 |
| $E_8$ | $\geq 4$ | 5 | 10 |

- $G_i$-brane loci $b_i$ ◂·······························▸

**Abelian U(1)$_A$ vectors**

- Sections $s_A : B_2 \dashrightarrow Y_3$ to the elliptic fibration
- U(1)$_A$-brane loci $b_A$ (height pairing)

# Gauge Algebras

## Physics of Axio-dilaton Profile

- **6d F-theory**
  - IIB string on compact 2-fold $B_2$ w/ 7-branes on complex curves (varying axio-dilaton)
  - 7-brane configuration encoded in an elliptic Calabi-Yau 3-fold $\pi : Y_3 \to B_2$
    - $y^2 = x^3 + f_4\, x + g_6$ ⋯⋯▸ discriminant $\Delta := 4f^3 + 27g^2 = 0$ ◂⋯▸ 7-brane loci $b_i$

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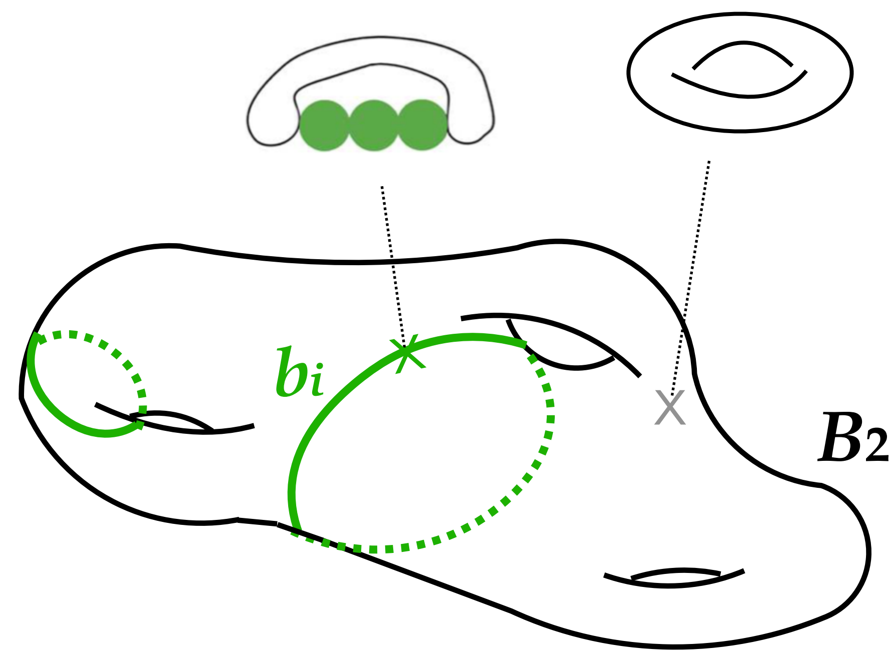


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# Non-abelian Group and Abelian Rank

Physics of the Arithmetic

- **The Mordell-Weil (MW) Group**
  - The rational sections to an elliptic fibration $\pi : Y_3 \to B_2$ form a group
  - **MW theorem** (1929). The MW group is a finitely-generated abelian group.

$$\mathrm{MW}(Y_3) \simeq \mathbb{Z}^N \oplus \mathrm{MW}(Y_3)_{\mathrm{Tors}}$$

- **The Abelian Gauge Sectors**
  - The continuous abelian gauge sector
    - encoded in the free part of the MW group:
      $\Rightarrow \mathrm{rank}(\mathrm{MW}(Y_3)) = N_{U(1)}$
  - The global structure of the gauge group
    - encoded in the torsional part of the MW group:
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    - rationale: analysis of the Shioda image of an $n$-torsional section reveals that the coweight lattice of the group is finer by order $n$ than that of the universal cover [Mayrhofer, Morrison, Till, Weigand '14]

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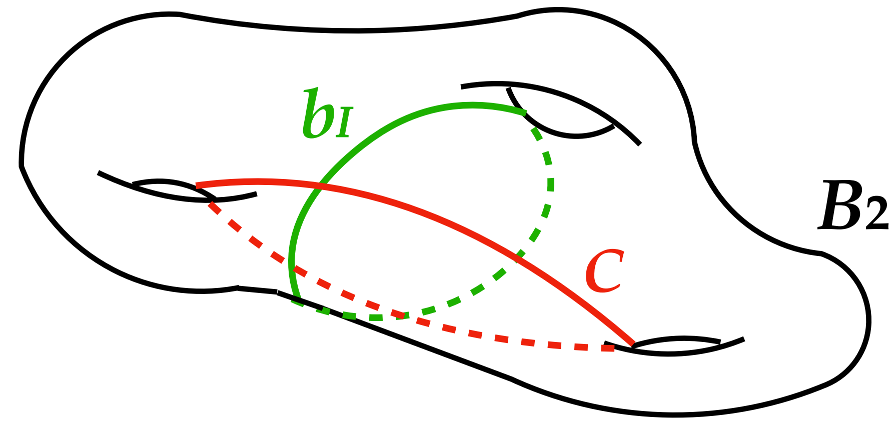
# Solitonic Strings

## Heterotic String

- **Branes on Curves**
  - 7-brane on $b_I$: 6d vector multiplet
  - 3-brane on $C$: effective string ($Q = [C]$)




- **Solitonic Strings**
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    [Martucci '14], [Haghighat, Murthy, Vafa, Vandoren '15], [Lawrie, Schafer-Nameki, Weigand '16]

    - the 3-7 modes present at $k_I$ ($:= C \cdot b_I$) points are charged under $G_I$
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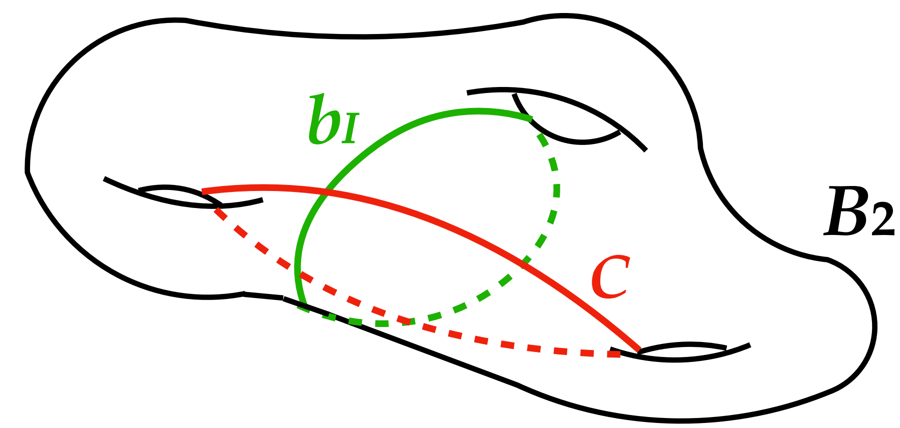


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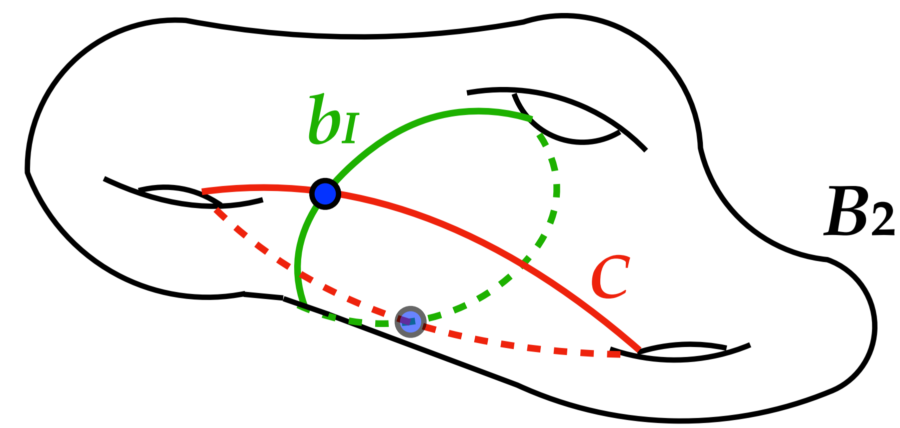


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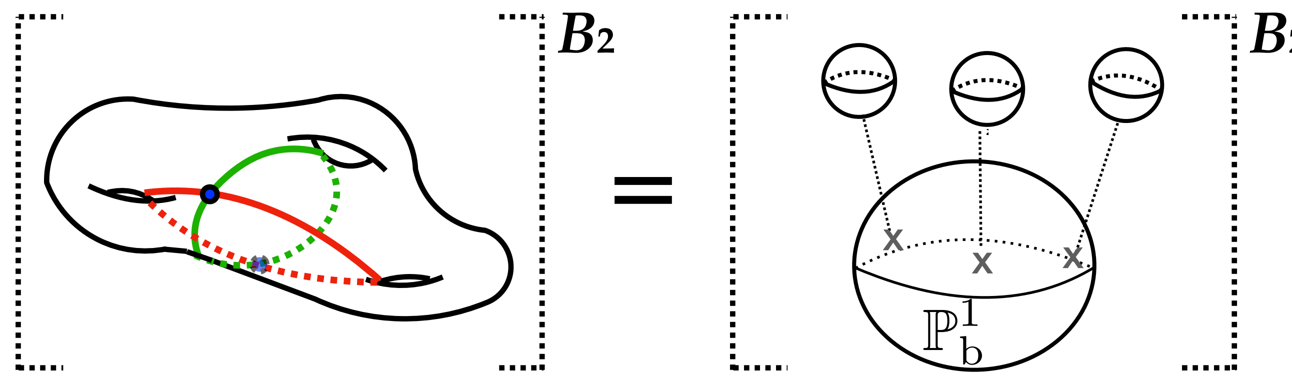


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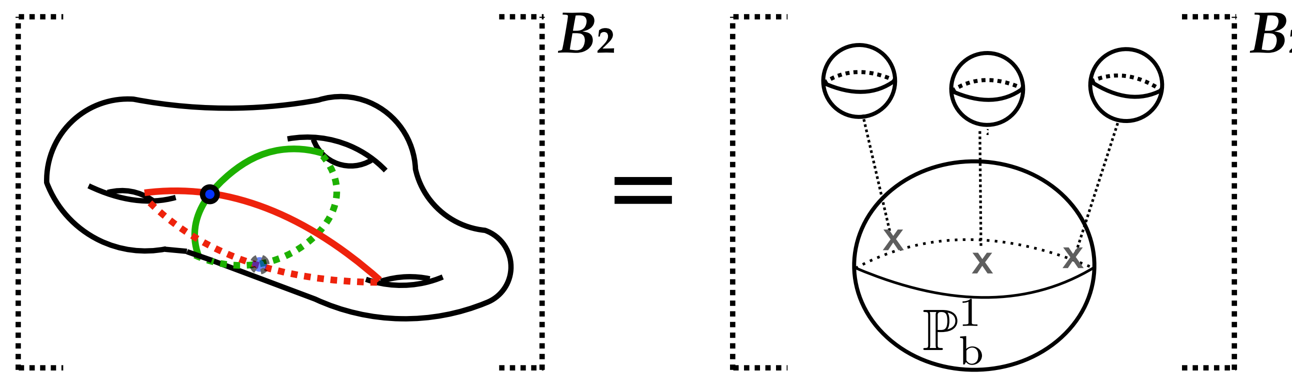


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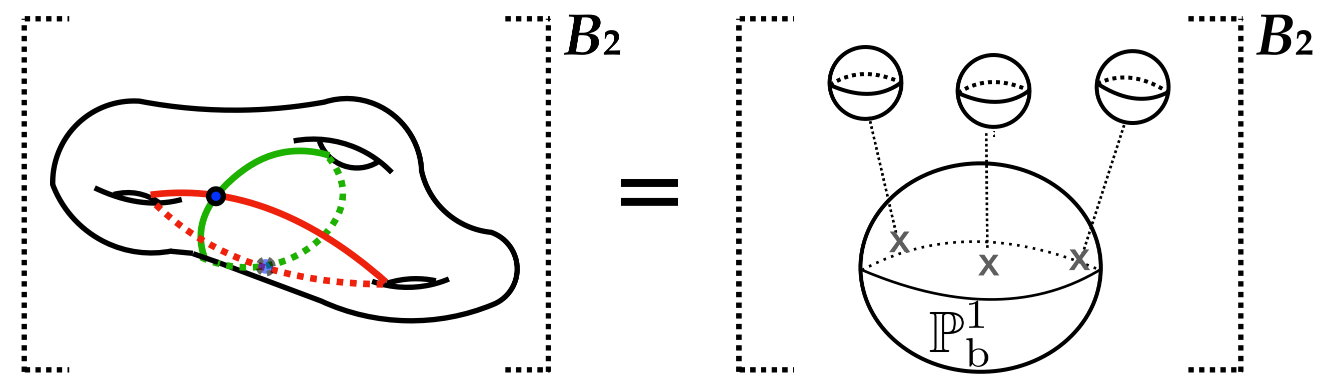


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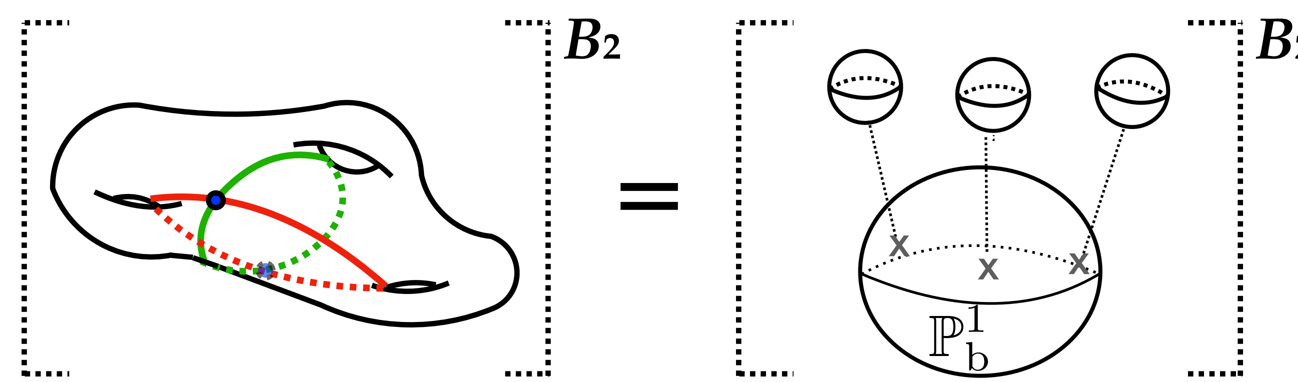


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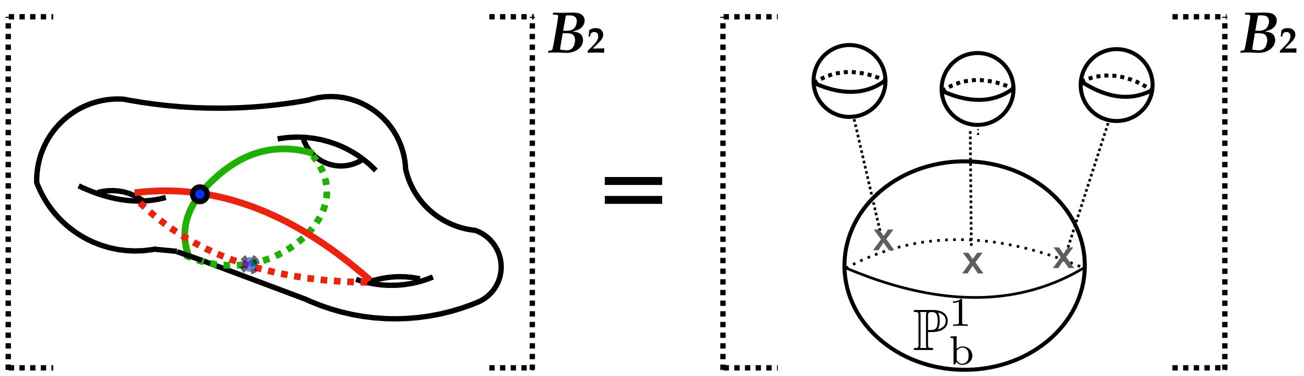


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# Outline

## Introduction

Motivation and Main Results

## Rudiments of 6d F-theory

Gauge Sector: Non-abelian and Abelian Sectors

Solitonic Strings: Heterotic String

## Part I. Bounds

Global Structure of the Non-abelian Sector

Rank of the Abelian Sector

## Part II. Limits

Non-minimal Brane Stack at Infinite vs. Finite Distance

## Conclusions

Summary and Discussion

# Anomalies of 1-Form Gauge Group

Global Structure Bound on the (0-Form) Gauge Sector

- **1-Form Symmetries**
  - Higher-form symmetries [Gaiotto, Kapustin, Seiberg, Willet '14]
    - extended objects are charged; necessarily abelian (gauged in quantum gravity)
  - General constraints on discrete 1-form gauge sector to constrain the global structure
    - specified by the embedding vector $s = (s_i \in \mathbb{Z}_{n_i})$ for each generator of $\Gamma$
- **Anomaly Constraints**
  - Center 1-form symmetry and its gauging [Apruzzi, Dierigl, Lin '20]
    - requires absence of mixed anomaly involving the 1-form symmetry & the tensors (2-form in 6d):

$$\sum_i \frac{n_i - 1}{2n_i}\, s_i^2\, k_i(Q) \equiv 0 \pmod 1 \qquad \text{e.g., for } \hat{G}_i = SU(n_i)$$

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global form of the 0-form gauge sector

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Level *given as* $k_i(Q) = C \cdot b_i$ *for a solitonic string w/ charge* $Q = [C]$

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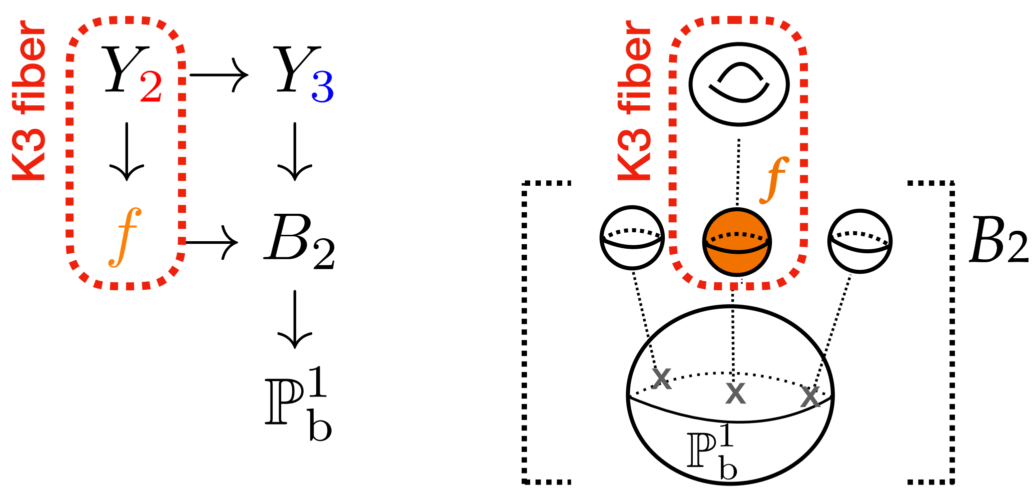


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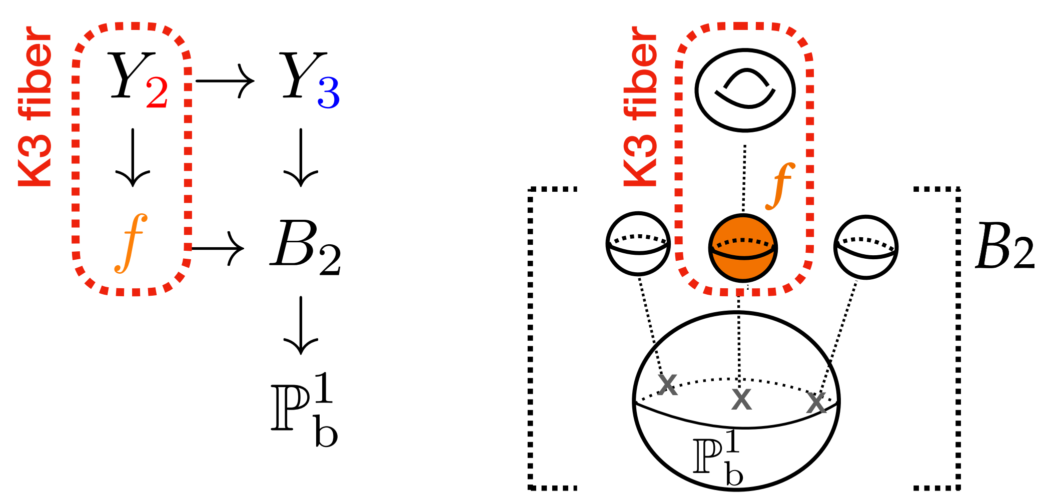


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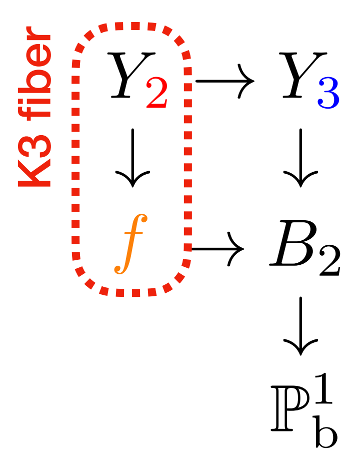


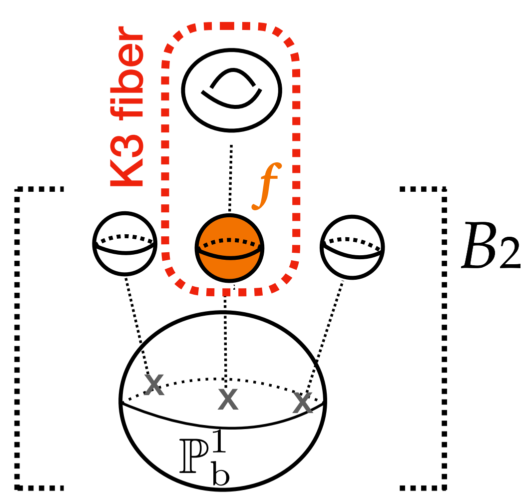


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- **Insight from Heterotic String/Curve** [S.-J.L., Oehlmann '22]

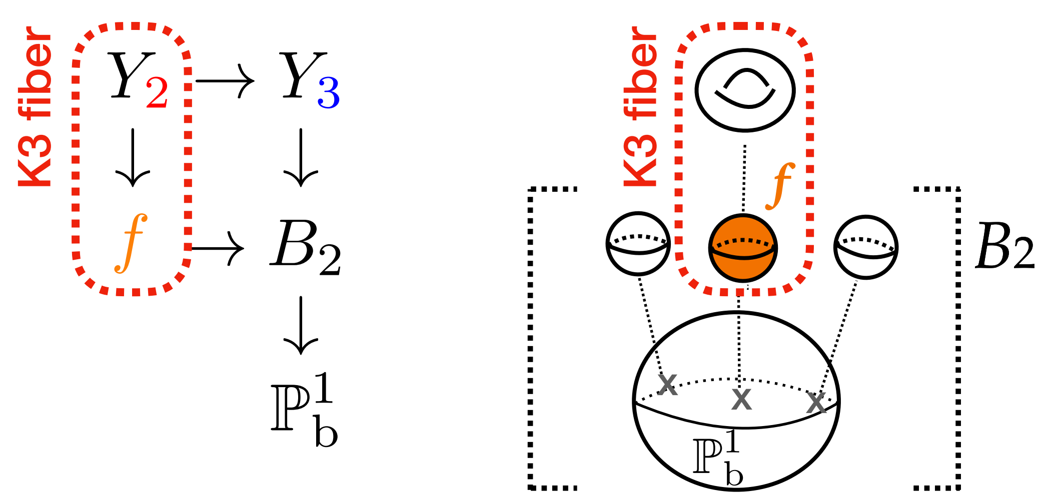
  - **Claim:** 1-form gauge sector $\Gamma$ of 6d F-theory w/ $n_T > 0$ can only take the form:

    $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_m \in \mathcal{T} \subsetneq \mathcal{T}^*$, *i.e.*, $(n, m)$=(1,1), ..., (1,6); (2,2), (2,4); (3,3).

  - **Heterotic Insight:** sever global structure bound for 8d F-theory persist in 6d F-theory!

  - **CY 3-folds as a Nested $T^2$-/K3-fibration** (cf.) ubiquity of nesting [Anderson, Gao, Gray, S.-J.L. '17]

    - $n_T > 0 \iff$ 2-fold base $B_2$ is $\mathbb{P}^1$-fibered
    - (tensionful) heterotic string as D3 on a $\mathbb{P}^1$-fiber (cf.) tensionless at infinite distance [S.-J.L., Lerche, Weigand '18]




  - **MW inclusion:** $\mathrm{MW}(Y_3) \subseteq \mathrm{MW}(Y_2) \Rightarrow \Gamma = \mathrm{MW}(Y_3)_{\mathrm{Tors}} \subseteq \mathrm{MW}(Y_2)_{\mathrm{Tors}} \in \mathcal{T}^*$

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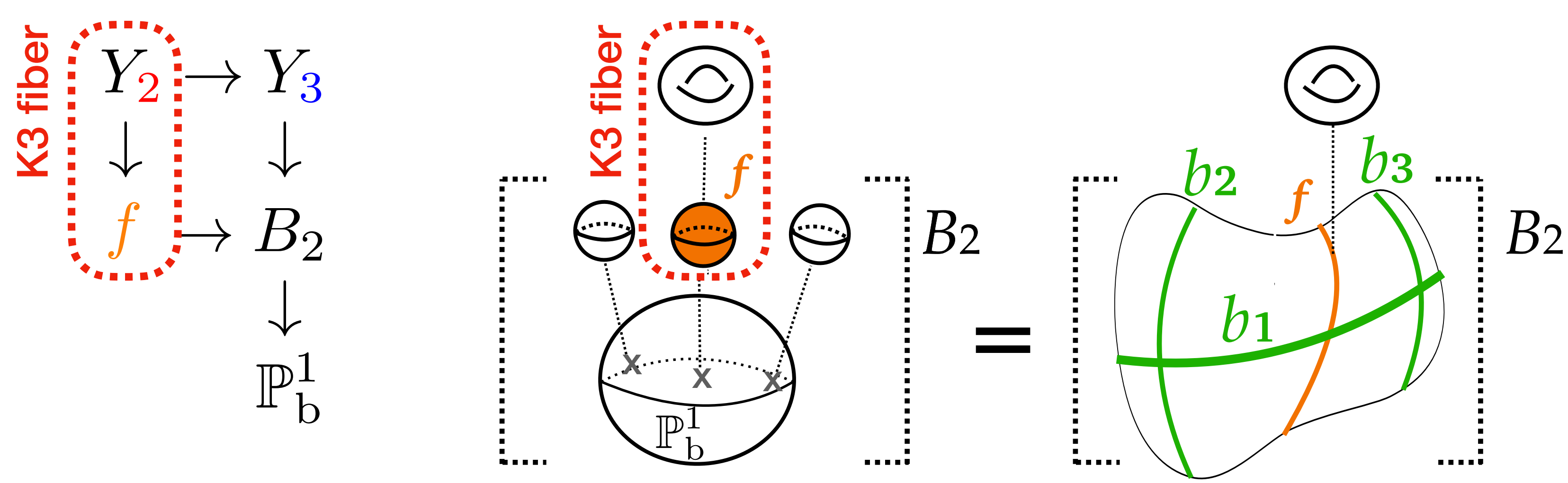


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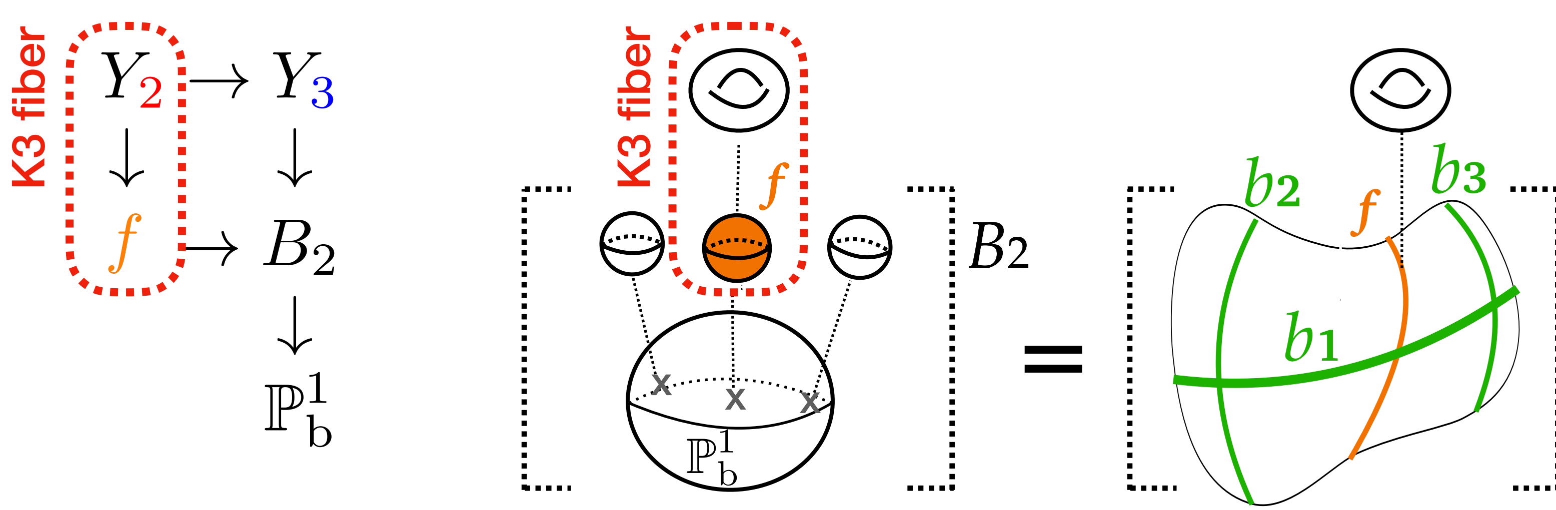


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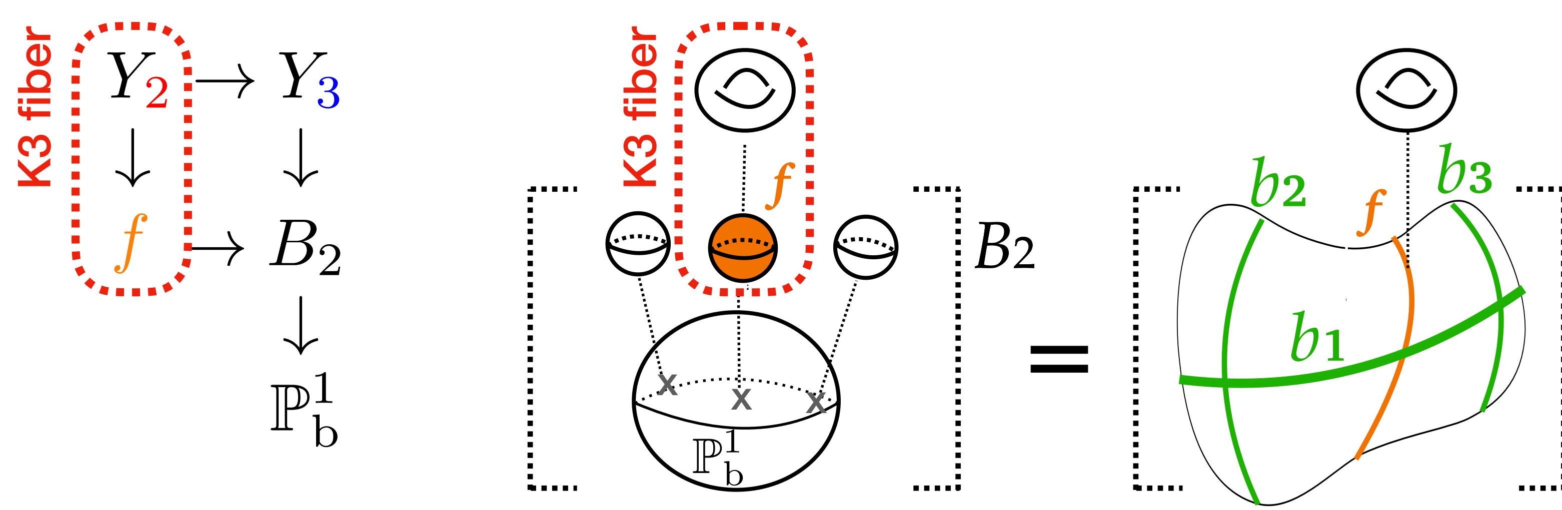


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# Ruling out the Outliers in $\mathcal{T}^* - \mathcal{T}$

Geometry and Physics

- **Geometric Viewpoint**
  - The K3 surfaces w/ $\mathrm{MW}(Y_2) \in \mathcal{T}^* - \mathcal{T}$ are extremal [Miranda, Persson '89]
    - **no complex structure deformations** for a (non-trivial) fibration.

- **Physical Viewpoint**
  - Each outlier leads to a **unique gauge group** (rank-18 non-abelian) [Miranda, Persson '89]
    - e.g., for $\Gamma = \mathbb{Z}_7$: $G = SU(7)^3/\mathbb{Z}_7$ w/ the embedding s=(1,2,3) (cf.) [Cvetic et al. '20]
  - Charged matter would render the **7-brane configuration inconsistent**
    - e.g., for $\Gamma = \mathbb{Z}_7$:

1. **Matter** must be charged under **multiple SU(7)s** in a special manner (for $\mathbb{Z}_7$-neutralness).
2. **Matter multiplicities** are subject to a certain divisibility condition for **anomaly cancellation**
3. The divisibility criterion is only fulfilled if **each brane locus has a positive genus.**
4. The genus constraint leads to $7(b_1 + b_2 + b_3) > 12\bar{K}_{B_2} = [\Delta]$ ⟶ *contradiction*!

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# Models w/o Tensor Multiplets

## E-String Transitions and Validity of the Bound

- **Claim** [S.-J.L., Oehlmann '22]
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- **Sketch of Derivation** — E-string transition & Tuning

  $$\Gamma = \mathrm{MW}(Y_3)_{\mathrm{Tors}} \subseteq \mathrm{MW}(Y_3^{(\mathrm{tune})})_{\mathrm{Tors}} = \mathrm{MW}(\hat{Y}_3)_{\mathrm{Tors}} \in \mathcal{T}$$

  developing conformal matter    E-string transition

  - **Recall: MW torsion via a specific tuning** [Aspinwall, Morrison '98]

    e.g., $\Gamma = \mathbb{Z}_2$ case: $\begin{cases} f = a_4 - 1/3\, a_2^2 \\ g = 1/27\, a_2(2a_2^2 - 9a_4) \end{cases}$ w/ $a_i \in H^0(\mathbb{P}^2, \mathcal{O}(3i))$

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  - **E-string transition: gain a tensor multiplet** w/ MW torsion kept intact

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# Connection to the Abelian Rank Bound

6d F-theory with $n_T > 0$

- **Heterotic Insight — Global Structure Bound**
  - If $n_T > 0$: global structure of non-abelian sector visible to (bounded by) heterotic string
  - If $n_T = 0$: the same heterotic bound applicable via E-string transition

- **Heterotic Insight — Abelian Rank Bound**
  - If $n_T > 0$: abelian gauge sector visible to heterotic string [S.-J.L., Weigand '19]
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    - unitarity [Kim, Shiu, Vafa '19] on the heterotic string: $N_{U(1)} \leq c_L(f) = 20 \quad \text{for } Q = f$
  - New insight from MW inclusion [S.-J.L., Oehlmann '22]
    - $N_{U(1)} = \text{rk}(\text{MW}(Y_3)) \leq \text{rk}(\text{MW}(Y_2)) \leq 20 - 2 = 18$
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(cf.) U(1)s still visible to the string with $Q = L$ (line in $\mathbb{P}^2$) [S.-J.L., Weigand '19]

- conservative bound is $c_L(L) = 32$ but the generic, stronger bound is conjectured to work:

$Q = L$ string on $\mathbb{P}^2$ ⟷ $Q = f + h$ string ("H + E" string) on $dP_1$ - overcounting!

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$c_L(Q) = 3\,(Q \cdot Q) - 9\,(K_{B_2} \cdot Q) + 2$

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(cf.) U(1)s still visible to the string with $Q = L$ (line in $\mathbb{P}^2$) [S.-J.L., Weigand '19]

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# Connection to the Abelian Rank Bound

6d F-theory with $n_T > 0$

- **Heterotic Insight — Global Structure Bound**
  - If $n_T > 0$: global structure of non-abelian sector visible to (bounded by) heterotic string
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# Outline

## Introduction

Motivation and Main Results

## Rudiments of 6d F-theory

Gauge Sector: Non-abelian and Abelian Sectors

Solitonic Strings: Heterotic String

## Part I. Bounds

Global Structure of the Non-abelian Sector

Rank of the Abelian Sector

## Part II. Limits

Non-minimal Brane Stack at Infinite vs. Finite Distance

## Conclusions

Summary and Discussion

# Going Beyond the Global Structure Bound

Geometry and Physics

- **Recall: F-theory on a Weierstrass model** $Y_d = \{y^2 = x^3 + f_4\, x + g_6\}$
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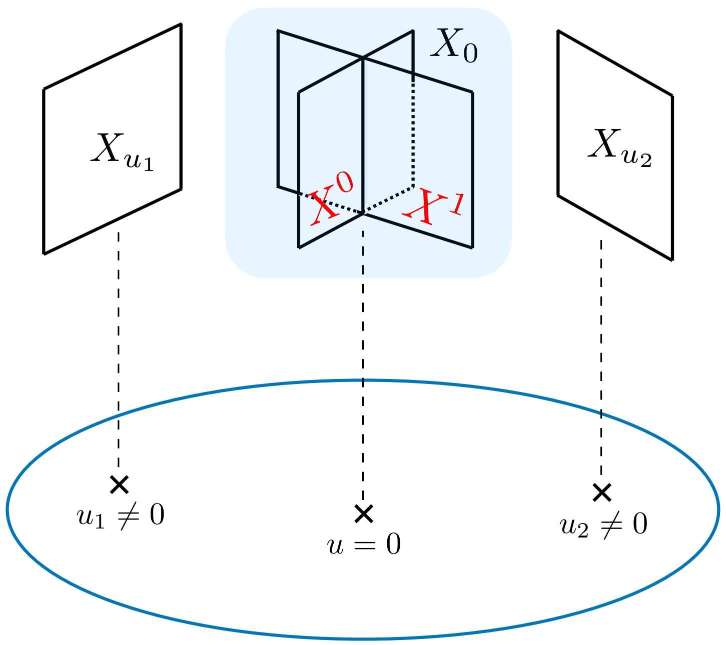
**Focus today**: **non-minimal brane stacks (codim-1)** — intuitions from **8d** F-theory (elliptic **K3**)

(cf.) novelties in 6d/CY3s [Alvarez-Garcia, S.-J.L., Weigand '23]

# Degeneration of K3

## Kulikov Models and Their Properties

- **Kulikov Models** [Kulikov '77], [Persson '77], [Friedman, Morrison '81]




  - **Degeneration - setup**
    - family of K3s $X_u$ degenerating at $u=0$: $X_0 = \cup X^i$
  - **Kulikov Model - definition and existence**
    - reduced, normal-crossing & trivial canonical bundle
    - achievable via base changes ($u \to u^\kappa$) and blow-ups/downs
  - **Classification of Kulikov Models: Type I** (finite distance) vs. **II/III** (infinite distance)
    - Type II: $X^i$s form a chain, $X^i \cap X^{i+1}$ are elliptic, and 2 transcendental 2-tori shrink
    - Type III: $X^i \cap X^j$ are rational and 1 transcendental 2-torus shrinks

- **8d F-theory w/ a Non-mininal Brane Stack**
  - Aim: modify the degenerate Weierstrass K3s to a Kulikov form, keeping them elliptic
    - How: base changes ($u \to u^\kappa$) & blow-ups/downs in the base
    - Why: non-minimal fibers improved; infinite-distance nature tested; universal physics manifest

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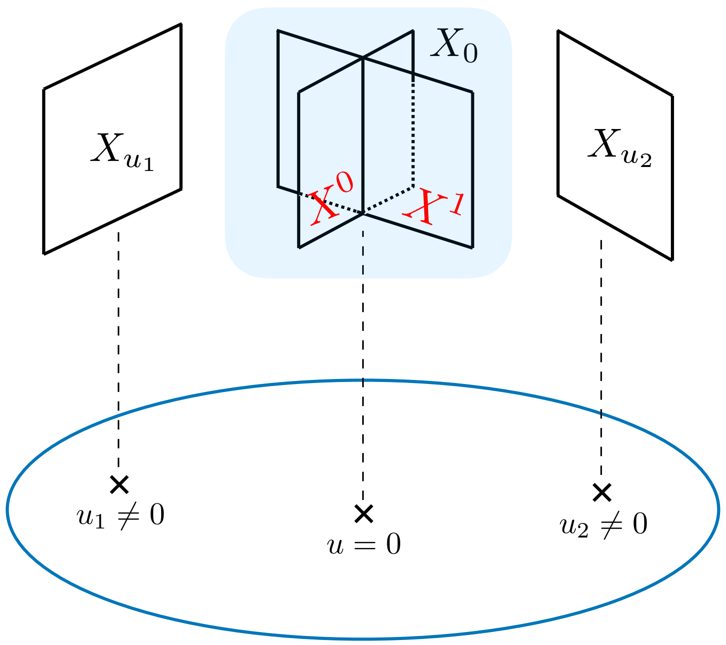


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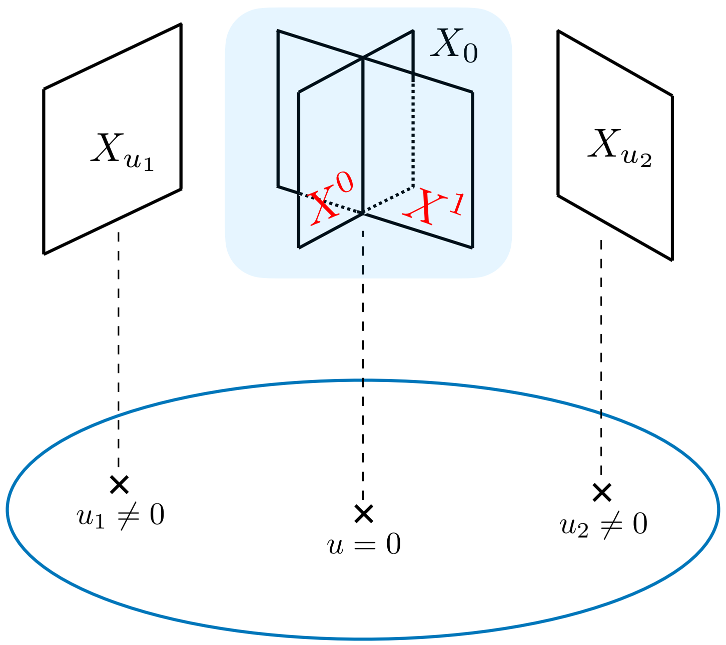


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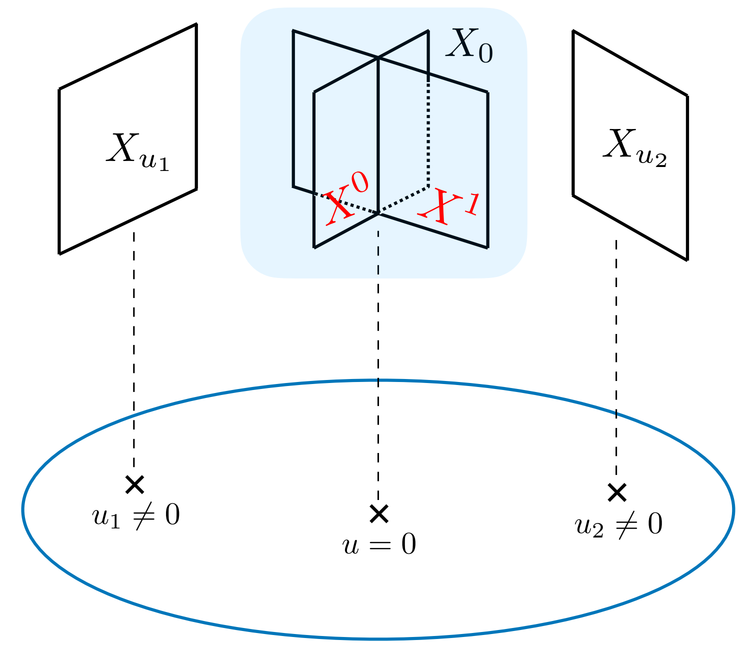


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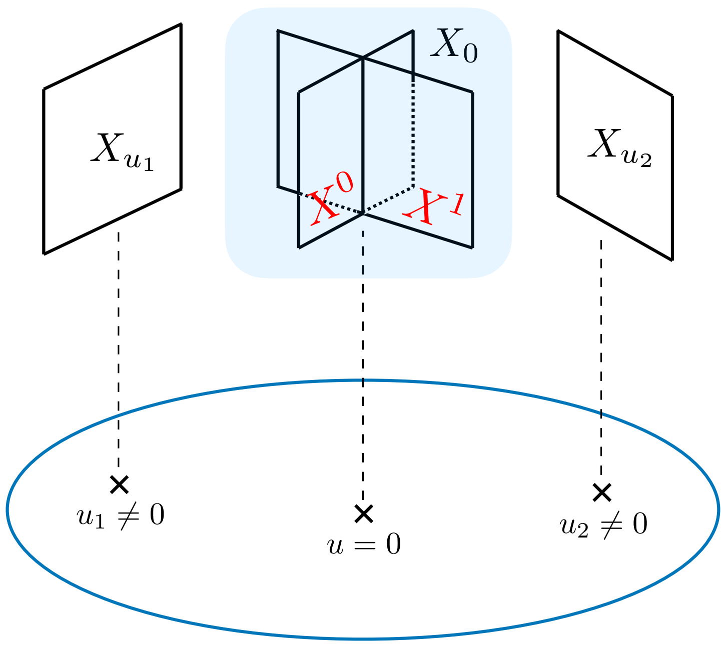


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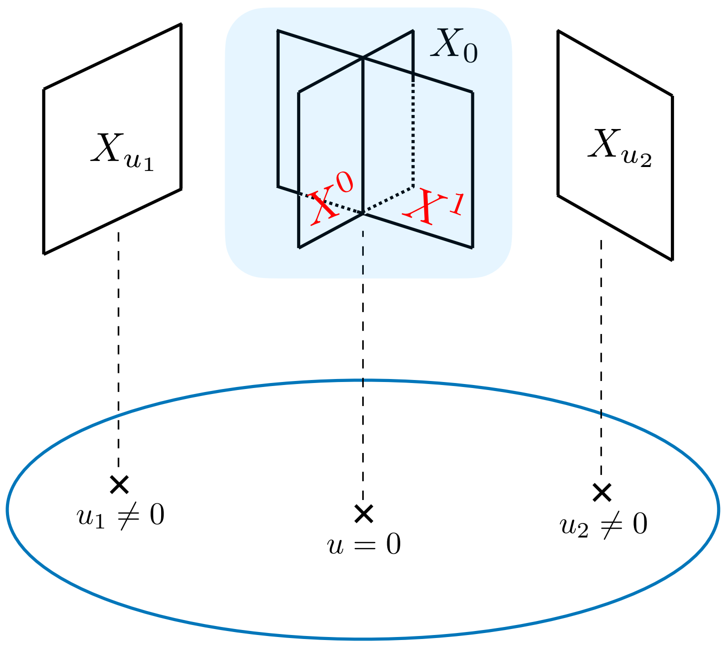


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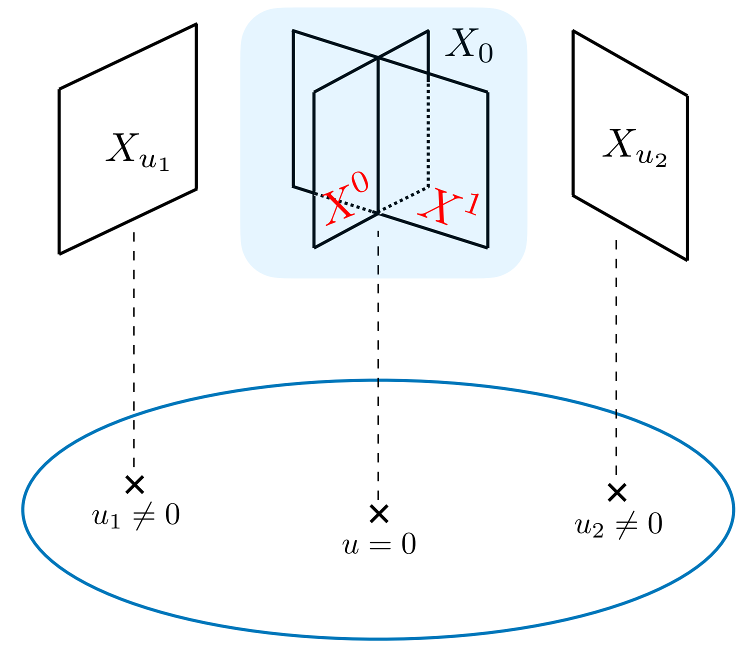


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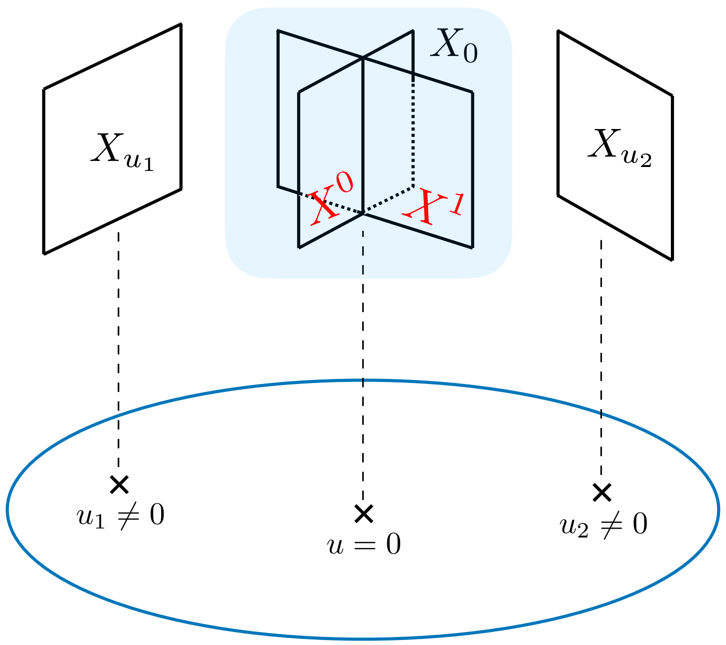


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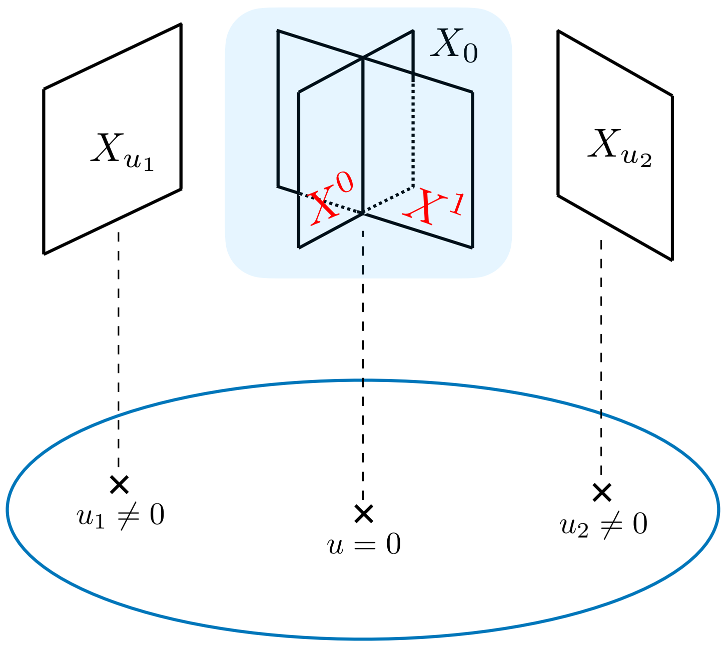


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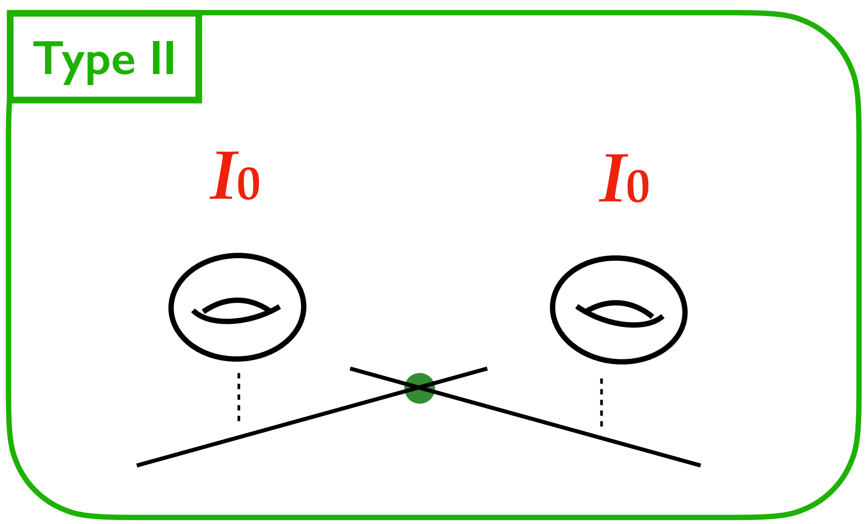


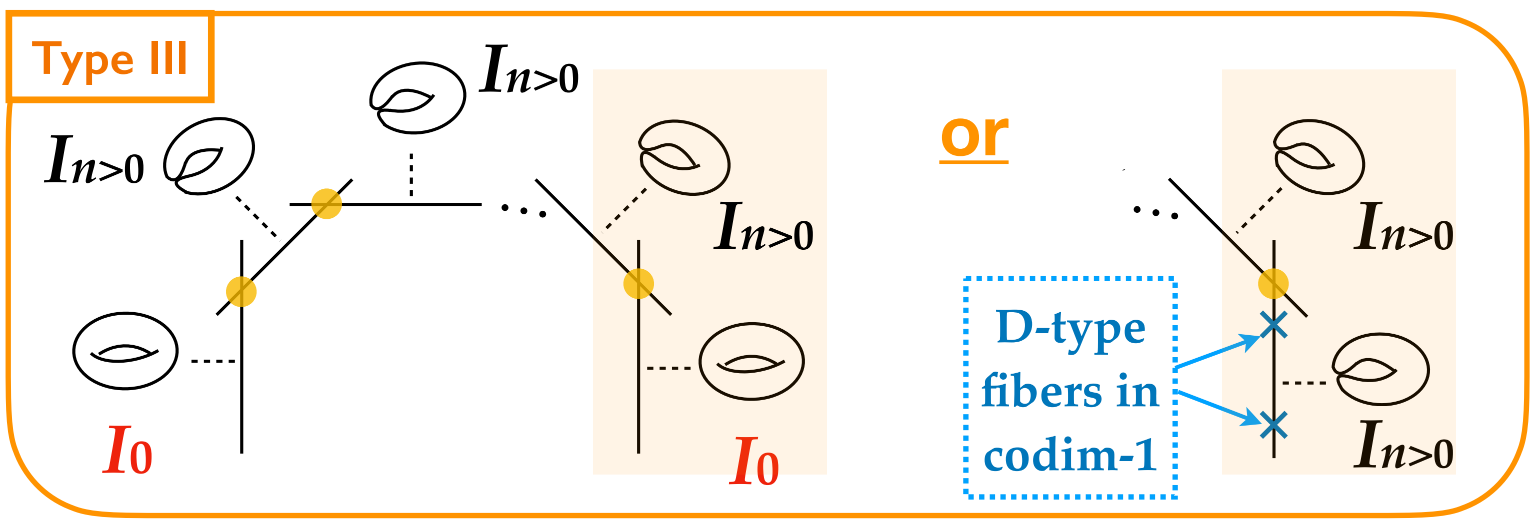

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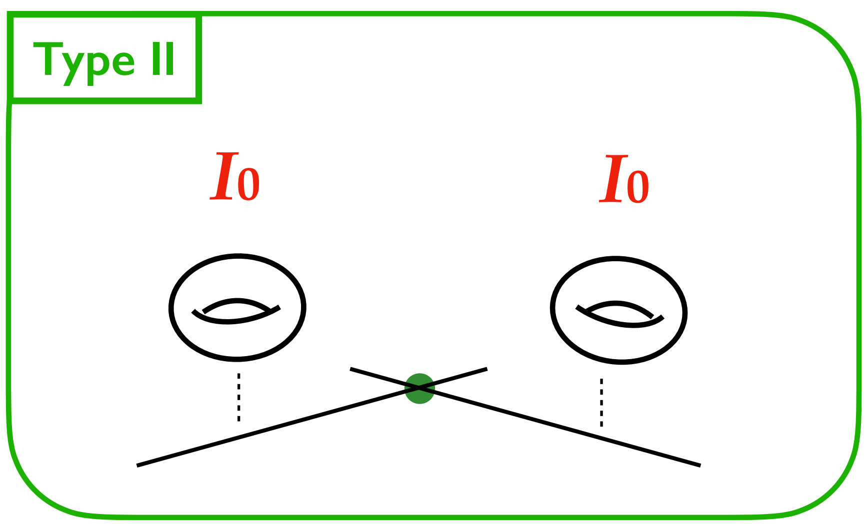


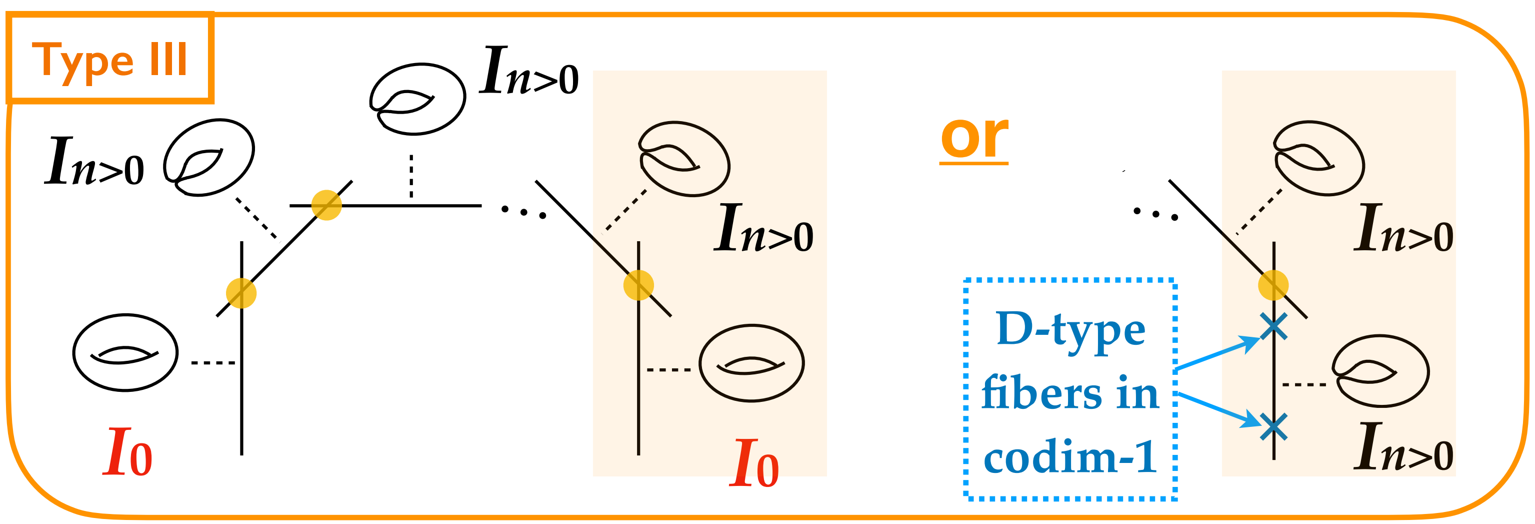


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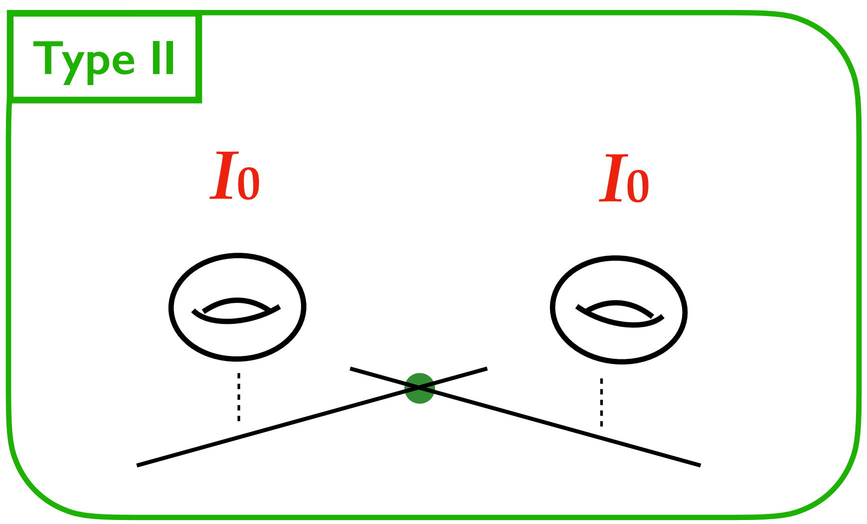


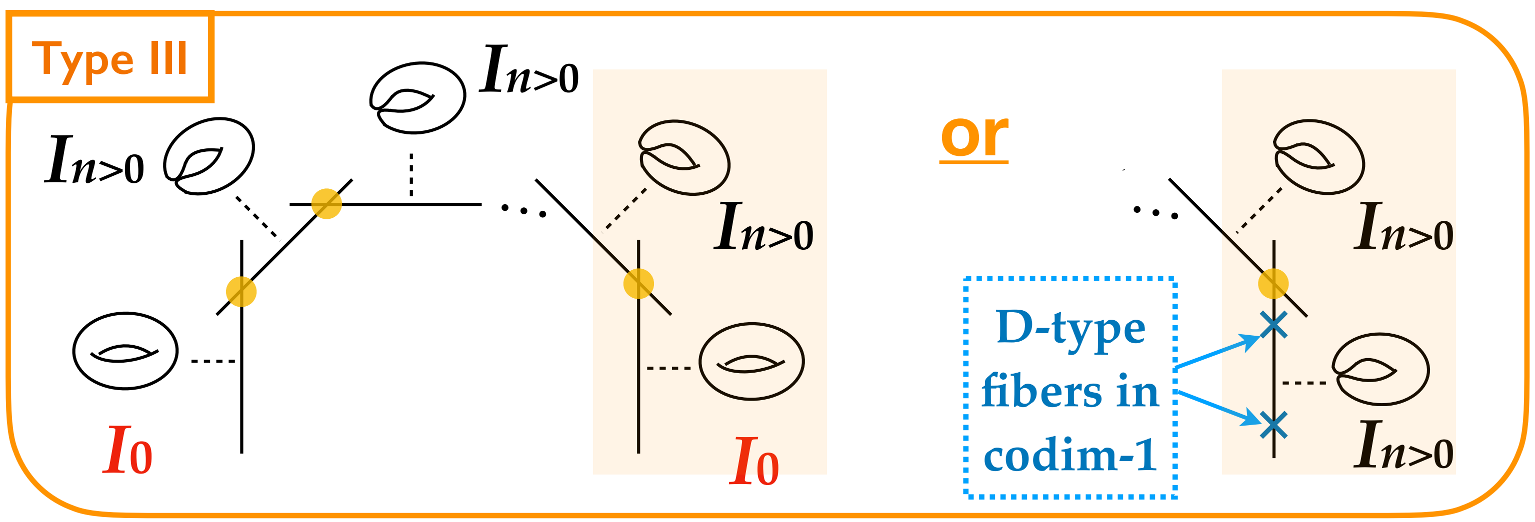


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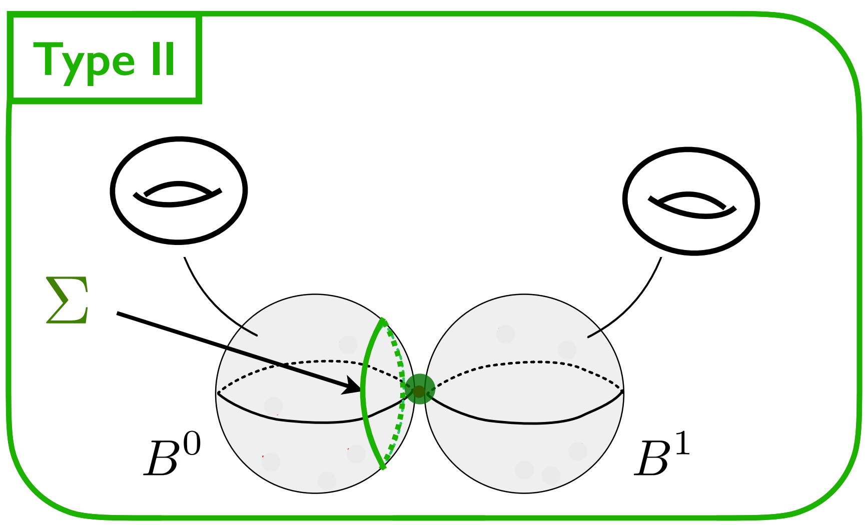


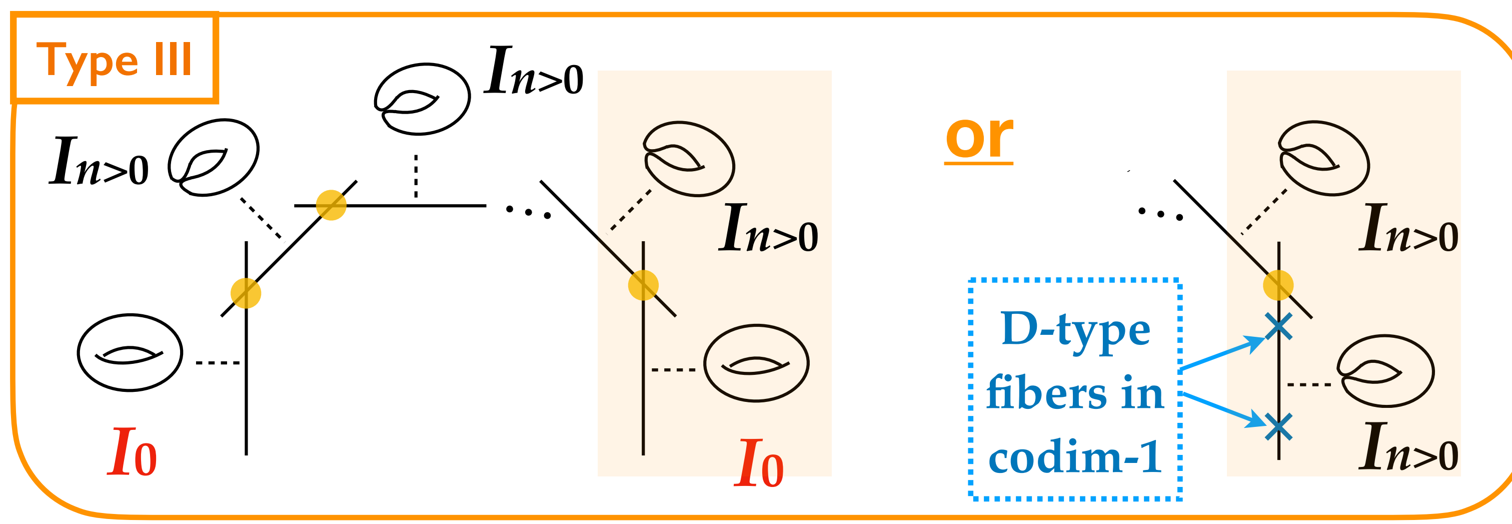


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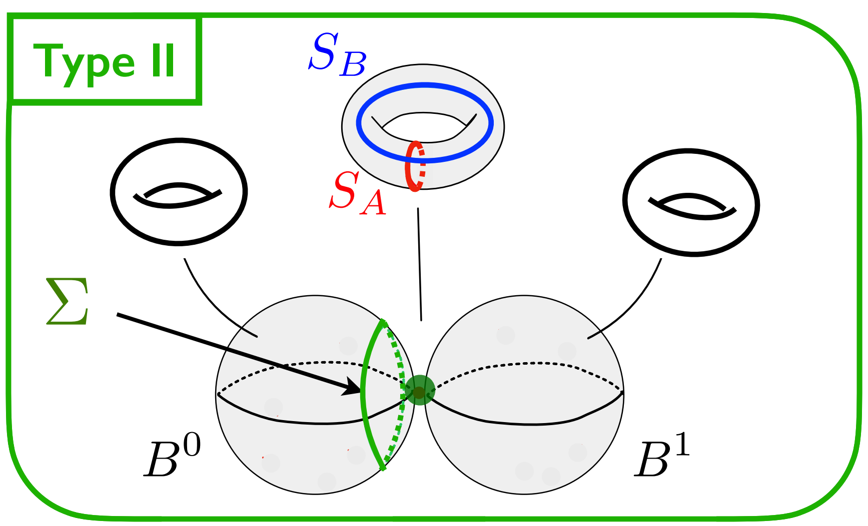


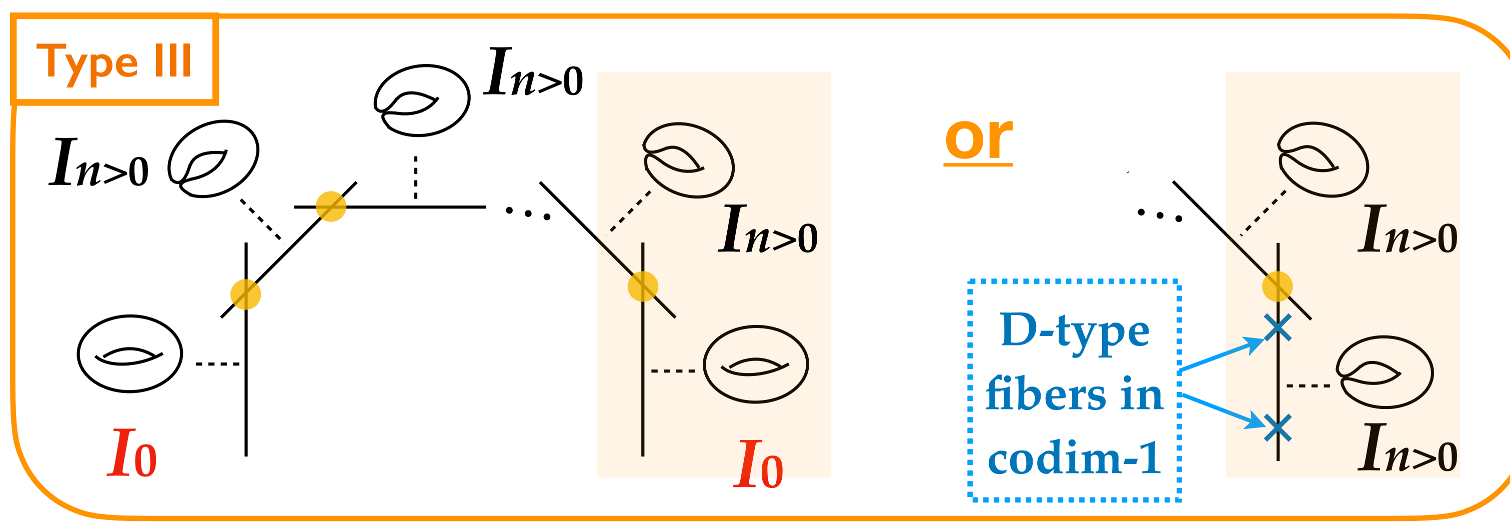


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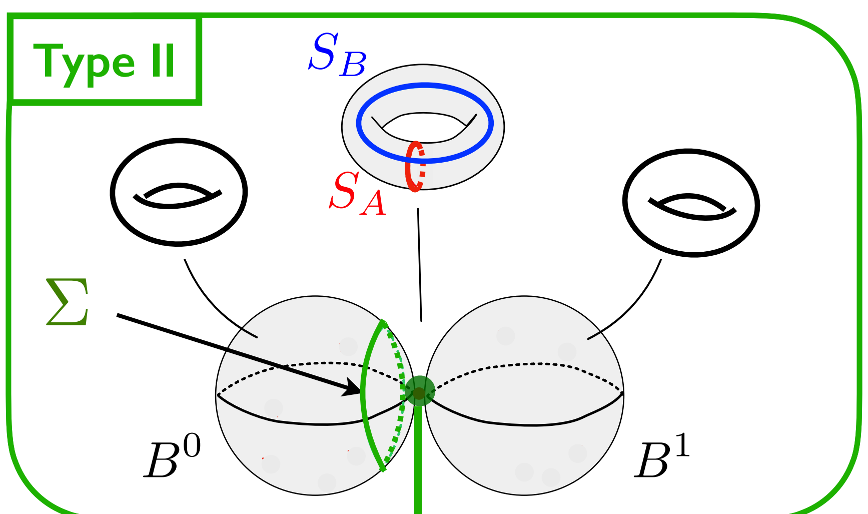


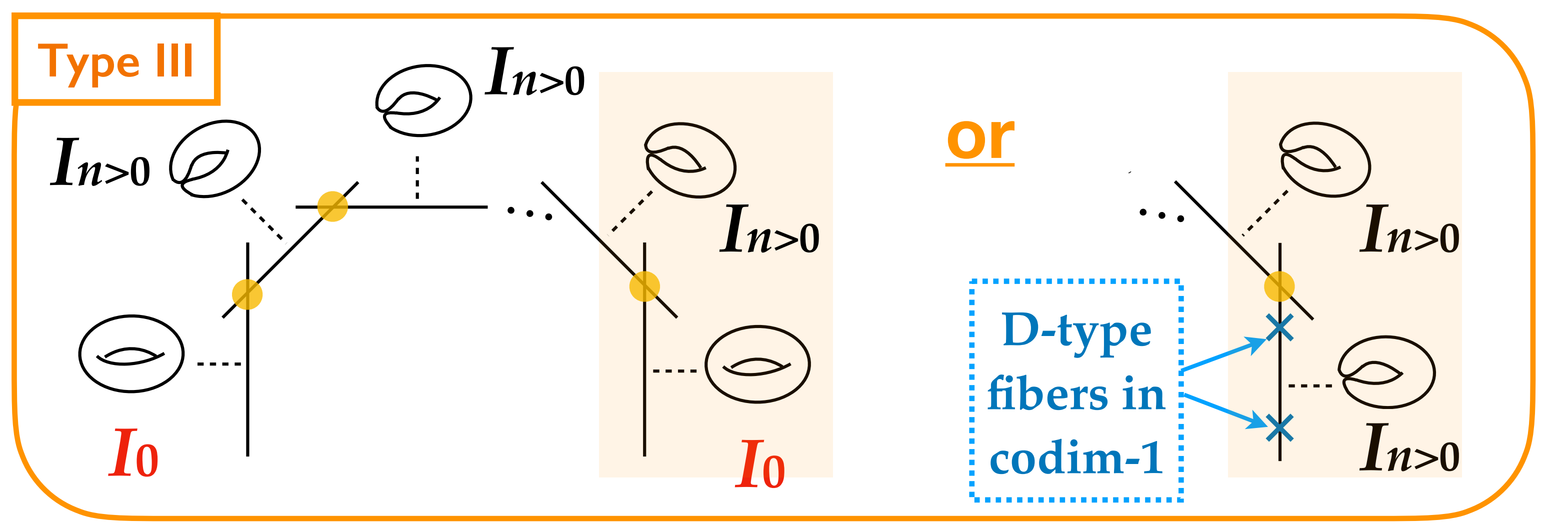


**Vanishing 2-tori**

$\gamma_A = S_A \times \Sigma$

$\gamma_B = S_B \times \Sigma$

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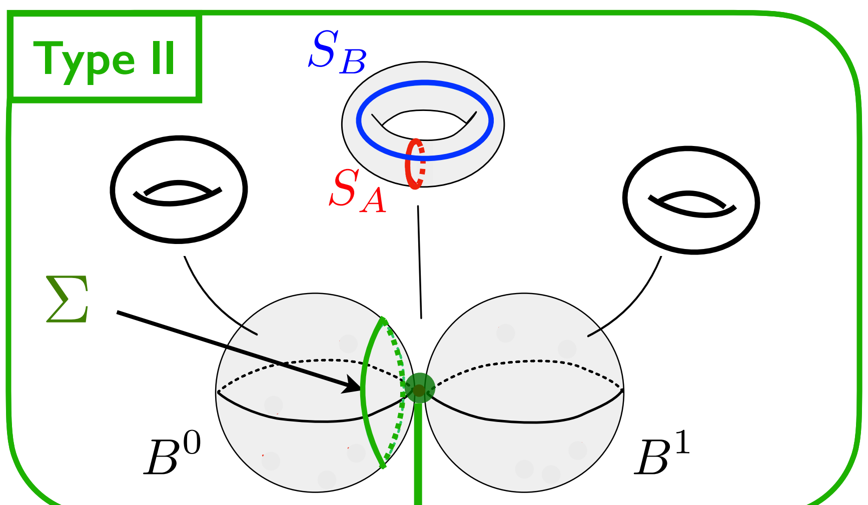


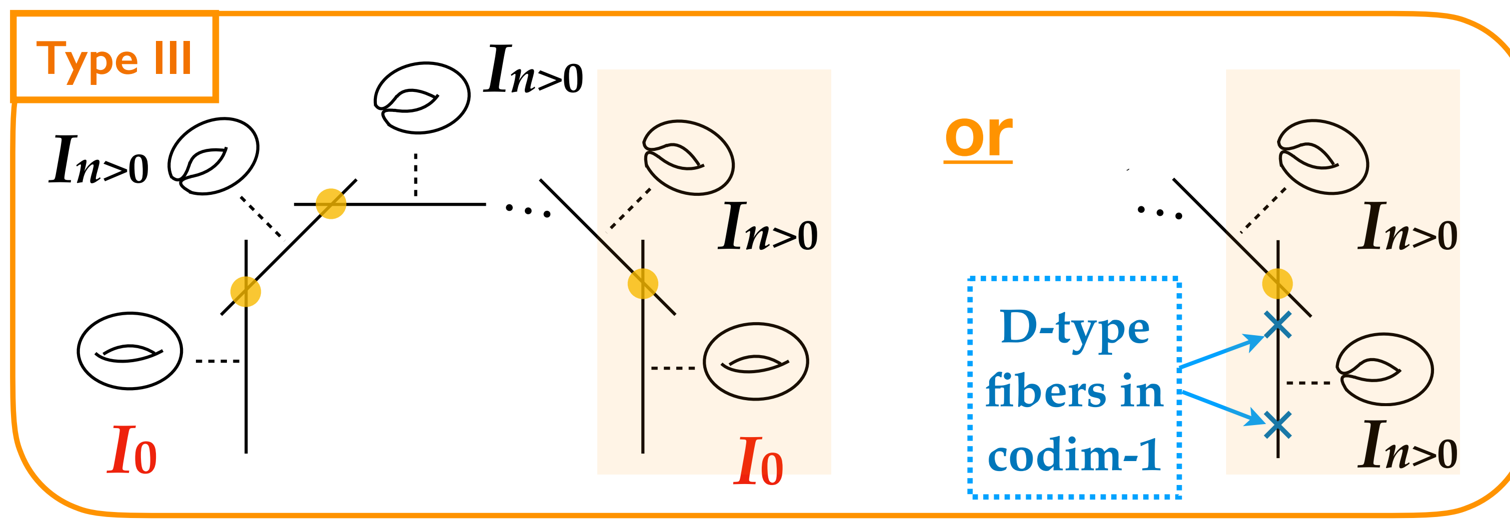


**Vanishing 2-tori** → **Light particle towers**

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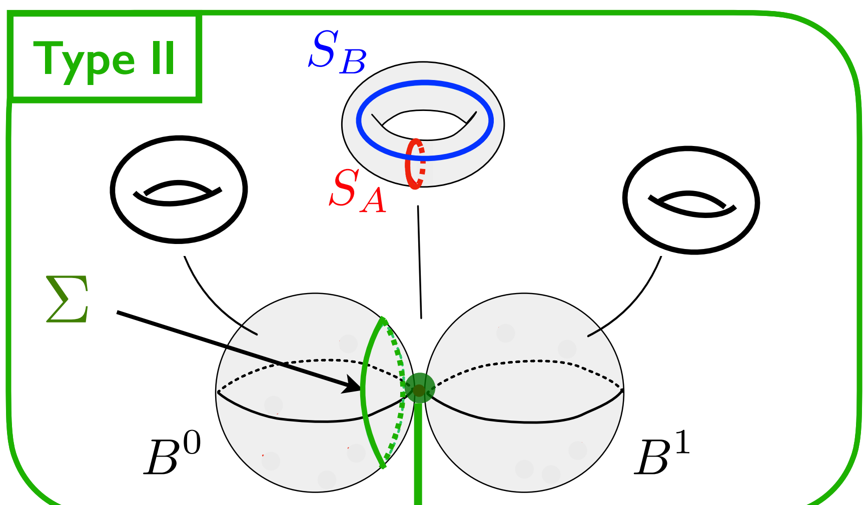


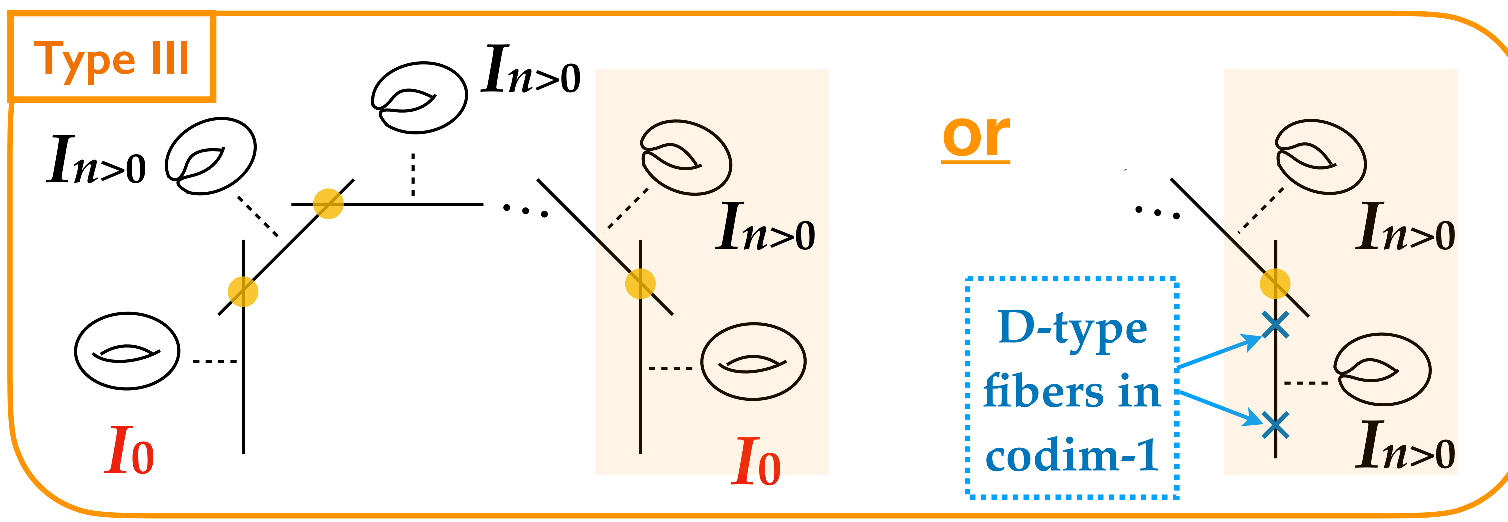


**Vanishing 2-tori** → **Light particle towers** → **Decompactification to 10d**

$\gamma_A = S_A \times \Sigma$

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- M2 branes on $\gamma_{A,B}$
- F/D-strings on $\Sigma$

- affinization [DeWolfe, Hauer, Iqbal, Zwiebach '98]

(cf.) heterotic [Collazuol, Grana, Herraez '22]

CHL vacua [Cvetic, Dierigl, Lin, Zhang '22]

# Extending the Kodaira-Neron Classification

## Kulikov Types via Vanishing Orders

[S.-J.L., (Lerche,) Weigand '21] & [Alvarez-Garcia, S.-J.L., Weigand] to appear

- **Inclusion of Non-minimal Codim-1 Fibers/Brane Stacks**

$$\mathrm{ord}_{\hat{Y}_0}(f, g, \Delta)\big|_{\mathrm{pt}\,\in\,\mathbb{P}^1} = \begin{cases} (\geq 4, \geq 6, \quad 12) \\ (\quad 4, \quad 6, > 12) \end{cases} \begin{array}{l} \Rightarrow \text{Type II} \\ \Rightarrow \text{Type III} \end{array}$$

"Given a complex structure limit of 8d F-theory, whether at ***finite*** or ***infinite*** distance, we can read off its **Kulikov type** and **characteristic physics** just* from the **vanishing order triple!**"

* in presence of **strictly** non-minimal fibers, we first need to improve them, for which full Weierstrass data should be used

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Strictly Non-minimal

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- **Inclusion of Non-minimal Codim-1 Fibers/Brane Stacks**

$$\mathrm{ord}_{\hat{Y}_0}(f,g,\Delta)\big|_{\mathrm{pt}\,\in\,\mathbb{P}^1} = \begin{cases} (\geq 4, \geq 6, \quad 12) & \Rightarrow \text{Type II} \\ (\quad 4, \quad 6, > 12) & \Rightarrow \text{Type III} \\ (> 4, > 6, > 12) & \Rightarrow^{*} \text{Type I or Type II} \end{cases}$$

Strictly Non-minimal

"Given a complex structure limit of 8d F-theory, whether at ***finite*** or ***infinite*** distance, we can read off its **Kulikov type** and **characteristic physics** just* from the **vanishing order triple!**"

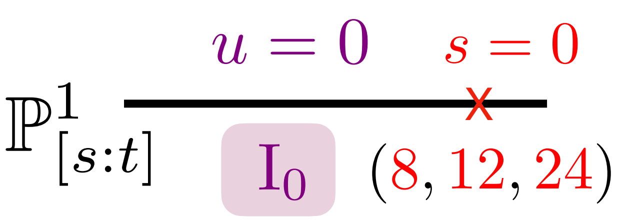
* in case the generic fibers at the end of "(f,g)-scaling-out" procedure are singular, even type III could arise

* in presence of **strictly** non-minimal fibers, we first need to improve them, for which full Weierstrass data should be used

# Non-minimal Brane Stack at Finite Distance

## Example: Type I Model in Disguise

- **Codim-1 Non-minimal Fiber w/ (>4, >6, >12) Vanishing**
  - Non-minimal fibers, if **strict**, may as well arise at **finite** distance
  - An "alerting" example
    - involves a non-minimal fiber w/ $\mathrm{ord}_{\hat{Y}_0}(f, g, \Delta)|_{s=0} = (8, 12, 24)$
    - turns into a **Type I** Kulikov model

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$$\Delta = 31 s^{24} + 12 s^{16} t^8 u^8 + 54 s^{12} t^{12} u^{12} + 12 s^8 t^{16} u^{16} + 31 t^{24} u^{24}$$

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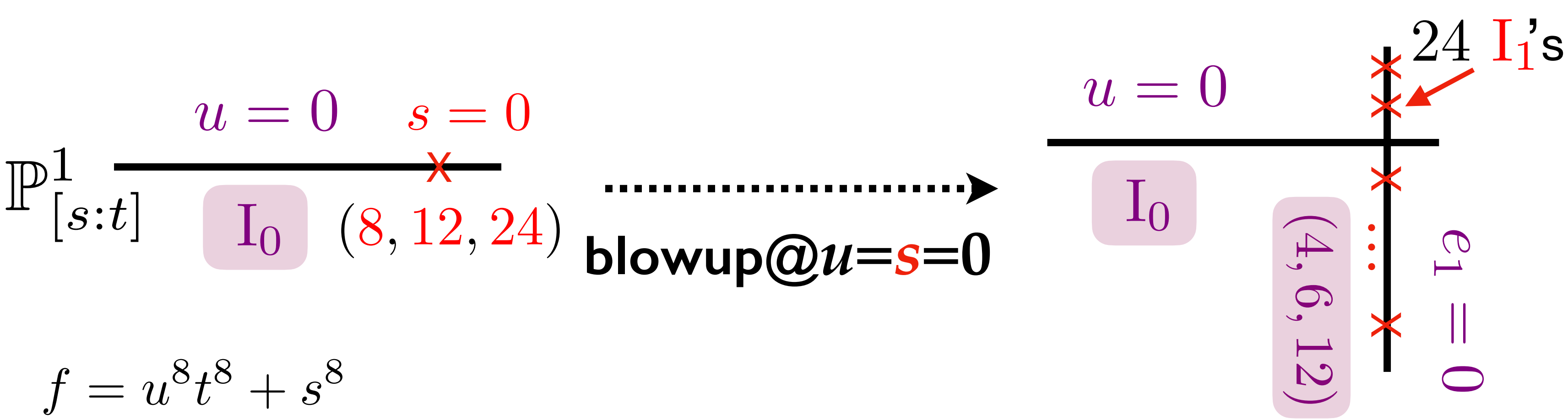


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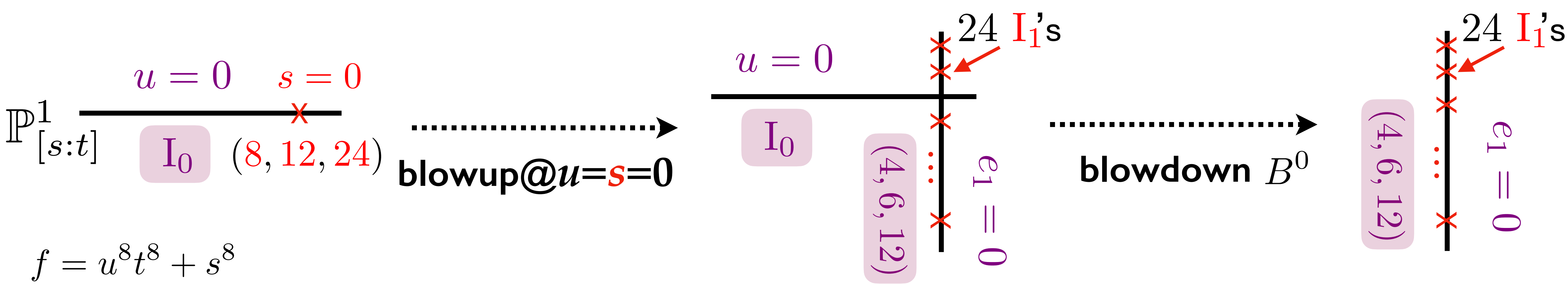


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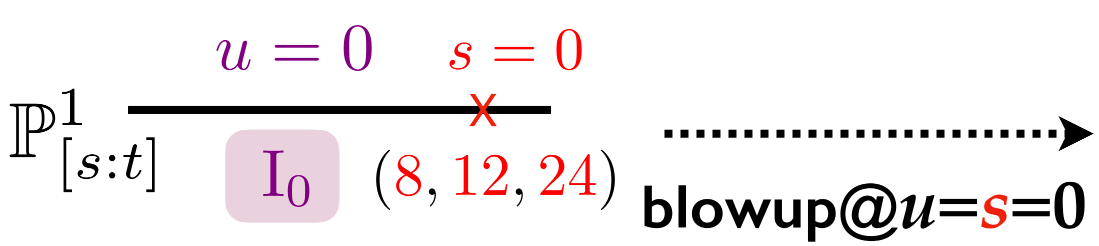


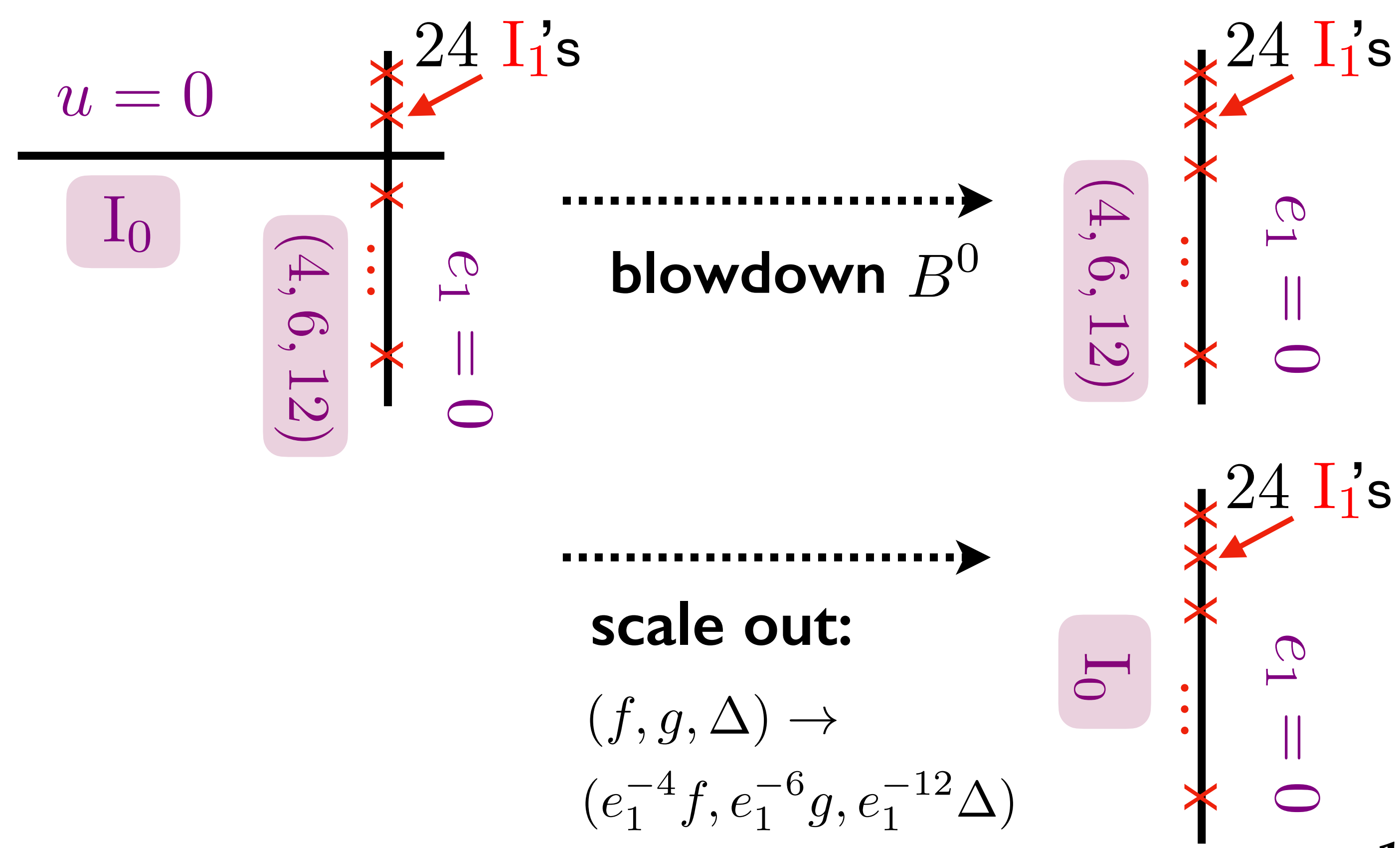


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# Outline

## Introduction

Motivation and Main Results

## Rudiments of 6d F-theory

Gauge Sector: Non-abelian and Abelian Sectors

Solitonic Strings: Heterotic String

## Part I. Bounds

Global Structure of the Non-abelian Sector

Rank of the Abelian Sector

## Part II. Limits

Non-minimal Brane Stack at Infinite vs. Finite Distance

## Conclusions

# *Summary and Discussion*

- **Bounds on gauge sector of 6d F-theory** (the global structure of the non-abelian sector & the rank of the abelian sector) have been (re-)derived via the **heterotic insight**:
  - The 1-form gauge sector & the 0-form U(1) gauge sector are both **visible** to the **heterotic string**!
- **Extreme limits of gauge sector of 6d F-theory**, potentially sitting at infinite distance, have been classified and analyzed:
  - Non-minimal brane stacks may sit either at **infinite** distance (**Type II** or **III**; **decompactifications**) or at **finite** distance (**Type I**; **standard gauge enhancements**)
- **The derived bounds** naturally connect to the **universal limiting behavior of EFTs**:
  - The heterotic string is the string whose "presence" & "criticality" are inferred at infinite distance
  - Violation of the global structure bound results in decompactification at infinite distance
- For a 6d N=(1,0) EFT, one may generally argue via charge completeness that a "heterotic" string exists, whose unitarity [Kim, Shiu, Vafa '19] leads to an abelian rank bound [S.-J.L., Weigand '19]
- It is desirable to understand/interpret/derive bounds also on other physical quantities

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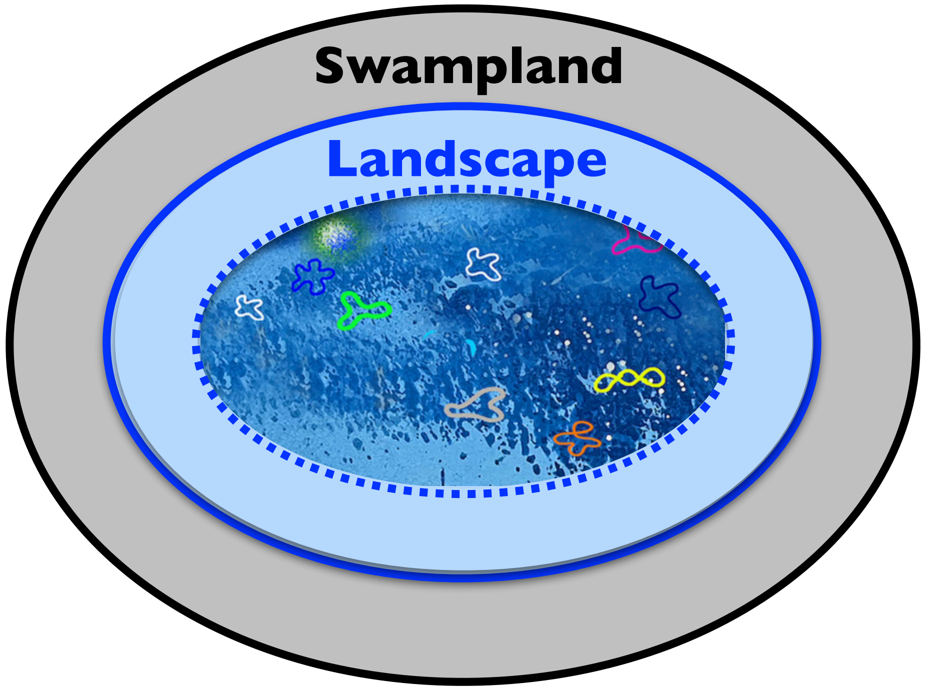
*Thank you*

# The Swampland Program

In Light of the String Landscape




- **Grouping the QFT Models**
  - Landscape (EFTs w/ a UV completion into QG)
    - contains the String Landscape (String EFTs)
  - **Swampland** (EFTs w/o a UV completion)

- **The Swampland Program**
  - Goal: distinguish EFTs in the Landscape from those in the **Swampland** [Vafa '05]
    - reveal common properties of quantum gravity theory (aka **Quantum Gravity Conjectures**)
  - String Landscape **as a guiding principle**
    - extract common physical properties of String EFTs
    - establish universal behaviors of the internal geometry

  → insights on quantum gravity conjectures!

# EFTs in the Moduli Space

Limits at Infinite Distance vs. Limitations at Finite Distance

- **Universal Features of EFTs at Infinite Distance**

**A tower of states become light w/ the mass scale** $m_0 \sim e^{-\alpha \frac{\Delta\phi}{M_{\rm Pl}}} M_{\rm Pl}$ [Ooguri, Vafa '06]

**EFTs *either* decompactify *or* reduce to a weakly-coupled string theory** [S.-J.L., Lerche, Weigand '19]

⇢ **Light tower of particles furnished by *either* Kaluza-Klein *or* string excitations!**

* Top-down evidence in [S.-J.L., Lerche, Weigand '18-'19], [(Baume,) Marchesano, Wiesner '19], [Xu '20], [Klawer, S.-J.L., Weigand, Wiesner '20], … (bulk Kahler); [Grimm, Palti, Valenzuela '18], [Grimm, Li, Palti '18], [Klemm, Joshi '19], [Grimm, Li, Valenzuela '19], … (bulk complex structure)

* Bottom-up intuitions e.g. in [Basile, Lust, Montella, '23], [Bedroya, Mishra, Wiesner, '24]

- **F-theoretic Evidence – Geometric Classification of Limits**

| **Bulk Geometry** (Kahler) | **Brane Configuration** (complex structure) |
|---|---|
| unique tensionless critical string<br>[S.-J.L., Lerche, Weigand '18-'19], [Klawer, S.-J.L., Weigand, Wiesner '20] | light dual-KK modes<br>[S.-J.L., Lerche, Weigand '21], [Alvarez-Garcia, S.-J.L., Weigand '23][2] |

⬇

| | |
|---|---|
| general constraints from the same string? | decompactification beyond expected bounds? |

# Main Results on Gauge Sector: Bounds & Limits

What this talk will be about

- **EFTs in Scrutiny**
  - Supergravity EFTs w/ minimal SUSY (main focus: EFTs of F-theory, mostly in 6d)

- **Bounds on the Gauge Sector (6d)** as seen by a solitonic string
  - Goal: constrain the global structure of the non-abelian gauge sector
    - **order of each cyclic quotient:** $m \leq 6$
  - Bonus: find a connection to the rank bound on the abelian gauge sector
    - **number of U(1) factors:** $N_{U(1)} \leq 18$

- **Limits of the Gauge Sector (6d/8d)** as what sets the bound
  - Goal: classify unconventional "non-minimal" stacks of coalescing 7-branes
    - non-minimal stacks at ***infinite*** distance: decompactifications
  - Alert: distinguish finite distance limits in disguise from the classification
    - non-minimal stacks at ***finite*** distance: standard gauge enhancement

# Gauge Algebras

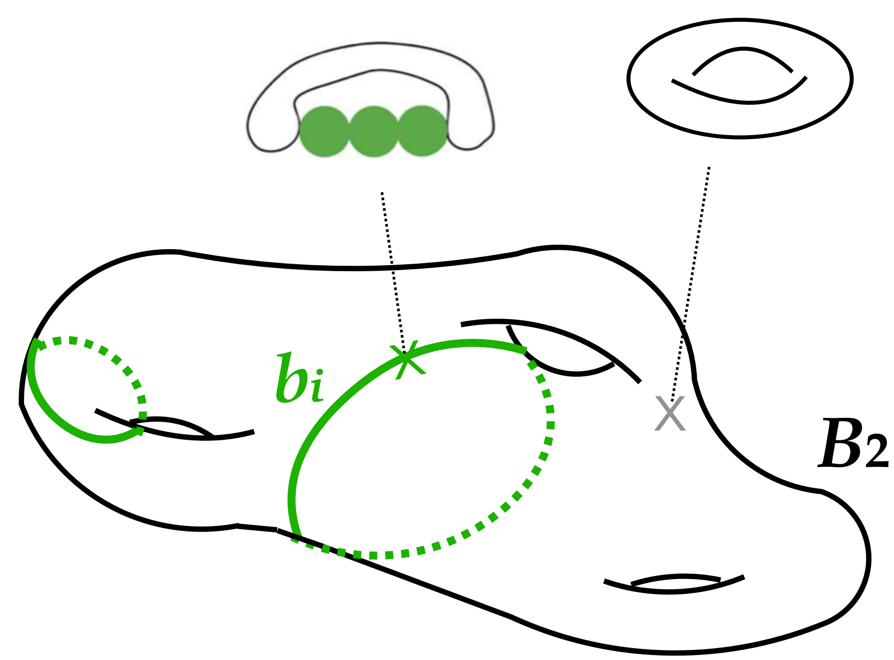
Physics of Axio-dilaton Profile

- **6d F-theory**
  - IIB string on compact 2-fold $B_2$ w/ 7-branes on complex curves (varying axio-dilaton)
  - 7-brane configuration encoded in an elliptic Calabi-Yau 3-fold $\pi : Y_3 \to B_2$

- $y^2 = x^3 + f_4\, x + g_6$ ⋯⋯➝ discriminant $\Delta := 4f^3 + 27g^2 = 0$ ⟷ 7-brane loci $b_i$

$f_4 \sim \bar{K}_{B_2}^{\otimes 4}$, $g_6 \sim \bar{K}_{B_2}^{\otimes 6}$

- **Gauge Algebras in F-theory**




**Non-abelian $G_i$ vectors**

- Codim-1 singular fibers over curves $b_i$

| Algebra $G$ | ord($f$) | ord($g$) | ord($\Delta$) |
|---|---|---|---|
| $A_N$ | 0 | 0 | $N+1$ |
| $D_N$ | 2 | 3 | $N+2$ |
| $E_6$ | $\geq 3$ | 4 | 8 |
| $E_7$ | 3 | $\geq 5$ | 9 |
| $E_8$ | $\geq 4$ | 5 | 10 |

**Abelian U(1)$_A$ vectors**

- Sections $s_A : B_2 \dashrightarrow Y_3$ to the elliptic fibration

- $G_i$-brane loci $b_i$ ⟷ U(1)$_A$-brane loci $b_A$ (height pairing) $:= -\pi_*(\sigma(s_A)\cdot\sigma(s_A))$

# Non-abelian Group and Abelian Rank

Physics of the Arithmetic

- **The Mordell-Weil (MW) Group**
  - The rational sections to an elliptic fibration $\pi : Y_3 \to B_2$ form a group
  - **MW theorem** (1929). The MW group is a finitely-generated abelian group.

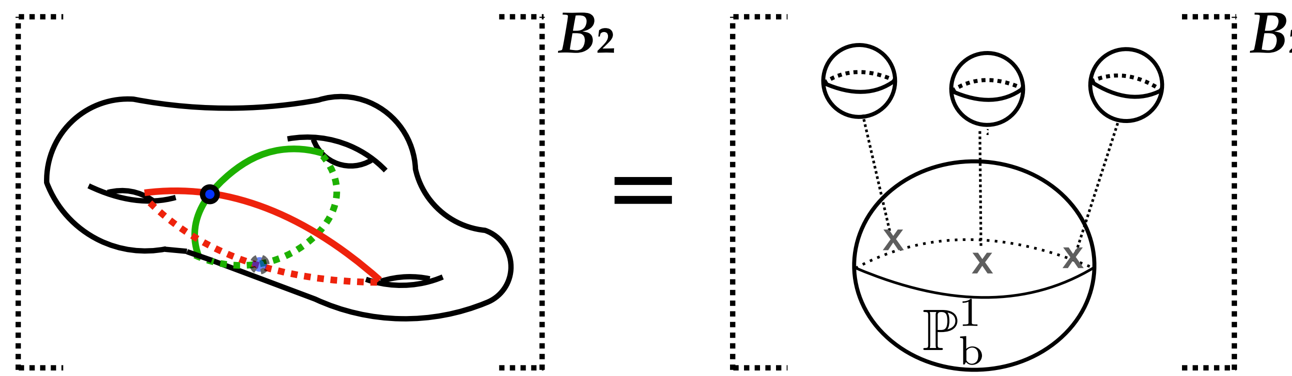
$$\mathrm{MW}(Y_3) \simeq \boxed{\mathbb{Z}^N} \oplus \boxed{\mathrm{MW}(Y_3)_{\mathrm{Tors}}}$$

- **The Abelian Gauge Sectors**
  - The continuous abelian gauge sector
    - encoded in the free part of the MW group:
      $N := \mathrm{rank}(\mathrm{MW}(Y_3)) = N_{U(1)}$
  - The global structure of the gauge group
    - encoded in the torsional part of the MW group:
      $\mathrm{MW}(Y_3)_{\mathrm{Tors}} \simeq \mathbb{Z}_n \oplus \mathbb{Z}_m \quad \Rightarrow \quad G = \hat{G}/(\mathbb{Z}_n \times \mathbb{Z}_m)$
    - **rationale**: analysis of the Shioda image of an $n$-torsional section reveals that the coweight lattice of the group is finer by order $n$ than that of the universal cover [Mayrhofer, Morrison, Till, Weigand '14]

# Solitonic Strings

Heterotic String

- **Branes on Curves**
  - 7-brane on $b_I$ : 6d vector multiplet
  - 3-brane on $C$: effective string (Q = [$C$])




- **Solitonic Strings**
  - N=4 SYM along $C$ w/ varying coupling → N=(0,4) worldsheet theory of a 6d string

    [Martucci '14], [Haghighat, Murthy, Vafa, Vandoren '15], [Lawrie, Schafer-Nameki, Weigand '16]
    - the 3-7 modes present at $k_I$ ($:= C \cdot b_I$) points are charged under $G_I$
    - topology of the curve $C$ in $B_2$ determines the associated string type
  - Essentially all 2-fold bases are $\mathbb{P}^1$-fibered, $p : B_2 \to \mathbb{P}^1_{\mathrm{b}}$ $\Longleftrightarrow$ #(tensor multiplets) $= n_T > 0$
    - **full list**: $\mathbb{P}^2, \mathbb{F}_k,$ and blowups thereof [Grassi '91], [Gross '93] (all but $\mathbb{P}^2$ is $\mathbb{P}^1$-fibered & has more than one 2-forms)
  - A distinguished string from the $\mathbb{P}^1$-fiber $f = p^{-1}(\mathrm{pt.} \in \mathbb{P}^1_{\mathrm{b}})$ $\Longleftrightarrow$ heterotic string
    - focus first on **generic F-theory vacua w/ heterotic string** (w/ $n_T > 0$)
    - separately analyze models on $B_2 = \mathbb{P}^2$ (w/ $n_T = 0$) afterwards

# Anomalies of 1-Form Gauge Group

Global Structure Bound on the (0-Form) Gauge Sector

- **1-Form Symmetries**
  - Higher-form symmetries [Gaiotto, Kapustin, Seiberg, Willet '14]
    - extended objects are charged; necessarily abelian (gauged in quantum gravity)
  - General constraints on discrete 1-form gauge sector to constrain the global structure
    - discrete 1-form gauge sector $\Gamma \subset \prod_i Z(\hat{G}_i) = \prod_i \mathbb{Z}_{n_i}$ ⟷ associated w/ global form of the 0-form gauge sector $G = (\prod_i \hat{G}_i)/\Gamma \times U(1)^N$

      **Our Focus**: *non-abelian*
    - specified by the embedding vector $s = (s_i \in \mathbb{Z}_{n_i})$ for each generator of $\Gamma$

- **Anomaly Constraints**
  - Center 1-form symmetry and its gauging [Apruzzi, Dierigl, Lin '20]
    - requires absence of mixed anomaly involving the 1-form symmetry & the tensors (2-form in 6d):

$$\sum_i \frac{n_i - 1}{2n_i} s_i^2 k_i(Q) \equiv 0 \pmod 1 \qquad \text{e.g., for } \hat{G}_i = SU(n_i)$$

Level *given as* $k_i(Q) = C \cdot b_i$ *for a solitonic string w/ charge* $Q = [C]$

# Geometric Bound via Anomalies

Success in 8d & Attempt in 6d

- ## Geometric Constraints

  - 1-form gauge sector of an F-theory vacuum
    - encoded in the **MW torsion**: $\Gamma = \mathrm{MW}(Y_d)_{\mathrm{Tors}} = \mathbb{Z}_n \times \mathbb{Z}_m \;\; (n \leq m)$ ⟶ constraints on $(n, m)$?
  - Previously known **geometric constraints** for elliptic Calabi-Yau $d$-folds

| $n$ | 1 | 2 | 3 | 1 | 2 | 4 |
|---|---|---|---|---|---|---|
| $m$ | $1,\ldots,6$ | $2,4$ | $3$ | $7,8$ | $6$ | $4$ |

$\mathcal{T}$ (for $d \geq 3$) — columns $n = 1, 2, 3$ with $m = 1,\ldots,6$; $2,4$; $3$

$\mathcal{T}^*$ (for $d=2$) — all columns

"**non-minimal**" fiber at codim-2 (codim-1) would arise for $\Gamma$ beyond $\mathcal{T}$ (resp., $\mathcal{T}^*$)

[Hajouji, Oehlmann '19], [Dierigl, Heckman '20]

  - For $d=2$: classification of the MW groups realizes all torsions in $\mathcal{T}^*$ [Miranda, Persson '89]

- ## EFT Constraints

**8d N=1 EFTs** [Cvetic, Dierigl, Lin, Zhang '20]

- anomaly $\sum_i \frac{n_i - 1}{2n_i} s_i^2 \, k_i \equiv 0 \pmod 1$
- gauge group rank can only be 18, 10, or 2 [Montero, Vafa '20]
- w/ rk=18 (as in 8d F-theory): consistent $\Gamma$s not beyond $\mathcal{T}^*$

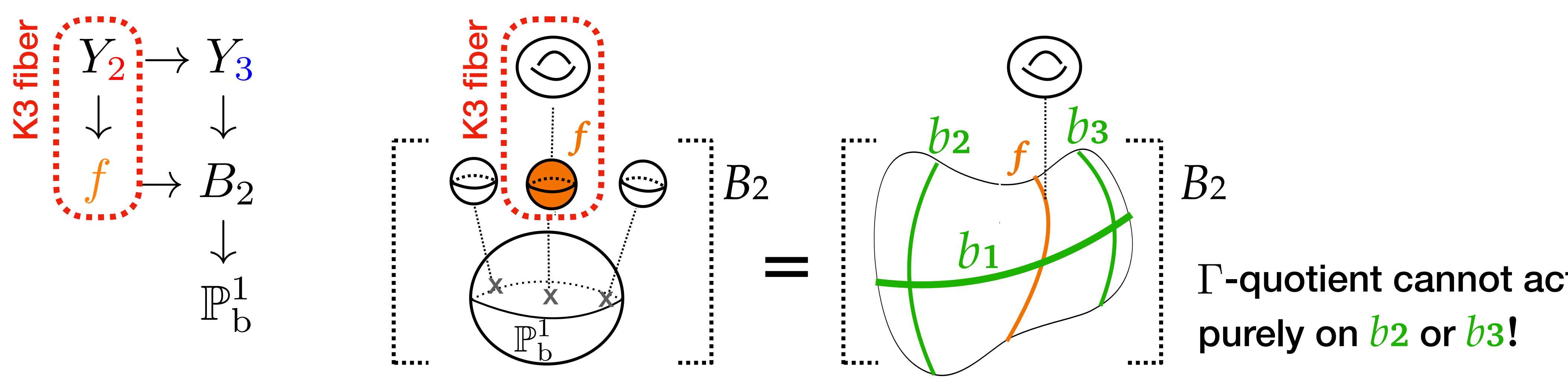
**6d N=(1,0) EFTs**

- anomaly not constraining $\Gamma$ as much
- severe(r) constraints on $\Gamma$ still expected in 6d F-theory! *why*?

# Geometric Bound via Heterotic String

Success in 6d F-theory with $n_T > 0$

- **Insight from Heterotic String/Curve** [S.-J.L., Oehlmann '22]
  - **Claim:** 1-form gauge sector $\Gamma$ of 6d F-theory w/ $n_T > 0$ can only take the form:
    $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_m \in \mathcal{T} \subsetneq \mathcal{T}^*$, *i.e.*, $(n,m)$=(1,1), …, (1,6); (2,2), (2,4); (3,3).
  - **Heterotic Insight:** sever global structure bound for 8d F-theory persist in 6d F-theory!
  - **CY 3-folds as a Nested $T^2$-/K3-fibration** (cf.) ubiquity of nesting [Anderson, Gao, Gray, S.-J.L. '17]
    - $n_T > 0 \iff$ 2-fold base $B_2$ is $\mathbb{P}^1$-fibered
    - (tensionful) heterotic string as D3 on a $\mathbb{P}^1$-fiber (cf.) tensionless at infinite distance [S.-J.L., Lerche, Weigand '18]




  - **MW inclusion:** $\mathrm{MW}(Y_3) \subseteq \mathrm{MW}(Y_2) \Rightarrow \Gamma = \mathrm{MW}(Y_3)_{\mathrm{Tors}} \subseteq \mathrm{MW}(Y_2)_{\mathrm{Tors}} \in \mathcal{T}^*$
    - outliers in $\mathcal{T}^* \setminus \mathcal{T}$ can be ruled out (next slide): $(n,m) \neq$ (1,7), (1,8); (2,6); (4,4)

# Ruling out the Outliers in $\mathcal{T}^* - \mathcal{T}$

Geometry and Physics

- **Geometric Viewpoint**
  - The K3 surfaces w/ $\mathrm{MW}(Y_2) \in \mathcal{T}^* - \mathcal{T}$ are extremal [Miranda, Persson '89]
    - **no complex structure deformations** for a (non-trivial) fibration.

- **Physical Viewpoint**
  - Each outlier leads to a **unique gauge group** (rank-18 non-abelian) [Miranda, Persson '89]
    - e.g., for $\Gamma = \mathbb{Z}_7$: $G = SU(7)^3/\mathbb{Z}_7$ w/ the embedding **s**$=(1,2,3)$ (cf.) [Cvetic et al. '20]
  - Charged matter would render the **7-brane configuration inconsistent**
    - e.g., for $\Gamma = \mathbb{Z}_7$:

1. **Matter** must be charged under **multiple SU(7)s** in a special manner (for $\mathbb{Z}_7$-neutralness).
2. **Matter multiplicities** are subject to a certain divisibility condition for **anomaly cancellation**
3. The divisibility criterion is only fulfilled if **each brane locus has a positive genus**.
4. The genus constraint leads to $7(b_1 + b_2 + b_3) > 12\bar{K}_{B_2} = [\Delta]$ ⟶ *contradiction*!

# Models w/o Tensor Multiplets

## E-String Transitions and Validity of the Bound

- **Claim** [S.-J.L., Oehlmann '22]
  - 1-form gauge sector $\Gamma$ of 6d F-theory w/ $n_T = 0$ (i.e. $B_2 = \mathbb{P}^2$) can only take the form:
    $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_m \in \mathcal{T} \subsetneq \mathcal{T}^*$, *i.e.*, $(n,m)$=(1,1), …, (1,6); (2,2), (2,4); (3,3).

- **Sketch of Derivation** — E-string transition & Tuning

$$\Gamma = \mathrm{MW}(Y_3)_{\mathrm{Tors}} \subseteq \mathrm{MW}(Y_3^{(\mathrm{tune})})_{\mathrm{Tors}} = \mathrm{MW}(\hat{Y}_3)_{\mathrm{Tors}} \in \mathcal{T}$$

developing conformal matter ($Y_3 \to Y_3^{(\mathrm{tune})}$)  E-string transition ($Y_3^{(\mathrm{tune})} \to \hat{Y}_3$)

  - **Recall: MW torsion via a specific tuning** [Aspinwall, Morrison '98]

    e.g., $\Gamma = \mathbb{Z}_2$ case: $\begin{cases} f = a_4 - 1/3\, a_2^2 \\ g = 1/27\, a_2(2a_2^2 - 9a_4) \end{cases}$ w/ $a_i \in H^0(\mathbb{P}^2, \mathcal{O}(3i))$

  - **Further tune: develop conformal matter** w/o reducing MW torsion

    e.g., $\Gamma = \mathbb{Z}_2$ case: $\begin{cases} a_2 = z_0^2\, b_2 \\ a_4 = z_1^4\, b_4 \end{cases}$ w/ $b_i \in H^0(\mathbb{P}^2, \mathcal{O}(2i)) \Rightarrow \begin{cases} f = z_1^4\, b_4 - 1/3\, z_0^4\, b_2^2 \\ g = 1/27\, z_0^2(2z_0^4\, b_2^2 - 9z_1^4\, b_4) \end{cases}$

    $\mathrm{ord}(f,g) = (4,6)$ @ [0:0:1] $\in \mathbb{P}^2_{z_0:z_1:z_2}$

  - **E-string transition: gain a tensor multiplet** w/ MW torsion kept intact

# Connection to the Abelian Rank Bound

6d F-theory with $n_T > 0$

- **Heterotic Insight — Global Structure Bound**
  - If $n_T > 0$: global structure of non-abelian sector visible to (bounded by) heterotic string
  - If $n_T = 0$: the same heterotic bound applicable via E-string transition

- **Heterotic Insight — Abelian Rank Bound**
  - If $n_T > 0$: abelian gauge sector visible to heterotic string [S.-J.L., Weigand '19]
    - U(1) loci hit the $\mathbb{P}^1$-fiber of $B_2$! [S.-J.L., Regalado, Weigand '18]
    - unitarity [Kim, Shiu, Vafa '19] on the heterotic string: $N_{U(1)} \leq c_L(f) = 20 \;\; \text{for } Q = f$

$$c_L(Q) = 3\,(Q \cdot Q) - 9\,(K_{B_2} \cdot Q) + 2$$

  - New insight from MW inclusion [S.-J.L., Oehlmann '22]
    - $N_{U(1)} = \mathrm{rk}(\mathrm{MW}(Y_3)) \leq \mathrm{rk}(\mathrm{MW}(Y_2)) \leq 20 - 2 = 18$
  - If $n_T = 0$: no control of the MW rank under the prereq tuning (for E-string transition)

(cf.) U(1)s still visible to the string with $Q = L$ (line in $\mathbb{P}^2$) [S.-J.L., Weigand '19]

- conservative bound is $c_L(L) = 32$ but the generic, stronger bound is conjectured to work:

$Q = L$ string on $\mathbb{P}^2$ ←···→ $Q = f + h$ string ("H + E" string) on $dP_1$ - overcounting!

# Going Beyond the Global Structure Bound

## Geometry and Physics

- **Recall: F-theory on a Weierstrass model** $Y_d = \{y^2 = x^3 + f_4\, x + g_6\}$
  - Geometric constraints on $\Gamma = \mathrm{MW}(Y_d)_{\mathrm{Tors}} = \mathbb{Z}_n \times \mathbb{Z}_m$ bound the global structure

| $n$ | 1 | 2 | 3 | 1 | 2 | 4 |
|---|---|---|---|---|---|---|
| $m$ | $1,\ldots,6$ | $2,4$ | $3$ | $7,8$ | $6$ | $4$ |

$\mathcal{T}$ (for $d \geq 3$) — columns $n=1,2,3$ (first three); $\mathcal{T}^*$ (for $d=2$) — all columns

"**non-minimal**" fiber at codim-2 (codim-1) would arise for $\Gamma$ beyond $\mathcal{T}$ (resp., $\mathcal{T}^*$)

[Hajouji, Oehlmann '19], [Dierigl, Heckman '20]

  - 7-brane algebra on a base divisor via codim-1 fiber types, i.e., via $\mathrm{ord}(f, g, \Delta)$

| Algebra $G$ | $\mathrm{ord}(f)$ | $\mathrm{ord}(g)$ | $\mathrm{ord}(\Delta)$ |
|---|---|---|---|
| $A_N$ | $0$ | $0$ | $N+1$ |
| $D_N$ | $2$ | $3$ | $N+2$ |
| $E_6$ | $\geq 3$ | $4$ | $8$ |
| $E_7$ | $3$ | $\geq 5$ | $9$ |
| $E_8$ | $\geq 4$ | $5$ | $10$ |
| non-minimal | $\geq 4$ | $\geq 6$ | $\geq 12$ |

◆ Minimal Kodaira fibers: Lie algebra G at **finite** distance

◆ Non-minimal fibers: potentially at **infinite distance**

- analysis via zoom-in on the brane collision (base blowups)

**(cf.)** codim-2 $\mathrm{ord}(f, g, \Delta) \geq (8, 12, 24)$

$\Leftrightarrow$ codim-1 $\mathrm{ord}(f, g, \Delta) \geq (4, 6, 12)$

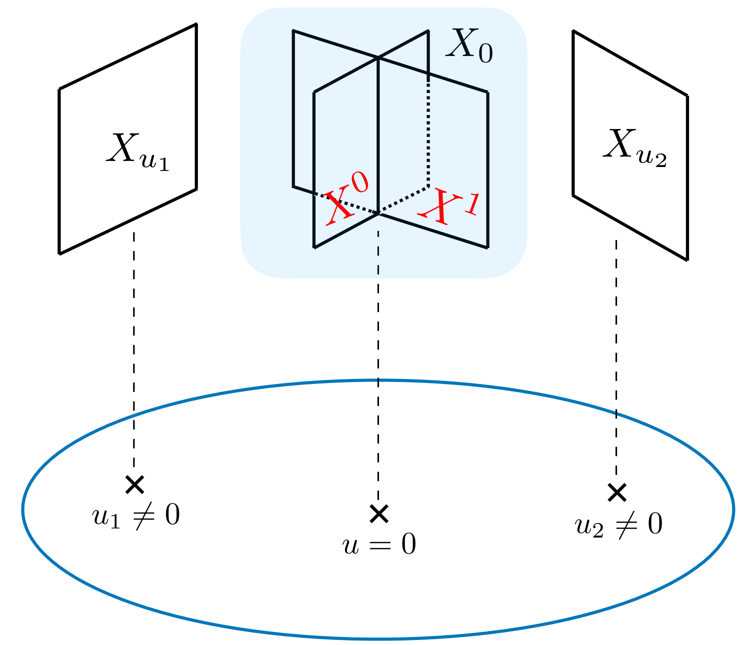
**Focus today**: **non-minimal brane stacks (codim-1)** — intuitions from **8d** F-theory (elliptic **K3**)

(cf.) novelties in 6d/CY3s [Alvarez-Garcia, S.-J.L., Weigand '23]

# Degeneration of K3

## Kulikov Models and Their Properties

- **Kulikov Models** [Kulikov '77], [Persson '77], [Friedman, Morrison '81]




  - **Degeneration - setup**
    - family of K3s $X_u$ degenerating at $u=0$: $X_0 = \cup X^i$
  - **Kulikov Model - definition and existence**
    - reduced, normal-crossing & trivial canonical bundle
    - achievable via base changes ($u \to u^\kappa$) and blow-ups/downs
  - **Classification of Kulikov Models: Type I (finite distance) vs. II/III (infinite distance)**
    - Type II: $X^i$s form a chain, $X^i \cap X^{i+1}$ are elliptic, and 2 transcendental 2-tori shrink
    - Type III: $X^i \cap X^j$ are rational and 1 transcendental 2-torus shrinks

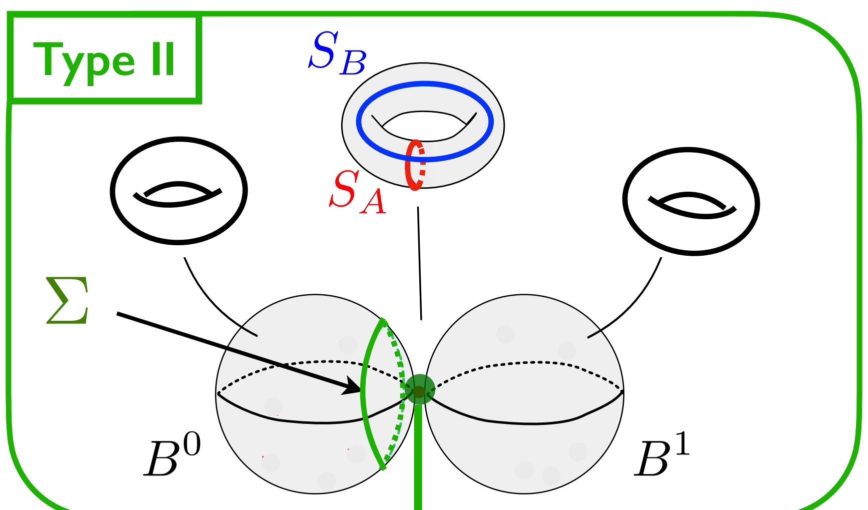
- **8d F-theory w/ a Non-mininal Brane Stack**
  - Aim: modify the degenerate Weierstrass K3s to a Kulikov form, keeping them elliptic
    - **How**: base changes ($u \to u^\kappa$) & blow-ups/downs in the base
    - **Why**: non-minimal fibers improved; infinite-distance nature tested; universal physics manifest

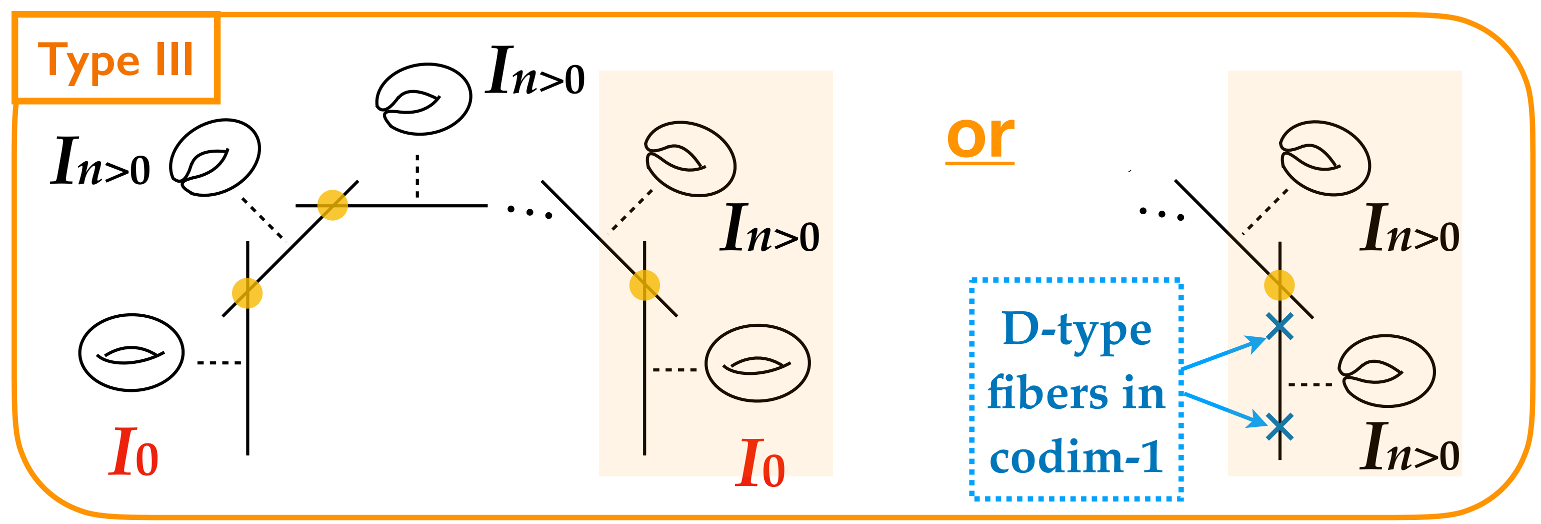
# The Fate of being Non-minimal

## Geometry and Physics of Non-minimal Brane Stacks

[S.-J.L., (Lerche,) Weigand '21] & [Alvarez-Garcia, S.-J.L., Weigand] to appear

- **Arena**: 8d F-theory = Weierstrass model of elliptic K3
- **Limits**: extreme configurations of 7-branes involving non-minimal brane stacks
  = degenerations involving codim-1 non-minimal fibers
- **Birational Analysis**: modification of the degenerations via allowed operations
- **Classification of Limits**: explicit derivation of the Kulikov Types II & III (those limits@infinite distance)




**Vanishing 2-tori** → **Light particle towers** → **Decompactification to 10d**

$\gamma_A = S_A \times \Sigma$
$\gamma_B = S_B \times \Sigma$

- M2 branes on $\gamma_{A,B}$
- F/D-strings on $\Sigma$

- affinization [DeWolfe, Hauer, Iqbal, Zwiebach '98]
(cf.) heterotic [Collazuol, Grana, Herraez '22]
CHL vacua [Cvetic, Dierigl, Lin, Zhang '22]

# Extending the Kodaira-Neron Classification

## Kulikov Types via Vanishing Orders

[S.-J.L., (Lerche,) Weigand '21] & [Alvarez-Garcia, S.-J.L., Weigand] to appear

- **Inclusion of Non-minimal Codim-1 Fibers/Brane Stacks**

$$\mathrm{ord}_{\hat{Y}_0}(f,g,\Delta)\big|_{\mathrm{pt}\,\in\,\mathbb{P}^1} = \begin{cases} (\geq 4, \geq 6, \; 12) & \Rightarrow \text{Type II} \\ (\;\; 4, \;\; 6, > 12) & \Rightarrow \text{Type III} \\ (> 4, > 6, > 12) & \Rightarrow^{*} \text{Type I or Type II} \end{cases}$$

Strictly Non-minimal

"Given a complex structure limit of 8d F-theory, whether at *finite* or *infinite* distance, we can read off its **Kulikov type** and **characteristic physics** just* from the **vanishing order triple!**"

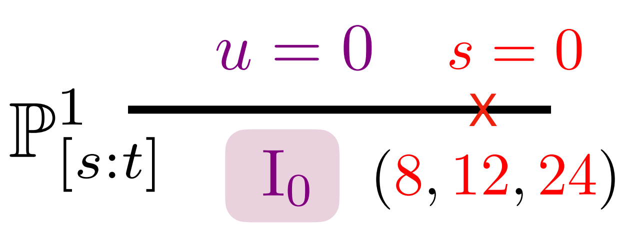
* in case the generic fibers at the end of "(f,g)-scaling-out" procedure are singular, even type III could arise

* in presence of **strictly** non-minimal fibers, we first need to improve them, for which full Weierstrass data should be used

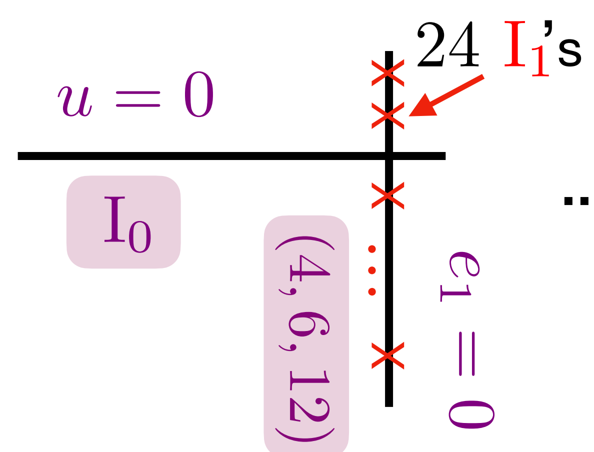
# Non-minimal Brane Stack at Finite Distance

## Example: Type I Model in Disguise

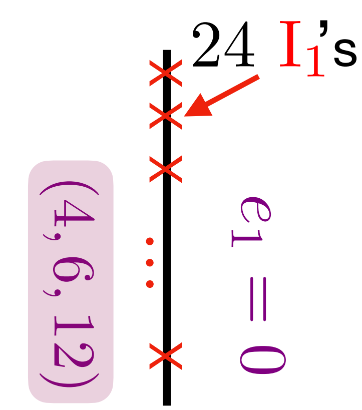
- **Codim-1 Non-minimal Fiber w/ (>4, >6, >12) Vanishing**
  - Non-minimal fibers, if **strict**, may as well arise at **finite** distance
  - An "alerting" example
    - involves a non-minimal fiber w/ $\mathrm{ord}_{\hat{Y}_0}(f, g, \Delta)|_{s=0} = (8, 12, 24)$
    - turns into a **Type I** Kulikov model




**blowup@$u$=$s$=0**

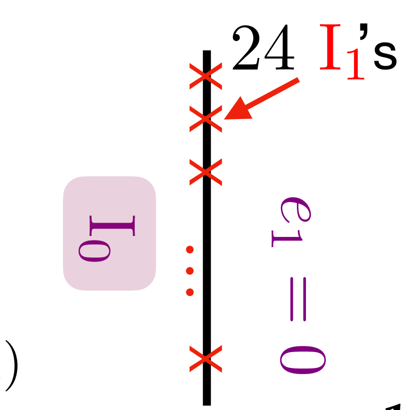



**blowdown** $B^0$




**scale out:**

$(f, g, \Delta) \to$
$(e_1^{-4} f, e_1^{-6} g, e_1^{-12} \Delta)$




$$f = u^8 t^8 + s^8$$

$$g = u^{12} t^{12} + s^{12}$$

$$\Delta = 31 s^{24} + 12 s^{16} t^8 u^8 + 54 s^{12} t^{12} u^{12} + 12 s^8 t^{16} u^{16} + 31 t^{24} u^{24}$$

# *Summary and Discussion*

- **Bounds on gauge sector of 6d F-theory** (the global structure of the non-abelian sector & the rank of the abelian sector) have been (re-)derived via the **heterotic insight**:
  - The 1-form gauge sector & the 0-form U(1) gauge sector are both **visible** to the **heterotic string**!

- **Extreme limits of gauge sector of 6d F-theory**, potentially sitting at infinite distance, have been classified and analyzed:
  - Non-minimal brane stacks may sit either at **infinite** distance (**Type II** or **III**; **decompactifications**) or at **finite** distance (**Type I**; **standard gauge enhancements**)

- **The derived bounds** naturally connect to the **universal limiting behavior of EFTs**:
  - The **heterotic string** is **the** string whose "presence" & "criticality" are inferred at infinite distance
  - Violation of the global structure bound results in decompactification at infinite distance

- For a 6d N=(1,0) EFT, one may generally argue via charge completeness that a "heterotic" string exists, whose unitarity [Kim, Shiu, Vafa '19] leads to an abelian rank bound [S.-J.L., Weigand '19]

- It is desirable to understand/interpret/derive bounds also on other physical quantities

*Thank you*